Naive Bayes Classification

Machine Learning and Deep Learning

Aniello Panariello, Matteo Boschini, Lorenzo Bonicelli, Angelo Porrello, Simone Calderara

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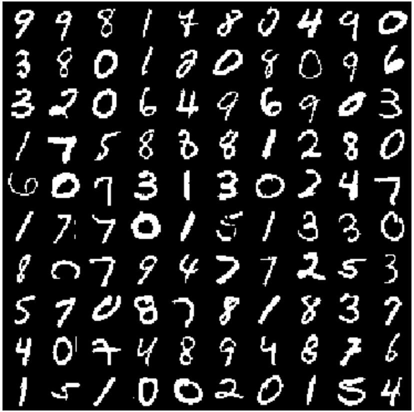
University of Modena and Reggio Emilia

MNIST digits classification

Today we meet MNIST for the first time:



- 70000 images of handwritten images;
- available grayscale, we will use a binarized version;
- the task is to classify each image into the right digit.

Supervised learning setting

We are given a training set $\{X_i, Y_i\}_{i=1}^{n}$, with $X_i \in \mathbb{R}^m$ and $Y_i \in \mathbb{R}$ for each $i = 1, \ldots, n$.

- n is the number of training images;
- each image $X_i = \{x_i^{(1)}, \ldots, x_i^{(m)}\}$ is a vector of m pixels;
- each label Y_i is just a number among $\{1, \ldots, d\}$.

Classification using bayes rule

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- $P(Y)$ is the prior probability of the class Y;
- $P(X/Y)$ is the likelyhood of the image X under the model Y;
- $P(X)$ is the probability of the image X under the whole dataset. Does not influence the argmax so we can drop it.

Prior $P(Y)$

Finding the prior of a class Y is easy.
Simply count the number of examples of each class and divide by the number of total examples.

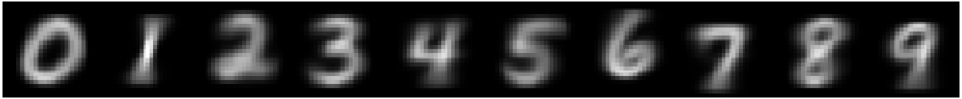
$$P(Y = c) = \frac{\sum_{i=1}^{n} \mathbf{1}\{Y_i == c\}}{n}$$

Likelyhood $P(X/Y)$

Naive assumption: all pixels are independent given the class. The probability of an image is the product of the probability of every single pixel.

$$P(X_i/Y_c) = \prod_{j=1}^{m} P(x_i^{(j)}/Y_c)$$

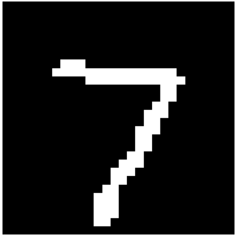
During training, we need to model this for each possible class:



Likelyhood $P(X/Y)$

During testing, the likelihood of an image under a class is built by taking the class model (the one built before, during training) and:

- for each active pixel, account for the probability of being active given class;
- for each zero pixel, account for one minus the probability of being active given class;



Inference

Once we can model $P(Y/X)$ for each class, we can classify an image by choosing the class that maximizes it:

$$\tilde{Y} = \arg\max_{Y} P(Y/X) = P(X/Y)P(Y)$$

Use logarithms!

All those products of probabilities can result in nothing. Use log-probabilities instead!

$$\log P(X_i / Y_c) = \sum_{j=1}^{m} \log P(x_i^{(j)} / Y_c)$$

$$\log P(Y / X) = \log P(X / Y) + \log P(Y)$$

Algorithm 1 pseudocode for training

for $j = 1$ to d **do**
  compute the class prior $P(Y_j)$
  compute a model for $P(X/Y_j)$
end for

Algorithm 2 pseudocode for inference

for $i = 1$ to n **do**
  for $j = 1$ to d **do**
    compute $\log P(X_i, Y_j)$
    compute $\log P(Y_j/X_i) = \log P(X_i, Y_j) + \log P(Y_j)$
  end for
  $Y_i = \arg\max_{Y_j} \log P(Y_j/X_i)$
end for