Hashing with SL_2

Hash Functions as Monoid Homomorphisms

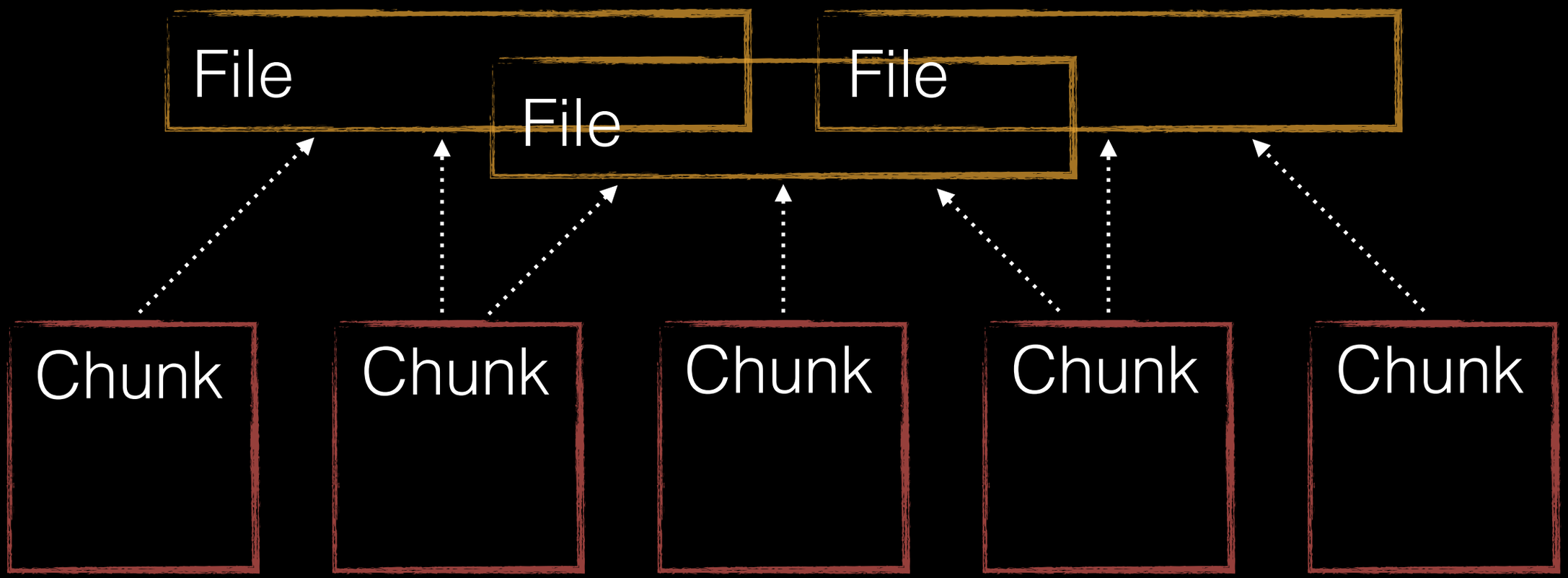
Based on the eponymous paper by Jean-Pierre Tillich and Gilles Zemor

Agenda

- Motivation
- Map-Reduce with Monoids
- Monoid Homomorphisms
- Hash Functions
- Cayley Graphs
- Practical Application and Results

Motivation

Distributed File Storage



Distributed File Storage

Chunk

- size
- hash

0100100101001001
0111010111010101

n:m

File

- name
- ...

Distributed File Storage

Chunk

- size
- hash

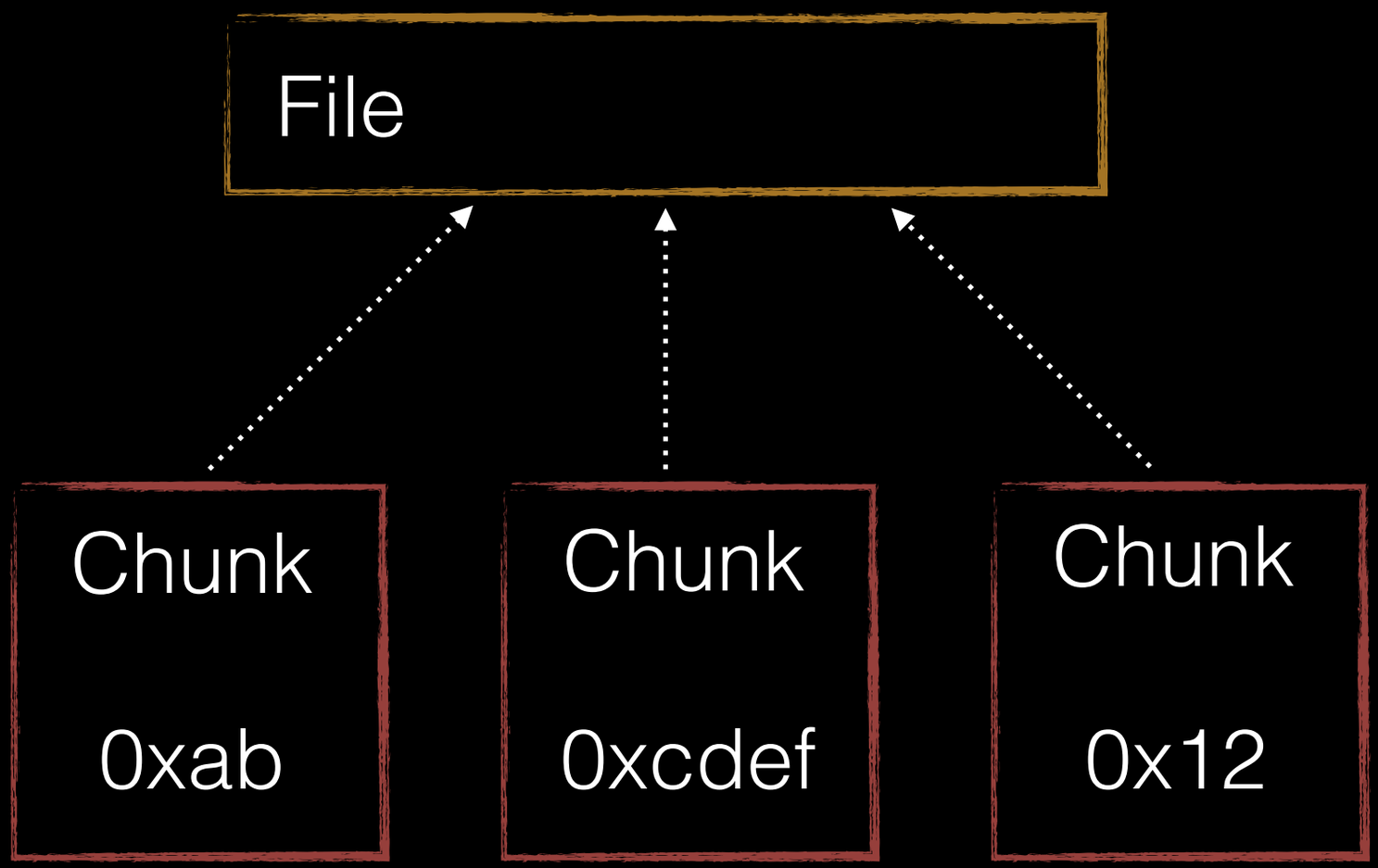
0100100101001001
0111010111010101

n:m

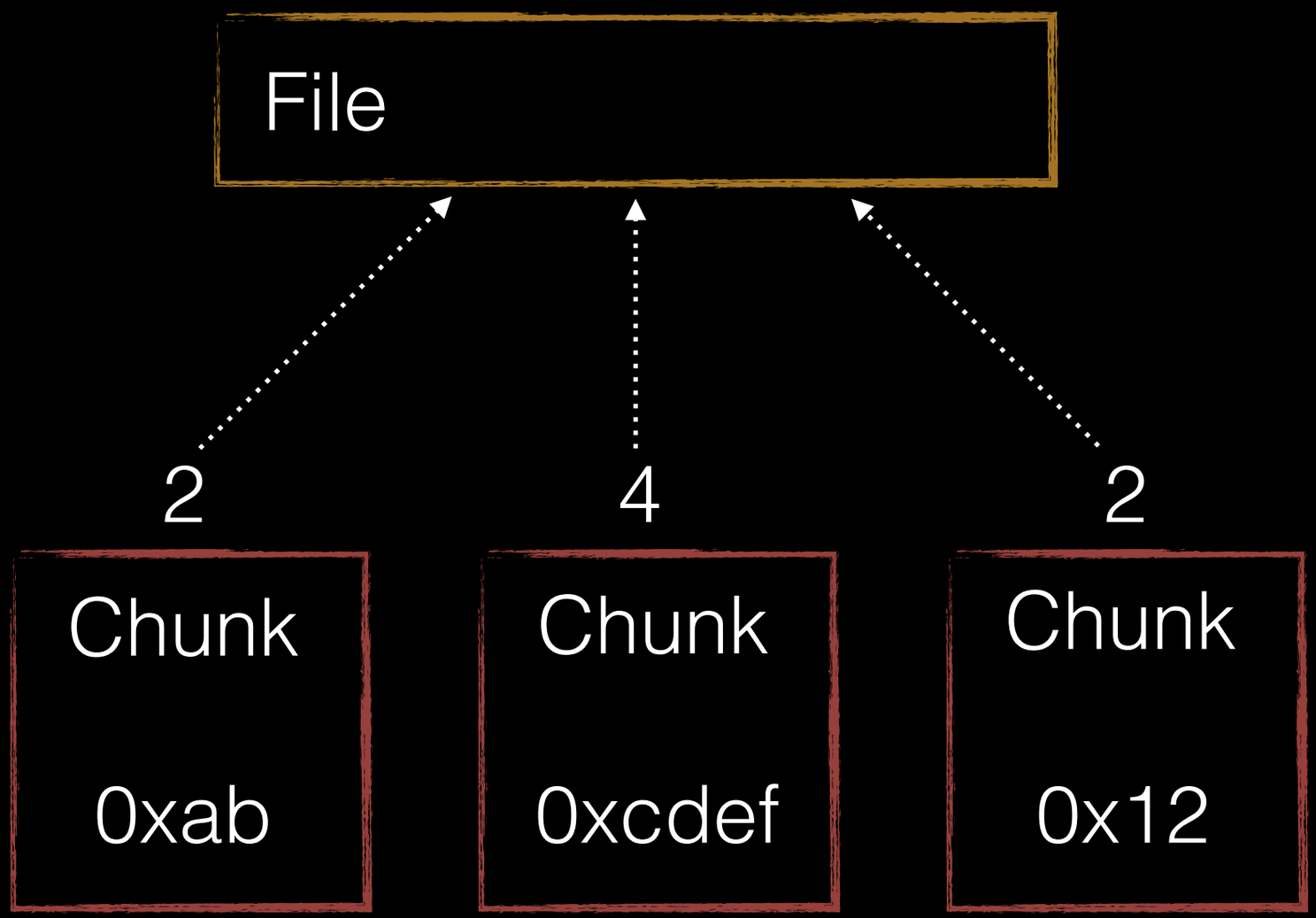
File

- name
- size?

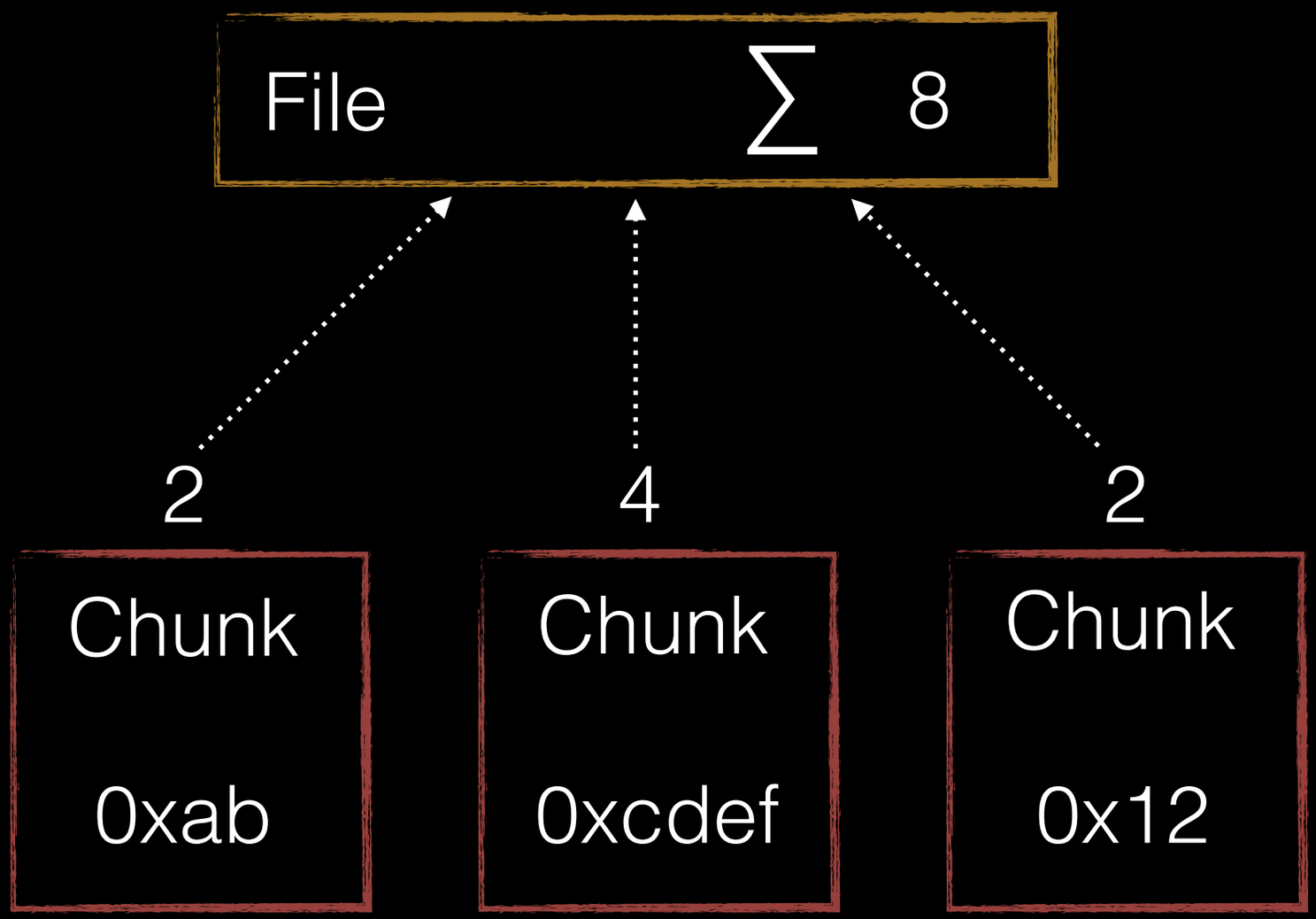
Distributed File Storage



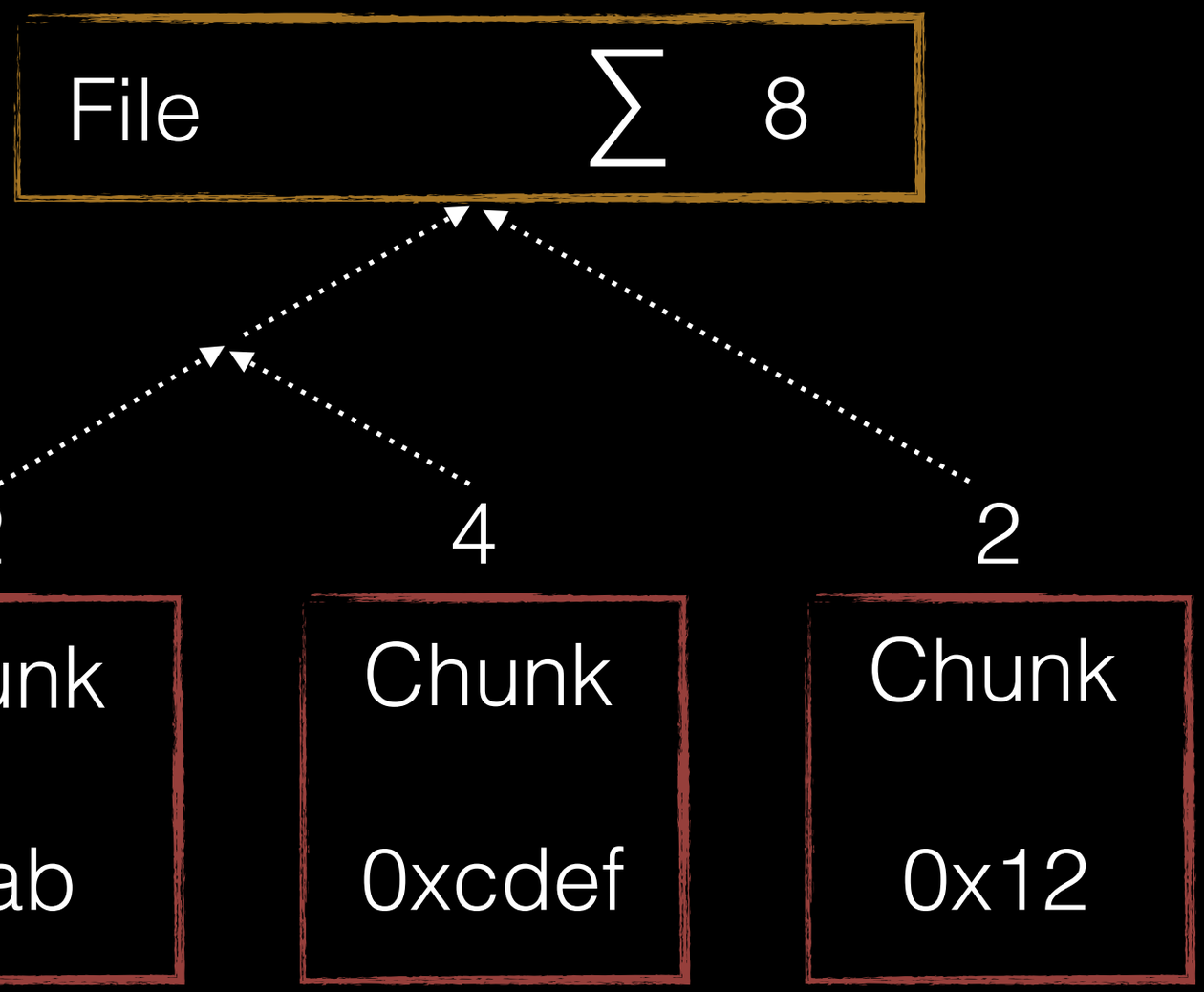
Distributed File Storage



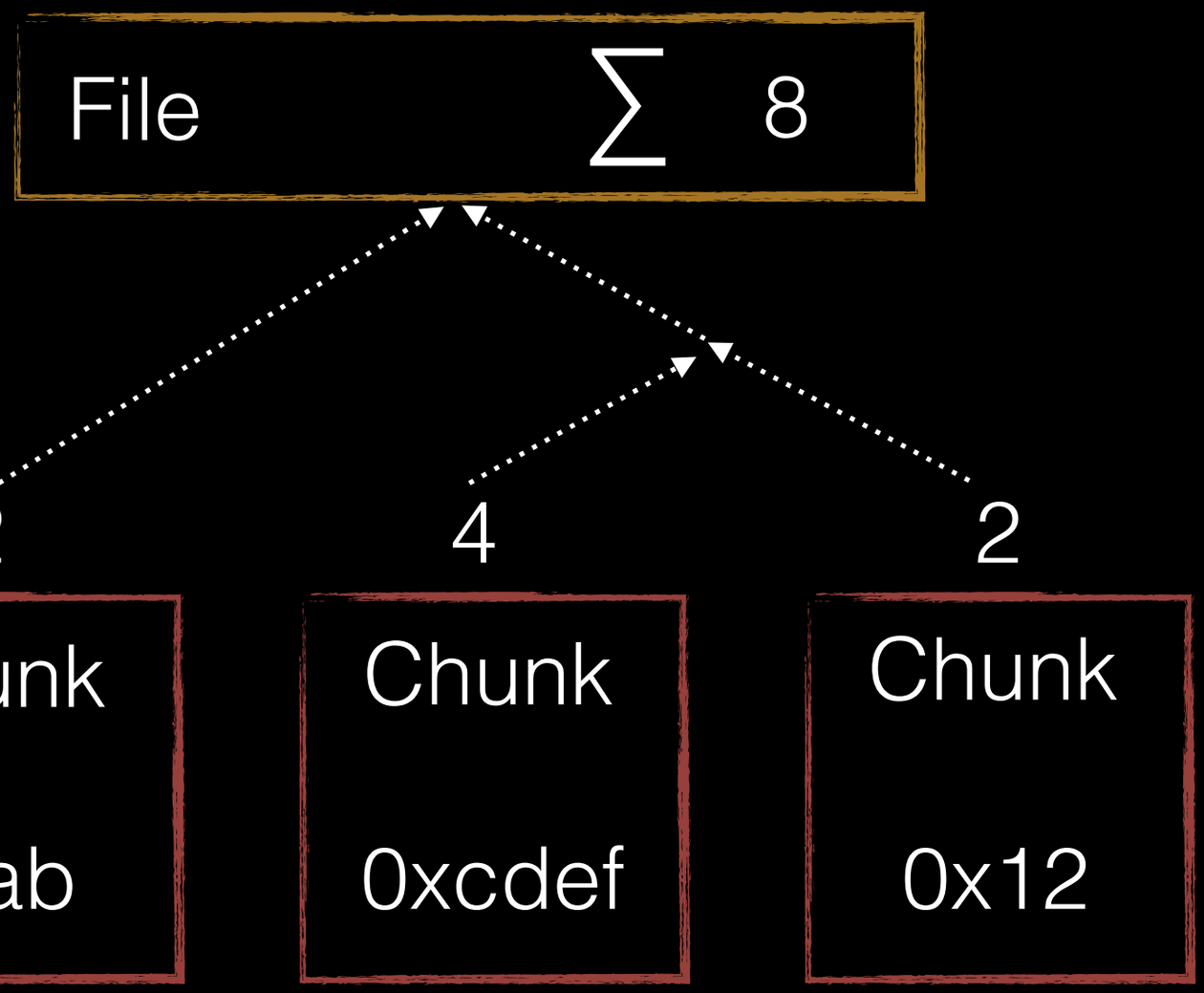
Distributed File Storage



Distributed File Storage



Distributed File Storage



Distributed File Storage

File $\sum$ 0

Map-Reduce with Monoids

Monoids

Type `M`

Identity element

```
zero: M
```

Binary operation

```
|+|: (M, M) => M
```

Monoid Laws

Left identity

```
forAll { (x: M) => zero |+| x == x }
```

Right identity

```
forAll { (x: M) => x |+| zero == x }
```

Associativity

```
forAll { (x: M, y: M, z: M) =>
x |+| (y |+| z) == (x |+| y) |+| z }
```

Concatenation

Type `Chunk = Array[Byte]`

Identity element

```
def zero: Chunk = Array()
```

Binary operation

```
def |+|: (Chunk, Chunk) => Chunk = _++_
```

Concatenation

Left identity

```
forAll { (x: Chunk) => zero |+| x == x }
```

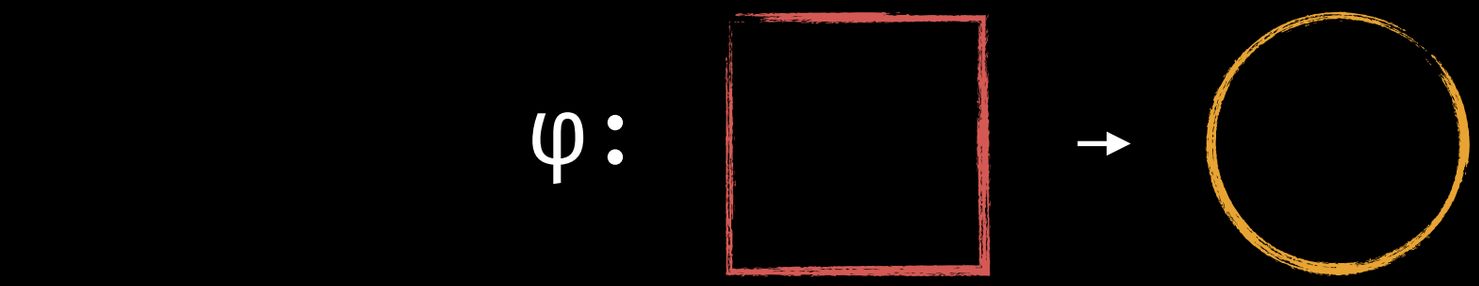
Right identity

```
forAll { (x: Chunk) => x |+| zero == x }
```

Associativity

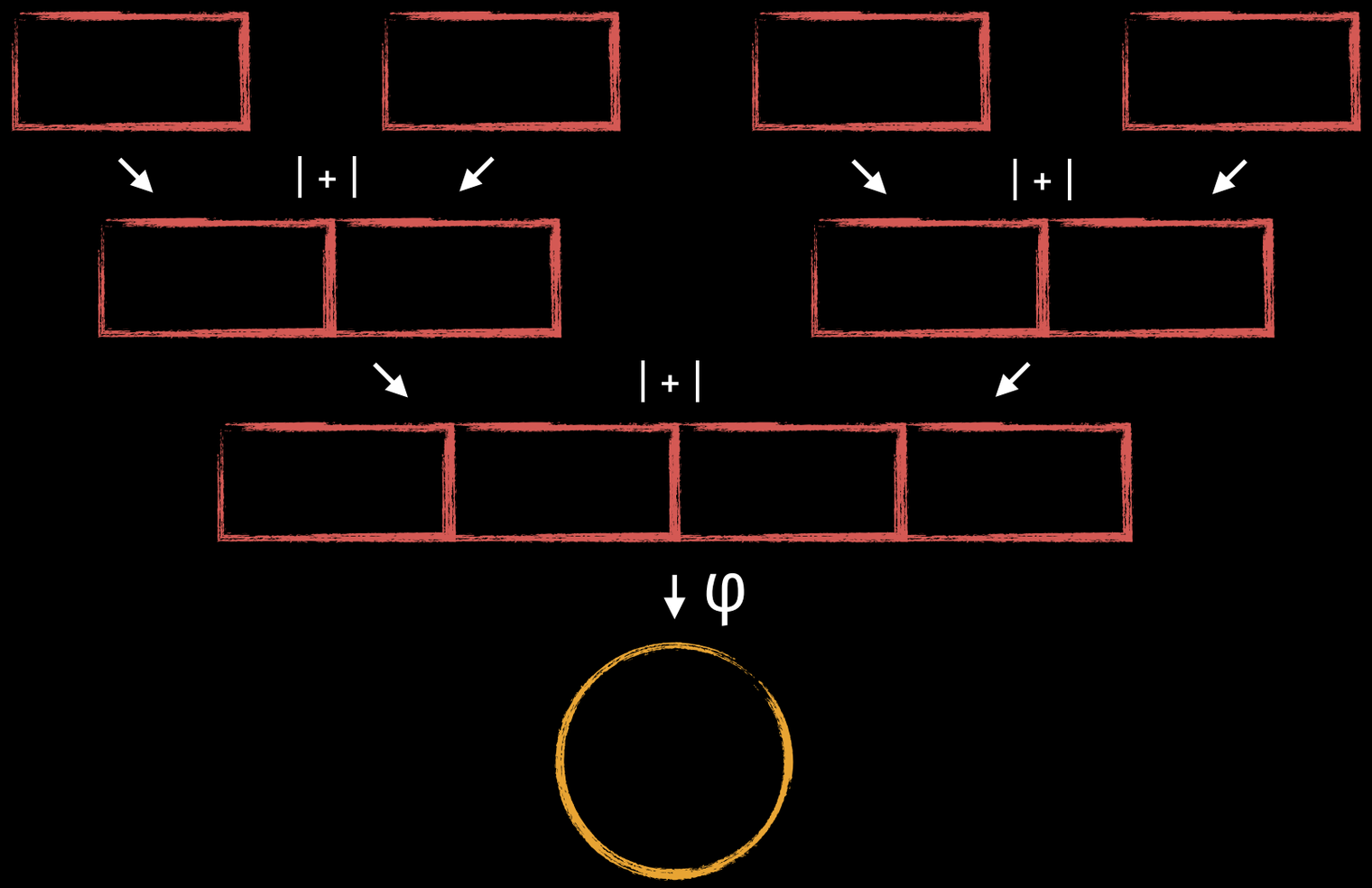
```
forAll { (x: Chunk, y: Chunk, z: Chunk) =>
  x |+| (y |+| z) == (x |+| y) |+| z }
```

Calculating size



Projection `size: Chunk => Size`

Calculating size



Calculating size

Type `Size = Long`

Identity element

```
def zero: Size = 0
```

Binary operation

```
def |+|: (Size, Size) => Size = _+_
```

Calculating size

Left identity

```
forAll { (x: Size) => zero |+| x == x }
```

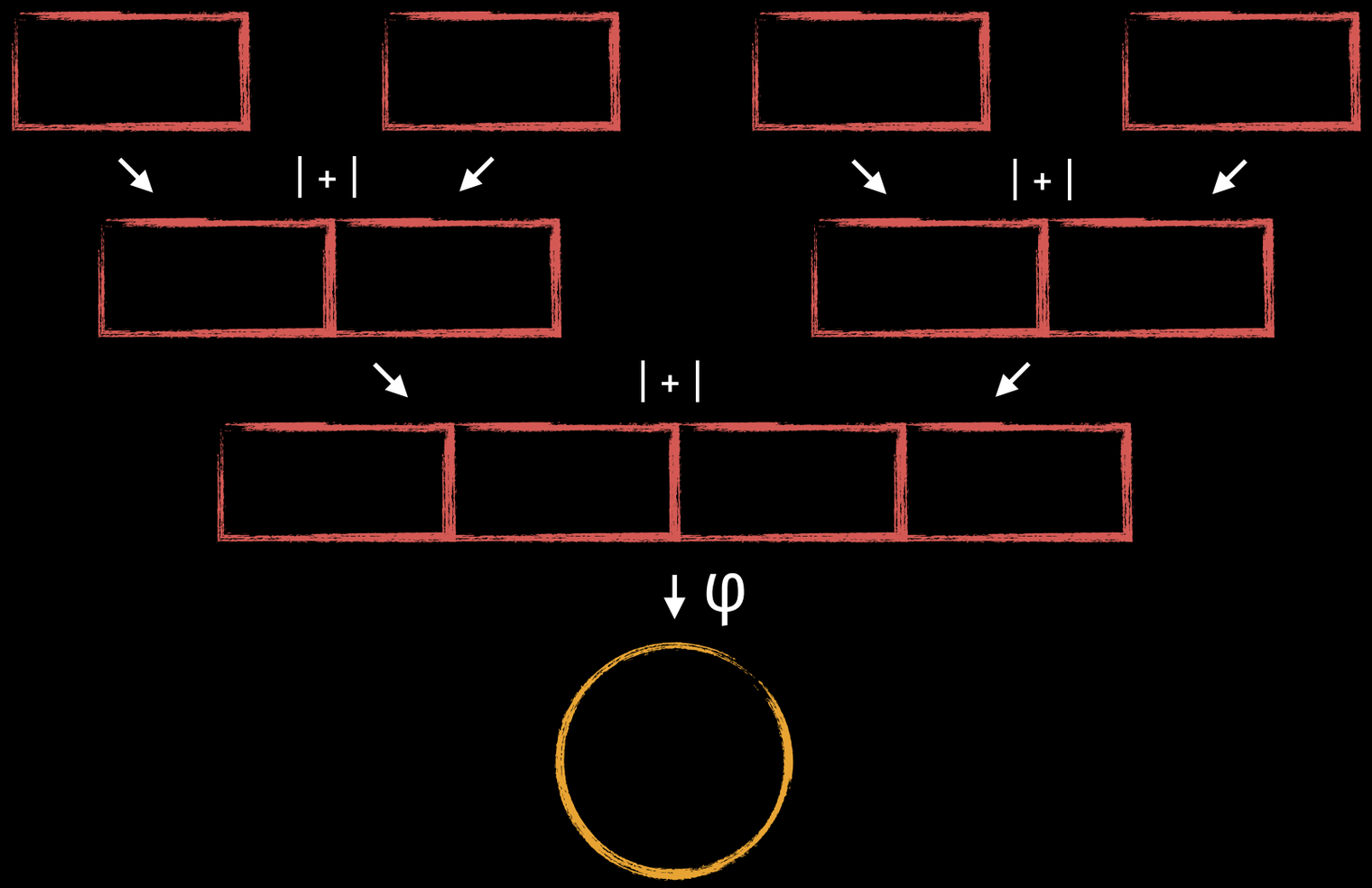
Right identity

```
forAll { (x: Size) => x |+| zero == x }
```

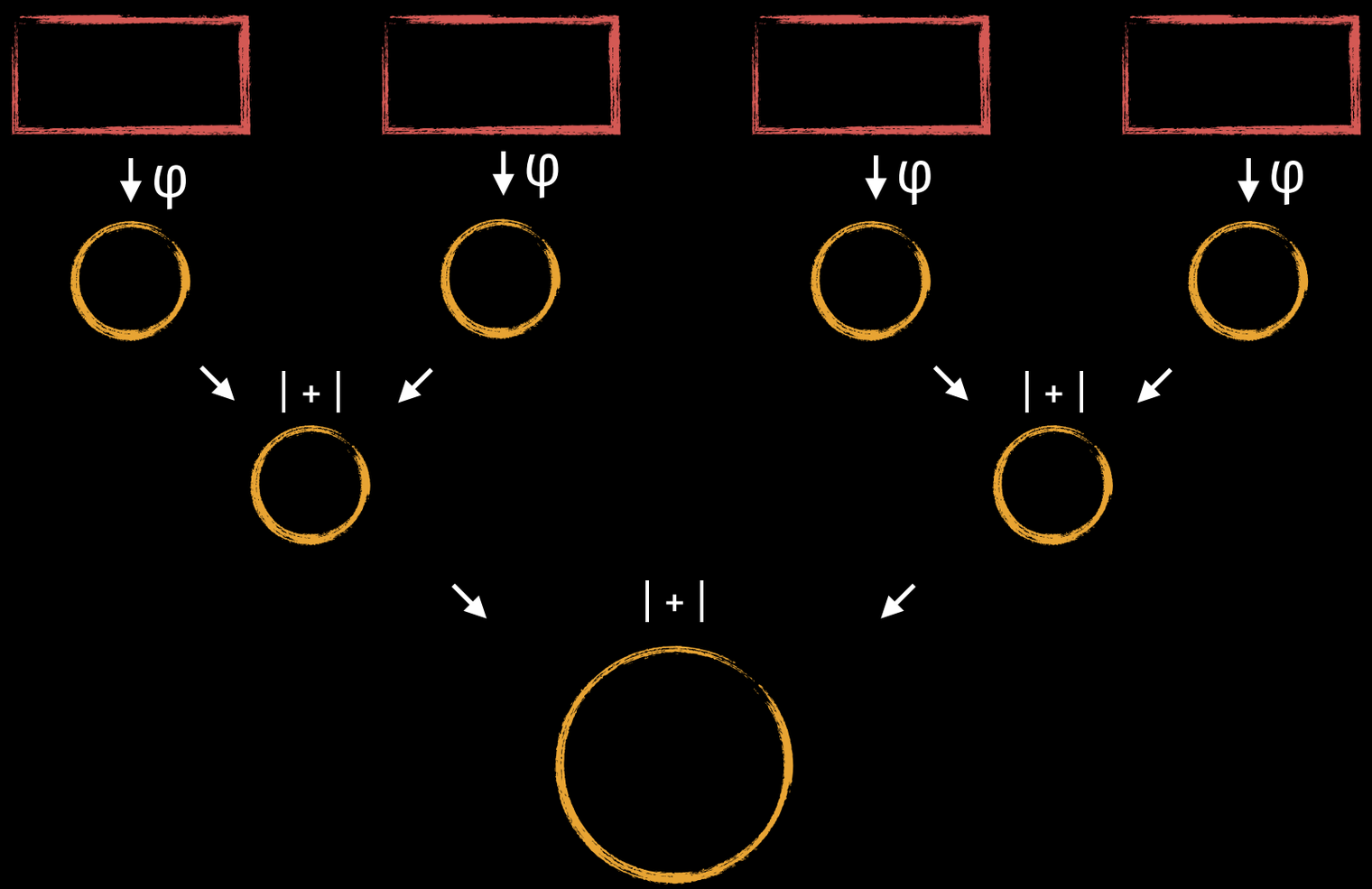
Associativity

```
forAll { (x: Size, y: Size, z: Size) =>
  x |+| (y |+| z) == (x |+| y) |+| z }
```

Calculating size



Calculating size



Monoid Homomorphisms

Monoid Homomorphisms

Monoid $(A, |+|_A, zero_A)$

Monoid $(B, |+|_B, zero_B)$

Morphism $\varphi: A \Rightarrow B$

Monoid Homomorphisms

Preserves identity

$$\varphi(\texttt{zero}_A) == \texttt{zero}_B$$

Preserves associativity

```
forAll { (x: A, y: A) =>

φ(x |+|A y) == φ(x) |+|B φ(y) }
```

Monoid Homomorphisms

Monoid `(Array[Byte], _++_, Array())`

Monoid `(Long, _+_, 0)`

Morphism `size: Array[Byte] => Long`

Monoid Homomorphisms

Preserves identity

```
size(Array()) == 0
```

Preserves associativity

```
  forAll { (x: Chunk, y: Chunk) =>
size(x ++ y) == size(x) + size(y) }
```

Formulating the **reduction step** as a **lawful monoidal concatenation**, we can **ensure the correctness** of the map-reduce computation regardless of grouping and empty elements.

Formulating the **mapping step** as a **lawful monoid homomorphism**, we can **ensure** that while introducing optimisations we **maintain that correctness**.

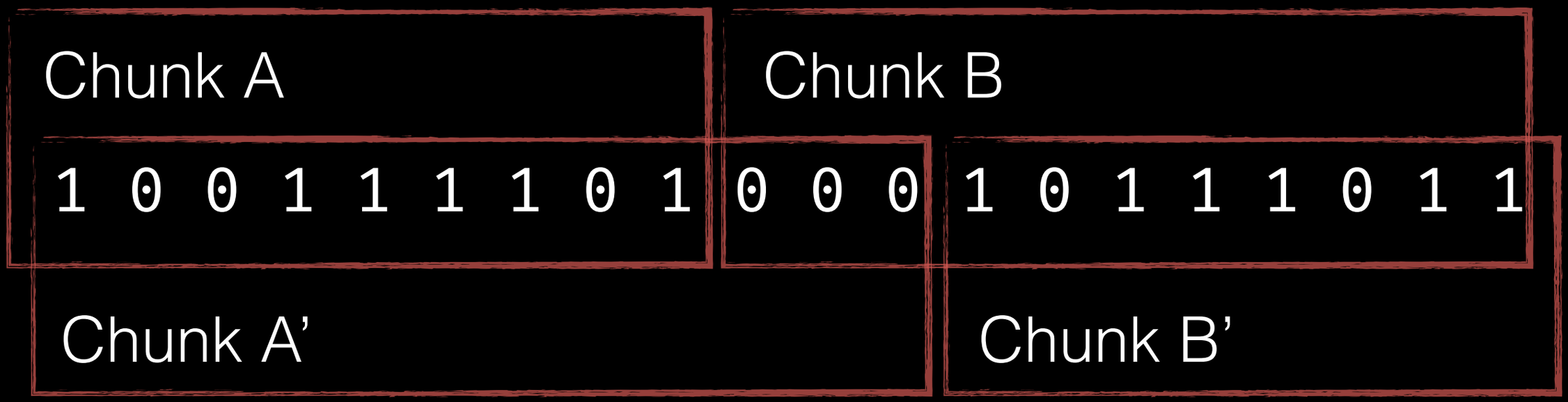
Chunk Sequence Equivalence

1 0 0 1 1 1 1 0 1 0 0 0 1 0 1 1 1 0 1 1

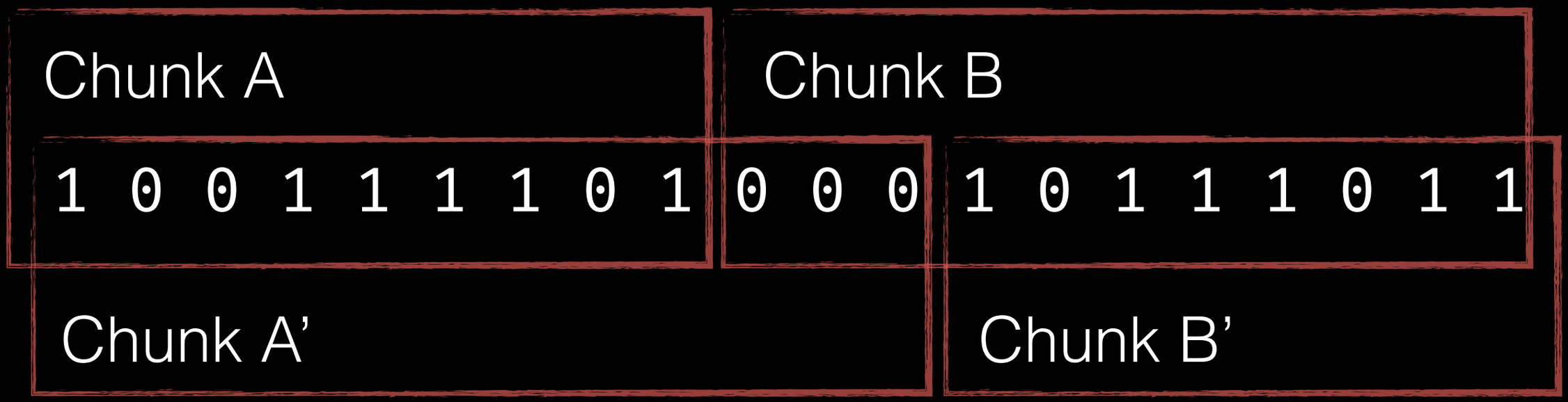
Chunk Sequence Equivalence

Chunk A	Chunk B
1 0 0 1 1 1 1 0 1	0 0 0 1 0 1 1 1 0 1 1

Chunk Sequence Equivalence

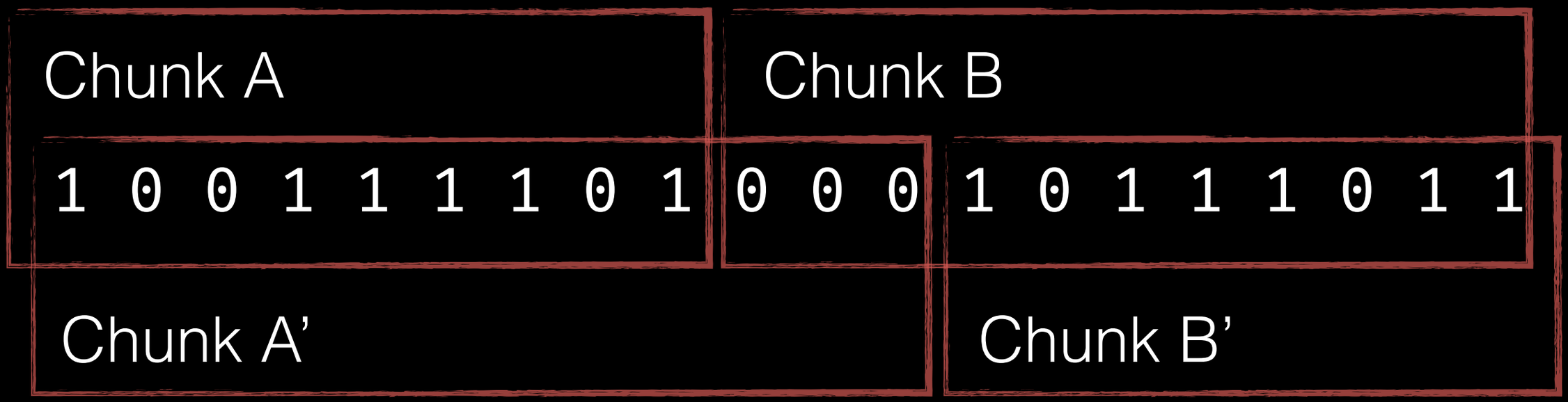


Chunk Sequence Equivalence



$$A \neq A'$$
$$B \neq B'$$
$$A + B = A' + B'$$

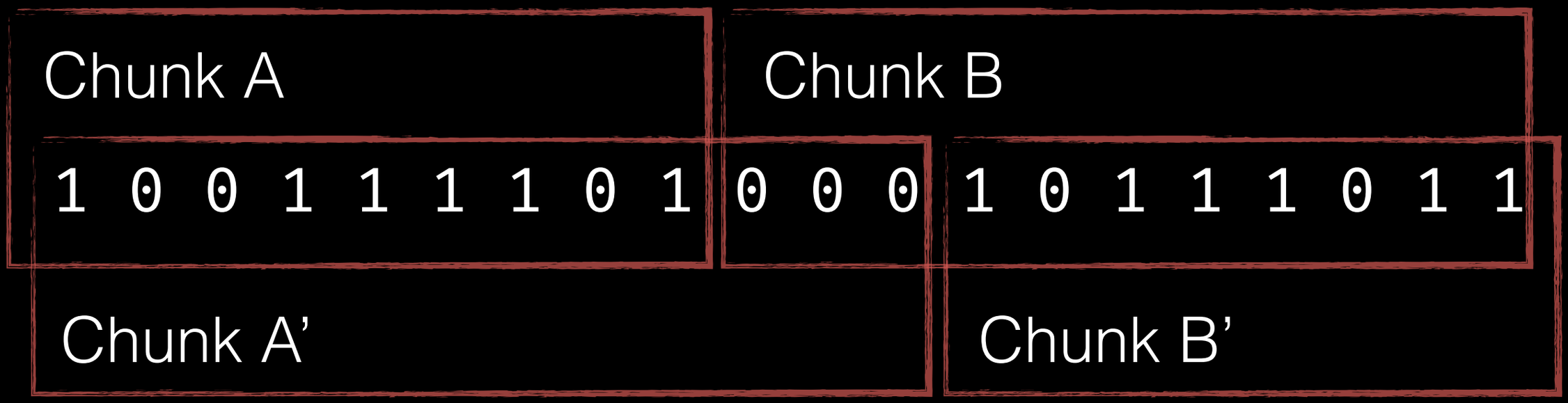
Chunk Sequence Equivalence



$$A \neq A'$$
$$B \neq B'$$
$$A + B = A' + B'$$

$$h(A) \neq h(A')$$
$$h(B) \neq h(B')$$
$$h(A + B) = h(A' + B')$$

Chunk Sequence Equivalence



$$A \neq A'$$
$$B \neq B'$$
$$A + B = A' + B'$$

$$h(A) \neq h(A')$$
$$h(B) \neq h(B')$$
$$h(A + B) = h(A' + B')$$

$$\mathbf{h(A) + h(B) = h(A') + h(B')}$$

Hash Functions

SYSV $ sum

```
$ echo -n | sum -s

0 0

$ echo -n c | sum -s

99 1

$ echo -n ab | sum -s

195 1

$ echo -n abc | sum -s

294 1
```

Cryptographic Hash Functions

1. Collision resistance: it is hard to find inputs $x_1 \neq x_2$ such that $H(x_1) = H(x_2)$.

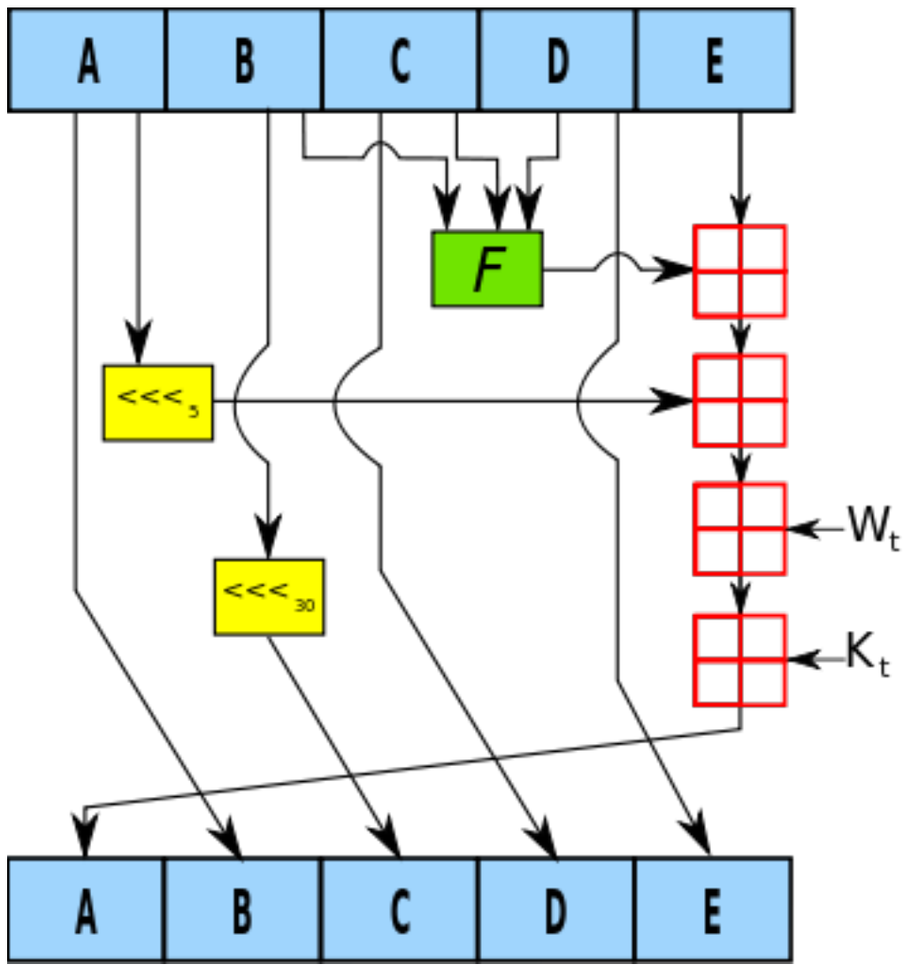
2. Second preimage resistance: given input x_1, it is hard to find another input $x_2 \neq x_1$ such that $H(x_1) = H(x_2)$.

3. Preimage resistance: given output y, it is hard to find input x such that $H(x) = y$.

SHA1 $ shasum

```
$ echo -n | shasum
da39a3ee5e6b4b0d3255bfef95601890afd80709  -
$ echo -n c | shasum
84a516841ba77a5b4648de2cd0dfcb30ea46dbb4  -
$ echo -n ab | shasum
da23614e02469a0d7c7bd1bdab5c9c474b1904dc  -
$ echo -n abc | shasum
a9993e364706816aba3e25717850c26c9cd0d89d  -
```

SHA1 $ shasum



“The properties we
require are mainly a certain kind of weak
collision-freeness and some limited
“unpredictability.””

–Goldwasser, S. and Bellare, M., Lecture Notes on Cryptography

The **unlawfulness and complexity** of the compression function is the property at the **core of the security** of Merkle–Damgård constructions.

This construction is a **pinnacle of the imperative programming paradigm**. Introducing lots of state, and using low-level bit shuffling to solve a problem.

As **functional** programmers, we know the **value** of a **carefully constructed** solution that is based on **logically-founded laws**.

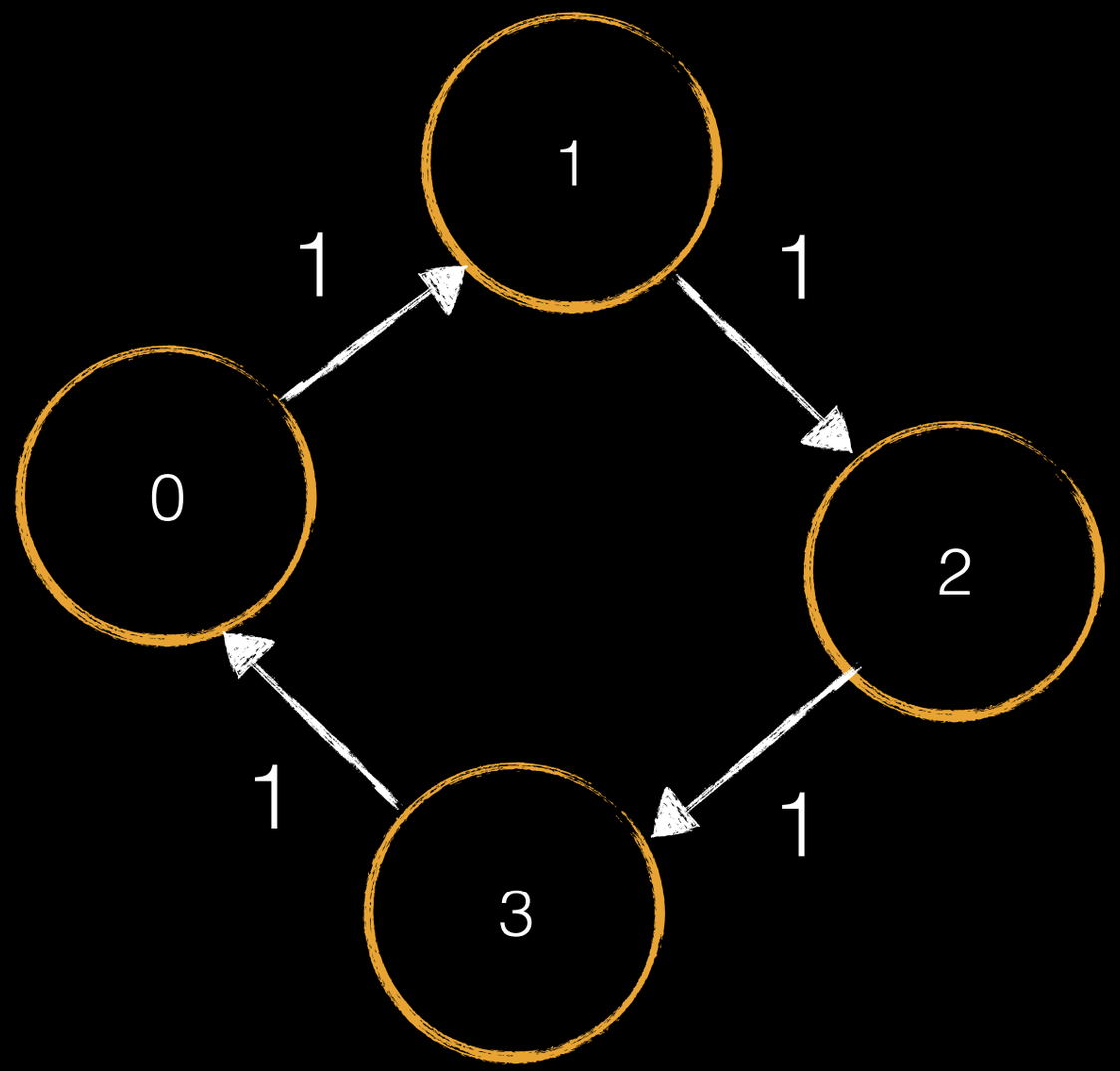
"… the SHA family […] [does] not really use any mathematical ideas to ensure the [desired] properties were met; the main idea was just to 'create a mess' by using complex iterations."

–Vladimir Shpilrain, 2015, Navigating in the Cayley graph of $SL_2(F_p)$ and applications to hashing

Cayley Graphs

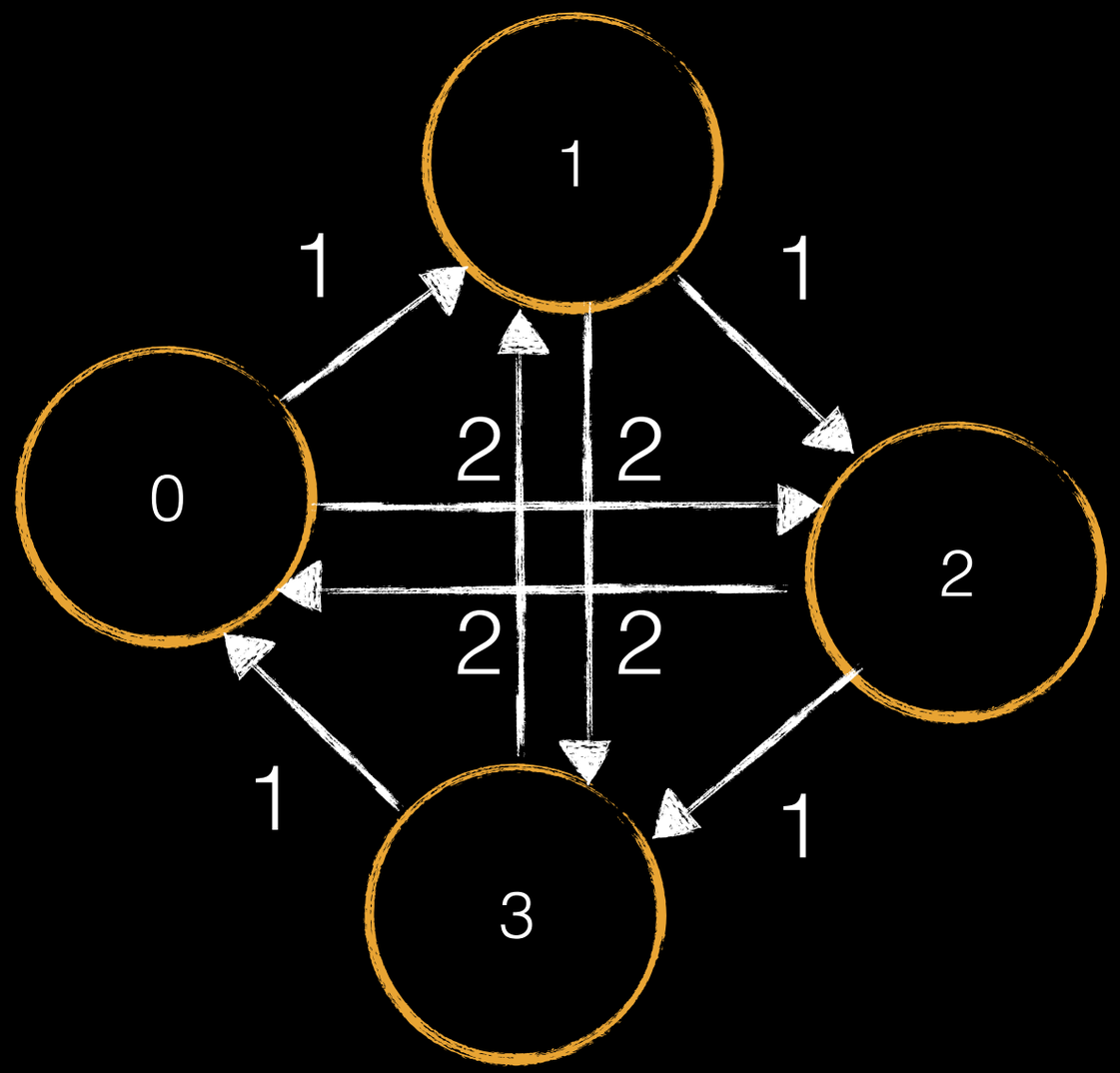
Cayley Graphs

$\Gamma(G = (\mathbb{N}/4, +), S = \{1\})$



Cayley Graphs

$\Gamma(G = (\mathbb{N}/4, +), S = \{1, 2\})$



Cayley Graphs

Associate:

$$\pi : \{0, 1\} \longrightarrow \{A, B\}$$

$$0 \longmapsto A$$

$$1 \longmapsto B$$

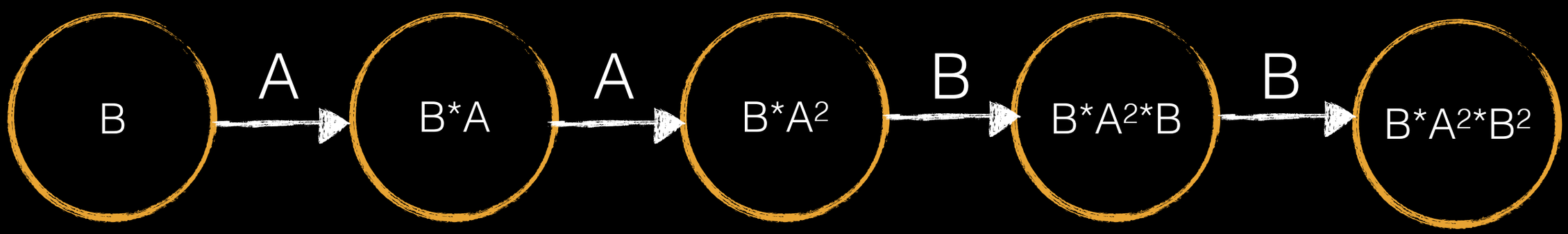
Then the hash code of a binary message $x_1x_2x\ldots x_n$ is the product:

$$\pi(x_1)\ \pi(x_2)\ \ldots\ \pi(x_n)$$

Cayley Graphs

Hashing a binary message represents a walk on the Cayley graph where the endpoint is the hash value.

Message: 10011



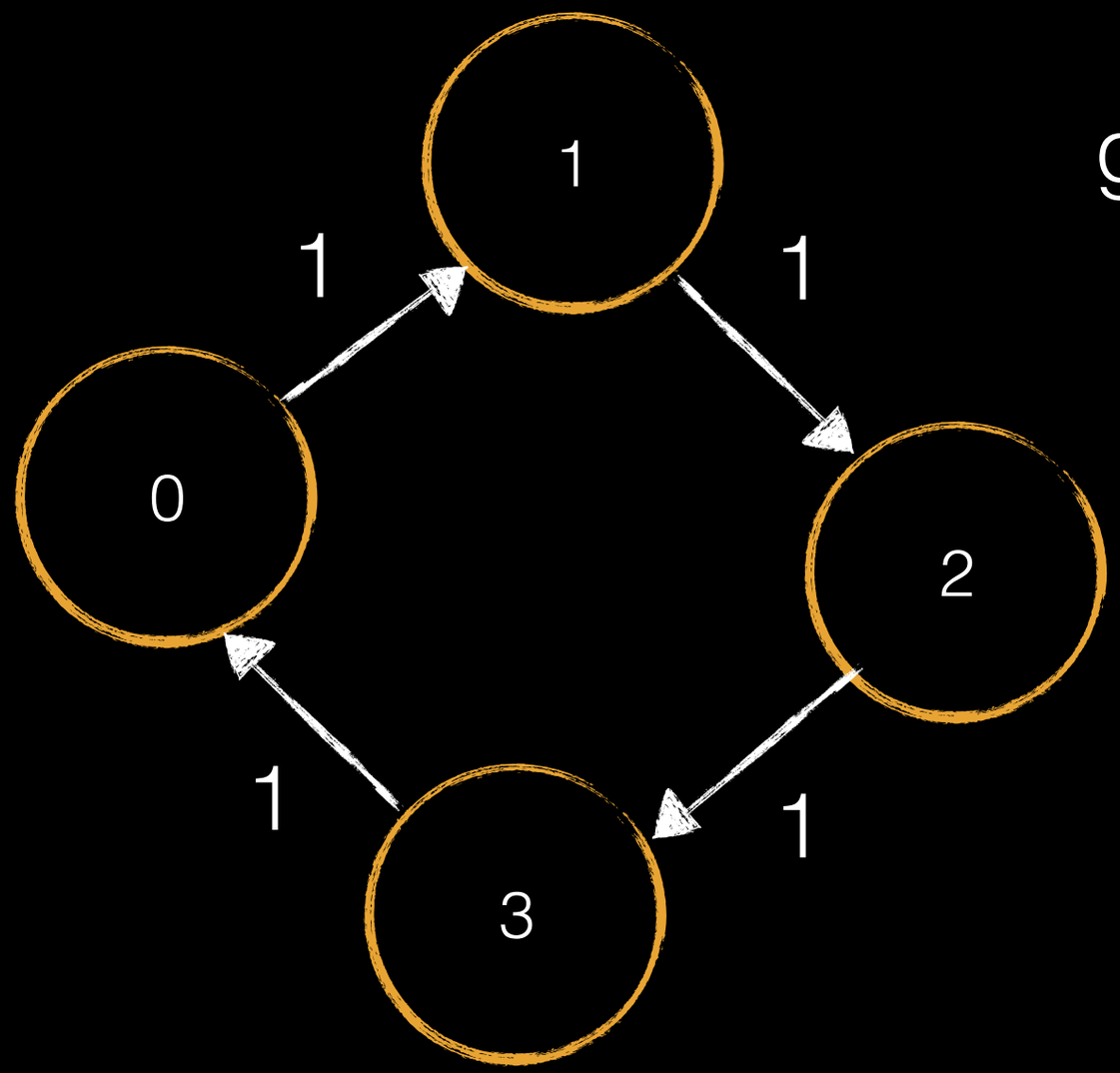
Cayley Graphs

Pick generators A and B of a semigroup S

…so that the Cayley graph generated by A and B has a large girth and therefore there will be no short relations (“collisions”).

Cayley Graphs

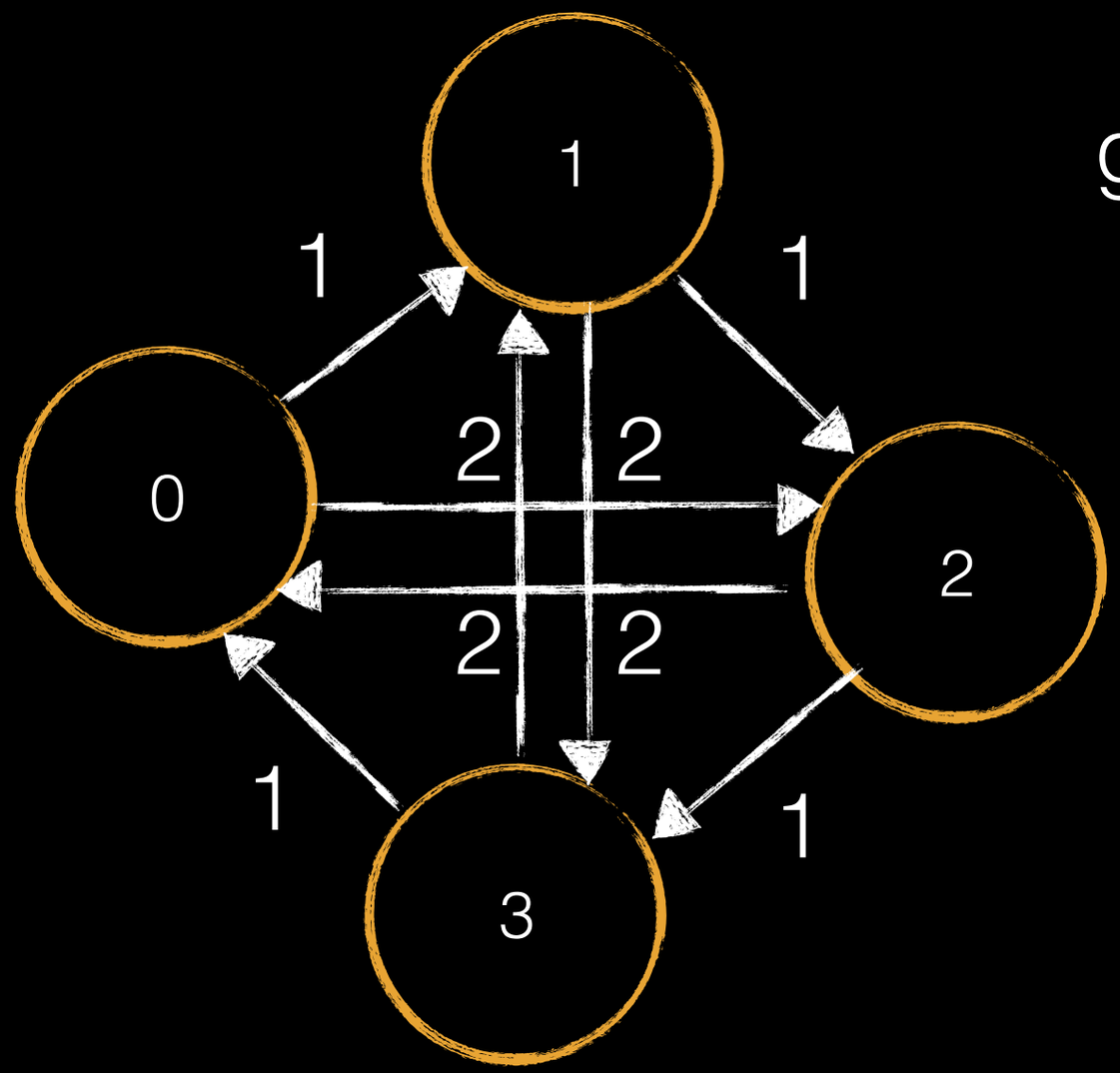
$\Gamma(G = (\mathbb{N}/4, +), S = \{1\})$



girth = 4

Cayley Graphs

$\Gamma(G = (\mathbb{N}/4, +), S = \{1, 2\})$



girth = 2

Cayley Graphs

What do we pick as our semigroup?

Needs to be **non-commutative** (otherwise collisions are trivial to produce).

Needs to be **hard to find inverses** (otherwise collisions are trivial to produce).

Hashing with SL_2

Let's pick a finite binary field (modulo an irreducible polynomial):

$$F_2^n = F_2[X] / P_n(X)$$

Finding the inverse requires finding short factorisations in the finite group. This is know to be difficult for large cardinalities (Pspace-complete).

Finite field arithmetic is already used in real-world applications (CRC, ECC), and has hardware support on modern processors.

Hashing with SL_2

Let's pick the special linear group of matrices over that finite binary field:

$$A = \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} X & X+1 \\ 1 & 1 \end{pmatrix}$$

Matrix multiplication is non-commutative.

Embedding commutativity into non-commutativity has some cool properties to make cryptographic attacks harder.

Practical Application and Results

Demo

Results

- A hash function which is
  - strong
  - monoidal
  - composable
  - parallelisable

Performance

- Currently ~3x slower than OpenSSL SHA-256
- Producing 512 bit digests, so actually 1.5x faster per bit of output ;)
- And we can parallelise!
- Given a threaded runtime across >=4 cores, we can trivially chunk and hash a byte array in parallel, outperforming SHA-256

References

- **Paper**: https://www.rocq.inria.fr/secret/Jean-Pierre.Tillich/publications/HashingSL2.pdf

- **Scala** bindings: https://github.com/srijs/hwsl2-scala

- **Haskell** bindings: https://github.com/srijs/hwsl2-haskell

- Low-level **SIMD** code: https://github.com/srijs/hwsl2-core