Hodge Theory

by Prof. Dr. Philip Engel

notes by Stefan Albrecht

University Bonn – winter term 2023/24

Contents

1 Overview and basic definitions

The aim of Hodge theory is to try to understand non-linear objects (e.g. projective varieties or Kähler manifolds) using linear objects (vector spaces, subspaces, lattices, etc.).

We will move freely between Algebraic Geometry (polynomial functions on $\mathbb{C}^n$, $\mathbb{C}[x_1, \dots, x_n]$) and Complex Geometry (holomorphic functions on $\mathbb{C}^n$ or open subsets $U \subseteq \mathbb{C}^n$).

Definition 1.1. An *affine algebraic variety* is a vanishing locus

$$V(f_1, \dots, f_m) = \{x \in \mathbb{C}^n \mid f_i(x) = 0 \text{ for all } i\}.$$

of some polynomials $f_i \in \mathbb{C}[x_1, \dots, x_n]$.

Example 1.2. $y^2 = x(x-1)(x-2)$ in $\mathbb{C}^2$.

In general, an algebraic variety is covered by affine algebraic varieties, whose transition functions are polynomial maps.

Definition 1.3. $\mathbb{CP}^n = \{\text{lines through the origin in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1} \setminus \{0\}/x \sim \lambda \times x$.

Consider $f_i \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous. Then $f_i(\lambda x) = \lambda^{\deg f_i} f_i(x)$, so it makes sense to talk about zeroes of homogeneous polynomials in $\mathbb{CP}^n$.

Definition 1.4. A *projective variety* is $V(f_1, \dots, f_m) \subseteq \mathbb{CP}^n$, $f_i \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous.

Example 1.5. $V(xy) \subseteq \mathbb{C}^2$ is the union of the two coordinate axes.

Definition 1.6. A *complex manifold* is a topological space X with local homeomorphisms onto open sets in $\mathbb{C}^n$, such that transition functions are holomorphic. In the case of $n = 1$, X is called a *Riemann surface*.

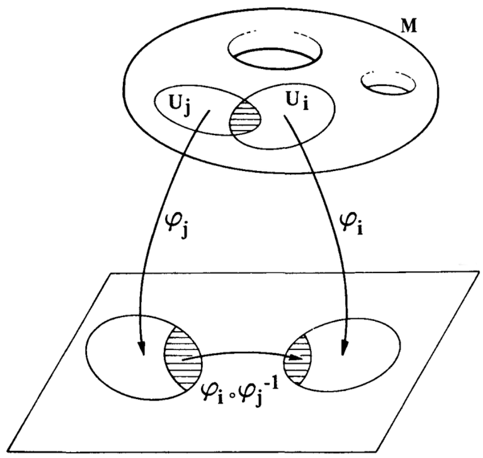


Figure 1: Two charts φ_i, φ_j of a manifold M

Example 1.7. $\mathbb{CP}^1 = \{[1 : y] \mid y \in C\} \cup \{[x : 1] \mid x \in \mathbb{C}\} =: U_1 \cup U_2$, where both factors are clearly isomorphic to $\mathbb{C}$. Now $[1 : y] = [x : 1]$ iff $xy = 1$. Now under the isomorphisms $U_1 \cap U_2$ gets identified with $\mathbb{C}^\times$, and $t \mapsto t^{-1}$ is holomorphic on $\mathbb{C}^\times$. This also shows that $\mathbb{CP}^1$ is homeomorphic to S^2.

2 Riemann surfaces of algebraic curves

2.1 The genus one case: Complex Tori

Example 2.1 (Complex Tori)**.** Consider $\mathbb{C}/\Lambda$ where Λ is a subgroup of $\mathbb{Z}$ isomorphic to $\mathbb{Z}^2$ and discrete, e.g. take $\Lambda = \mathbb{Z}[i]$. Focusing on the fundamental region $[0,1]+[0,1]i$, one sees that $\mathbb{C}/\Lambda$ topologically is a torus. For charts, for a point $z \in \mathbb{C}/\Lambda$ pick a representative in $\mathbb{C}$ with a neighbourhood. The transition maps then work out to be simple translations.

From a different point of view, homogenize the equation $y^2 = x(x-1)(x-\lambda)$, $\lambda \neq 0, 1$ from example 1.2 to $y^2 z = x(x-z)(x-\lambda z)$ to get a projective variety in $\mathbb{CP}^2$, which adds a unique additional point $[0:1:0]$.

Consider the "multiform function" $f(x) = \sqrt{x(x-1)(x-\lambda)}$. This clearly has zeroes at 0, 1 and λ, but its other values are not uniquely specified[1]. Picking one value, say $f(\frac{1}{2})$, also determines the value of f in a neighbourhood of that point, if we want f to be continuous. In fact, if one analytically continues f along the circle $x = \frac{1}{2}e^{i\theta}$, $\theta \in [0, 2\pi]$, we get $f(x) = \frac{1}{\sqrt{2}} e^{i\theta/2} \sqrt{(x-1)(x-\lambda)}$, where the latter square root can be chosen to be well-defined on, say, $|z| < \frac{2}{3}$. Hence $f(e^{2\pi i}x) = -f(x)$, which is a problem. To fix this, Riemann's idea was to enlargen the region of definition to two linked complex planes so one can circle around the origin twice without running into problems. This introduces cuts in the planes where they are connected, but on this object f is a well-defined function. Topologically, a plane with two cuts (one from 0 to 1 and one from λ to ∞) is a open cylinder, and glueing two of these together yields, again, a torus.

In conclusion, we came up with different ways to construct a compact Riemann surface of genus 1: The quotient $\mathbb{C}/\Lambda$ versus the projective variety $y^2 z = x(x-z)(x-\lambda z)$ or the "domain" of the function $\sqrt{x(x-1)(x-\lambda)}$ in the above sense. When are $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $zy^2 = x(x-z)(x-\lambda z)$ the same Riemann surface?

Definition 2.2. An *isomorphism of Riemann surfaces* $f : X \to Y$ is a homeomorphism which is biholomorphic in local charts.

Question: Given a one-dimensional complex torus $\mathbb{C}/\Lambda$, can we find polynomial equations describing the same Riemann surface?

Weierstrass answered this question by building functions x and y on $\mathbb{C}/\Lambda$.

Proposition 2.3. *There does not exist a holomorphic nonconstant function $f : \mathbb{C}/\Lambda \to \mathbb{C}$.*

Proof. Any such f gives $\widetilde{f} : \mathbb{C} \to \mathbb{C}/\Lambda \to \mathbb{C}$ with $\widetilde{f}$ bounded and entire, hence constant. □

Building a meromorphic function on $\mathbb{C}/\Lambda$ is equivalent to finding $f : \mathbb{C} \to \mathbb{CP}^1$ such that $f(x+\lambda) = f(x)$ for $\lambda \in \Lambda$. Define

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

This function converges and is invariant under the action of the lattice. One computes its derivative as $\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$. Note that $\wp$ is even and $\wp'$ is odd. For the series expansion around 0 one gets

$$\wp(z) = \frac{1}{z^2} + c_1 z^2 + c_2 z^4 + \ldots \quad \text{and} \quad \wp'(z) = -2(\frac{1}{z^3} - c_1 z - \ldots)$$

and one can verify $\wp'(z)^2 = 4\wp(z)^3 + g_2 \wp(z) + g_3$ for $g_2 = -20c_1$ and some constant $g_3 \in \mathbb{C}$ (verify using the series expansion that $\wp'^2 - 4\wp^3 - g_2\wp$ is biperiodic and holomorphic).

[1] Assume λ is in a general position

Proposition 2.4. *There exists a polynomial relation* $\wp'(z)^2 = 4\wp(z)^3 + g_2\wp(z) + g_3$ *for some constants* $g_2, g_3 \in \mathbb{C}$. □

Consider the map $\varphi : \mathbb{C}/\Lambda \to \mathbb{CP}^2$, $z \mapsto [\wp(z) : \wp'(z) : 1]$. (For $z = 0$, we get $0 \mapsto [0 : 1 : 0]$.) Now $\operatorname{im} \varphi \subseteq V(x_1^2 x_2 - 4x_0^3 - g_2 x_0 x_2^2 - g_3 x_2^3) =: V(f)$. We claim that φ is injective and surjective on $V(f)$.

Proof. $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is 2 to 1 because $\wp^{-1}(\infty) = 2[0]$ and the multiplicity is the number of inverse images of $\wp$ near ∞. So $\mathbb{C}/\Lambda \to \mathbb{CP}^1$ is the quotient map by the $\mathbb{Z}^2$-action $z \mapsto -z$. Assume $\wp(z) = \wp(w)$ and $\wp'(z) = \wp'(w)$ for some $z \neq w$. By the above, $z = -w$ and $\wp'(z) = 0$. If $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$, since $\wp'$ is odd we have $\wp'(\frac{1}{2}v_1) = \wp'(\frac{1}{2}v_2) = \wp'(\frac{1}{2}(v_1 + v_2)) = 0$. Since $\wp'^{-1}(\infty) = 3[0]$, by the same argument as before 0 has at most 3 preimages, hence $z \in \{\frac{1}{2}v_1, \frac{1}{2}v_2, \frac{1}{2}(v_1 + v_2)\}$, and hence $z = -z = w$. This proves that φ is injective.

For surjectivity, we use the open mapping theorem: If $f : C \to D$ is a holomorphic map of Riemann surfaces, then $\operatorname{im} f$ is open. Hence $\operatorname{im} \varphi$ is open. Since $\mathbb{C}/\Lambda$ is compact, we also have that $\operatorname{im} \varphi$ is closed. Thus φ is surjective. □

This answers the question how to go from a lattice to a cubic. Now let us think about the reverse direction.

Definition 2.5. A holomorphic 1-form ω on a Riemann surface Σ is a compatible collection of expressions $\{f(z)dz\}$ f holomorphic, ranging over the charts of Σ.

Spelt out, this means whenever we have charts $\varphi_1 : U_1 \to \mathbb{C}$ and $\varphi_2 : U_2 \to \mathbb{C}$ with expressions $f_1(z)dz$ and $f_2(z)dz$ on U_1 and U_2, respectively, with transition map $w = \varphi_2 \circ \varphi_1$, we want $f_2(w(z))d(w(z)) = f_1(z)dz$, i.e. $f_1(z) = f_2(w(z))w'(z)$.

Now define a holomorphic 1-form on $V(y^2 - x(x-1)(x-\lambda))$ by $\omega = \frac{dx}{y}$. When $x \neq 0, 1, \lambda, \infty$, then x is a local coordinate. Then $y \neq 0$ and everything is fine. If $x = 0$, then $w = \sqrt{x}$ is a local holomorphic coordinate. Then $x = w^2$ and $y = w\sqrt{(w^2-1)(w^2-\lambda)}$ as well as $dx = 2w dw$. Together,

$$\frac{dx}{y} = \frac{2}{\sqrt{(w^2-1)(w^2-\lambda)}} dw,$$

where the fraction is a holomorphic function of w near 0. The same arguments work for $x = 1$ and $x = \lambda$. At ∞, we had $w = x^{-\frac{1}{2}}$ as a holomorphic function and similar calculations show that everything works out. ω is nowhere vanishing: In a local chart z, $\omega = f(z)dz$, then $f(z) \neq 0$.

Proposition 2.6. *Any holomorphic* 1*-form on a Riemann surface* Σ *is closed as a* $\mathbb{C}$*-valued differentiable* 1*-form.*

There is a map $d : \{\text{diff. } p\text{-forms}\} \to \{\text{diff. } p+1\text{-forms}\}$ given by

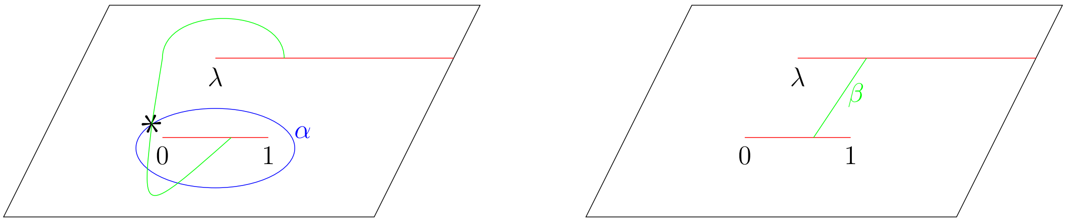
$$f dx_1 \wedge \cdots \wedge dx_p \mapsto \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p.$$

Write $\omega = f(z)dz = f(x+iy)(dx + idy)$. Then $d\omega$ computes as

$$d\omega = i\frac{\partial f}{\partial x} dx \wedge dy + \frac{\partial f}{\partial y} dy \wedge dx = \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy = 0$$

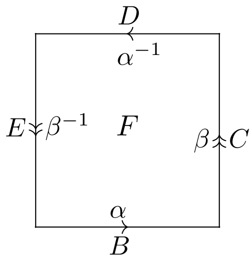
Consider $A(p) = \int_*^p \omega$ as a "function" on $\Sigma = V(y^2 - x(x-1)(x-\lambda)$. $A(p)$ depends on the chosen path. If γ_1, γ_2 are two homotopic paths from $*$ to p, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ by Stokes theorem. Hence A depends only on the homotopy class of the chosen path. If γ_1, γ_2 are two homotopy classes of paths from

$*$ to p, then $\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\gamma_2^{-1} \circ \gamma_1} \omega$ and $\gamma_2^{-1} \circ \gamma_1 \in \pi_1(\Sigma, *) \cong \mathbb{Z}^2$, since Σ is a torus. Set $v_1 = \int_\alpha \omega$, $v_2 = \int_\beta \omega$, where α, β are generators of $\pi_1(\Sigma, *)$, as indicated in the picture:



Then A is a single valued function with target $\mathbb{C}/\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$. v_1 and v_2 are called the "Abelian" integrals. We can explicitly write $v_1 = 2\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$ and $v_2 = 2\int_0^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$. Claim: $v_1, v_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

Proof. Cut along α, β. You get a square F, denote its sides as in the figure.



Then $-i\int_\Sigma \omega \wedge \bar{\omega} > 0$, since locally, if $\omega = f(z)dz$, then

$$-i\omega \wedge \bar{\omega} = -if\bar{f}dz \wedge d\bar{z} = 2f\bar{f}dx \wedge dy.$$

On the other hand, $\int_\Sigma \omega \wedge \bar{\omega} = \int_F \omega \wedge \bar{\omega} = \int_F d(A) \wedge \bar{\omega} = \int_F d(A \cdot \bar{\omega}) = \int_{\partial F} A\bar{\omega}$ by Stokes. Note $\bar{\omega}|_B = \bar{\omega}|_{-D}$ and the same for C, E. Similarly $A|_B - A|_{-D}$ is equal to the constant function $\int_\beta \omega$ and $A|_C - A|_{-E} = \int_{-\alpha} \omega$. Hence

$$\begin{aligned} \int_\Sigma \omega \wedge \bar{\omega} &= \int_B A\bar{\omega} - \int_{-D} A\bar{\omega} + \int_C A\bar{\omega} - \int_{-E} A\bar{\omega} = \int_B \left(\int_\beta \omega\right)\bar{\omega} + \int_C \left(\int_{-\alpha} \omega\right)\bar{\omega} \\ &= \int_\alpha \bar{\omega} \int_\beta \omega - \int_\alpha \omega \int_\beta \bar{\omega} = \bar{v}_1 v_2 - v_1 \bar{v}_2. \end{aligned}$$

Putting everything together, we have $-i(\bar{v}_1 v_2 - v_1\bar{v}_2) > 0$, i.e. $\mathrm{Im}(v_1\bar{v}_2) > 0$. □

So $\Lambda := \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ is a lattice and $A : \Sigma \to \mathbb{C}/\Lambda$ is a locally invertible map into a torus. Hence A is a covering map and Σ compact implies the fibres of A are finite, i.e. Σ is a finite covering of $\mathbb{C}/\Lambda$. With some covering theory, this implies $\Sigma = \mathbb{C}/\Lambda'$, where $\Lambda' \subseteq \Lambda$ is a finite index sublattice.

Next we ask: Given lattices $\Lambda, \Lambda' \subseteq \mathbb{C}$, when are $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ isomorphic as Riemann surfaces?

Proposition 2.7. $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ *if and only if there exists* $c \in \mathbb{C}^\times$ *s.t.* $c\Lambda = \Lambda'$.

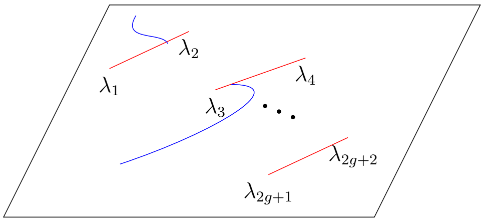
Proof. Let $i : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be an iso. and assume $i(0) = 0$. Lift i to the universal cover $\tilde{\imath} : \mathbb{C} \to \mathbb{C}$ isomorphism with $\tilde{\imath}(0) = 0$. This implies that $\tilde{i}$ is linear, i.e. $\tilde{i}(z) = cz$. Since $i(\Lambda) = \Lambda'$, it follows that $c\Lambda = \Lambda'$. The converse follows similarly. □

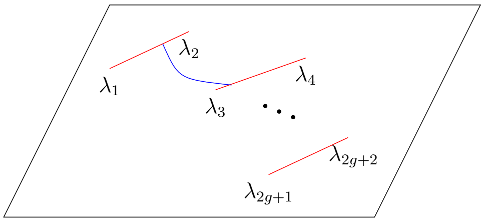
Given any lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \subseteq \mathbb{C}$, multiply it by $\frac{1}{v_2}$ to get $\Lambda' \cong \mathbb{Z}\tau \oplus \mathbb{Z}$ where $\tau = \frac{v_1}{v_2}$. Assume $\mathrm{Im}(\tau) > 0$ (otherwise replace τ by $-\tau$). Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$ be the upper half-plane. Then $\tau, \tau' \in \mathbb{H}$ define the same complex torus if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Hence the space of 1-dimensional complex tori is in bijection to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

2.2 Curves of higher genus

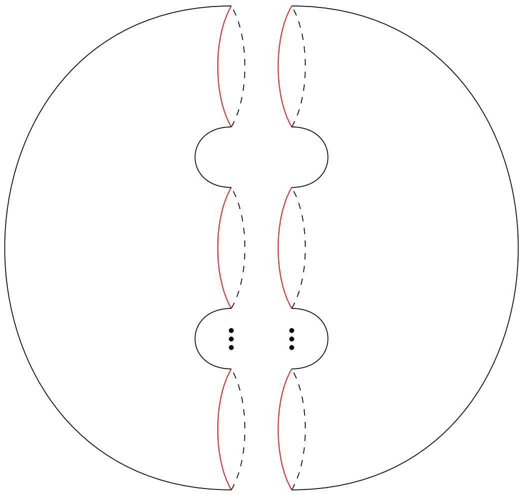
Consider a Riemann surface Σ of genus g.

Example 2.8 (Hyperelliptic curves)**.** $y^2 = (x-\lambda_1)\cdots(x-\lambda_{2g+2})$. This corresponds to the Riemann surface Σ of the function $f(x) = \sqrt{(x-\lambda_1)\cdots(x-\lambda_{2g+2})}$, which has a unique analytic continuation to $\mathbb{C}$ without the $g+1$ cuts between λ_{2i-1} and λ_{2i}, $i = 1, \dots, g+1$.





The result is a genus g Riemann surface and the local chart near $x = \lambda_i$ is $\sqrt{x-\lambda_i}$.



Again consider $\omega = \frac{dx}{y}$. For the same reasons as before, ω is holomorphic at $y = 0$. Near ∞, $w = \frac{1}{x}$ is a local corrdinate, and

$$\omega = \frac{d(1/w)}{\sqrt{(1/w-\lambda_1)\cdots(1/w-\lambda_{2g+2})}} = \frac{-1/w^2 dw}{1/w^{g+1}\sqrt{h(w)}} = -w^{g-1}h(w)^{-1/2}dw$$

for some holomorphic function h. Hence for $0 \le r \le g-1$, even $\omega_r = \frac{x^r dx}{y}$ is a holomorphic 1-form on Σ, so there is a g-dimensional vector space $\bigoplus_{r=0}^{g-1} \mathbb{C}\omega_r$ of holomorphic 1-forms.
Fact: Let $\Omega^1(\Sigma)$ be the $\mathbb{C}$-vector space of holomorphic 1-forms on a genus g compact Riemann surface Σ. Then $\dim_{\mathbb{C}} \Omega^1(\Sigma) = g$. We will prove this later.

Exercise 2.9. What is the genus of $y^3 = x^6 - 1$?

2.3 De-Rham Cohomology

Let M be a real manifold of dimension d. Let $\bigwedge^p(M) := \{\text{smooth } p\text{-forms on } M\}$, that is smooth p-forms on an open cover that agree on intersections, where $\bigwedge^p(U) = \{\sum_{|I|=p} f_I dx_i \wedge \cdots \wedge dx_p\}$ with the f_I smooth. Now consider the de-Rham complex

$$0 \to \bigwedge^0(M) \xrightarrow{d_0} \bigwedge^1(M) \xrightarrow{d_1} \bigwedge^2(M) \to \cdots \to \bigwedge^d(M) \to 0$$

where

$$d(f dx_I) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

Now define the de-Rham cohomology as the homology of this complex, i.e.

$$H^p_{\text{dR}}(M, \mathbb{C}) = \ker(d_p)/\operatorname{im}(d_{p-1}).$$

Theorem 2.10 (De Rham). $H^p_{dR}(M, \mathbb{C}) \cong H^p_{sing}(M, \mathbb{C})$.

Here, the map is defined as follows: Let $[\omega] \in H^p_{\text{dR}}(M, \mathbb{C})$ be represented by $\omega \in \bigwedge^p(M)$ which is exact: $d_p\omega = 0$. Then define $[\omega] \mapsto (\sigma \mapsto \int_\sigma \omega)$.

Now consider M a complex manifold of $\mathbb{C}$-dimension d.

Definition 2.11. The *smooth (p, q)-forms* on $U \subseteq \mathbb{C}^n$ are defined as

$$\bigwedge^{p,q}(U) = \left\{ \sum_{\substack{|I|=p,\\ |J|=q}} f_{I\bar{J}} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge d\bar{x}_{j_1} \cdots \wedge d\bar{x}_{j_q} \right\}$$

where the $f_{I\bar{J}}$ are smooth functions and the x_i are the coordinates of $\mathbb{C}^n$. The smooth $(p+q)$-forms of a manifold M are forms locally of the type $\bigwedge^{p,q}(U)$.

This is well-defined because the pullback of a (p, q)-form under a holomorphic map is a (p, q)-form.

Example 2.12. $z\bar{z}dz$ is a smooth $(1, 0)$-form on $\mathbb{C}$. It corresponds to $(x^2 + y^2)(dx + idy) \in \bigwedge^1(\mathbb{R}^2)$. Similarly, $\bar{z}d\bar{z} \in \bigwedge^{0,1}(\mathbb{C})$. If (z, w) are the coordinates of $\mathbb{C}^2$, then $dz \wedge d\bar{w} + d\bar{z} \wedge dw \in \bigwedge^{1,1}(\mathbb{C})$.

Lemma 2.13. $\bigwedge^k(M_\mathbb{R}) = \bigoplus_{p+q=k} \bigwedge^{p,q}(M)$*, where* $M_\mathbb{R}$ *is* M *considered as a real manifold.*

Proof. If $z_1, \ldots, z_n$ are local complex coordinates and $x_1, y_1, \ldots, x_n, z_n$ the corresponding local real coordinates, then $z_i = x_i + iy_i$ and $x_i = \frac{1}{2}(z_i + \bar{z}_i)$, $y_i = \frac{1}{2}(z_i - \bar{z}_i)$, so one can directly translate expressions from each set into an expression from the other set. □

On a one-dimensional complex manifold Σ, the only spaces of (p, q)-forms to consider are $\bigwedge^{0,0}(\Sigma) = \{\text{loc. smooth functions}\}$, $\bigwedge^{1,0}(\Sigma)$, $\bigwedge^{0,1}(\Sigma)$ and $\bigwedge^{1,1}(\Sigma)$. Hence the de-Rham complex is

$$0 \to \bigwedge^{0,0}(\Sigma) \xrightarrow{d} \bigwedge^{0,1}(\Sigma) \oplus \bigwedge^{1,0}(\Sigma) \xrightarrow{d} \bigwedge^{1,1}(\Sigma)$$

and the first exterior derivative is $d = \partial_z \oplus \partial_{\bar{z}}$, i.e. given by $f \mapsto \frac{\partial f}{\partial z} d\bar{z} + \frac{\partial f}{\partial \bar{z}} dz$. A holomorphic 1-form on Σ is locally expressible as $\omega = f(z)dz$ with f holomorphic, $\omega \in \bigwedge^{1,0}(\Sigma)$. As before, we see that ω is closed:

$$d(f(z, \bar{z})dz) = \frac{\partial f}{\partial z} \underbrace{dz \wedge dz}_{=0} + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{=0} d\bar{z} \wedge dz = 0.$$

For a smooth complex-valued function $f \in \bigwedge^{0,0}(M)$ write

$$df = \underbrace{\frac{\partial f}{\partial z_1}dz_1 + \ldots + \frac{\partial f}{\partial z_d}dz_d}_{=:\partial f} + \underbrace{\frac{\partial f}{\partial \bar{z}_1}d\bar{z}_1 + \ldots + \frac{\partial f}{\partial \bar{z}_d}dz_d}_{=:\overline{\partial} f}$$

Then $d : \bigwedge^k(M) \to \bigwedge^{k+1}(M)$ decomposes into a sum $d = \partial + \overline{\partial}$ where

$$\partial(f_{I,\bar{J}}dz_I \wedge d\bar{z}_J) = \partial f_{I,\bar{J}} \wedge dz_I \wedge d\bar{z}_J$$

and similarly for $\overline{\partial}$. We get a double complex

$$\begin{array}{ccccc}
\bigwedge^{0,2}(M) & & & & \\
\bar{\partial}\Big\uparrow & & & & \\
\bigwedge^{0,1}(M) & \xrightarrow{\partial} & \bigwedge^{1,1}(M) & & \\
\bar{\partial}\Big\uparrow & & \bar{\partial}\Big\uparrow & & \\
\bigwedge^{0,0}(M) & \xrightarrow{\partial} & \bigwedge^{1,0}(M) & \xrightarrow{\partial} & \bigwedge^{2,0}(M)
\end{array}$$

with the squares commuting up to sign. Also it is easy to check that $\bar{\partial} \circ \bar{\partial} = 0 = \partial \circ \partial$. The total complex of a double complex $(E^{p,q}, d_1, d_2)$ is $(\bigoplus_{p+q=k} E^{p,q}, d_1 + d_2)$. In this case, we get

$$\left(\bigoplus_{p+q=k} \bigwedge^{p,q}(M), \partial + \bar{\partial}\right) = (\bigwedge^k(M), d)$$

the original de Rham complex on M.

Example 2.14.

$$\bar{\partial}(z\bar{z}d\bar{w} + wd\bar{z}) = zd\bar{z} \wedge d\bar{w} + 0 \in \bigwedge^{0,2}(\mathbb{C}^2).$$

Definition 2.15. The *Dolbeault cohomology*

$$H^{p,q}(M) := \frac{\ker(\bar{\partial} : \bigwedge^{p,q}(M) \to \bigwedge^{p,q+1}(M))}{\mathrm{im}(\bar{\partial} : \bigwedge^{p,q-1}(M) \to \bigwedge^{p,q}(M))}$$

This is an analogue of de Rham cohomology when M is a complex manifold.

Theorem 2.16 (Poincaré $\bar{\partial}$-lemma)**.** *Let $0 \in U \subseteq \mathbb{C}^n$ be an open set, $\omega \in \bigwedge^{p,q}(U)$. Then, if $\bar{\partial}\omega = 0$, then there exists an open subset $0 \in V \subseteq U$ and an $\alpha \in \bigwedge^{p,q-1}(V)$ such that $\bar{\partial}\alpha = \omega|_V$.*

Proof. (for $n = 1$). Let $\Delta \subseteq \mathbb{C}$ be the unit disk. Let $gd\bar{z} \in \bigwedge^{0,1}(\overline{\Delta})$. Then $\bar{\partial}gd\bar{z}$ is automatically 0. Then

$$f(z, \bar{z}) := \frac{1}{2\pi i}\int_\Delta \frac{g(w, \bar{w})}{w - z} dw \wedge d\bar{w}$$

satisfies $\bar{\partial}f = gd\bar{z}$: Write $g = g_1 + g_2$ such that $\operatorname{supp} g_1 \subseteq B_{2\varepsilon}(z)$ and $\operatorname{supp}(g_2) \subseteq B_\varepsilon(z)^c$. Now

$$f = \frac{1}{2\pi i}\left(\int_\Delta \frac{g_1}{w - z}dw \wedge d\bar{w} + \underbrace{\int_\Delta \frac{g_2}{w - z}dw \wedge d\bar{w}}_{\bar{\partial}(-)=0}\right),$$

and

$$\int_\Delta \frac{g_1}{w - z}dw \wedge d\bar{w} = \int_{B_{2\varepsilon}(0)} \frac{g_1(z + u)}{u}du \wedge d\bar{u} = \frac{i}{2}\int_0^{2\pi}\int_0^{2\varepsilon} g_1(z + u)e^{-i\theta}drd\theta$$

is clearly the integral of a smooth function, hence smooth. One calculates

$$\begin{aligned}
2\pi i\bar\partial f &= \bar\partial \int_\Delta \frac{g_1}{w-z} dw \wedge d\bar w = \lim_{\mu\to 0} \bar\partial \int_{B_{2\varepsilon}(z)-B_\mu(z)} \frac{g_1}{w-z} dw \wedge d\bar w \\
&= \lim_{\mu\to 0} \left(\int_{B_{2\varepsilon}(z)-B_\mu(z)} \underbrace{\frac{\partial g_1}{\partial \bar w}(w) \frac{1}{w-z} dw \wedge d\bar w}_{d\eta \text{ where } \eta = -\frac{g(w)dw}{w-z}} \right) d\bar z \\
&\overset{\text{Stokes}}{=} \lim_{\mu\to 0} \left(\int_{C_{2\varepsilon}(z)} \frac{-g_1(w)dw}{w-z} + \int_{C_{\mu(z)}} \frac{g_1(w)}{w-z} dw \right) d\bar z \\
&= 2\pi i g_1(z) d\bar z = 2\pi i g(z) d\bar z
\end{aligned}$$

□

Let Σ be a Riemann surface of genus g. Let α_i, β_i, $i = 1, \dots, g$ be the loops as indicated in the picture.

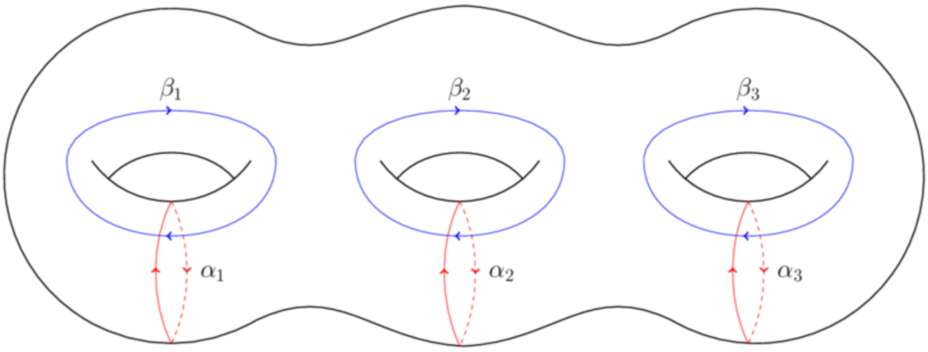


Figure 2: A genus 3 surface with a basis for its first homology

They form a basis of $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. By Poincaré duality, $H_1(\Sigma, \mathbb{Z}) \overset{PD}{\cong} H^1(\Sigma, \mathbb{Z})$. From the cohomological product structure one gets a pairing on $H_1(\Sigma, \mathbb{Z})$ by $\alpha \cdot \beta = \int_\Sigma PD(\alpha) \smile PD(\beta)$, which is the intersection form: Represent α, β by transversely intersecting cycles. Then $\alpha \cdot \beta = \sum_{p \in \alpha\cap\beta} \underbrace{\mathrm{or}_p(\alpha, \beta)}_{\in \pm 1}$.

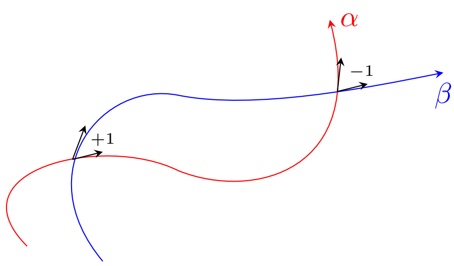


Figure 3: Two cycles α, β with intersection form $\alpha \cdot \beta = 1 - 1 = 0$.

The Intersection matrix of the chosen basis $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ is given by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathbb{Z}^{2g\times 2g}$. Thus $H_1(\Sigma, \mathbb{Z})$ has the structure of a symplectic lattice, with an alternating map $(-,-) : \Lambda \otimes \Lambda \to \mathbb{Z}$.

Choose generators α_i, β_i of $\pi_1(\Sigma, *)$ that are homologous to the α_i, β_i from before. Then $\pi_1(\Sigma, *) = \langle \alpha_i, \beta_i \mid \prod_i [\alpha_i, \beta_i] \rangle$ Recall $\dim \Omega^1(\Sigma) = g$. Let $\omega, \omega' \in \Omega^1(\Sigma)$.

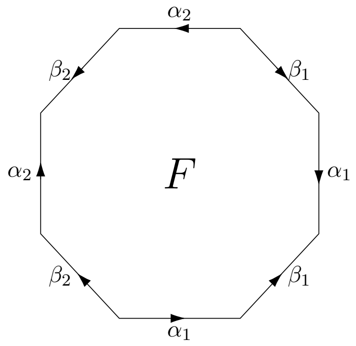


Figure 4: Cut up torus of genus 2

Theorem 2.17 (Riemann Bilinear Relations)**.** *There exists a unique basis $\Omega^1(\Sigma) = \bigoplus_{i=1}^g \mathbb{C}\omega_i$ such that $\int_{\alpha_j} \omega_i = \delta_{ij}$ and $(\int_{\beta_j} \omega_i)_{i,j}$ is a symmetric $g \times g$-matrix with positive definite imaginary part.*

Lemma 2.18.

$$\textit{Symmetry:} \qquad \sum_{i=1}^g \int_{\alpha_i} \omega \int_{\beta_i} \omega' - \int_{\alpha_i} \omega' \int_{\beta_i} \omega = 0.$$

$$\textit{Positivity:} \qquad i\sum_{i=1}^g \int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\beta_i} \omega \int_{\alpha_i} \bar{\omega} > 0.$$

Proof. Let $A(p) = \int_*^p \omega$ where $*$ is one of the vertices of F. $A(p)$ is holomorphic on F and welldefined since $d\omega = 0$. Further $dA = \omega$ on F.

$$0 = \int_\Sigma \omega \wedge \omega' = \int_F \omega \wedge \omega' = \int_F d(A\omega') = \int_{\partial F} A\omega'.$$

Now, as in the case of elliptic curves,

$$\int_{\alpha_1} A\omega' + \int_{\alpha_1^{-1}} A\omega' = \int_{\alpha_1} (A|_{\alpha_1} - A|_{\alpha^{-1}})\omega' = (A|_{\alpha_1} - A|_{\alpha_1^{-1}}) \int_{\alpha_1} \omega' = -\int_{\beta_1} \omega \int_{\alpha_1} \omega'.$$

Doing this for all i gives

$$\int_{\partial F} A\omega' = -\sum_{i=1}^g \int_{\beta_i} \omega \int_{\alpha_i} \omega' - \int_{\alpha_i} \omega \int_{\beta_i} \omega'.$$

This shows symmetry. For positivity, we have

$$0 < i\int_\Sigma \omega \wedge \bar{\omega} = i\int_F \omega \wedge \bar{\omega} = i\int_F d(A\bar{\omega}) = i\int_{\partial F} A\bar{\omega}$$

As before,

$$\int_{\alpha_i} A\bar{\omega} + \int_{\alpha^{-1}} A\bar{\omega} = -\int_{\beta_i} \omega \int_{\alpha_i} \bar{\omega}.$$

Doing this for all sides of F gives the result. □

Corollary 2.19. *There is no $\omega \in \Omega^1(\Sigma) \setminus 0$ such that $\int_{\alpha_i} \omega = 0$ for all i.* □

Corollary 2.20. $\dim_{\mathbb{C}} \Omega^1(\Sigma) \leq g$.

Proof. $\Omega^1(\Sigma) \to \mathbb{C}^n$, $\omega \mapsto (\int_{\alpha_i} \omega)_i$ is an injective linear map. □

Corollary 2.21. *If Σ is the Riemann surface of $\sqrt{(x-\lambda_1)\cdots(x-\lambda_{2g+2}} = y$, then*

$$\Omega^1(\Sigma) = \bigoplus_{r=0}^{g} \mathbb{C}\frac{x^r\,dr}{y}.$$

□

Proof. (of 3.8) Assume again $\Omega^1(\Sigma) = g$, which we will prove later. The map in the proof of 3.10 is then an iso, hence we can choose a basis that satisfies $\int_{\alpha_j} \omega_i = \delta_{ij}$.

Consider the "period matrix of Σ" $P = (P_{ij}) = (\int_{\beta_j} \omega_i)$. P is symmetric: Let ω_k, ω_l be elements of the normalized basis. Then

$$0 = \sum_{j=1}^{g} \int_{\alpha_j} \omega_k \int_{\beta_j} \omega_l - \int_{\alpha_j} \omega_l \int_{\beta_j} \omega_l = \sum_{j=1}^{g} \delta_{jk} \int_{\beta_j} \omega_l - \delta_{jl} \int_{\beta_l} \omega_k = \int_{\beta_k} \omega_l - \int_{\beta_l} \omega_k = P_{lk} - P_{kl}.$$

Let $\omega = c_1\omega_1 + \ldots + c_g\omega_g$ with $c_i \in \mathbb{R}$ not all 0. Then

$$\begin{aligned} 0 < i \sum_{j=1}^{g} \int_{\alpha_j} \omega \int_{\beta_j} \bar{\omega} - \int_{\beta_j} \omega \int_{\alpha_j} \bar{\omega} &= i \sum_{j,k=1}^{g} c_j \int_{\beta_j} \bar{c}_k \bar{\omega}_k - c_k \int_{\beta_j} \bar{\omega}_k \bar{c}_j \\ &= 2\operatorname{Im}\left(c_j \bar{c}_k \int_{\beta_j} \omega_k \right) = 2(\vec{c})^\dagger \operatorname{Im} P\vec{c} \end{aligned}$$

So $\operatorname{Im} P$ is positive definite as a bilinear form. □

Lemma 2.22. *Assuming* $\dim_{\mathbb{C}} \Omega^1(\Sigma) = g$, *then* $H^1_{dR}(\Sigma, \mathbb{C}) \cong \Omega^1(\Sigma) \oplus \overline{\Omega^1(\Sigma)}$

Proof. Every $\omega \in \Omega^1(\Sigma)$ is d-closed, hence so is every $\bar{\omega} \in \overline{\Omega^1(\Sigma)}$. So we have a map

$$\Omega^1(\Sigma) \oplus \overline{\Omega^1(\Sigma)} \xrightarrow{\varphi} H^1_{dR}(\Sigma, \mathbb{C}) \cong H_1(\Sigma, \mathbb{C})^* = \operatorname{Hom}(H_1(\Sigma, \mathbb{Z}), \mathbb{C})$$

given by $\omega \mapsto \gamma \mapsto \int_\gamma \omega$ It is represented by the $2g \times 2g$-matrix (rows $\omega_i, \bar{\omega}_i$, columns α_j, β_j)

$$\begin{pmatrix} I & P \\ I & \bar{P} \end{pmatrix} \rightsquigarrow \begin{pmatrix} I & P \\ 0 & \bar{P} - P \end{pmatrix}$$

where $\bar{P} - P = -2i \operatorname{Im} P$ is positive definite, hence has full rank. Thus φ is injective and since the dimensions agree, it is an isomorphism. □

Let $* \in \Sigma$ be a base point. As in the genus 1 case, we can define a multivalued holomorphic map $p \mapsto (\int_*^p \omega_i)_i \in \mathbb{C}^g$. If γ, γ' are paths from $*$ to p, then $\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\gamma - \gamma'} \omega$ with $\gamma - \gamma' \in H_1(\Sigma, \mathbb{Z})$. Thus the value of our function is unique up to the free abelian group generated by $(\int_{\alpha_i} \omega_j)_j$ and $(\int_{\beta_i} \omega_j)_j$, $i, = 1, \ldots, g$. This is exactly $\mathbb{Z}^g \oplus P\mathbb{Z}^g \subseteq \mathbb{C}^g$. Since $\operatorname{Im} P > 0$, this is a discrete subgroup of $\mathbb{C}^g$ (Exercise).

Definition 2.23. The *Jacobian* of Σ is $\operatorname{Jac}(\Sigma) = \mathbb{C}^g / \mathbb{Z}^g \oplus P\mathbb{Z}^g$. This is a compact complex manifold of dimension g. Further, the *Abel-Jacobi map*

$$AJ : \Sigma \to \operatorname{Jac}(\Sigma), \quad p \mapsto \left(\int_*^p \omega_1, \ldots, \int_*^p \omega_g \right)$$

is single-valued, well-defined and holomorphic.

Note that $\operatorname{Jac}(\Sigma)$ is diffeomorphic to $(S^1)^{2g}$.

3 Holomorphic Vector Bundles

Definition 3.1. Let M be a complex manifold. A holomorphic vector bundle $\pi : \mathcal{E} \to M$ is a complex manifold $\mathcal{E}$ of rank r that has local trivializations

$$\begin{tikzcd} \mathcal{E}|_U \arrow[rr, "\pi"] \arrow[dr, "h_U"'] & & U \\ & U \times \mathbb{C}^r \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

with biholomorphic maps h_U such that $h_V \circ h_U^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r$ is linear on every fibre and the induced map $t_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ is holomorphic.

Example 3.2. $\mathbb{C}^r \times M \to M$ is the trivial vector bundle.

Definition 3.3. A *section* of $\mathcal{E}$ over $U \subseteq M$ is a holomorphic map $s : U \to \mathcal{E}|_U$ such that $\pi \circ s = \mathrm{id}$. Denote the space of sections over U by $\mathcal{E}(U)$.

There is a holomorphic vector bundle Ω^p on M such that $\Omega^p(U)$ consists of the holomorphic p-forms on U: On a coordinate chart $U \hookrightarrow \mathbb{C}^n$, $\Omega^p(U) = \{\sum_{|I|=p} f_I dz_I \mid f_I \text{ holomorphic}\} = \bigoplus_{|I|=p} \mathrm{Hol}(U) \cdot dz_I$ with the transition functions the usual coorinate change. This is a holomorphic vector bundle of rank $r = \binom{\dim_{\mathbb{C}} M}{k}$ with $\Omega^k|_U \cong C^r \times U$ trivialized by $\{dz_{i_1} \wedge \cdots \wedge dz_{i_k}\}$ in a local coordinate chart $(z_1, \ldots, z_d)$ with $d = \dim M$. If $\varphi_U : U \to \mathbb{C}^d, \varphi_V : V \to \mathbb{C}^d$ are two charts with local coordinates z, w, respectively, let us compute the coorinate change t_{UV} for Ω^1. This is the change of coordinates on 1-forms $\sum f_i dz_i \rightsquigarrow \sum g_i dw_i$. One obtains

$$\begin{pmatrix} \frac{dw_1}{dz_1} & \frac{dw_2}{dz_1} & \cdots & \\ \vdots & & & \vdots \\ & & \cdots & \frac{dw_d}{dz_d} \end{pmatrix} = \mathrm{Jac}(\varphi_V \circ \varphi_U^{-1}).$$

Note that the usual constructions on vector spaces, like $\mathrm{Hom}, \otimes, (-)^\vee$, exist for vector bundles. With this in mind, we can write $\Omega^k = \bigwedge^k \Omega^1$.

Note: Similarly to holomorphic vector bundles, one can define e.g. smooth vector bundles by requiring that the manifolds, trivializations and transition maps involved are smooth.

Given a complex manifold M, then $\bigwedge^{p,q}(M) \to M$ is a smooth vector bundle with trivialization given by $\{dz_I \wedge d\bar{z}_J \mid |I| = p, |J| = q\}$. In particular, $\bigwedge^{p,0}(M)$ is not the same as ω^p, since the first is considered as a smooth manifold: Looking at sections,

$$\textstyle\bigwedge^{p,0} = \{\sum f dz_I \mid f \text{ smooth}\} \quad \text{but} \quad \omega^p(U) = \{\sum f_I dz_I \mid f_I \text{ holomorphic}\}.$$

Definition 3.4. The holomorphic line bundle $\Omega^{\dim M}$ is called the canonical bundle K_M.

Crash Course on Sheaves

Let X be a topological space. A sheaf of abelian groups (or with values in a category $\mathcal{C}$) F on X is an assignment $\{\text{open sets of } X\} \to \mathtt{Ab}$, $U \mapsto F(U)$, together with restriction maps $\rho_{UV} : F(U) \to F(V)$ for all open $V \subseteq U$ such that $\rho_{UU} = \mathrm{id}_{F(U)}$, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$, $\rho(\emptyset) = 0$, and such that given $\{s_i \in F(U_i)\}$ with $\rho_{ij}(s_i) = \rho_{ji}(s_j)$ there exists a unique $s \in F(\bigcup U_i)$ with $s|_{U_i} = s_i$. Sheaves of abelian groups on X form an abelian category. A homomorphism $\varphi : F \to G$ is a collection of homomorphisms $\varphi(U)$ in $\mathcal{C}$ such that $\rho_{UV}\varphi(U) = \varphi(V)\rho(UV)$.

Any complex holomorphic vector bundle gives a sheaf E via $E(U)$ the holomorphic sections over U. This is even a sheaf of $\mathbb{C}$-vector spaces. For $\Omega^0 = \mathbb{C} \times M$, the trivial bundle, the corresponding sheaf $\mathcal{O} = \Omega^0$ is the sheaf of holomorphic functions $U \mapsto \mathcal{O}(U)$. Let $\mathcal{O}^*$ be the sheaf of nonvanishing holomorphic functions

$$\mathcal{O}^*(U) = \{f : U \to \mathbb{C}^* \mid f \text{ holomorphic}\}.$$

This is a sheaf of abelian groups.

Example 3.5. Let X be any topological space, A an abelian group. Then the constant sheaf on X with value A is the sheaf

$$\underline{A} : U \mapsto A(U) = \{\text{loc. const. functions } U \to A\}$$

For instance, if $X = \mathbb{R}; A = \mathbb{Z}$, then $\underline{\mathbb{Z}}((0,1) \cup (2,3)) = \mathbb{Z} \oplus \mathbb{Z}$. For M a complex manifold, define a morphism $\exp : \mathcal{O} \to \mathcal{O}^\times$ via $\mathcal{O}(U) \to \mathcal{O}^*(U)$, $f \mapsto e^f$. We want to compute the kernel K and cokernel Q of this map. For the cokernel, first set $Q^{\text{pre}}(U) = \mathcal{O}^*(U)/\exp(\mathcal{O}(U))$. For example, if $M = \mathbb{C}$ and $U = \mathbb{C}^*$, then $\mathcal{O}^*(\mathbb{C}^*)/\exp(\mathcal{O}(\mathbb{C}^*)) \cong \mathbb{Z}$, since given $f \in \mathcal{O}^*(\mathbb{C}^*)$ one can take the logarithm locally and analytically continue. Walking around the origin once, the difference is a logarithm of 1, i.e. an element of $2\pi i\mathbb{Z}$. This shows that there exists a unique $n \in \mathbb{Z}$ such that fz^n has a well-defined $\log$ on $\mathbb{C}^*$. On the other hand, for all $V \subseteq \mathbb{C}^*$ sufficiently small, $Q^{\text{pre}}(V) = 1$. Hence Q^{pre} is not a sheaf. Hence sheafify Q^{pre} to get

$$Q(U) = \{\text{compatible sections of } Q^{\text{pre}}(V_i) \text{ for a suff. small cover } U = \bigcup V_i\}.$$

Hence $Q = 1$. Similiarly, $K = K^{\text{pre}} = \underline{2\pi i\mathbb{Z}}$, since the kernel presheaf is already a sheaf. This gives a short exact sequence of abelian groups

$$0 \to 2\pi i\underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^\times \to 0,$$

the exponential exact sequence.

Čech Cohomology

Let $\mathcal{U} = \{U_i\}$ be an open cover of X and F a sheaf on X. Set

$$C^p_{\mathcal{U}}(X, F) = \bigoplus_{i_0 < \ldots < i_p} F(U_{i_0} \cap \cdots \cap U_{i_p}).$$

For example, if $X = S^1$, $F = \underline{\mathbb{Z}}$ and $\mathcal{U} = \{U_0, U_1\}$ with $U_0 = S^1 \setminus \{-1\}$, $U_1 = S^1 \setminus \{1\}$, then $C^0_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_0) \oplus \underline{\mathbb{Z}}(U_1) = \mathbb{Z} \oplus \mathbb{Z}$, and $C^1_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_0 \cap U_1) = \mathbb{Z}^2$. There is a coboundary $\partial^p : C^p_{\mathcal{U}}(X, F) \to C^{p+1}_{\mathcal{U}}(X, F)$ given by

$$(s_{i_0,\ldots,i_p}) \mapsto \left(\sum_{j=0}^{p+1} (-1)^j s_{i_0,\ldots,\hat{i}_j,\ldots,i_{p+1}}\right)|_{U_{i_0}\cap\ldots\cap U_{i_{p+1}}}$$

In the example, the map $C^0_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) \to C^1_{\mathcal{U}}(S^1, \underline{\mathbb{Z}})$ is given by $(a, b) \mapsto a|_{U_{01}} - b|_{U_{01}} = (a - b, a - b)$. Define the Čech cohomology with respect to $\mathcal{U}$ as the cohomology of this complex, i.e. $H^p_{\mathcal{U}}(X, F) = H^p(C^\bullet_{\mathcal{U}}(X, F), \partial^\bullet)$. In our case, $H^p_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \mathbb{Z}$ if $p = 0, 1$ and 0 else. This cohomology is dependent ono $\mathcal{U}$. For example, if $\mathcal{U} = \{S^1\}$, then $H^p_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \mathbb{Z}$ for $p = 0$ and 0 otherwise. If $\mathcal{V}$ refines $\mathcal{U}$, then one gets a natural map $H^p_{\mathcal{U}}(X, F) \to H^p_{\mathcal{V}}(X, F)$ by using the restriction maps.

Definition 3.6. The Čech cohomology is

$$H^p(X, F) = \operatorname{colim}_{\mathcal{U}} H^p_{\mathcal{U}}(X, F)$$

If $\mathcal{U}$ is a good cover of M, i.e. all intersections of the U_i are contractible, then $H^p(X,F) = H^p_{\mathcal{U}}(X,F)$. For any short exact sequence $0 \to F \to G \to H \to 0$ of sheaves on X, we get a long exact sequence

$$0 \to H^0(X,F) \to H^0(X,G) \to H^0(X,H) \to H^1(X,F) \to \cdots$$

Let $E \to M$ be a holomorphic vector bundle. Consider the smooth vector bundlees $E \otimes \bigwedge^{0,p}$ for $p \geq 0$. There is an exact sequence

$$0 \to E \to E \otimes_{\mathcal{O}} \bigwedge^{0,0} \xrightarrow{1\otimes\bar{\partial}} E \otimes_{\mathcal{O}} \bigwedge^{0,1}$$

where the middle term is just the sheaf of smooth sections of E. Remember that the Poincare $\bar{\partial}$-lemma says that if $\alpha \in \bigwedge^{0,p}(U)$ s.t. $\bar{\partial}\alpha = 0$, then there exists $V \subseteq U$ open and $\beta \in \bigwedge^{0,p-1}(V)$ such that $\bar{\partial}\beta = \alpha|_V$. Hence on $U \subseteq M$ a trivializing chart for E, we have $E|_U = \mathcal{O}_U^{\oplus r}$, hence restricted to U, the Poincare lemma says that the sequence

$$0 \to \mathcal{O}_U^{\oplus r} \to \bigwedge^{0,0}(U)^{\oplus r} \to \bigwedge^{0,1}(U)^{\oplus r} \to \cdots$$

is exact. Hence this works globally and we obtain

Proposition 3.7. *There exists an exact sequence of sheaves*

$$0 \to E \to E \otimes_{\mathcal{O}} \bigwedge^{0,0} \xrightarrow{1\otimes\bar{\partial}} E \otimes_{\mathcal{O}} \bigwedge^{0,1} \xrightarrow{1\otimes\bar{\partial}} E \otimes_{\mathcal{O}} \bigwedge^{0,2} \xrightarrow{1\otimes\bar{\partial}} \cdots$$

This complex is called the Dolbeault complex *for E.*

Before that, we should check

Proposition 3.8. *$\bar{\partial}$ is well-defined, independent of the coordinate chart.*

Proof. Let $s_U \otimes \omega_U \in E \otimes \bigwedge^{0,q}(U)$ and $s_V \otimes \omega_V \in E \otimes \bigwedge^{0,q}(V)$. Let $\varphi_U : U \to \mathbb{C}^d$ and $\varphi_V : V \to \mathbb{C}^d$ be the coordinate charts and $t_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ the transition function of E, which is holomorphic. We have $s_V \otimes \omega_W = t_{UV} s_U \otimes (\varphi_U \circ \varphi_V^{-1})^* \omega_U$. Now $\bar{\partial}\omega_v = \bar{\partial}(\varphi_U \circ \varphi_V^{-1})^* \omega_U$ because $\bar{\partial}(\varphi_U \circ \varphi_V^{-1}) = 0$. ☐

We want to prove the following

Theorem 3.9. *Let V be any C^∞-vector bundle on a smooth manifold X. Then $H^i(X,V) = 0$ for $i > 0$.*

Definition 3.10. A sheaf F on X is *flasque* if for any $V \subseteq U$ opens, ρ_{UV} is surjective. F is *soft* if for any $Z \subseteq X$ closed, $F(X) \to F(Z)$ is surjective, where $F(Z) := \mathrm{colim}_{U \supseteq Z} F(U)$.

For example, take $V = C^\infty$ the trivial bundle over $\mathbb{R}$, then $\frac{1}{x}$ does not lie in the image of $C^\infty(\mathbb{R}) \to C^\infty((0,1))$. So C^∞-vector bundles are usually not flasque. However, C^∞ is soft on $\mathbb{R}$ (exercise). This also holds for any C^∞-vector bundle on a manifold.

Lemma 3.11. *A flasque sheaf is soft.* ☐

Lemma 3.12. *Let $0 \to F \to G \to H \to 0$ be an exact sequence with F soft. Then $0 \to F(X) \to G(X) \to H(X) \to 0$ is exact.* ☐

Corollary 3.13. *If F and G are soft, so is H.*

Proof. Let $Z \subseteq X$ be closed and $s \in H(Z)$. By lemma 3.12, there exists $t \in G(Z)$ that maps to s. By assumption, there is a $\widetilde{t} \in G(X)$ restricting to t. Then the image of $\widetilde{t}$ in $H(X)$ restricts to s. ☐

Proposition 3.14. *If $0 \to F_0 \to F_1 \to \ldots$ is an exact sequence of soft sheaves then $0 \to F_0(X) \to F_1(X) \to \ldots$ is also exact.* ☐

Recall: An injective sheaf I is one such that for any $\varphi : A \to I$ and any inclusion $A \to B$, there is an extionsion $\widetilde{\varphi} : B \to I$ such that $\widetilde{\varphi}|_A = \varphi$.

Theorem 3.15. *Sheaves of abelian groups on a paracompact space admit injective resolutions, and sheaf cohomology can be computed as the homology of any such resolution.*

Proof. Omitted. □

Injective sheaves are flasque (exercise) and hence soft. Thus by proposition 3.14, $H^i(X, F) = 0$ if F is soft. This proves theorem 3.9. In particular, $H^i(X, E \otimes \bigwedge^{0,q}) = 0$ for all $i > 0$. So the Dolbeault resolution is an acyclic resolution of E.

Proposition 3.16. *An acyclic resolution of E computes $H^i(E)$: Given $0 \to E \to F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots$ acyclic, then*

$$H^i(E) = \frac{\ker(F_i(X) \to F_{i+1}(X))}{\operatorname{im}(F_{i-1}(X) \to F_i(X))}$$

Proof. Split up the resolution into short exact sequences $0 \to \ker d_i \to F_i \to \ker d_{i+1} \to 0$ and use the associated long exact sequences in cohomology. □

This implies *Dolbeault's theorem*

$$H^{p,q}(X) = \frac{\ker(\bar{\partial} : \bigwedge^{p,q}(X) \to \bigwedge^{p,q+1}(X)}{\operatorname{im}(\bar{\partial} : \bigwedge^{p,q-1}(X) \to \bigwedge^{p,q}(X))} \cong H^q(X, \Omega^p)$$

from the Dolbeault complex for $E = \Omega^p$.

Line Bundles on $\mathbb{CP}^n$

Definition 3.17. The *tautological line bundle* $\mathcal{O}(-1)$ of $\mathbb{CP}^n$ is the total space of lines through 0 in $\mathbb{C}^{n+1}$

As complex manifolds, we have $\mathcal{O}(-1) \cong \mathrm{Bl}_0\, \mathbb{C}^{n+1} \xrightarrow{\pi} \mathbb{CP}^n$, where π is the blow-up map followed by the natural projection.

Consider the dual $\mathcal{O}(1) \cong \mathcal{O}(-1)^*$. Let $U \subseteq \mathbb{CP}^n$ be open. Then $\mathcal{O}(1)(U)$ consists of holomorphically varying families of linear functions on the tautological lines through U. That is,

$$\mathcal{O}(1)(U) = \{f : \pi^{-1}(U) \to \mathbb{C} \text{ holomorphic} \mid f(\lambda x) = \lambda f(x)\}.$$

In particular, $\mathcal{O}(1)(\mathbb{CP}^n)$ is the set of linear forms on $\mathbb{C}^{n+1}$.

Next we can define $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ and $\mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k}$ for $k > 0$. This yields a collection of line bundles such that

$$\mathcal{O}(m)(U) = \{f : \pi^{-1}(U) \to \mathbb{C} \mid f(\lambda x) = \lambda^m f(x)\}.$$

For $U = \mathbb{CP}^n$ one has $\mathcal{O}(m)(\mathbb{CP}^n) \cong \mathbb{C}[x_0, \ldots, x_n]^{(m)}$.

More generally, if $X \subseteq \mathbb{CP}^n$ is a projective variety, we can restrict $\mathcal{O}(m)$ to X, which we denote $\mathcal{O}_X(m)$.

Line Bundles associated to divisors

Definition 3.18. Let X be a smooth projective variety. A divisor on X is a $\mathbb{Z}$-linear combination of irreducible, codimension 1 subvarieties. More generally, if X is a complex manifold, take the $\mathbb{Z}$-linear combination of all closed subsets of X that are locally cut out by a simgle holomorphic function.

If X is a Riemann surface, then codimension 1 subvarieties are points, hence $\mathrm{Div}(X) = \bigoplus_{p \in X} \mathbb{Z}[p]$.

Associate to any divisor D on X the holomorphic line bundle $\mathcal{O}_X(D)$ constructed as follows: For curves, declare $\mathcal{O}_X(D)|_U = \mathcal{O}_U$ where $U = X \setminus \operatorname{supp} D$. Let V_i be an open neighbourhood of $P_i \in \operatorname{supp} D$ that doesn't contain any other element of $\operatorname{supp} D$. Now $\mathcal{O}_X(D)|_{V_i} \cong \mathcal{O}_V$. Take $t_{UV_i} : U \cap V \to \mathbb{C}^\times$ to be $t_{UV_i} = z_i^{n_i}$, where $z_i : V_i \to \mathbb{C}$ is the local coordinate $P_i \to 0$ and n_i is the coefficient of $[P_i]$ in D.

In general, if f is a meromorphic function, define $\mathrm{div}(f) = \sum_P \mathrm{ord}_p(f)[P] \in \mathrm{Div}(X)$. Then if V is a small open, write $D \cap V = \mathrm{div}(f)$ for a suitable meromorphic function f and use this f to define the transition function. (Fact from Hartshorne: If X is smooth, every Weil divisor is Cartier.)

Definition 3.19. $D = \sum n_i[X_i] \in \mathrm{Div}(X)$ is effective if $n_i \geq 0$ for all i.

If D is effective, there is a section $s_D \in \mathcal{O}_X(D)(X)$ given by $h_U(s_D|_U) = 1$ where h_U is the trivialization to $U \times \mathbb{C}$. On $U \cap V_i$, we need $h_V(s_D|_V) = t_{UV}(1) = z_i^{n_i}$, which by assumption extends holomorphically over $P_i \in V_i$. If D is not effective, s_D defines a meromorphic section.

Proposition 3.20. *Let $\mathcal{L} \to X$ be a holomorphic line bundle over a Riemann surface X. Then if $s \in \mathcal{L}(X)$ (or if s is a meromorphic section),* $\mathrm{ord}_p(s)$ *is well-defined, where* $\mathrm{ord}_p(s) := \mathrm{ord}_p(h_U(s))$ *for any trivializing chart $U \ni p$.*

Proof. If V is any other such chart, then

$$\mathrm{ord}_p(h_V(s)) = \mathrm{ord}_p(t_{UV}(h_U(s))) = \mathrm{ord}_p t_{UV} + \mathrm{ord}_p h_U(s) = \mathrm{ord}_p h_U(s)$$

since t_{UV} is invertible. □

Proposition 3.21. *$\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X(D')$ iff there is a meromorphic function f on X such that* $\mathrm{div}(f) = D - D'$.

Proof. Let $s, s' \in \mathcal{L}(X)$ for $\mathcal{L}$ a holomorphic line bundle, $s, s' \neq 0$. Then s/s' is a well-defined meromorphic function, defined on an open chart U as $h_U(s)/h_U(s')$, since on a chart V,

$$\frac{h_V(s)}{h_V(s')} = \frac{t_{UV} h_U(s)}{t_{UV} h_U(s')} = \frac{h_U(s)}{h_U(s')}.$$

If $\mathcal{L} = \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, then $\mathrm{div}(s_D/s_{D'}) = D - D'$.

Conversely, suppose there is a meromorphic $f : X \to \mathbb{C}$ such that $\mathrm{div}(f) = D - D'$. On $(\operatorname{supp} D \cup \operatorname{supp} D')^c$, multiplication by f induces an isomorphism. One checks that this extends to the whole line bundles. □

Definition 3.22. Let $\mathrm{PDiv}(X) = \{\mathrm{div}\, f \mid f \in \mathbb{C}(X)^*\} \subseteq \mathrm{Div}(X)$ be the subgroup of *principal divisors*. The *divisor class group* is $\mathrm{Cl}(X) = \mathrm{Div}(X)/\,\mathrm{PDiv}(X)$.

Aside: Let $X = \operatorname{Spec} \mathcal{O}_K$ for $K/\mathbb{Q}$ a number field. As in number theory, $\mathrm{Div}(X) = \bigoplus_{0 \neq \mathfrak{p} \subseteq \mathcal{O}_k} \mathbb{Z}[\mathfrak{p}]$ and $\mathrm{PDiv}(X) = \{\sum n_\mathfrak{p}[\mathfrak{p}] \mid \prod \mathfrak{p}^{n_\mathfrak{p}} = (a) \text{ for some } a \in K\}$. Then $\mathrm{Cl}(X)$ is the class group of the number field, e.g. $\mathrm{Cl}(\operatorname{Spec} \mathbb{Z}[i]) = 1$ or $\mathrm{Cl}(\operatorname{Spec} \mathbb{Z}[\sqrt{-5}]) = \{1, (1 + \sqrt{-5}, 2)\}$.

Is every holomorphic line bundle $\mathcal{L} \to X$ of the form $\mathcal{L} \cong \mathcal{O}_X(D)$ for some divisor D? Equivalently, does every $\mathcal{L} \to X$ admit a meromorphic section $s \in \mathrm{Mero}(X, \mathcal{L})$? (If so, $\mathcal{L} \cong \mathcal{O}(\mathrm{div}\, s)$.)

Definition 3.23. Let $\mathrm{Pic}\, X$ denote the set of isomorphism classes of holomorphic line bundles $\mathcal{L} \to X$, which is a group under the tensor product.

Indeed, if $\mathcal{L}_1 \to X$ and $\mathcal{L}_2 \to X$ are line bundles, their tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2 \to X$ naturally admits the structure of a line bundle: For U, V trivializing charts of both $\mathcal{L}_i$, given the transition functions $t^i_{UV} : U \cap V \to \mathcal{C}^\times$ for $\mathcal{L}^i$, the transition function for $\mathcal{L}_1 \otimes \mathcal{L}_2$ is $t_{UV} = t^1_{UV} t^2_{UV}$.

Note that by proposition 3.21, the map $\mathrm{Cl}(X) \to \mathrm{Pic}(X), [D] \mapsto \mathcal{O}_X(D)$ is a well-defined group homomorphism.

Example 3.24. $\mathrm{Cl}(\mathbb{CP}^n) = \mathbb{Z}$. Let $D \subsetneq \mathbb{CP}^n$ be a divisor. It has the form $D = V(f_d)$ where $f_d \in \mathbb{C}[x_0, \dots, x_n]^{(d)} \setminus 0$. (see below) If $D' = V(g_d)$ is another divisor of the same degree, then $D - D' = \div(f_d/g_d)$ with $f_d/g_d \in \mathbb{C}(\mathbb{CP}^n)^\times$, so $[D] = [D']$.

Chow's Lemma: If X is a projective variety over $\mathbb{C}$, every closed analytic subspace of X is algebraic.

Example 3.25. $y - e^x = 0$ in $\mathbb{C}^2$ is not algebraic, but $\mathbb{C}^2$ is not projective. What about its closure in $\mathbb{CP}^1 \times \mathbb{CP}^1$? Since e^x has a transcentental pole at ∞, it obtains almost all values in any neighbourhood of ∞. Therefore, the closure contains $\infty \times \mathbb{CP}^1$. So it is not an analytic subspace of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Hermitian Metrics

Let $E \to X$ be a holomorphic vector bundle.

Definition 3.26. A *hermitian metric* is a smoothly varying hermitian metric $h_x.\overline{E}_x \otimes E_x \to \mathbb{C}$. In other words, $h \in C^\infty(X, \overline{E}^* \otimes E^*)$, where E^* is the dual line bundle, with transition functions t^{-1}_{UV}, if t_{UV} is the transition function on E, and $h(\overline{e}, e) > 0$ for $e \in E(U)$ nonvanishing.

Example 3.27. Let $D = \sum n_p[P] \in \mathrm{Div}(C)$, C a Riemann surface. For $P_i \in \operatorname{supp} D$, let $p \in W_i \subset V_i$ be opens and $U = (\bigcup W_i)^c$. Then $\mathcal{O}(D)|_U = \mathcal{O}_U$.

Let $s_D \in \mathrm{Mero}(C, \mathcal{O}(D))$ such that $\mathrm{div}(s_D) = D$. Define $h_x(\overline{s}_D, s_D) = 1$ on $x \in (\bigcup V_i)^c$ and $h_x(\overline{s}_D, s_D) = |x|^{2n_i}$ in the chart $W_i \ni P_i$ with local coordinate x. In the trivialization $\mathcal{O}(D)|_{V_i} \cong \mathcal{O}_{V_i}$, $h_x(1,1) = 1$. Then smoothly interpolate $h(\overline{s}_D, s_D)$ on the annuli $V_i \setminus W_i$

If h is a hermitian metric $E \to X$, then $h(\bar{s}, t) \in C^\infty(X)$, for $s, t \in C^\infty(X, E)$.

Proposition 3.28. *Let $\mathcal{L} \to X$ be a holomorphic line bundle with a hermitian metric h. Let $s \in \mathcal{L}(U)$ be a local holomorphic nonvanishing section (s generates $\mathcal{L}|_U$). Then $\frac{i}{2\pi}\partial\bar\partial \log h(s, \bar{s})$ is independent of s.*

Proof. If $s' \in \mathcal{L}(U)$ is some other generator, then $s' = fs$ for some $f \in \mathcal{O}^\times(U)$. Then

$$\frac{i}{2\pi}\partial\bar\partial \log h(\overline{fs}, fs) = \frac{i}{2\pi}\partial\bar\partial\big(\log h(\bar{s}, s) + \log \bar{f} + \log f\big) = \frac{i}{2\pi}\partial\bar\partial \log h(\bar{s}, s),$$

since $\log f$ is killed by $\bar\partial$ and $\log \bar{f}$ by ∂. □

Set $c_1(\mathcal{L}) = [\frac{i}{2\pi}\partial\bar\partial \log h(s, \bar{s})] \in H^2_{dR}(X, \mathbb{C})$, the *first Chern class* of $\mathcal{L}$. This is independent of h: If h' is another hermitian metric on $\mathcal{L}$, then $h' = ch$ for $c \in C^\infty(X)$, $c > 0$. Hence $\frac{i}{2\pi}\partial\bar\partial \log h'(s, \bar{s}) = \frac{i}{2\pi}\partial\bar\partial(\log c + \log h(s, \bar{s})$ and $\partial\bar\partial \log c = d\bar\partial \log c$ is exact.

Recall the exponential exact sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 1$. A holomorphic line bundle defines a class $[t] \in H^1(X, \mathcal{O}^\times)$: Let t_{UV} be the transition functions. These satisfy $t_{U_0U_1} t^{-1}_{U_0U_2} t_{U_1U_2} = 1$, so $t = \{t_{U_iU_j}\} \in Z^1(X, \mathcal{O}^*)$. The long exact sequence in cohomology associated to the exponential exact sequence maps $[t]$ to $c_1(\mathcal{L})$, in particular, $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$.