# Algebraic Topology
# Serre spectral sequence, characteristic classes and bordism

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## Contents

**1 Informal introduction** **2**

**2 Spectral sequences** **4**

# 1 Informal introduction

One of the big goals of homotopy theory is to compute

$$[X,Y]_\bullet = \{\text{base-point preserving cont. maps } X \to Y\}/\text{homotopy}$$

for $X$ and $Y$ pointed CW-complexes. CW-complexes are build out of spheres, hence the building blocks are the sets $[S^n, S^k]_\bullet = \pi_n(S^k,*)$. For $n \geq 1$, there are groups, abelian if $n > 1$. What do we know about these groups?

- $\pi_n(S^k,*) = 0$ for $n < k$ by cellcular approximation.
- $\pi_n(S^n,*) \cong \mathbb{Z}$ by the Hurewicz theorem and $H_n(S^n,\mathbb{Z}) \cong \mathbb{Z}$
- $X$ is $(n-1)$-connected CW-complex: Then $\pi_n(X,*) \cong H_n(X,\mathbb{Z})$.
- $\pi_k(S^1,*) = 0$ for $k \geq 2$ by covering space theory (universal cover of $S^1$ is $\mathbb{R}$, which is contractible).
- $\pi_3(S^2,*) \neq 0$, since the attaching map of the 4-cell for $\mathbb{CP}^2$ is a map $\eta : S^3 \to S^2 \cong \mathbb{CP}^1$. If this was null-homotopic, then we would have $\mathbb{CP}^2 \sim S^2 \vee S^4$, which contradicts the ring structure on $H^\bullet(\mathbb{CP}^2,\mathbb{Z}) \cong \mathbb{Z}[x]/x^3$.
- $\pi_k(S^n,*) \to \pi_{k+1}(S^{n+1},*) \to \pi_{k+2}(S^{n+2},*) \to \cdots$ eventually stabilizes by the Freudenthal suspension theorem.

To go beyond this, we need a new tool, the Serre spectral sequence. To motivate its usefulness, consider the following strategy: There exists a map $f : S^2 \to K(\mathbb{Z},2)$ which induces an isomorphism $f_* : \pi_2(S^2,*) \to \pi_2(K(\mathbb{Z},2),*)$. We can take its homotopy fibre $H = \mathrm{hofb}_x(f)$ (2-connected cover of $S^2$). Then there is a fiber sequence $H \to S^2 \xrightarrow{f} K(\mathbb{Z},2)$ and a long exact sequence in homotopy

$$\cdots \to \pi_4(K(\mathbb{Z},2),*) \to \pi_3(H,*) \to \pi_3(S^2,*) \to \pi_3(K(\mathbb{Z},2),*) \to \pi_2(H,*) \to \pi_2(S^2,*) \to$$

$$\to \pi_2(K(\mathbb{Z},2),*) \to \pi_1(H,*) \to \pi_1(S^2,*) \to \cdots$$

from which we conclude $\pi_3(H,*) \cong \pi_3(S^2,*)$ and $\pi_1(H,*) = \pi_2(H,*) = 0$, i.e. $H$ is 2-connected and the higher homotopy groups agree with the ones of $S^2$. By the Hurewicz theorem, $\pi_3(S^2,*) = H_3(H,\mathbb{Z})$. Hence we want to find a way to compute $H_*(H,*)$ from $H_*(S^2,\mathbb{Z})$ and $H_*(K(\mathbb{Z},2),\mathbb{Z})$.

This will also help to compute $\pi_n(S^k,*)$ in other ways (for example we will show that $\pi_n(S^k,*)$ is finite unless $n = k$ or $n = 2k-1$ and $k$ even). Furthermore, the Serre spectral sequence will allow us to compute the (co-)homology of spaces like $U(n), SU(n), \Omega S^n, K(\mathbb{Z}/2,n)$ etc. and (re-)prove structural theorems like Hurewicz, Freudenthal suspension, Thom isomorphisms and more.

So, given a fiber sequence $F \to Y \to X$, what could the relationship between the homology groups of $F, Y$ and $X$ be?

**Example 1.1.** Consider the easiest case $F \to X \times F \xrightarrow{\mathrm{pr}_X} X$, the trivial filtration. Then the Alexander-Whitney map induces an isomorphism

$$H_n(X \times F,\mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X, H_q(F)).$$

This is the kind of result we want: It computes the homology of the total space in terms of the homology of $X$ and $F$.

**Example 1.2** (Hopf fibration). $S^1 \to S^3 \xrightarrow{\eta} S^2$:

| $n$ | $H_n(S^3, \mathbb{Z})$ | $\bigoplus_{p+q=n} H_p(S^2, H_q(S^1, \mathbb{Z})$ |
|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | 0 | $\mathbb{Z}$ |
| 2 | 0 | $\mathbb{Z}$ |
| 3 | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 4 | 0 | 0 |

Hence clearly the Künneth formula from the previous example is "too big" to describe the homology in this case. However, consider the "2-step"-filtration $S^1 \subseteq S^3$ which satisfies $\widetilde{H}_n(S^3/S^1, \mathbb{Z}) \cong \mathbb{Z}$ for $n = 2, 3$ and $0$ otherwise. Hence $H_\bullet(S^1, \mathbb{Z}) \oplus H_\bullet(S^3/S^1, \mathbb{Z})$ agrees with the right-hand side of the table above. This does not agree with $H_*(S^3, \mathbb{Z})$, because the long exact sequence corresponding to $S^1 \to S^3 \to S^3/S^1$ does not split into nice short exact sequences. Concretely, the boundary map $\widetilde{H}_2(S^3/S^1, \mathbb{Z}) \to H_1(S^1, \mathbb{Z})$ is an isomorphism, hence these two terms do not contribute to $H_\bullet(S^3, \mathbb{Z})$.

It turns out that something similar holds for all fibre sequences $F \to Y \to X$: There exists a filtration

$$F_0 \subseteq F_1 \ldots \subseteq F_m \subseteq \ldots \subseteq C_*(Y, \mathbb{Z})$$

on $C_*(Y, \mathbb{Z})$ such that $H_{p+q}(F_p/F_{p-1}) \cong C_p^{\text{cell}}(X, H_q(F, \mathbb{Z})$. To then understand $H_\bullet(Y, \mathbb{Z})$, one needs to understand the cancellations in the associated long exact sequences. This is best encoded in a spectral sequence.

# 2 Spectral sequences

**Definition 2.1.** A (homologically, Serre-graded) *spectral sequence* is a triple $(E^\bullet, d^\bullet, h^\bullet)$, where

- $(E^r)_{r\geq 2}$ is a sequence of $\mathbb{Z}$-bigraded abelian groups. We write $E^r_{p,q}$ for the $(p,q)$-graded part of $E^r$. $E^r$ is called the $r$-th *page* of the spectral sequence.
- $(d^r : E^r \to E^r)$ is a sequence of morphisms, called *differentals*, of bidegree $(-r, r-1)$ satisfying $d^r \circ d^r = 0$.
- $h^r : H_\bullet(E^r) \to E^{r+1}$ is a sequence of bigrading-preserving isomorphisms. Here $H_\bullet(E^r)$ denotes the homology with respect to $d^r$, which inherits a bigrading from $E^r$.

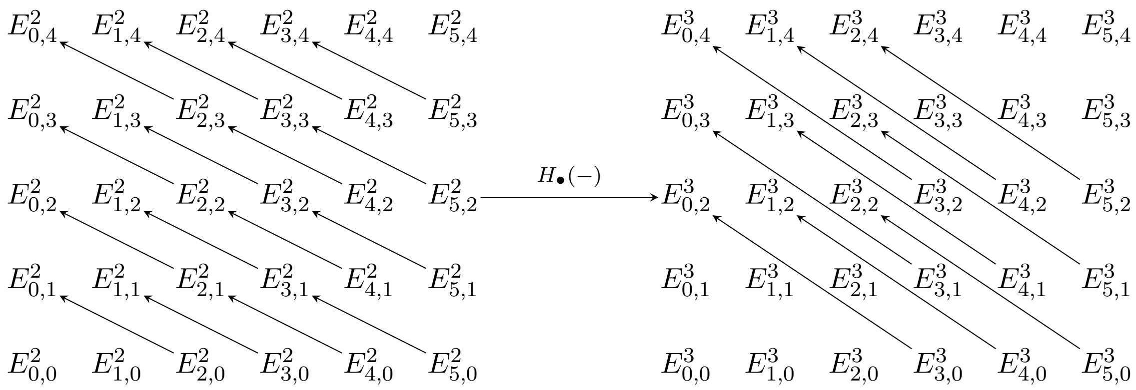

Figure 1: The second and third page of a spectral sequence

**Definition 2.2.** We say that a spectral sequence is *1st quadrant* if all abelian groups $E^2_{p,q}$ are trivial whenever $p < 0$ or $q < 0$.

**Lemma 2.3.** *For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we have $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$ for all $r \geq 2$. Moreover, for a given $(p,q) \in \mathbb{Z}^2$, the map $h$ induces an isomorphism $E^r_{p,q} \to E^{r+1}_{p,q}$ for $r > r_0 = \max(p, q+1)$, i.e. the groups $E^r_{p,q}$ stabilize as $r \to \infty$.*

*Proof.* The first statement follows directly from the existence of the isomorphisms $h$ by induction on $r$. For the second statement, if $r > r_0$, then the target of the differental $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial, hence every element of $E^r_{p,q}$ is a cycle. Moreover, the domain of the incoming differential $d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}$ is trivial. Hence $E^r_{p,q} \cong H_\bullet(E^r_{p,q}) \xrightarrow[h]{\cong} E^{r+1}_{p,q}$ □

**Definition 2.4.** For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we define the $E^\infty$-*page* as the bigraded abelian group $E^\infty_{p,q} = E^{r_0+1}_{p,q}$ with $r_0 = \max(p, q+1)$. By the previous lemma, $E^\infty_{p,q} \cong E^r_{p,q}$ whenever $r > r_0$

By a filterted object in an abelian category $\mathcal{A}$ we mean an object $H \in \mathcal{A}$ with a sequence of inclusions

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^n \subseteq \ldots \subseteq H.$$

We will apply this to $\mathcal{A}$ the category of graded abelian groups and $H = H_*(E, \mathbb{Z})$.

**Definition 2.5.** A first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$ is said to *converge* to a filtered object in graded abelian groups $(H, F)$ if there is a chosen isomorphism $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$ for all $p, q$ and $F^p_n = H_n$ if $n \leq p$. In this case we write $E^2_{p,q} \Rightarrow H$.

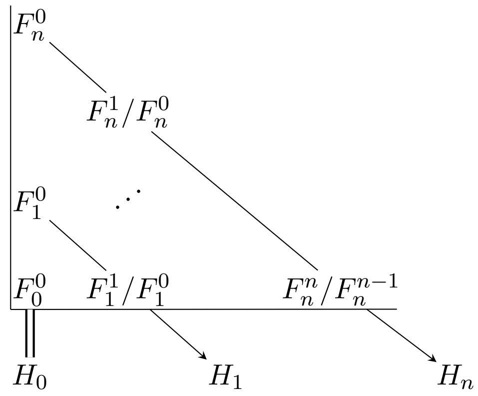


Figure 2: Visualization of $E^\infty$ as filtrations of the $H_i$ for a convergent spectral sequence $E^2_{p,q} \Rightarrow H$

**Remark.** Convergence is really a *datum* of the necessary isomorphism and not a property. Convergent spectral sequences are often simply encoded as $E^2_{p,q} \Rightarrow H$, but this suppresses not only this data, but also the higher pages, the differentials, and the filtration on $H$.

We now want to introduce the Serre spectral sequence for the homology of fibre sequences.

**Definition 2.6.** Let $f : Y \to X$ be a continuous map of topological spaces and $x \in X$ a point. The *homotopy fibre* $\mathrm{hofb}_x(f)$ of $f$ at $x$ is defined to be

$$\mathrm{hofb}_x(f) = P_x X \times_X Y$$

where $P_x X = \{\gamma : [0,1] \to X \mid \gamma(1) = x\}$ is the based path space of $X$ It comes with a map $P_x X \to X$ given by $\gamma \mapsto \gamma(0)$. In words: $\mathrm{hofb}_x(f)$ is the space of pairs $(\gamma, y)$ where $y \in Y$ and $\gamma$ is a path in $X$ from $f(y)$ to $x$. We note that $P_x X$ is contractible by the homotopy

$$H : P_x X \times [0,1] \to P_x X, \quad (\gamma, t) \mapsto s \mapsto \gamma((1-t)s + t)$$

**Example 2.7.** If $f : * \to X$, then $\mathrm{hofb}_x(f) = \Omega_x X$.

**Definition 2.8.** A *fibre sequence* of topological spaces is a sequence $F \xrightarrow{i} Y \xrightarrow{f} X$, a basepoint $x \in X$, a homotopy $h : F \to X^{[0,1]}$ from the composite $f \circ i$ to the constant map with value $c_x : F \to X$ and such that the induced map $F \to \mathrm{hofb}_x(f),\ z \mapsto (h(z), i(z))$ is a weak homotopy equivalence.

Recall: A weak homotopy equivalence is a map inducing isomorphisms on $\pi_n(-, x)$ for all $n \in \mathbb{N}$ and all basepoints $x$.

**Example 2.9.** 1. Let $f : Y \to X$ be any continuous map, $x \in X$. Then the pair $(\mathrm{hofb}_x f \to Y \to X, H)$, where $H$ is the homotopy from the definition of the homotopy fibre above. Every fibre sequence is equivalent to this in the following sense: Given $(F \to Y \to X, h)$, there is a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\simeq"] \arrow[d] & \mathrm{hofb}_x(f) \arrow[d] \\
Y \arrow[r, equal] \arrow[d, "f"] & Y \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

In particular, $\Omega_x X \to x \to X$ is a fibre sequence where $h : \Omega_x X \times [0,1] \to X$ is the evaluation map. If one instead chooses the constant homotopy, one does not obtain a fibre sequence (unless the space is contractible). This is because the induced map $\Omega_x X \to \mathrm{hofb}_x(f) = \Omega_x X$ is constant and hence usually not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces $F$ and $X$, $x \in X$, the pair $(F \to F \times X \to X, \mathrm{const})$ is a fibre sequence, the *trivial fibre sequence*. To see that, note that $\mathrm{hofb}_x(\mathrm{pr}_X) = F \times P_x X$ with induced map

$$F \to F \times P_x X, \quad y \mapsto (y, \mathrm{const}),$$

which is a homotopy equivalence as $P_x X$ is contractible.

3. Let $p : E \to B$ be a fibre bundle with fibre $F = p^{-1}(b)$ for some $b \in B$. Then the sequence $F \to E \to B$ with the constant homotopy is a fibre sequence. This is a special case of the next example.

4. Recall that $p : E \to B$ is a Serre fibration if in every commutative diagram of the form

$$\begin{tikzcd} D^n \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a lift $D^n \times I \to E$ making both triangles commute. Given a Serre fibration $p : E \to B$ and $b \in B$, the sequence $F = p^{-1}(b) \to E \to B$ with the constant homotopy is a fibre sequence. (see exercises) Note: Every fibre sequence is also equivalenct to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle $S^1 \to S^3 \xrightarrow{\eta} S^2$. It arises by letting $S^1 = U(1)$ act on $S^2 \subseteq \mathbb{C}^2$ via $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$, with quotient space $\mathbb{CP}^1 \cong S^2$.

6. Example 5 generalizes to fibre bundles $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ with limit case $S^1 \to S^\infty \to \mathbb{CP}^\infty$, which is equivalent to $\Omega\mathbb{CP}^\infty \to * \to \mathbb{CP}^\infty$.