Algebraic Topology
Serre spectral sequence, characteristic classes and bordism

by Prof. Dr. Markus Hausmann

notes by Stefan Albrecht

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1 Informal introduction

One of the big goals of homotopy theory is to compute

$$[X,Y]_\bullet = \{\text{base-point preserving cont. maps } X \to Y\}/\text{homotopy}$$

for X and Y pointed CW-complexes. CW-complexes are build out of spheres, hence the building blocks are the sets $[S^n, S^k]_\bullet = \pi_n(S^k, *)$. For $n \geq 1$, there are groups, abelian if $n > 1$. What do we know about these groups?

- $\pi_n(S^k, *) = 0$ for $n < k$ by cellcular approximation.
- $\pi_n(S^n, *) \cong \mathbb{Z}$ by the Hurewicz theorem and $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$
- X is $(n-1)$-connected CW-complex: Then $\pi_n(X, *) \cong H_n(X, \mathbb{Z})$.
- $\pi_k(S^1, *) = 0$ for $k \geq 2$ by covering space theory (universal cover of S^1 is $\mathbb{R}$, which is contractible).
- $\pi_3(S^2, *) \neq 0$, since the attaching map of the 4-cell for $\mathbb{CP}^2$ is a map $\eta : S^3 \to S^2 \cong \mathbb{CP}^1$. If this was null-homotopic, then we would have $\mathbb{CP}^2 \sim S^2 \vee S^4$, which contradicts the ring structure on $H^\bullet(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}[x]/x^3$.
- $\pi_k(S^n, *) \to \pi_{k+1}(S^{n+1}, *) \to \pi_{k+2}(S^{n+2}, *) \to \cdots$ eventually stabilizes by the Freudenthal suspension theorem.

To go beyond this, we need a new tool, the Serre spectral sequence. To motivate its usefulness, consider the following strategy: There exists a map $f : S^2 \to K(\mathbb{Z}, 2)$ which induces an isomorphism $f_* : \pi_2(S^2, *) \to \pi_2(K(\mathbb{Z}, 2), *)$. We can take its homotopy fibre $H = \mathrm{hofb}_x(f)$ (2-connected cover of S^2). Then there is a fiber sequence $H \to S^2 \xrightarrow{f} K(\mathbb{Z}, 2)$ and a long exact sequence in homotopy

$$\cdots \to \pi_4(K(\mathbb{Z},2),*) \to \pi_3(H,*) \to \pi_3(S^2,*) \to \pi_3(K(\mathbb{Z},2),*) \to \pi_2(H,*) \to \pi_2(S^2,*) \to$$
$$\to \pi_2(K(\mathbb{Z},2),*) \to \pi_1(H,*) \to \pi_1(S^2,*) \to \cdots$$

from which we conclude $\pi_3(H, *) \cong \pi_3(S^2, *)$ and $\pi_1(H, *) = \pi_2(H, *) = 0$, i.e. H is 2-connected and the higher homotopy groups agree with the ones of S^2. By the Hurewicz theorem, $\pi_3(S^2, *) = H_3(H, \mathbb{Z})$. Hence we want to find a way to compute $H_*(H, *)$ from $H_*(S^2, \mathbb{Z})$ and $H_*(K(\mathbb{Z}, 2), \mathbb{Z})$.

This will also help to compute $\pi_n(S^k, *)$ in other ways (for example we will show that $\pi_n(S^k, *)$ is finite unless $n = k$ or $n = 2k - 1$ and k even). Furthermore, the Serre spectral sequence will allow us to compute the (co-)homology of spaces like $U(n), SU(n), \Omega S^n, K(\mathbb{Z}/2, n)$ etc. and (re-)prove structural theorems like Hurewicz, Freudenthal suspension, Thom isomorphisms and more.

So, given a fiber sequence $F \to Y \to X$, what could the relationship between the homology groups of F, Y and X be?

Example 1.1. Consider the easiest case $F \to X \times F \xrightarrow{\mathrm{pr}_X} X$, the trivial filtration. Then the Alexander-Whitney map induces an isomorphism

$$H_n(X \times F, \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X, H_q(F)).$$

This is the kind of result we want: It computes the homology of the total space in terms of the homology of X and F.

Example 1.2 (Hopf fibration)**.** $S^1 \to S^3 \xrightarrow{\eta} S^2$:

n	$H_n(S^3,\mathbb{Z})$	$\bigoplus_{p+q=n} H_p(S^2, H_q(S^1,\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}$
2	0	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0

Hence clearly the Künneth formula from the previous example is "too big" to describe the homology in this case. However, consider the "2-step"-filtration $S^1 \subseteq S^3$ which satisfies $\widetilde{H}_n(S^3/S^1,\mathbb{Z}) \cong \mathbb{Z}$ for $n = 2, 3$ and 0 otherwise. Hence $H_\bullet(S^1,\mathbb{Z}) \oplus H_\bullet(S^3/S^1,\mathbb{Z})$ agrees with the right-hand side of the table above. This does not agree with $H_*(S^3,\mathbb{Z})$, because the long exact sequence corresponding to $S^1 \to S^3 \to S^3/S^1$ does not split into nice short exact sequences. Concretely, the boundary map $\widetilde{H}_2(S^3/S^1,\mathbb{Z}) \to H_1(S^1,\mathbb{Z})$ is an isomorphism, hence these two terms do not contribute to $H_\bullet(S^3,\mathbb{Z})$.

It turns out that something similar holds for all fibre sequences $F \to Y \to X$: There exists a filtration

$$F_0 \subseteq F_1 \ldots \subseteq F_m \subseteq \ldots \subseteq C_*(Y,\mathbb{Z})$$

on $C_*(Y,\mathbb{Z})$ such that $H_{p+q}(F_p/F_{p-1}) \cong C_p^{\text{cell}}(X, H_q(F,\mathbb{Z})$. To then understand $H_\bullet(Y,\mathbb{Z})$, one needs to understand the cancellations in the associated long exact sequences. This is best encoded in a spectral sequence.

2 The Serre Spectral Sequence

Definition 2.1. A (homologically, Serre-graded) *spectral sequence* is a triple $(E^\bullet, d^\bullet, h^\bullet)$, where

- $(E^r)_{r\geq 2}$ is a sequence of $\mathbb{Z}$-bigraded abelian groups. We write $E^r_{p,q}$ for the (p,q)-graded part of E^r. E^r is called the r-th *page* of the spectral sequence.
- $(d^r : E^r \to E^r)$ is a sequence of morphisms, called *differentals*, of bidegree $(-r, r-1)$ satisfying $d^r \circ d^r = 0$.
- $h^r : H_\bullet(E^r) \to E^{r+1}$ is a sequence of bigrading-preserving isomorphisms. Here $H_\bullet(E^r)$ denotes the homology with respect to d^r, which inherits a bigrading from E^r.

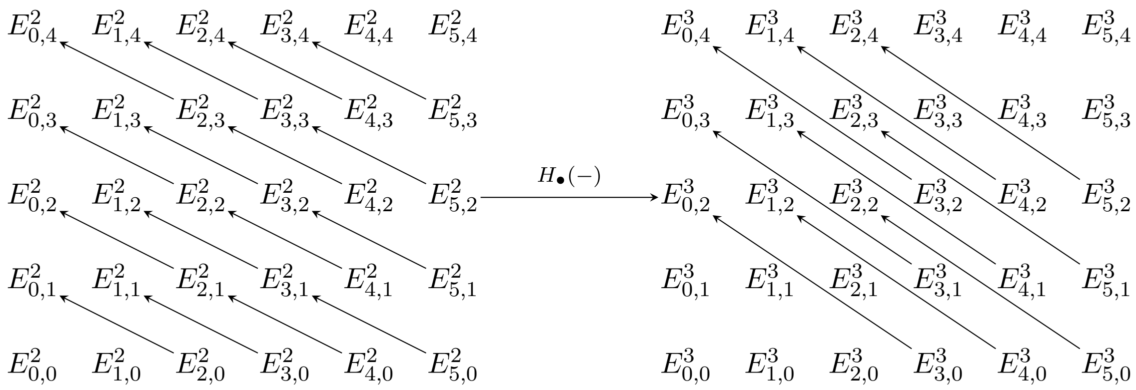


Figure 1: The second and third page of a spectral sequence

Definition 2.2. We say that a spectral sequence is *1st quadrant* if all abelian groups $E^2_{p,q}$ are trivial whenever $p < 0$ or $q < 0$.

Lemma 2.3. *For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we have $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$ for all $r \geq 2$. Moreover, for a given $(p,q) \in \mathbb{Z}^2$, the map h induces an isomorphism $E^r_{p,q} \to E^{r+1}_{p,q}$ for $r > r_0 = \max(p, q+1)$, i.e. the groups $E^r_{p,q}$ stabilize as $r \to \infty$.*

Proof. The first statement follows directly from the existence of the isomorphisms h by induction on r. For the second statement, if $r > r_0$, then the target of the differental $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial, hence every element of $E^r_{p,q}$ is a cycle. Moreover, the domain of the incoming differential $d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}$ is trivial. Hence $E^r_{p,q} \cong H_\bullet(E^r_{p,q}) \xrightarrow[h]{\cong} E^{r+1}_{p,q}$ □

Definition 2.4. For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we define the E^∞-*page* as the bigraded abelian group $E^\infty_{p,q} = E^{r_0+1}_{p,q}$ with $r_0 = \max(p, q+1)$. By the previous lemma, $E^\infty_{p,q} \cong E^r_{p,q}$ whenever $r > r_0$

By a filterted object in an abelian category $\mathcal{A}$ we mean an object $H \in \mathcal{A}$ with a sequence of inclusions

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^n \subseteq \ldots \subseteq H.$$

We will apply this to $\mathcal{A}$ the category of graded abelian groups and $H = H_*(E, \mathbb{Z})$.

Definition 2.5. A first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$ is said to *converge* to a filtered object in graded abelian groups (H, F) if there is a chosen isomorphism $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$ for all p, q and $F^p_n = H_n$ if $n \leq p$. In this case we write $E^2_{p,q} \Rightarrow H$.

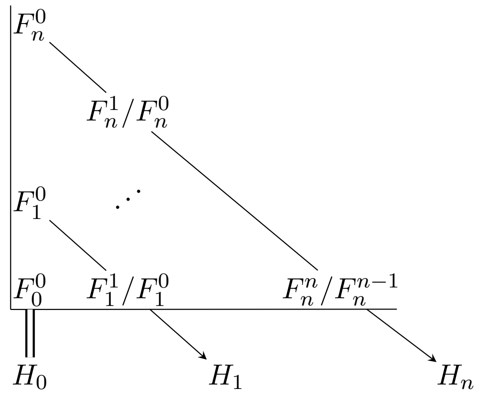


Figure 2: Visualization of E^∞ as filtrations of the H_i for a convergent spectral sequence $E^2_{p,q} \Rightarrow H$

Remark. Convergence is really a *datum* of the necessary isomorphism and not a property. Convergent spectral sequences are often simply encoded as $E^2_{p,q} \Rightarrow H$, but this suppresses not only this data, but also the higher pages, the differentials, and the filtration on H.

We now want to introduce the Serre spectral sequence for the homology of fibre sequences.

Definition 2.6. Let $f : Y \to X$ be a continuous map of topological spaces and $x \in X$ a point. The *homotopy fibre* $\mathrm{hofb}_x(f)$ of f at x is defined to be

$$\mathrm{hofb}_x(f) = P_xX \times_X Y$$

where $P_xX = \{\gamma : [0,1] \to X \mid \gamma(1) = x\}$ is the based path space of X. It comes with a map $P_xX \to X$ given by $\gamma \mapsto \gamma(0)$. In words: $\mathrm{hofb}_x(f)$ is the space of pairs (γ, y) where $y \in Y$ and γ is a path in X from $f(y)$ to x. We note that P_xX is contractible by the homotopy

$$H : P_xX \times [0,1] \to P_xX, \quad (\gamma, t) \mapsto s \mapsto \gamma((1-t)s + t)$$

Example 2.7. If $f : * \to X$, then $\mathrm{hofb}_x(f) = \Omega_x X$.

Definition 2.8. A *fibre sequence* of topological spaces is a sequence $F \xrightarrow{i} Y \xrightarrow{f} X$, a basepoint $x \in X$, a homotopy $h : F \to X^{[0,1]}$ from the composite $f \circ i$ to the constant map $c_x : F \to X$ such that the induced map $F \to \mathrm{hofb}_x(f),\ z \mapsto (h(z), i(z))$ is a weak homotopy equivalence.

Recall: A weak homotopy equivalence is a map inducing isomorphisms on $\pi_n(-, x)$ for all $n \in \mathbb{N}$ and all basepoints x.

Example 2.9. 1. Let $f : Y \to X$ be any continuous map, $x \in X$. Then the pair $(\mathrm{hofb}_x f \to Y \to X, H)$, where H is the homotopy from the definition of the homotopy fibre above, is a fibre sequence. Every fibre sequence is equivalent to this in the following sense: Given $(F \to Y \to X, h)$, there is a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\simeq"] \arrow[d] & \mathrm{hofb}_x(f) \arrow[d] \\
Y \arrow[r, equal] \arrow[d, "f"] & Y \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

In particular, $\Omega_x X \to x \to X$ is a fibre sequence, where $h : \Omega_x X \times [0,1] \to X$ is the evaluation map. If one instead chooses the constant homotopy, one does not obtain a fibre sequence (unless the space is contractible). This is because the induced map $\Omega_x X \to \mathrm{hofb}_x(f) = \Omega_x X$ is constant and hence usually not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces F and X, $x \in X$, the pair $(F \to F \times X \to X, \mathrm{const})$ is a fibre sequence, the *trivial fibre sequence*. To see that, note that $\mathrm{hofb}_x(\mathrm{pr}_X) = F \times P_x X$ with induced map

$$F \to F \times P_x X, \quad y \mapsto (y, \mathrm{const}),$$

which is a homotopy equivalence as $P_x X$ is contractible.

3. Let $p : E \to B$ be a fibre bundle with fibre $F = p^{-1}(b)$ for some $b \in B$. Then the sequence $F \to E \to B$ with the constant homotopy is a fibre sequence. This is a special case of the next example.

4. Recall that $p : E \to B$ is a Serre fibration if in every commutative diagram of the form

$$\begin{tikzcd} D^n \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a lift $D^n \times I \to E$ making both triangles commute. Given a Serre fibration $p : E \to B$ and $b \in B$, the sequence $F = p^{-1}(b) \to E \to B$ with the constant homotopy is a fibre sequence. (see exercises) Note: Every fibre sequence is also equivalenct to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle $S^1 \to S^3 \xrightarrow{\eta} S^2$. It arises by letting $S^1 = U(1)$ act on $S^2 \subseteq \mathbb{C}^2$ via $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$, with quotient space $\mathbb{CP}^1 \cong S^2$.

6. Example 5 generalizes to fibre bundles $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ with limit case $S^1 \to S^\infty \to \mathbb{CP}^\infty$, which is equivalent to $\Omega\mathbb{CP}^\infty \to * \to \mathbb{CP}^\infty$.

We are now ready to state the existence of the Serre spectral sequence.

Theorem 2.10 (Serre). *For every fibre sequence $(F \xrightarrow{\iota} Y \xrightarrow{p} X, h)$ with X simply-connected and abelian group A, there exists a spectral sequence of the following form*

$$E^2_{p,q} = H_p(X, H_q(F, A)) \implies H_{p+q}(Y, A)$$

As noted before, this information does not include the differentials and the higher pages, as well as the filtrations on $H_\bullet(Y, A)$ and the identifications of its subquotients with the E^∞-page.

One edge case is easy to state: The map

$$H_n(F, A) = H_0(X, H_n(F, A)) = E^2_{0,n} \twoheadrightarrow E^\infty_{0,n} \hookrightarrow H_n(Y, A)$$

agrees with the factorization $H_n(F, A) \twoheadrightarrow \mathrm{im}\, \iota_* \hookrightarrow H_n(Y, A)$.

We now assume this theorem and give some sample computations.

Example 2.11. We revisit the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. S^2 is simply connected, so we get a spectral sequence. The E^2-page is $H_p(S^2, H_q(S^1, A))$, which looks like

$$\begin{tikzcd} q & & & & \\ & 0 & 0 & 0 & 0 \\ & A & 0 & A & 0 \\ & A & 0 & A \arrow[ull] & 0 \\ & & & & p \end{tikzcd}$$

There is one potentially non-trivial d^2-differential, namely $d^2 : E^2_{2,0} \to E^2_{0,1}$. All higher differentials d^r, $r > 2$, are trivial for degree reasons. Hence the E^∞-page looks as follows:

$$\begin{array}{c|cccc} q & & & & \\ & 0 & 0 & 0 & 0 \\ & \operatorname{coker}(d^2) & 0 & A & 0 \\ & A & 0 & \ker(d^2) & 0 \\ \hline & & & & p \end{array}$$

We know that $H_n(S^3, A) = A$ for $n = 0, 3$ and 0 else. From the E^∞-page we thus get $H_0(S^3, A) = A$, $H_1(S^3, A) = \operatorname{coker}(d^2)$, $H_2(S^3, A) = \ker(d^2)$, $H_3(S^3, A) = A$. Hence d^2 must be an isomorphism.

Lemma 2.12. *There is a fibre bundle*

$$U(n-1) \overset{i}{\hookrightarrow} U(n) \to S^{2n-1},$$

where $U(n)$ denotes the topological group of unitary $n \times n$-matrices and i is the standard inclusion which adds a trivial $\mathbb{C}$-summand.

Proof. The gorup $U(n)$ acts on $\mathbb{C}^n$ by definition. This action restricts to the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Furthermore, this action is transitive, because every vector of length 1 can be extended to an orthonormal basis. Hence S^{2n-1} is in bijection with the orbit space $U(n)/\operatorname{Stab}(x)$, for any $x \in S^{2n-1}$. For $x = (0, \ldots, 0, 1)$, the stabalizer equals $i(U(n-1))$. We obtain a continuous bijective map $U(n)/U(n-1) \to S^{2n-1}$, $[A] \mapsto A(0, \ldots, 0, 1)^t$, which is a homeomorphism since its domain is quasi-compact and its codomain is Hausdorff. Finally, we use the fact that for a smooth, free action of a compact Lie group G on a manifold M, the map $M \to M/G$ is always a fibre bundle (in fact a G-principal bundle). $\square$

Example 2.13. We consider the case $n = 2$, i.e. the fibre sequence $S^1 \cong U(1) \hookrightarrow U(2) \to S^3$. We want to compute the homology of $U(2)$ via the Serre spectral sequence $E^2_{p,q} = H_p(S^3, H_q(S^1, \mathbb{Z}))$ All differentials on all pages have to be trivial for degree reasons. (The spectral sequence "collapses".) Hence $E^\infty = E^2$ and every antidiagonal has at most one non-trivial term, so we can read off $H_n(U(2), \mathbb{Z}) = \mathbb{Z}$ for $n = 0, 1, 3, 4$ and 0 else. In fact, one can show that $U(2) \cong S^3 \times U(1)$, so the homology could alternatively be computed with the Künneth theorem.

Example 2.14. Next we consider the fibre sequence $U(2) \hookrightarrow U(3) \to S^5$ with E^2-page $E^2_{p,q} = H_p(S^5, H_q(U(2), \mathbb{Z}))$, which looks like

$$\begin{array}{c|cccccc} q & & & & & & \\ & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & Z & 0 & 0 & 0 & 0 & \mathbb{Z} \\ & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ \hline & & & & & & p \end{array}$$

The first potentially non-trivial differential is $d^5 : E^5_{0,5} \to E^5_{0,4}$. At this point we cannot decide what this differential is. All higher differentials are again trivial for degree reasons, and all filtrations collapse to at most one entry. We obtain

$$H_n(U(3),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0,1,3,5,8,9, \\ \operatorname{coker}(d^5) & \text{for } n = 4, \\ \ker(d^5) & \text{for } n = 5, \\ 0 & \text{else.} \end{cases}$$

This example illustrates a typical situation, namely that one can often not fully determine all differentials but still deduce a lot. We will soon see that $d^5 = 0$ and hence $H_4(U(3),\mathbb{Z}) \cong H_5(U(3),\mathbb{Z}) \cong \mathbb{Z}$

Example 2.15. We consider $U(3) \to U(4) \to S^7$. The $E^2_{p,q} = H_p(S^7, H_q(U(3),\mathbb{Z}))$-page is

$$\begin{array}{c|ccccc}
q & & & & & \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & 0 & & & & 0 \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & ? & & \vdots & & ? \\
 & ? & \cdots & 0 & \cdots & ? \\
 & \mathbb{Z} & & \vdots & & \mathbb{Z} \\
 & 0 & & & & 0 \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
\hline
 & 0 & & & & 7 \quad p
\end{array}$$

The only possibly non-trivial differentials are $d^7 : E^7_{7,0} \to E^7_{0,6}$ and $d^7 : E^7_{7,3} \to E^7_{0,9}$ which we cannot compute at this point. Nevertheless, we can still deduce a lot, for example $H_1(U(4),\mathbb{Z}) = \mathbb{Z} = H_3(U(4),\mathbb{Z})$, $H_2(U(4),\mathbb{Z}) = 0$, $H_4(U(4),\mathbb{Z}) \cong H_4(U(3),\mathbb{Z})$,$H_5(U(4),\mathbb{Z}) \cong H_5(U(3),\mathbb{Z})$ and there is a short exact sequence $0 \to \mathbb{Z} \to H_8(U(4),\mathbb{Z}) \to \mathbb{Z} \to 0$, so $H_8(U(4),\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

In the previous examples we used the Serre spectral sequence to compute the homology of the total space of the fibre sequence. We now show that it can also be used to compute the homology of the base space or fibre.

Example 2.16. We consider the fibre sequence $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ for $n \geq 2$ and want to compute the homology of $\mathbb{CP}^n$. By path-connectedness, the E^2-page of $H_p(\mathbb{CP}^n, H_q(S^1,\mathbb{Z})) \Rightarrow H_{p+q}(S^{2n+1},\mathbb{Z})$ begins like this:

$$\begin{array}{c|cc}
q & & \\
 & 0 & 0 \\
 & \mathbb{Z} & ? \\
 & \mathbb{Z} & ? \\
\hline
 & & \quad p
\end{array}$$

Since $H_1(S^{2n+1}, \mathbb{Z}) = 0$, there must be a surjective d^2-differential $d^2 : E^2_{2,0} \to E^2_{0,1}$. But since $H_2(S^{2n+1}, \mathbb{Z}) = 0$, this differential must also be injective. Hence

$$\mathbb{Z} \cong E^2_{2,0} = H_2(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong H_2(\mathbb{CP}^n, \mathbb{Z}).$$

Furthermore, we see that $E^2_{1,0} = H_1(\mathbb{CP}^n, \mathbb{Z}) = 0$. Using $H_0(S^1, \mathbb{Z}) = H_1(S^1, \mathbb{Z})$, this implies $E^2_{1,1} = 0$ and $E_{2,1} = \mathbb{Z}$. Now we see that the 2-page looks like

$$\begin{array}{r|cccc} q & & & & \\ & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & 0 & \mathbb{Z} & ? \\ & \mathbb{Z} & 0 & \mathbb{Z} & ? \\ \hline & & & & p \end{array}$$

By the same argument, we can deduce $d^2 : E_{4,0} \to E_{2,1}$ is an isomorphism, i.e. $H_4(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong \mathbb{Z}$, and $E_{3,0} = E_{3,1} = 0$ and so on. Since $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$, we cannot conlude that the $\mathbb{Z}$ in bidegree $(2n, 1)$ must be the image of a differental. There are two possibilities: If $d^2 : E^2_{2n+2,0} \to E^2_{2n+1}$ is the trivial map, then $E^2_{2n+2,0} = 0$ and then by induction $E^2_{p,q} = 0$ for all $p > 2n$. If, on the other hand, $d^2 : E^2_{2n+2,0} \to E^2_{2n,1}$ is non-zero, it has to be surjective: Indeed, since the cokernel is isomorphic to the lowest term of the filtration on $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$, and no $\mathbb{Z}/n\mathbb{Z}$ embeds into $\mathbb{Z}$. This then implies $H_k(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$ for all $k > 2n$. This case can be ruled out using that $\mathbb{CP}^n$ is a $2n$-dimensional CW-complex and hence $H_n(\mathbb{CP}^n, \mathbb{Z}) = 0$ for $k > 2n$. In summary, we obtain

$$H_k(\mathbb{CP}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, \ldots, 2n, \\ 0 & \text{else.} \end{cases}$$

Next we turn to an example where the Serre spectral sequence can be used to compute the homology of the fibre

Example 2.17. We consider the fibre sequence $\Omega S^3 \to * \to S^3$. On the E^2-page, we have the entries $H^p(S^3, H_q(\Omega S^3, \mathbb{Z}))$, i.e.

$$\begin{array}{r|cccc} q & & & & \\ & \vdots & \vdots & \vdots & \vdots \\ & ? & 0 & 0 & ? \\ & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \hline & & & & p \end{array}$$

The homology of the point is 0 in positive degrees, so we must have $E^\infty_{p,q} = 0$ unless $p = q = 0$. The only non-trivial differentials are $d^3 : E^3_{3,q} \to E^3_{0,q+2}$, so we conclude that these are isomorphisms. Hence $H_q(\Omega S^3, \mathbb{Z}) \cong H_{q+2}(\Omega S^3, \mathbb{Z})$. Note $E^{0,1} = 0$ since this entry cannot be killed by any differential. This implies

$$H_k(\Omega S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

In particular, ΩS^3 is an infinite-dimensional space.

We now discuss the cohomological version of the Serre spectral sequence and its multiplicative structure. This multiplication also helps in determining differentials, for example for the spectral sequences computing (co-)homology of unitary groups as above.

Definition 2.18. A *cohomologically graded spectral sequence* is a triple $(E_\bullet, d_\bullet, h_\bullet)$ where $(E_r)_r$ is a sequence of bigraded abelian groups, $(d_r : E_r \to E_r)_r$ is a sequence of differentials ($d_r \circ d_r = 0$) of bidegree $(r, 1 - r)$, and $(h_r : H_\bullet(E_r) \to E_{r+1})_r$ a sequence of bigrading-preserving isomorphisms.

As before, one defines first quadrant ($E_2^{p,q} = 0$ if $p < 0$ or $q < 0$) spectral sequences and the E_∞-page.

Rather than the filtrations $0 = F^{-1} \subseteq F^0 \subseteq \cdots \subseteq H$, one now considers filtrations

$$H = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

Definition 2.19. A cohomological first quadrant spectral sequence is said to *converge* to a filtered object (H, F) in graded abelian groups if there are isomorphisms $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$ for all p, q, and $F_p^n = 0$ for all $p > n$. Again we write $E_2^{p,q} \implies H$.

Definition 2.20. A *(commutative) multiplicative structure* on a cohomologically graded spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ is a bigraded (commutative) ring structure on E_r, i.e. there are associative maps $E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$, such that d_r is a graded derivation, i.e.

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{p+q} x \cdot d_r(y)$$

for $x \in E_r^{p,q}$. Here (graded) commutative means $xy = (-1)^{(p+q)(p'+q')} yx$. As a result, $H_\bullet(E_r)$ is a bigraded ring and we further require that the h_r are isomorphisms of bigraded rings. Furthermore, the E_∞-page also inherits the structure of a (commutative) bigraded ring.

Definition 2.21. A filtration $\cdots \subseteq F_n \subseteq \cdots \subseteq F_1 \subseteq F_0 = H$ on a graded ring H is said to be *multiplicative* (or compatible with the multiplicative structure) if $F_s F_t \subseteq F_{s+t}$. We say that (H, F) is a *filtered graded ring.*

It follows that the associated graded object $\bigoplus F_p/F_{p+1}$ of a filtered graded (commutative) ring is a bigraded (commutative) ring.

Definition 2.22. A multiplicative first quadrant spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ is said to *converge* to a filtered graded ring (H, F) if it converges additively and the chosen isomorphism $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$ is compatible with the graded ring structure.

Theorem 2.23 (Serre)**.** *For every fibre sequence of spaces $(F \to Y \to X, h)$ with X simply connected and every abelian group A, there exists a cohomological first quadrant spectral sequence of the form*

$$E_2^{p,q} = H^p(X, H^q(F, A)) \quad \implies \quad H^{p+q}(Y, A).$$

If A is a (commutative) ring, then the spectral sequence is multiplicative and converges multiplicatively, where on the E_2-page the multiplication is given by $(-1)^{p'q}$-times the composite

$$\begin{aligned} H^p(X, H^q(F,R)) \otimes H^{p'}(X, H^{q'}(F,R)) &\to H^{p+p'}(X, H^q(F,R) \otimes H^{q'}(F,R)) \\ &\to H^{p+p'}(X, H^{q+q'}(F,R)). \end{aligned}$$

Note: If $H^\bullet(F, R)$ or $H^\bullet(X, R)$ is flat over R of finite type, then the E_2-page is isomorphic to the graded tensor product of $H^\bullet(X, R)$ and $H^\bullet(F, R)$.

Example 2.24. We reconsider the fibre sequence $U(1) \to U(2) \to S^3$ with $E_2^{p,q} = H^p(S^3, H^q(U(1), \mathbb{Z}))$, i.e.

$$\begin{array}{c|cccc} q & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \hline & & & & p \end{array}$$

There cannot be non-trivial differentials. As a graded ring, the E_2-page (and hence also the E_∞-page) is isomorphic to $H^\bullet(S^3, \mathbb{Z}) \otimes H^\bullet(U(1), \mathbb{Z})$. Let $x_1 \in H^1(U(1), \mathbb{Z})$ and $x_3 \in H^3(S^3, \mathbb{Z})$ be generators.

Then $H^\bullet(U(1), \mathbb{Z}) \cong \bigwedge(x_1)$ and $H^\bullet(S^3, \mathbb{Z}) \cong \bigwedge(x_3)$, where $\bigwedge(M)$ denotes the exterior algebra on a set M, i.e. the free algebra on $x \in M$ modulo the relations $x_i x_j = -x_j x_i$ and $x_i^2 = 0$ for all $x_i, x_j \in M$. Hence, the E_2-page is isomorphic to $\bigwedge(x_1, x_3)$. The $\mathbb{Z}$ in bidegree $(3, 1)$ is spanned by $x_1 x_3$. The filtration collapses degreewise and hence $H^*(U(2), \mathbb{Z})$ is exterior on classes $x_1 \in H^1(U(2), \mathbb{Z})$ and $x_3 \in H^3(U(2), \mathbb{Z})$ that are uniquely determined by the spectral sequence.

Example 2.25. We move on to the fibre sequence $U(2) \to U(3) \to S^5$. The E_2-page looks like the homological spectral sequence

$$
\begin{array}{c|cccccc}
q & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
\hline
 & & & & & & p
\end{array}
$$

Keeping the notation from the previous example, denote generators of the first column by $1, x_1, x_3, x_1 x_3$. Again, the E_2-page is given by

$$H^\bullet(S^5, \mathbb{Z}) \otimes H^\bullet(U(2), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5),$$

where $x_5 \in H^5(S^5, \mathbb{Z})$ is a generator. There is one possibly non-trivial differential $d_5 : E_5^{0,4} \to E_5^{5,0}$. However, the product rule implies

$$d_5(x_1 x_3) = d_5(x_1) x_3 + (-1)^{1+0} x_1 d_5(x_3) = 0 + 0 = 0.$$

Again, the filtration collapses and hence $H^*(U(3), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5)$.

Example 2.26. We revisit $U(3) \to U(4) \to S^7$. The E_2-page now is

$$
\begin{array}{c|ccc}
q & \mathbb{Z} & & \mathbb{Z} \\
 & \mathbb{Z} & & \mathbb{Z} \\
 & 0 & & 0 \\
 & \mathbb{Z} & & \mathbb{Z} \\
 & \mathbb{Z} & \vdots & \mathbb{Z} \\
 & \mathbb{Z} & \cdots\ 0\ \cdots & \mathbb{Z} \\
 & \mathbb{Z} & \vdots & \mathbb{Z} \\
 & 0 & & 0 \\
 & \mathbb{Z} & & \mathbb{Z} \\
 & \mathbb{Z} & & \mathbb{Z} \\
\hline
 & 0 & & 7 \quad p
\end{array}
$$

As before, the product rule implies that all d_7 differentials must be trivial, and $E_2 \cong \bigwedge(x_1, x_3, x_5, x_7)$. There is a non-trivial filtration on $H^8(U(4), \mathbb{Z})$ of the form

$$0 \to \mathbb{Z}(x_1 x_7) \to H^8(U(4), \mathbb{Z}) \to \mathbb{Z}(x_3 x_5) \to 0$$

Additively the sequence splits, but one has to be careful with the multiplicative structure. To resolve this, we need to be precise with the differentiation between the classes x_i on the E_∞-page and the corresponding classes $\overline{x_i} \in H^*(U(4), \mathbb{Z})$. Note that the choice of each $\overline{x_i}$ is unique since the filtration collapses in degrees 0 to 7. Furthermore, we record their filtrations $\overline{x_1}$ is in F_0^1, $\overline{x}_3$ is in F_0^3, $\overline{x}_5$ is in F_0^5 and $\overline{x}_7$ is in F_7^7. It follows that $\overline{x}_1\overline{x}_7$ is a generator of F_7^8, and $\overline{x}_3\overline{x}_5$ is a generator of F_0^8/F_1^8. Hence, $H^8(U(4), \mathbb{Z})$ is a free group on $\overline{x}_1\overline{x}_7$ and $\overline{x}_3\overline{x}_5$, and it follows that $H^\bullet(U(4), \mathbb{Z}) \cong \Lambda(x_1, x_3, x_5, x_7)$.

Theorem 2.27. *For all $n \in \mathbb{N}$, there is an isomorphism of graded rings*

$$H^\bullet(U(n), \mathbb{Z}) \cong \bigwedge(x_1, x_3, \dots, x_{2n-1})$$

with x_i of degree i.

Proof. By induction on n, the start $n = 1$ is clear. Let $n \geq 2$ and assume we know the statement for $n - 1$. We consider the Serre spectral sequence for the fibre sequence $U(n-1) \to U(n) \to S^{2n-1}$. By induction, its E_2-page is isomorphic to

$$E_2 \cong H^\bullet(S^{2n-1}, \mathbb{Z}) \otimes H^\bullet(U(n-1), \mathbb{Z}) = \bigwedge(x_{2n-1}) \otimes \bigwedge(x_1, x_3, \dots, x_{2n-3}).$$

Here, x_i is a generator of $E_2^{0,i}$ for $i \leq 2n-3$ and x_{2n-1} is a generator of $E_2^{2n-1,0}$. The only possibly non-trivial differentials are d_{2n-1}. For degree reasons, d_{2n-1} vanishes on all generators $x_1, x_3, \dots, x_{2n-1}$. By the product rule, all differentials are 0. Hence the E_∞-page is isomorphic to the E_2-page, and an exterior algebra $\bigwedge(x_1, x_3, \dots, x_{2n-1})$.

The filtrations on $H^\bullet(U(n), \mathbb{Z})$ collapse in degrees $0, \dots, 2n-2$, therefore we obtain unique lifts $\overline{x}_1, \dots, \overline{x}_{2n-1} \in H^\bullet(U(n), \mathbb{Z})$. We only know from the spectral sequence that x_i^2 is of lower filtration hence a multiple of $\overline{x}_{2n-1}$, but not necessarily that $\overline{x_i}^2 = 0$. However we know from the additive structure (all subquotients are free over $\mathbb{Z}$) that $H^\bullet(U(n), \mathbb{Z})$ is torsionfree. As the multiplication is graded commutative, we hence have $\overline{x}_i^2 = 0$. We obtain a ring map $f : \bigwedge(x_1, \dots, x_{2n-1}) \to H^\bullet(U(n), \mathbb{Z})$ by sending x_i to $\overline{x}_i$. To check that f is an isomorphism, define a grading on $\bigwedge(x_1, \dots, x_{2n-1})$ by setting the degree of $x_1, \dots, x_{2n-3}$ to be 0, and the degree of x_{2n-1} to be $2n-1$. This induces a filtrion by setting F_i to be the direct sum of the graded pieces of detree $2i$. Then f is filtration preserving and induces an isomorphism on associated graded pieces. The proof is finished by the following lemma. □

Lemma 2.28. *Let A and B be graded abelian groups equipped with filtrations*

$$\cdots F_2 \subseteq F_1 \subseteq F_0 = A \qquad \textit{and} \qquad \cdots G_2 \subseteq G_1 \subseteq G_0 = B$$

which are eventually 0 in every degree. If we have a graded filtration-preserving morphism $f : A \to B$ that induces an isomorphism on all associated graded pieces $F_i/F_{i+1} \cong G_i/G_{i+1}$, then it is an isomorphism.

Proof. This is an iterated 5-lemma argument. □

Example 2.29. We revisit the fibre sequence $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and use the Serre spectral sequence to compute the ring $H^\bullet(\mathbb{CP}^n, \mathbb{Z})$. Arguing analoguously to the homological case, the E_2-page looks as follows:

$$\begin{tikzcd}[row sep=small, column sep=small]
q & & & & & & & \\
& \mathbb{Z} \arrow[drr, "\cong"] & 0 & \mathbb{Z} \arrow[drr, "\cong"] & 0 & \cdots \arrow[drr, "\cong"] & 0 & \mathbb{Z} \\
& \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\
& 0 & & & & & & 2n \quad p
\end{tikzcd}$$

Let e be a generator for $E_2^{0,1}$. Then $x = d_2(e)$ is a generator of $E_2^{2,0}$ and ex is a generator for $E_2^{2,1}$. Hence $d_2(ex)$ generates $E_2^{4,0}$ By the product rule, we have $d_2(ex) = d_2(e)x + (-1)^1 ed_2(x) = x^2$. Hence x^2 generates $H^4(\mathbb{CP}^n, \mathbb{Z})$. Similarly, $d_2(ex^2) = x^3$ generates $E_2^{6,0} = H^6(\mathbb{CP}^n, \mathbb{Z})$, and so on. We obtain an isomorphism $H^\bullet(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$ and $H^\bullet(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[x]$.

Example 2.30. Next we compute the ring structure on $H^\bullet(\Omega S^2, \mathbb{Z})$. Dual to the homological case, we obtain the following E_2-page corresponding to the fibre sequence $\Omega S^3 \to * \to S^3$:

$$\begin{tikzcd}[row sep=small, column sep=small]
q & & & & \\
& \vdots & & & \vdots \\
& \mathbb{Z} \arrow[ddrrr, "\cong"] & 0 & 0 & \mathbb{Z} \\
& 0 & & & 0 \\
& \mathbb{Z} \arrow[ddrrr, "\cong"] & & & \mathbb{Z} \\
& 0 & & & 0 \\
& \mathbb{Z} \arrow[ddrrr, "\cong"] & & & \mathbb{Z} \\
& 0 & & & 0 \\
& \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
& & & & p
\end{tikzcd}$$

Let $x \in H^2(\Omega S^3, \mathbb{Z}) = E_3^{0,2}$ be a generator. Then $e = d_3(x)$ generates $E_3^{3,0}$, and ex generates $E_3^{3,2}$. We have $d_3(x^2) = d_3(x)x + (-1)^2 xd_3(x) = 2ex$. In particular, x^2 is twice a generator of $H^4(\Omega S^3)$, which we denote $x^2/2$. Again we repeat this: $d_3(x^3) = d_3(x)x^2 + xd_3(x^2) = 6e(x^2/2)$. Inductively, we get that x^n is $n!$ times a generator of $H^{2n}(\Omega S^3, \mathbb{Z})$. We get an isomorphism $H^\bullet(\Omega S^3, \mathbb{Z}) \cong \mathbb{Z}[x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}, \dots] \subseteq \mathbb{Z}[x]$ or $H^\bullet(\Omega S^3, \mathbb{Z}) \cong \mathbb{Z}[x_i]/\binom{m+n}{n} x_{m+n} = x_m x_n$. This is called the divided power algebra on one generator $\Gamma(x)$. This is not finitely generated as a ring.

Remark: There is also a ring structure on the homology of $H_\bullet(\Omega S^3, \mathbb{Z})$ induced by the H-space (in fact A_∞- or E_1-space) structure by concatenation of loops. One can show that with this ring structure, $H_\bullet(\Omega S^3, \mathbb{Z})$ is a polynomial ring on a generator in degree 2. More generally, if $H_\bullet(X, \mathbb{Z})$ is free over $\mathbb{Z}$, then $H_*(\Omega\Sigma X, \mathbb{Z})$ is the tensor algebra on $H_*(X, \mathbb{Z})$ (Bott-Samelson theorem) and the map $H_\bullet(X, \mathbb{Z}) \to H_*(\Omega\Sigma X, \mathbb{Z}) = T(H_\bullet(X, \mathbb{Z}))$ is induced by the natural map $X \to \Omega\Sigma X$.

Remark: ΩS^3 vs. $\mathbb{CP}^\infty$. They have isomorphic homology groups, they both have CW-structures (up to homotopy equivalence for ΩS^3) with one cell in every even dimension, but very different homotopy groups: $\pi_n(\mathbb{CP}^\infty, *) = 0$ for $n > 2$ and $\pi_n(\Omega S^3, *) \cong \pi_{n+1}(S^3, *) \neq 0$ for all $n > 2$. Their cohomology rings are non-isomorphic (see above). The attaching maps $S^3 \to S^2$ for 4-cell for $\mathbb{CP}^\infty$ is a generator of $\pi_3(S^2, *)$, for ΩS^3 it is twice a generator. Both are loopspaces ($\mathbb{CP}^2 \simeq \Omega K(\mathbb{Z}, 3)$), but $H_*(\Omega S^3, \mathbb{Z})$ is polynomial and $H_*(\mathbb{CP}^\infty, \mathbb{Z})$ is a divided power algebra.

Example 2.31. We consider the map $S^3 \to K(\mathbb{Z}, 3)$ classifying a generator of $H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$. (equivalently, inducing an isomorphism on $\pi_3(-, *)$). Let X denote the homotopy fibre, so that $X \to S^3 \to$

$K(\mathbb{Z},3)$ is a fibre sequence. By the long exact sequence in homotopy: X is 3-connected and $\pi_n(X,*) \cong \pi_n(S^3,*)$ for $n \geq 4$. We want to understand (co-)homology of X. As we don't know the (co-)homology of $K(\mathbb{Z},3)$ yet, we take a second homotopy fibre, yielding a fibre sequence $\mathbb{CP}^\infty \cong \Omega K(\mathbb{Z},3) \to X \to S^3$. We obtain the following Serre spectral sequence: $E_2^{p,q} = H^p(S^3, H^q(\mathbb{CP}^\infty,\mathbb{Z})) \cong H^p(S^3,\mathbb{Z}) \otimes H^q(\mathbb{CP}^\infty,\mathbb{Z})$.

$$\begin{tikzcd}
\vdots & & & \vdots \\
\mathbb{Z} \arrow[rrrdd] & 0 & 0 & \mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} \arrow[rrrdd] & & & \mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} \arrow[rrrdd] & & & \mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}
\end{tikzcd}$$

(axes: q vertical, p horizontal)

Since X is 3-connected, $d_3 : E_2^{0,2} \to E_2^{3,0}$ must be an isomorphism. Let $x \in E_2^{0,2}$ be a generator. Then $x^i \in E_2^{0,2i}$ will be a generator, and by the product rule $d_3(x^2) = 2d_3(x)x$ is twice a generator. Inductively, $d_3(x^n)$ is n times a generator of $E_2^{3,2n-2}$. Thus the E_∞-page is

$$\begin{array}{cccc}
\vdots & & & \vdots \\
0 & 0 & 0 & \mathbb{Z}/4\mathbb{Z} \\
0 & & & 0 \\
0 & & & \mathbb{Z}/3\mathbb{Z} \\
0 & & & 0 \\
0 & & & \mathbb{Z}/2\mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} & 0 & 0 & 0
\end{array}$$

(axes: q vertical, p horizontal)

and hence

$$\widetilde{H}^n(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if} n = 2k+1, \\ 0 & \text{else} \end{cases}$$

with trivial cup product. By the universal coefficient theorem we get $\widetilde{H}_n(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if } k = 2n, \\ 0 & \text{else.} \end{cases}$

Corollary 2.32. *We have $\pi_4(S^3,*) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_4(S^2,*) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. X is 3-connected, and $H_4(X,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Hurewicz theorem implies $\pi_4(X,*) = \mathbb{Z}/2\mathbb{Z}$ and we saw that $\pi_4(S^3,*) = \pi_4(X,*)$. By the Hopf fibration ($S^1 \to S^3 \to S^2$), the same is true for S^2. $\square$

We now know the following homotopy groups of spheres:

	π_1	π_2	π_3	π_4	π_5	π_6
S^1	$\mathbb{Z}$	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^6	0	0	0	0	0	$\mathbb{Z}$

where $\pi_5(S^4,*) = \pi_6(S^5,*) = \mathbb{Z}/2\mathbb{Z}$ follows from the Freudenthal suspension theorem.

3 Construction of the Serre spectral sequence

We focus on the cohomological version. Roughly speaking we proceed via

double complexes ⇒ filtered complexes ⇒ exact couple ⇒ spectral sequence.

Definition 3.1. An exact couple is a pair of abelian groups (A, E) together with a triangle

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which is exact, i.e. exact at each of the three corners.

Define $d_1 : E \to E$ via $d_1 = j \circ k$. We have $d_1^2 = j \circ k \circ j \circ k = j \circ 0 \circ k = 0$, so d_1 is a differential. We can define $H(E) = \ker(d_1)/\operatorname{im}(d_1)$, and claim that there is a new triangle

$$\begin{tikzcd} A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd} \tag{$*$}$$

defined via $E_2 = H(E)$, $A_2 = \operatorname{im}(i) \subseteq A$, $i_2 = i|_{A_2}$. For $a \in A_2$ write $a = i(b)$ for some $b \in A$. Then $j_2(a) := [j(b)]$. This is well-defined: $d_1(j(b)) = j(k(j(b))) = 0$, and if $i(b) = i(b')$, then $b - b' \in \ker i = \operatorname{im} k$, so $j(b - b') \in \operatorname{im}(j \circ k) = \operatorname{im} d_1$. Finally, $k_2([e]) := k(e)$: Since $j(k(e)) = 0$, exactness implies $k(e) \in \operatorname{im} i$, and if $e \in \operatorname{im} d_1$, then $e \in \operatorname{im} j$ and $k(e) = 0$.

Lemma 3.2. *The triangle ($*$) is again an exact couple.*

Proof. Diagram chase (omitted). □

Hence we can iterate and obtain a sequence of exact couples (A_n, E_n) with maps i_n, j_n, k_n. In particular we obtain a sequence of abelian groups E_n with differentials $d_n = j_n \circ k_n$ and isomorphisms $H(E_n) \cong E_{n+1}$. This is like a spectral sequence, except we are missing the bigrading.

For the Serre spectral sequence the two gradings play different roles: A filtration degree (x-axis), and the difference between the cohomological degree and the filtration degree.

Definition 3.3. An unrolled exact couple is a collection of pairs $(A^s, t^s)_{s\in\mathbb{Z}}$ of abelian groups with maps

$$\begin{tikzcd} \cdots \arrow[rr, "i"] & & A^{s+1} \arrow[rr, "i"] \arrow[dl, "j"] & & A^s \arrow[rr, "i"] \arrow[dl, "j"] & & A^{s-1} \arrow[r] \arrow[dl, "j"] & \cdots \\ & \cdots & & E^s \arrow[ul, "k"] & & E^{s-1} \arrow[ul, "k"] & & \end{tikzcd}$$

Every unrolled exact couple gives an exact couple via $A = \bigoplus_s A^s,\ E = \bigoplus_s E^s$ combined in a single triangle. We obtain a cochain complex

$$\cdots \xrightarrow{j\circ k} E^{s-1} \xrightarrow{j\circ k} E^s \xrightarrow{j\circ k} E^{s+1} \to \cdots$$

Hence $H_*(E)$ inherits a grading, i.e. $H(E) = \bigoplus_j H^j(E)$. Generally we can write $E_r = \bigoplus_s E_r^s \cong \bigoplus_s H^s(E_{r-1})$. Assume $e \in E^s$, we can chase through its "life" in the spectral sequence: If $d_1(e) \neq 0$, then e does not define a class in $H^\bullet(E)$. If $d_1(e) = 0$, then $[e] \in H^s(E) = E_2^s$. In the unrolled picture, $d_2[e] = j_2k_2(e)$ is computes as follows: By exactness, $k(e) \in \operatorname{im}(i)$, say $k(e) = i(b)$, and $d_2([e]) = [j(b)]$. If $d_2([e]) \neq 0$, e does not define a class in $H(E_2)$. If $d_2(e) = 0$, we can continue in this way. In general, if $k(e) = i^r(b)$ for some $r \geq 0$ and $b \in A^{s+r+1}$, then e defines a class in $E_{r+1}^s = H^s(E_r)$ and $d_{r+1}([e])$ is $j(b)$. If e survives in every step, it is called a permanent cycle. We note: If $e \in E_r^s$, then $d_r([e])$ is represented by some element of E_r^{s+r}, i.e. d_r raises the filtration degree by r.

A filtered cochain complex is a cochain complex $C^\bullet$ together with a sequence of subcomplexes

$$\ldots \subseteq F^2C^\bullet \subseteq F^1C^\bullet \subseteq F^0C^\bullet = C^\bullet$$

For convenience, we extend the filtration grading to $\mathbb{Z}$ via $F^sC^\bullet = C^\bullet$ for $s < 0$. The associated graded complex is the collection of subquotients

$$\operatorname{gr}^s C^\bullet = F^sC^\bullet / F^{s+1}C^\bullet.$$

The corresponding short exact sequences induce a long exact sequence

$$\cdots \to H^t(F^{s+1}C^\bullet) \to H^t(F^sC^\bullet) \to H^t(\operatorname{gr}^s(C^\bullet)) \to H^{t+1}(F^{s+1}C^\bullet) \to \cdots$$

Taking the direct sum over all t, we obtain

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "i"] & H^\bullet(F^{s+1}C^\bullet) \arrow[rr, "i"] \arrow[dl, "j"] & & H^\bullet(F^sC^\bullet) \arrow[rr, "i"] \arrow[dl, "j"] & & H^\bullet(F^{s-1}C^\bullet) \arrow[r] \arrow[dl, "j"] & \cdots \\
\cdots & & H^\bullet(\operatorname{gr}^s(C^\bullet)) \arrow[ul, "k"] & & H^\bullet(\operatorname{gr}^s(C^\bullet)) \arrow[ul, "k"] & &
\end{tikzcd}$$

in which each triangle is exact. We observe that i and j preserve the cohomological degree, but k raises it by 1. We hence obtain an unrolled exact couple with an additional cohomological degree, and an exact couple with

$$A = \bigoplus_{s,t} H^t(F^sC^\bullet), \qquad E = \bigoplus_{s,t} H^t(\operatorname{gr}^s C^\bullet)$$

and an associated spectral sequence. What does it converge to? We define a filtration on $H^\bullet(C)$ by setting $F^sH^t(C^\bullet) = \operatorname{im}(H^t(F^sC^\bullet) \to H^t(C^\bullet))$.

Theorem 3.4. *If for every t, the cohomology $H^t(F^sC^\bullet)$ is 0 for s large enough, the spectral sequence associated to the exact couple constructed above converges to $(H^\bullet C^\bullet, F^sH^\bullet(C^\bullet))$, with E_1-page $E_1^{s,t} = H^t(\operatorname{gr}^s(C^\bullet))$.*

The gradings of this spectral sequence are different to the one for the Serre spectral sequence: d_r raises the filtration degree by r and the cohomological degree by 1.

If $C^\bullet$ is concentrated in nonnegative degree, we get a first quadrant spectral sequence with the cohomological degree remaining constant along rows. For the Serre spectral sequence, we use cohomological degree minus filtration degree for the vertical axis. This regrading remains in the first quadrant because all terms with filtration degree larger than cohomological degree are trivial.

Proof. (of 3.4) In the exact couple, A_r is the direct sum over all s of

$$A_r^s := \operatorname{im}(i^{r-1} : H^\bullet(F^{s+r-1}C) \to H^\bullet(F^sC))$$

For $t \in \mathbb{Z}$ we set $n_t \in \mathbb{N}$ to be the minimum over all n such that $H^t(F^nC) = 0$. Then for $r \geq n_t + 1$, either $s > 0$, i.e. $s + r - 1 > n_t$, so $H^t(F^{s+r-1}C) = 0$, so $A_r^s = 0$, or $s \leq 0$, then $H^t(F^sC) = H^tC$, and $A_r^s = F^{s+r-1}H^tC$. So

$$A_r^t = \bigoplus_{s\leq 0} F^{s+r-1}H^t(C) = \bigoplus_{0\leq p\leq n_t} F^pH^t(C).$$

This is independent of r if $r \geq n_t+1$. Let A^t_∞ be this value. The map $i_r : A^t_r \to A^t_r$ is the direct sum over the inclusions $F^{p+1}H^t(C) \to F^pH^t(C)$. In particular, it is injective, so by exactness $k_r : E^{t-1}_r \to A^t_r$ must be zero. Thus for r large, all differentials are zero and the terms E^t_r stabilize as well. Moreover, by exactness of

$$\begin{tikzcd} A_\infty \arrow[rr, "i_r"] & & A_\infty \arrow[dl, "j_r"] \\ & E_\infty \arrow[ul, "0"] & \end{tikzcd}$$

we have

$$E^t_\infty \cong \operatorname{coker}(i_r) \cong \bigoplus_{p \leq n_t} F^pH^t(C)/F^{p+1}H^t(C).$$

□

Example 3.5. Let X be a CW-complex with skeleta

$$\operatorname{sk}_0 X \subseteq \operatorname{sk}_1 X \subseteq \ldots \subseteq \operatorname{sk}_n \subseteq \ldots$$

We can filter $C^\bullet(X, A)$ by

$$F^sC^\bullet(X, A) = \ker(C^\bullet(X, A) \to C^\bullet(\operatorname{sk}_s X, A) =: C^\bullet(X, \operatorname{sk}_s X, A)$$

Then we obtain a spectral sequence with E_1-page

$$H^\bullet(C^\bullet(X, \operatorname{sk}_s X, A)/C^\bullet(X, \operatorname{sk}_{s+1} X, A) \cong H^t(\operatorname{sk}_{s+1} X, \operatorname{sk}_s X, A) \cong \widetilde{H}^t(\operatorname{sk}_{s+1} X/\operatorname{sk}_s X, A)$$

It converges to $H^\bullet(C^\bullet(X, A)) = H^\bullet(X, A)$. Note that on the E_1-page only the entries on the main diagonal are nontrivial, and these are exactly the cellular cochain complex. For degree reasons, the spectral sequence degenerates at the E_2-page, where the entries along the main diagonals are the homology of the cellular cochain complex, i.e. the cellular cohomology. Thus, this reproves that the cellular cochain complex computes cohomology.

Example 3.6. Let $p : Y \to X$ be a Serre fibration with X a CW-complex. Then we can filter Y via the preimages $p^{-1}(\operatorname{sk}_s X)$ and obtain a filtration on $C^\bullet(Y, A)$. The resulting spectral sequence "is" the Serre spectral sequence. However, some aspects (in particular, the multiplicative structure) are clearer in the construction with double complexes.

The Spectral Sequence of a Double Complex

Definition 3.7. A double complex is a bigraded abelian group $C^{\bullet,\bullet}$ equipped with two differentials

$$\delta_h : C^{p,q} \to C^{p+1.q}, \quad \text{and} \quad \delta_v : C^{p,q} \to C^{p,q+1}$$

satisfying $\delta_h^2 = 0 = \delta_v^2$ and $\delta_h\delta_v = \delta_v\delta_h$, i.e. there is an infinite commutative diagram

$$\begin{tikzcd} C^{p-1,q+1} \arrow[r, "\delta_h"] & C^{p,q+1} \arrow[r] & C^{p+1,q+1} \arrow[r] & \cdots \\ C^{p-1,q} \arrow[r] \arrow[u, "\delta_v"] & C^{p,q} \arrow[r] \arrow[u] & C^{p+1,q} \arrow[r] \arrow[u] & \cdots \\ C^{p-1,q-1} \arrow[r] \arrow[u] & C^{p,q-1} \arrow[r] \arrow[u] & C^{p+1,q-1} \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

with all rows and columns complexes. The "vertical cohomology groups" $H^q_{\delta_v}(C^{p,\bullet})$ inherit a "horizontal" differential $\delta_h : H^q_{\delta_v}(C^{p,\bullet}) \to H^q_{\delta_v}(C^{p+1,\bullet})$. We write $H^p_{\delta_h}H^q_{\delta_v}(C^{\bullet,\bullet})$ for the resulting cohomology groups.

Example 3.8. D_1, D_2 cochain complexes. Then the tensor products $D_1^p \otimes D_2^q$ form a double complex with δ_h from D_1 and δ_v from D_2.

Definition 3.9. Let $(C^{\bullet,\bullet}, \delta_h, \delta_v)$ a double complex. Its *total complex* $\mathrm{Tot}(C)$ is the cochain complex $\mathrm{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q}$ and $\delta = \delta_h + (-1)^p \delta_v$.

Note: The $(-1)^p$ is needed to guarantee $\delta^2 = 0$.

A double complex $C^{\bullet,\bullet}$ can be filtered by

$$F^s(C)^{p,q} = \begin{cases} C^{p,q} & \text{if } p \geq s, \\ 0 & \text{else.} \end{cases}$$

This induces a filtration on $\mathrm{Tot}(C)$ via $F^s \mathrm{Tot}(C) = \mathrm{Tot}(F^s(C))$. Then $\mathrm{gr}_s \mathrm{Tot}(C)^t = C^{s,t-s}$ with differential $(-1)^s \delta_v$. The sign doesn't affect cohomology, hence $H^t(\mathrm{gr}_s \mathrm{Tot}(C)) = H^{t-s}_{\delta_v}(C^{s,\bullet})$. We hence get a spectral sequence with E_1-page $E_1^{s,t} = H^1(\mathrm{gr}_s \mathrm{Tot}(C)) = H^{t-s}_{\delta_v}(C^{s,\bullet})$. Moreover, d_1 agrees with the horizontal differentials δ_h (exercise), so $E_2^{s,t} \cong H^s_{\delta_h} H^{t-s}_{\delta_v}(C^{\bullet,\bullet})$. If $C^{p,q} = 0$ if $p < 0$ or $q < 0$, then the E_1-page is concentrated in degrees $t \geq s \geq 0$ and the spectral sequence converges to the $H^\bullet(\mathrm{Tot}(C))$. It is hence customary to reindex to $(s, t-s)$ and obtain a first quadrant spectral sequence with Serre grading $H^s_{\delta_h} H^t_{\delta_v}(C) \Rightarrow H^{s+t}(\mathrm{Tot}(C))$.

Remark: Swapping the horizontal and vertical direction yields a different spectral sequence also converging to $H^\bullet(\mathrm{Tot}(C))$.

Dress's construction of the Serre spectral sequence

Let $f : E \to B$ be a Serre fibration. A singular (p,q)-simplex of f is a commutative diagram

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "f"] \\ \Delta^p \arrow[r] & B \end{tikzcd}$$

Let $C_{p,q}(f)$ be the free abelian group on all singular (p,q)-simplices. There is a differential $\delta_h : C_{p,q}(f) \to C_{p-1,q}(f)$ by taking the alternating sum of the faces of the p-simplex, i.e. if $d_i : \Delta^{p-1} \to \Delta^p$ is the i-th face of Δ^p, we consider

$$\begin{tikzcd} \Delta^{p-1} \times \Delta^q \arrow[r, "d_i \times \mathrm{id}"] \arrow[d] & \Delta^p \times \Delta^q \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ \Delta^{p-1} \arrow[r, "d_i"] & \Delta^p \arrow[r] & B \end{tikzcd}$$

Similarly, one obtains a vertical differential $\delta_v : C_{p,q}(f) \to C_{p,q-1}(f)$ as the alternating sum of the faces of the q-simplex, obtained as

$$\begin{tikzcd} \Delta^{p} \times \Delta^{q-1} \arrow[r, "\mathrm{id} \times d_i"] \arrow[d] & \Delta^p \times \Delta^q \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ \Delta^{p} \arrow[r] & \Delta^p \arrow[r] & B \end{tikzcd}$$

By dualizing oe obtain a double complex $C^{\bullet,\bullet}(f, A) = \mathrm{Hom}(C_{\bullet,\bullet}(f), A)$ for every coefficient group A. We have $C^{p,q}(f, A) = 0$ for $p < 0$ or $q < 0$, hence we obtain a Serre-graded first quadrant spectral sequence with E_2-page $E_2^{s,t} = H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}(f, A))$ converging to $H^\bullet(\mathrm{Tot}(C^{\bullet,\bullet}(f, A)))$. This is the Serre spectral sequence. We have to show that the E_2-page and the limit agree with $H^\bullet(B, H^\bullet(F, A))$ and $H^\bullet(E, A)$, respectively.

We start with $H^\bullet(\mathrm{Tot}(C^\bullet(f,A))) \cong H^\bullet(E,A)$. For this we swap the horizontal and vertical directions and obtain another spectral sequence $H^s_{\delta_v} H^t_{\delta_h}(C(f,A)) \Rightarrow H^\bullet(\mathrm{Tot}(C(f,A)))$. Claim: The E_2-page is 0 if $s \neq 0$ and $H^0_{\delta_v} H^t_{\delta_h}(C(f,A)) \cong H^t(E,A)$.

We fix $s \geq 0$ and consider diagrams

$$\begin{tikzcd} \Delta^t \times \Delta^s \arrow[r] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "f"] \\ \Delta^t \arrow[r] & B \end{tikzcd}$$

We can rewrite this to

$$\begin{tikzcd} \Delta^t \arrow[r] \arrow[d] & \mathrm{map}(\Delta^s, E) \arrow[d, "\mathrm{map}(\Delta^s, f)"] \\ B \arrow[r, "\mathrm{const}"] & \mathrm{map}(\Delta^s, B) \end{tikzcd}$$

This in turn is equivalent to a single map

$$\Delta^t \to B \times_{\mathrm{map}(\Delta^s,B)} \mathrm{map}(\Delta^s, E) =: P,$$

i.e. a singular t-simplex of the space P. One checks that in fact $C^{s,\bullet}(f,A)$ is isomorphic to $C^\bullet(P,A)$. Now, Δ^s is contractible, hence $B \to \mathrm{map}(\Delta^s, B)$ is a homotopy equivalence. Since f is a Serre fibration, it follows that $P \to \mathrm{map}(\Delta^s, E)$ is a weak homotopy equivalence. In particular, $H^t_{\delta_h}(C^{\bullet,\bullet}(f,A)) \cong H^t(E,A)$ forall $t \geq 0$. Under these identifications, every face map of Δ^s induces the identity on these groups. Hence the complex computing $H^s_{\delta_v} H^t_{\delta_h}(E,A)$ equals

$$H^t(E,A) \xrightarrow{0} H^t(E,A) \xrightarrow{\mathrm{id}} H^t(E,A) \xrightarrow{0} H^t(E,A) \xrightarrow{\mathrm{id}} \cdots$$

thus

$$H^s_{\delta_v} H^t_{\delta_h}(C(f,A)) = \begin{cases} H^t(E,A) & s = 0 \\ 0 & s > 0. \end{cases}$$

It follows that $E_\infty = E_2$ and $H^t(\mathrm{Tot}(C(f,A))) \cong H^t(E,A)$.

It remains to compute the E_2-term.

Definition 3.10. The *fundamental grupoid* $\pi_1 X$ of a space X is the category with objects the points of X and morphisms $\mathrm{Hom}_{\pi_1 X}(x,y) = \{\text{homotopy classes of paths } \gamma \text{ from } x \text{ to } y\}$. Composition is the concatenation of paths.

$\pi_1 X$ is a groupoid, i.e. every morphism is invertible. Furthermore, $\mathrm{Hom}_{\pi_1 X}(x,x) = \pi_1(X,x)$.

Definition 3.11. A *local system* M on X is a functor $M : \pi_1 X \to \mathtt{Ab}$. We write M_x for $M(x)$

Note: If X is path-connected, $\pi_1 X$ is equivalent to the groupoid with one object $x \in X$ and automorphisms $\pi_1(X,x)$. Hence a local system is equivalent datum to an abelian group A with action by $\pi_1(X,x)$.

If X is even simply-connected, every local system is isomorphic to the constant local system for an abelian group A.

Example 3.12. Let $f : E \to B$ be a Serre fibration, A an abelian group and $q \in \mathbb{N}$. We write $F_x := f^{-1}(x)$ for the fibre of $x \in X$. We obtain a local system $x \mapsto H_q(F_x, A)$. On homotopy classes of paths

$[\gamma : x \to y]$, this is defined as follows: Consider the pullback

$$\begin{tikzcd} F_\gamma \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ I \arrow[r, "\gamma"] & B \end{tikzcd}$$

which comes with weak homotopy equivalences $F_x \to F_\gamma, F_y \to F_\gamma$. Hence on homology we obtain an induced map

$$H_p(\gamma, A) : H_q(F_x, A) \xrightarrow{\simeq} H_q(F_\gamma, A) \xleftarrow{\simeq} H_q(F_y, A).$$

To show compatability with composition and homotopy invariance, consider pullbacks of f along maps $\Delta^2 \to B$.

Similarly, one obtains a local system $x \mapsto H^q(F_x, A)$.

Example 3.13. Let M be a topological manifold of dimension n, then the assignment $x \mapsto M_x = H_n(M, M \setminus \{x\}, \mathbb{Z})$ extends to a local system. For a path $\gamma : x \to y$, cover the path by contractible open sets U_i and use that $H_n(M, M \setminus \{x\}, \mathbb{Z}) \cong H_n(M, M \setminus U_i, \mathbb{Z})$. iteratively. M is orientable iff this local system is isomorphic to the constant one on $\mathbb{Z}$.

Next we define (co-)homology with coefficients in a local system M on X. We set $C_n(X, M) = \bigoplus_{\sigma:\Delta^n \to X} M_{\sigma_0}$ where $\sigma_0 \in X$ is the image of the 0th vertex of Δ^n. There is a differential

$$d : C_n(X, M) \to C_{n-1}(X, M), \quad d(\sigma, m) = (\sigma \circ d_0, (\sigma_{0,1})_*(m)) + \sum_{i=1}^{n} (-1)^i (\sigma \circ d_i, m)$$

where $\sigma_{0,1}$ is the image under σ of any path from the 0th vertex to the 1st vertex. Note that $\sigma \circ d_0$ has 0th vertex equal to the first vertex of σ, while all the other $\sigma \circ d_i$ have $(\sigma \circ d_i)_0 = \sigma_0$.

We omit the proof that this defines a chain complex.

Definition 3.14. For a local system M, we define $H_\bullet(X, M)$ as the homology of this chain complex.

If M is constant, this recovers the ordinary homology. A map of local systems $M \to N$ induces maps $H_\bullet(X, M) \to H_\bullet(X, N)$. A map of spaces $f : X' \to X$ and a local system M on X, we obtain a pullback local system f^*M and a map $H_\bullet(X', f^*M) \to H_\bullet(X, M)$, where $(f^*M)_x = M_{f(x)}$.

Let M be a local system on a space X. Then $C^\bullet(X, M)$ is defined as $C^n(X, M) = \prod_{\sigma:\Delta^n \to X} M_{\sigma_0}$ with differential

$$df(\sigma) = M(\sigma_{01})^{-1}(f(d^0\sigma)) + \sum_{i=1}^{n} (-1)^i f(d^i\sigma).$$

Definition 3.15. Cohomology with local coordinates is defined as $H^\bullet(X, M) = H^\bullet(C^\bullet(X, M))$.

Example 3.16. One can show that using local coefficients there is a version of Poincaré duality without an orientation assumption: If M is a compact topological manifold of dimension n, there is a fundamental class $[M] \in H_n(M, \mathbb{Z}^{or})$, where $\mathbb{Z}^{or}$ is the local system from example 3.13, such that there are isomorphisms $H^\bullet(M, \mathbb{Z}) \to H_{n-\bullet}(M, \mathbb{Z}^{or})$ and $H^\bullet(M, \mathbb{Z}^{or}) \to H_{n-\bullet}(M, \mathbb{Z})$

We now go back to the spectral sequence constructed out of the double complex $C^{\bullet,\bullet}(p, A)$ for a Serre fibration $p : E \to B$. We already know that it is a first quadrant spectral sequence converging to $H^\bullet(E, A)$. It remains to understand the E_2-page. We fix a map $\sigma\Delta^p \to B$ and consider a square

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^p \arrow[r, "\sigma"] & B \end{tikzcd}$$

This is equivalent to a square

$$\begin{tikzcd} \Delta^q \arrow[r] \arrow[d] & E^{\Delta^p} \arrow[d] \\ * \arrow[r, "\sigma"] & B^{\Delta^p} \end{tikzcd}$$

This in turn is equivalent to a map $\Delta^q \to F_\sigma$, where F_σ is the fibre of the fibration $E^{\Delta^p} \to B^{\Delta^p}$ over the point σ. Thus the columns of the double complex are isomorphic to a product over all maps $\sigma : \Delta^p \to X$ of the singular cochain complex of F_σ. Again, F_σ is weakly homotopy equivalent to the fibre F_{σ_0} of the original fibration $p : E \to B$ over the 0th vertex σ_0, using that Δ^p is contractible. Hence, the vertical cohomology of the double complex are the products

$$\prod_{\sigma:\Delta^p\to B} H^q(F_\sigma, A) \cong \prod_{\sigma:\Delta^p\to B} H^q(F_{\sigma_0}, A).$$

In this decomposition, the horizontal differentials work as follows: For $i > 0$, the triangle

$$\begin{tikzcd} H^q(F_{\sigma_0}, A) & \\ H^q(F_\sigma, A) \arrow[u, "\cong"] \arrow[r, "d^i"] & H^q(F_{\sigma\circ d_i}, A) \arrow[ul, "\cong"'] \end{tikzcd}$$

commutes. For $i = 0$ we have a commuting square

$$\begin{tikzcd} H^q(F_{\sigma_0}, A) \arrow[r, "H^q(\gamma,A)", "\cong"'] & H^q(F_{(\sigma\circ d_0)_0}, A) \\ H^q(F_\sigma, A) \arrow[u, "\cong"] \arrow[r, "d_0"] & H^q(F_{\sigma\circ d_0}, A) \arrow[u, "\cong"] \end{tikzcd}$$

where γ is the image in X of any path in Δ^p from e_0 to e_1. Hence, the vertical cohomologies equipped with the horizontal differential are isomorphic to the cochain complex $C^\bullet(X, H^q(B, A))$. This shows that the E_2-page is given by $E_2^{p,q} \cong H^p(B, H^q(F_-, A))$ as claimed.

Analogously, the spectral sequence associated to the double complex $C_{\bullet,\bullet}(p, A)$ yields the homological Serre spectral sequence, with E_2-term the local system homology $H_p(B, H_q(F, A))$.

It remains to discuss multiplicative properties.

Definition 3.17. A multiplicative structure on a double complex $C^{\bullet,\bullet}$ is a collection of maps $\mu : C^{p,q} \otimes C^{p',q'} \to C^{p+p',q+q'}$ that are associative and unital (with unit 1 in degree $(0, 0)$). Moreover, the differential $\delta = \delta_h + (-1)^p\delta_v$ of $\mathrm{Tot}(C)$ satisfies the Leibniz rule, i.e. $\delta(xy) = \delta(x)y + (-1)^{p+q}x\delta(y)$.

Chasing through our constructions, we find

Proposition 3.18. *The spectral sequence associated to a multiplicative double complex is multiplicative.*

To apply this, we define a multiplicative structure on $C^{\bullet,\bullet}(p, R)$ where R is a ring, and $p : E \to B$ a Serre fibration. Recall that for cochains $\varphi \in C^p(X, R), \psi \in C^q(X, R)$ and $\sigma : \Delta^{p+q} \to X$ one defines

$$(\varphi \smile \psi)(\sigma) = \varphi(d^*_{p\text{-front}}\sigma)\psi(d^*_{q\text{-back}}\sigma).$$

Similarly, if $\varphi \in C^{p,q}(p, R), \psi \in C^{p',q'}(p, R)$ and a $(p + p', q + q')$-simplex σ represented by

$$\begin{tikzcd} \Delta^{p+p'} \times \Delta^{q+q'} \arrow[r, "d"] \arrow[d] & E \arrow[d] \\ \Delta^{p+p'} \arrow[r, "\sigma"] & B \end{tikzcd}$$

are given, we set
$$(\varphi \smile \psi)(\sigma) := \varphi(d^*_{(p,q)\text{-front}}\sigma) \cdot \psi(d^*_{(p',q')\text{-back}}\sigma)$$
where $d^*_{(p,q)\text{-front}}\sigma$ is the (p,q)-simplex given by

$$\begin{tikzcd}
\Delta^p \times \Delta^q \arrow[r, "d_{p\text{-front}} \times d_{q\text{-front}}"] \arrow[d] & \Delta^{p+p'} \times \Delta^{q+q'} \arrow[r, "d"] \arrow[d] & E \arrow[d] \\
\Delta^p \arrow[r, "d_{p\text{-front}}"] & \Delta^{p+p'} \arrow[r, "\sigma"] & B
\end{tikzcd}$$

and similarly for $d_{(p',q')\text{back}}$

Lemma 3.19. *Both the horizontal and the vertical differential on $C^{\bullet,\bullet}(p,R)$ satisfy the graded Leibniz rule with respect to this cup product. Hence $C^{\bullet,\bullet}(p,R)$ becomes a multiplicative double complex.*

Proof. Analogous to the Leibniz rule for the ordinary cup product. □

Hence, the cohomological Serre spectral sequence becomes multiplicative, and one checks that the identifaction $E_2^{p,q} \cong H^p(B, H^q(F_-, R))$ is multiplicative with respect to the multiplication on $H^\bullet(B, H^\bullet(F_-, R))$ described earlier, and that the convergence to $(H^\bullet(E,R), F_\bullet)$ is multiplicative.

Finally, we record the naturality of the Serre spectral sequence.

Definition 3.20. A morphism of cohomologically graded spectral sequences $f : (E_r, d_r, h_r) \to (E'_r, d'_r, h'_r)$ is a collection of bigraded maps $f_r : E^r \to E'_r$ that commute with differentials and satisfy $h'_r \circ f_r = H^\bullet(f_r) \circ h_r$.

Note: f is determined by f_2, but it is a condition that the higher f_r commute with the differentials. Analogously one can define morphisms of homologically graded spectral sequences.

We now consider the category Fib with objects the Serre fibrations $p : E \to B$ and morphisms the commutative squares

$$\begin{tikzcd}
E \arrow[r, "g^E"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\
B \arrow[r, "g^B"] & B'
\end{tikzcd}$$

(not necessarily pullback squares). Then $C_{\bullet,\bullet}(-,A)$ becomes a functor from Fib to double complexes by postcomposition, hence the assignment sending $p : E \to B$ to its Serre spectral sequence also becomes a functor (similarly for the cohomological version). The identification $E^2_{p,q} \cong H_p(B, H_q(F_-, A))$ is a natural isomorphism of functors $\mathrm{Fib} \to \mathtt{Ab}$. The maps $H_\bullet(g^E, A)$ and $H^\bullet(g^E, A)$ preserve the filtrations, and $E^\infty_{p,q} \cong F^p(H_{p+q}(E,A))/F^{p-1}(H_{p+q}(E,A))$ is natural, similarly for cohomology.

We give a simple application of naturality. Let $p : E \to B$ be a Serre fibration. We have surjections $E_2^{p,0} \twoheadrightarrow E_3^{p,0} \cdots \twoheadrightarrow E_\infty^{p,0}$, an inclusion $E_\infty^{p,0} = F^p H^p(E,A) \hookrightarrow H^p(E,A)$. and a map $H^p(B,A) \to H^p(B, H^0(F_-, A)) \cong E_2^{p,0}$ induced by the map of local systems $A \to H^0(F_-, A)$ from the projection $F \to *$. The composite $e(p : E \to B) : H^p(B,A) \to H^p(E,A)$ is called the edge homomorphism.

Lemma 3.21. *The morphism $e(p)$ agrees with $H^p(p, A)$.*

Proof. By naturality of the Serre spectral sequence, the edge homomorphism is also natural. We consider the square of fibrations

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \big\downarrow p & & \big\downarrow \mathrm{id} \\ B & \xrightarrow{\mathrm{id}} & B \end{array}$$

and obtain a commutative square

$$\begin{array}{ccc} H^\bullet(B,A) & \xrightarrow{\mathrm{id}} & H^\bullet(B,A) \\ \big\downarrow e(\mathrm{id}) & & \big\downarrow e(p) \\ H^\bullet(B,A) & \xrightarrow{p^*} & H^\bullet(E,A) \end{array}$$

It hence suffices to check that $e(\mathrm{id}) = \mathrm{id}$, which one easily checks directly. □

There is also an edge homomorphism of the form

$$H^q(E,A) \twoheadrightarrow F_0H^q(E,A)/F_1H^q(E,A) \cong E_\infty^{0,q} \hookrightarrow E_2^{0,q} \cong H^0(B, H^q(F_-,A)) \xrightarrow{(x\hookrightarrow B)^*} H^q(F_x,A)$$

which one can show timilarly to agree with $H^q(F_x \hookrightarrow E, A)$. There are also homological versions of these two edge homomorphisms.

This finishes the construction of the Serre spectral sequence. We now turn to more structural applications. As a warm-up, we reprove the Hurewicz theorem, using only the case $\pi_1(X,x)^{\mathrm{ab}} \xrightarrow{\cong} H_1(X,\mathbb{Z})$ for path-connected X as input.

Proposition 3.22. *Let $n > 1$ and X $(n-1)$-connected. Then $H_k(X,\mathbb{Z}) = 0$ for $0 < k < n$ and the Hurewicz map $\pi_n(X,x) \to H_n(X,\mathbb{Z})$ is an isomorphism.*

Proof. By induction on n. The loop space ΩX is $(n-2)$-connected and satisfies the relation $\pi_k(X,x) \cong \pi_{k-1}(\Omega X, x)$. By the induction hypothesis, $H_k(\Omega X,\mathbb{Z}) = 0$ for $0 < k < n-1$, and $\pi_{n-1}(\Omega X,x) \cong H_{n-1}(\Omega X,\mathbb{Z})$ is an isomorphism. We apply the Serre spectral sequence for the fibre sequence $\Omega X \to x \to X$. The E^∞-page must by 0 away from $(0,0)$ hence we see directly that $H_k(X,\mathbb{Z}) = 0$ for $0 < k < n$. Moreover, $d^n : H_n(X,\mathbb{Z}) \to H_{n-1}(\Omega X,\mathbb{Z})$ must be an isomorphism. Hence we obtain an isomorphism $c_X : \pi_n(X,x) \xrightarrow{\cong} \pi_{n-1}(\Omega X,x) \xrightarrow{\cong} H_{n-1}(\Omega X,\mathbb{Z}) \xleftarrow{\cong} H_n(X,\mathbb{Z})$. Why does this composite c_X agree with the Hurewicz map up to a sign? Note that c_X is natural in $(n-1)$-connected spaces. Reduce to the universal case $X = S^n$: Let $y \in \pi_n(X,x)$, represented by $f : S^n \to X$. We obtain a square

$$\begin{array}{ccc} \pi_n(S^n,*) & \xrightarrow{c_{S^n}} & H_n(S^n,\mathbb{Z}) \\ \big\downarrow f_* & & \big\downarrow H_n(f) \\ \pi_n(X,x) & \xrightarrow{c_X} & H_n(X) \end{array}$$

Note that c_{S^n} must send [id] to one of the two orientations for S^n. Further $c_X(y) = H_n(f)(c_{S^n}([\mathrm{id}])) = H_n(f)(\pm h([\mathrm{id}])) = \pm h([f]) = \pm h(y)$, where h is the Hurewicz map and the sign is determined by c_{S^n}, thus in particular the same for all $(n-1)$-connected X and $y \in \pi_n(X,x)$. □

4 Serre classes

Definition 4.1. Let $\mathcal{C} \subseteq \mathtt{Ab}$ be a non-empty full subcategory. $\mathcal{C}$ is called a *Serre class* if it is closed under extensions, subgroups, quotient groups and isomorphisms, i.e. given $0 \to A \to B \to C \to 0$ short exact, then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$.

Example 4.2. Examples of Serre classes include finitely generated abelian groups $\mathcal{C}^{f.g.}$, torsion abelian groups $\mathcal{C}^{tor}$, p-power torsion groups $\mathcal{C}^{p}$, finite abelian groups $\mathcal{C}^{fin}$ and finite p-power torsion groups $\mathcal{C}^{p,tor}$.

On the other hand, torsion-free groups, p-torsion groups or rational/uniquely divisible abelian groups do not form Serre classes.

Lemma 4.3. *Let $\mathcal{C}$ be a Serre class. Then*

a) Given an exact sequence $0 \to A_1 \to A_2 \to \ldots \to A_n \to 0$, if all but one of the A_i are in $\mathcal{C}$, so is the last.

b) If $C_\bullet$ is a chain complex with $C_n \in \mathcal{C}$ for all n, then $H_n C_\bullet \in \mathcal{C}$ for all n.

c) If $0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$ is a finite filtration, then $A \in \mathcal{C}$ if and only if $A_i/A_{i-1} \in \mathcal{C}$ for all i.

Proof. a) From $A_{i-1} \xrightarrow{f} A_i \xrightarrow{g} A_{i+1}$ we get a short exact sequence $0 \to \operatorname{im}(f) \to A_i \to \operatorname{im}(g) \to 0$, where $\operatorname{im}(f)$ is a quotient of A_{i-1} and $\operatorname{im}(g)$ is a subgroup of A_{i+1}.

b) Similarly, we get $0 \to \operatorname{im}(d_{n+1}) \to \ker(d_n) \to H_n(C_\bullet) \to 0$ with $\ker(d_n) \subseteq C_n$.

c) Follows by induction on n. □

We sometimes require stronger axioms.

Definition 4.4. A Serre class $\mathcal{C}$ satisfies

- the *tensor axiom* if $A \otimes B \in \mathcal{C}$ and $\operatorname{Tor}(A, B) \in \mathcal{C}$ whenever $A, B \in \mathcal{C}$,
- the *group homology axiom* if $H_n(K(A, 1), \mathbb{Z}) \in \mathcal{C}$ for all $n \geq 1$ whenever $A \in \mathcal{C}$.

Recall: $\operatorname{Tor}(A, B)$ computes as the kernel of $P_1 \otimes B \to P_0 \otimes B$ where $0 \to P_1 \to P_0 \to A \to 0$ is a free resolution of A, and $A \otimes B$ is the cokernel of this map. Hence the tensor axiom is automatic if $\mathcal{C}$ is closed under infinite direct sums (e.g. $\mathcal{C}^{tor}$) or if $\mathcal{C}$ is a subclass of finitely generated abelian groups (since then P_0, P_1 can be chosen free of finite rank). Furthermore, we have:

Lemma 4.5. *Every Serre class $\mathcal{C}$ of finitely generated abelian groups also satisfies the group homology axiom.*

Proof. We have $\widetilde{H}_n(K(\mathbb{Z}, 1), \mathbb{Z}) = H_n(S^1, \mathbb{Z}) = \mathbb{Z}$ for $n = 1$ and 0 else, and

$$\widetilde{H}_n(K(\mathbb{Z}/m, 1), \mathbb{Z}) = \begin{cases} \mathbb{Z}/m & m \text{ odd}, \\ 0 & \text{else} \end{cases}$$

This can be seen e.g. via Lens spaces, yielding S^∞ as universal cover with $\mathbb{Z}/m$ acting by multiplication with an m-th root of unity. Alternatively, use group homology. A general finitely generated abelian group is a finite direct sum of $\mathbb{Z}/m$'s and $\mathbb{Z}$'s. The claim then follows by induction using the Künneth short exact sequence

$$0 \to H_\bullet(K(A,1),\mathbb{Z}) \otimes H_\bullet(K(B,1),\mathbb{Z}) \to H_\bullet(K(A\times B,1),\mathbb{Z}) \\ \to \operatorname{Tor}(H_\bullet(K(A,1),\mathbb{Z}), H_\bullet(K(B,1),\mathbb{Z}))[-1] \to 0$$

We already saw that $\mathcal{C}$ satisfies the tensor axiom, hence $H_\bullet(K(A\times B,1),\mathbb{Z}) \in \mathcal{C}$. $\square$

Our next goal is to show the following

Theorem 4.6. *Let $\mathcal{C}$ be a Serre class satisfying the tensor and group homology axioms, and X a simple space. Then the following are equivalent:*

1) $H_n(X,\mathbb{Z}) \in \mathcal{C}$ for all $n \geq 1$,

2) $\pi_n(X,*) \in \mathcal{C}$ for all $n \geq 1$.

Recall: X is simple if it is path-connected, $\pi_1(X,*)$ is abelian, and $\pi_1(X,*)$ acts trivially on $\pi_n(X,*)$ for all $n > 1$. For example, every H-space is simple.

Corollary 4.7. *$\pi_k(S^n,*)$ is finitely generated for all k,n.*

Corollary 4.8. *More generally, $\pi_k(X,*)$ is finitely generated for every simple finite CW-complex.*

Theorem 4.6 holds more generally for *nilpotent* spaces. On the other hand, $X = S^1 \vee S^2$ shows that theorem 4.6 does not hold for all spaces, because $\pi_2(X,*) \cong H_2(\widetilde{X},\mathbb{Z}) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$ is not finitely generated.

Lemma 4.9. *Let $\mathcal{C}$ be a Serre class satisfying the tensor axiom, $F \to Y \to X$ a fibre sequence of path-connected spaces and $\pi_1(X,*)$ acting trivially on $H_*(F,\mathbb{Z})$. (i.e. the local system $H_\bullet(F_-,\mathbb{Z})$ is isomorphic to the trivial one.) Then, if two out of F,Y,X have homology groups in $\mathcal{C}$, so does the third.*

Proof. We abbreviate $H_\bullet(-,\mathbb{Z})$ to $H_\bullet$.
Case 1: $H_kF, H_kX \in \mathcal{C}$ for all $k > 0$. Then

$$E^2_{p,q} \cong H_p(X, H_q(F)) \cong H_pX \otimes H_qF \oplus \operatorname{Tor}(H_{p-1}X, H_qF) \in \mathcal{C}$$

for $(p,q) \neq (0,0)$. Note that for $p=1$ we have $\operatorname{Tor}(H_{p-1}X, H_qF) = 0$ since H_0X is free. Hence, Lemma 4.3 implies first $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq 0,0$ and then $H_kY \in \mathcal{C}$ for $k > 0$.
Case 2: $H_kF, H_kY \in \mathcal{C}$ for all $k > 0$. Then by Lemma 4.3 we have $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. We now show that $H_nX \in \mathcal{C}$ by induction on n. For $n=1$, we have $H_1X = E^2_{1,0} = E^\infty_{1,0}$. We now choose $n > 1$ and assume $H_kX \in \mathcal{C}$ for $k = 1,\ldots,n-1$. From the hypothesis we get $E^r_{p,q} \in \mathcal{C}$ for $p < n$, $(p,q) \neq (0,0)$ and all r, as in case 1. We have a filtration $H_nX \supseteq E^3_{p,q} \supseteq E^4_{p,q} \supseteq \ldots \supseteq E^{n+1}_{n,0} = E^\infty_{n,0}$ and short exact sequences $0 \to E^{i+1}_{n,0} \to E^i_{n,0} \to \operatorname{im}(d^i) \to 0$. Since $\operatorname{im}(d^i) \subseteq E^i_{n-i,i-1}$ and $n-i < n$, we have $E^i_{n-i,i-1} \in \mathcal{C}$ and $\operatorname{im}(d^i) \in \mathcal{C}$. By backwards induction it follws that $H_nX \in \mathcal{C}$.
Case 3: $H_kX, H_kY \in \mathcal{C}$ for all $k > 0$. Similar to case 2 (exercise). $\square$

Corollary 4.10. *Let $\mathcal{C}$ be a Serre class satisfying the tensor and group homology axioms. Then, if $A \in \mathcal{C}$, we have $H_k(K(A,n),\mathbb{Z}) \in \mathcal{C}$ for all $k,n \geq 1$.*

Proof. Induction on n starting with the group homology axiom and using the fibre sequences $K(A,n-1) \simeq \Omega K(A,n) \to * \to K(A,n)$ $\square$

Recall the Postnikov tower. Let X be a space, then the Postnikov tower of X is a diagram

$$\begin{tikzcd} & \vdots \arrow[d] \\ & \tau_{\leq 1}X \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0}X \end{tikzcd}$$

such that $\pi_k(\tau_{\leq n}X) = 0$ for $k > n$ and the map $X \to \tau_{\leq n}X$ induces an isomorphism on $\pi_k(-)$ for $k \leq n$. In particular, $X \to \operatorname{holim}_n \tau_{\leq n}X$ is a weak homotopy equivalence.

For simplicity, we assume that X is path-connected. Then each $\tau_{\leq n}X$ is also path-connected and there is a fibre sequence $K(\pi_n X, n) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$. The key observation is that if X is simple, the action of $\pi_1(\tau_{\leq n}X) \cong \pi_1(X)$ on $H_\bullet(K(\pi_n X, n))$ is trivial. This follows from the fact that any self-map of $K(A, n)$ which induces the identity on π_n is homotopic to the identity. (In fact, π_n induces a bijection $[K(A, n), K(A, n)] \to \operatorname{Hom}(A, A)]$.)

We want to prove a more general version of theorem 4.6. For that we make the following definition.

Definition 4.11. Let $\mathcal{C}$ be a Serre class. We say that a map $f : A \to B$ of abelian groups is an isomorphism $\bmod \mathcal{C}$, if $\ker(f), \operatorname{coker}(f) \in \mathcal{C}$. Similarly, f is a mono- or epimorphism $\bmod \mathcal{C}$ if $\ker(f) \in \mathcal{C}$ or $\operatorname{coker}(f) \in \mathcal{C}$, respectively.

f is an isomorphism $\bmod \mathcal{C}^{tor}$ if and only if $f \otimes \mathbb{Q}$ is an isomorphism. Multiplication by 2 is an automorphism of $\mathbb{Z} \bmod \mathcal{C}^{fin,2}$.

Theorem 4.12 (Hurewicz theorem $\bmod \mathcal{C}$)**.** *Let $\mathcal{C}$ be a good (i.e. satisfying the tensor and group homology axioms) Serre class. For $n > 0$ and a simple space X, one has $H_k(X) \in \mathcal{C}$ for all $0 < k < n$ iff $\pi_k(X) \in \mathcal{C}$ for all $0 < k < n$. In this case, the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism $\bmod \mathcal{C}$.*

Proof. We first assume $\pi_k(X) \in \mathcal{C}$ for $0 < k < n$. We consider the Postinkov tower of X. Since $H_k(X) \cong H_k(\tau_{\leq k}X)$, it suffices to show that $H_m(\tau_{\leq k}X) \in \mathcal{C}$ for all $k < n$ and $m > 0$. This follows inductively from the fibre sequences $K(\pi_k(X, k)) \to \tau_{\leq k}X \to \tau_{\leq k-1}X$ via Lemma 4.9 and Corollary 4.10.

Next we show by induction on n that if $H_k(X) \in \mathcal{C}$ for all $0 < k < n$, then $\pi_n(X) \to H_n(X)$ is an isomorphism $\bmod \mathcal{C}$. This proves the general statement by applying it inductively for smaller n. By naturality, the Hurewicz map is equivalent to the map

$$\pi_n(X) \cong \pi_n(K(\pi_n X, n)) \xrightarrow{h, \cong} H_n(K(\pi_n X, n)) \to H_n(\tau_{\leq n}X) \cong H_n(X),$$

where h is the classic Hurewicz morphism. Now the induction hypothesis implies that $\pi_k(X) \in \mathcal{C}$ for $k < n$. By the first part of the proof, we hence know that $H_n(\tau_{\leq n-1}X) \in \mathcal{C}$. We analyse the Serre spectral sequence for the fibre sequence $K(\pi_n X, n) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$. The E^2-page has 0 outside the zeroth and n-th row, so the first possible differential is $d^{n+1} : H_{n+1}(\tau_{\leq n-1}X) \to H_n(K(\pi_n X, n))$ with cokernel $E^\infty_{0,n}$, which sits in a short exact sequence

$$0 \to E^\infty_{0,n} \to H_n(\tau_{\leq n}X) \to H_n(\tau_{\leq n-1}X) \to 0.$$

Putting these together, we obtain an exact sequence

$$H_{n+1}(\tau_{n-1}X) \xrightarrow{d} H_n(K(\pi_n X, n)) \to H_n(\tau_{\leq n}X) \to H_n(\tau_{\leq n-1}X) \to 0.$$

As the first and last term are in $\mathcal{C}$, this finishes the proof. □

Corollary 4.13. *Let p be a prime. Then the first p-prime torsion in $\pi_*(S^3)$ is a copy of $\mathbb{Z}/p$ in degree $2p$.*

Proof. Recall that we computed the homology of the homotopy fibre F of $S^3 \to K(\mathbb{Z}, 3) = \tau_{\leq 3} S^3$ (called Postnikov section) as $H_k(F) \cong \mathbb{Z}/n$ if $k = 2n$ and 0 if k is odd. Apply the $\bmod \mathcal{C}$ Hurewicz theorem for the Serre class of finite abelian groups with order coprime to p. This shows that $\pi_k(F)$ is finite with no p-torsion for $k < 2p$ and that the kernel and cokernel of $\pi_{2p}(F) \to H_{2p}(F) \cong \mathbb{Z}/p$ have order coprime to p. It follows that the p-power torsion of $\pi_{2p}(F) \cong \pi_{2p}(S^3)$ is a copy of $\mathbb{Z}/p$. □

Definition 4.14. A good Serre class $\mathcal{C}$ is called a Serre ideal, if for every $A \in \mathcal{C}$ and any abelian group B the tensor product $A \otimes B$ lies in $\mathcal{C}$.

Lemma 4.15. *$\mathcal{C}$ is a Serre ideal if and only if $\bigoplus_X A \in \mathcal{C}$ whenever $A \in \mathcal{C}$ for all sets X.*

Proof. We have $\bigoplus_X A \cong (\bigoplus_X \mathbb{Z}) \otimes A$. For the other direction, we use that any abelian group is a quotient of a free group, and that $\otimes$ is right-exact. □

By the same argument, $\mathrm{Tor}(-, -)$ is also in $\mathcal{C}$. $\mathcal{C}^p, \mathcal{C}^{tor}$ are Serre ideals, but $\mathcal{C}^{fg}$ is not.

Lemma 4.16. *Let $\mathcal{C}$ be a Serre ideal, and $f : X \to Y$ be a map of 1-connected spaces and $n > 0$. Then the following are equivalent:*

- *$\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism $\bmod \mathcal{C}$ for $0 < k < n$ and an epimorphism $\bmod \mathcal{C}$ for $k = n$,*
- *$H_k(f) : H_k(X) \to H_k(Y)$ is an isomorphism $\bmod \mathcal{C}$ for $0 < k < n$ and an epimorphism $\bmod \mathcal{C}$ for $k = n$.*

Proof. First assume the second condition. We consider the fibre sequence $F \to X \xrightarrow{f} Y$. By the long exact sequence on $\pi_\bullet$ reduced $\bmod \mathcal{C}$, the first condiiton is equivalent to saying that F is $(n-1)$ connected $\bmod \mathcal{C}$, that is $\pi_k(F) \in \mathcal{C}$ for $0 < k < n$.

By the Hurewicz theorem, this is equivalent to saying $H_k(F) \in \mathcal{C}$ for $0 < k < n$. We claim that $H_p(Y, H_q(F)) \in \mathcal{C}$ for $0 < q \leq n-1$ or in fact $H_p(Y, A) \in \mathcal{C}$ for any $A \in \mathcal{C}$. This follows from the ideal property, which implies that $C_m(Y, A)$ lies in $\mathcal{C}$ for all m, and hence so does its homology group.

Hence, the region $0 < q < n$ of the Serre E^2-page lies in $\mathcal{C}$ and likewise for the E^∞-page. Puttin that together, $H_k(X)$ admits a filtration $0 \subseteq F_0 \subseteq \ldots \subseteq F_k = H_k(X)$ such that $F_i/F_{i-1} \in \mathcal{C}$ where $k - n + 1 \leq i < k$. Moreover, the last quotient F_k/F_{k-1} is isomorphic to $H_k(Y) \cong E^2_{k,0} \bmod \mathcal{C}$ for $k \leq n$. This implies the result via the edge homomorphism.

For the converse, assume that $H_k(X) \to H_k(Y)$ is an isomorphism $\bmod \mathcal{C}$ for $0 < k < n$ and an epimorphism $\bmod \mathcal{C}$ for $k = n$. We want to show that $\pi_k(F) \in \mathcal{C}$ for $k < n$. Suppose otherwise, then there is a minimal $0 < k < n$ for which $\pi_k(F) \notin \mathcal{C}$. Inspecting the Serre spectral sequence similarly to before, we find that $H_k(F) \cong E^2_{0,k} \twoheadrightarrow E^3_{0,k} \twoheadrightarrow \ldots \twoheadrightarrow E^{k+1}_{0,k}$ are all isomorphisms $\bmod \mathcal{C}$, as well as $E^{k+1}_{k+1,0} \hookrightarrow E^k_{k+1,0} \hookrightarrow \ldots \hookrightarrow E^2_{k+1,0} \cong H_{k+1}(Y)$. By the Hurewicz theorem, $H_k(F) \notin \mathcal{C}$ and hence $E^{k+1}_{0,k} \notin \mathcal{C}$. By assumption, $H_{k+1}(X) \to H_{k+1}(Y)$ is an epimorphism $\bmod \mathcal{C}$. By the description of the edge homomorphism, $E^{k+1}_{0,k} \to EE^\infty_{0,k}$ must be an isomorphism $\bmod \mathcal{C}$, so $E^\infty_{0,k} \notin \mathcal{C}$. But this group lies in the kernel of $H_k(X) \to H_k(Y)$, contradicting the latter being an isomorphism $\bmod \mathcal{C}$. □

There exists a space $\mathrm{Sp}^\infty(X)$ and a map $X \to \mathrm{Sp}^\infty(X)$, such that $\pi_n(\mathrm{Sp}^\infty(X)) \cong H_n(X)$.

Example 4.17. The Whitehead theorem fails in general for good Serre classes which are not Serre ideals: Consider $\mathbb{CP}^\infty \times X \to X$ be the projection where X is 1-connected and $H_2(X)$ is not finitely generated (e.g. $X = \bigvee_{\mathbb{N}} S^2$). Then $\pi_k(\mathbb{CP}^\infty \times X) \cong \pi_k(\mathbb{CP}^\infty) \times \pi_k(X) \to \pi_k(X)$ is an isomorphism $\mathrm{mod}\ \mathcal{C}^{fg}$ for all $k > 1$, but $H_4(\mathbb{CP}^\infty \times X) \cong \mathbb{Z} \oplus H_2(X) \oplus H_4(X) \to H_4(X)$ is not an isomorphism $\mathrm{mod}\ \mathcal{C}^{fg}$, since $H_2(X)$ is not finitely generated.

We now turn to the rational homotopy groups of spheres. We already saw that $\pi_k(S^3)$ is finite for $k > 3$, hence $\pi_k(S^3) \otimes \mathbb{Q} = 0$ for $k > 3$. In particular, the 3rd Postnikov section $S^3 \to K(\mathbb{Z}, 3)$ is an isomorphism on $\pi_\bullet(-) \otimes \mathbb{Q}$ (or equialently an isomorphism $\mathrm{mod}\ \mathcal{C}$). By the Whitehead theorem, $S^3 \to K(\mathbb{Z}, 3)$ also induces an isomorphism $H_\bullet(-, \mathbb{Q})$. In particular, $H^\bullet(K(\mathbb{Z}, 3), \mathbb{Q}) = H^\bullet(S^3, \mathbb{Q}) \cong \bigwedge_{\mathbb{Q}}(x_3)$. The idea is to build up to the rationalised sphere by knowledge of Eilenberg-MacLane spaces. That is, we study $H^\bullet(K(\mathbb{Z}, n), \mathbb{Q})$ to compute $\pi_\bullet(S^n) \otimes \mathbb{Q}$.

Lemma 4.18. *We have*

$$H^\bullet(K(\mathbb{Z}, n), \mathbb{Q}) \cong \begin{cases} \bigwedge(x) & \text{if } n \text{ is odd,} \\ \mathbb{Q}[x] & \text{otherwise} \end{cases}$$

for $x \in H^n(K(\mathbb{Z}, n), \mathbb{Q})$ the image of the tautological (=fundamental) class in $H^n(K(\mathbb{Z}, n), \mathbb{Z})$.

Proof. For an induction on n, use the fibre sequence $K(\mathbb{Z}, n-1) \cong \Omega K(\mathbb{Z}, n) \to * \to K(\mathbb{Z}, n)$. The case $n = 1, 2$ is clear. The step $(n-1)$ odd to n even is entirely analogous to the previously studied $S^1 \to * \to \mathbb{CP}^\infty$. Now assume n is even.

The Serre spectral sequence with coefficient group $\mathbb{Q}$ has a copy of $\mathbb{Q}$ at position (p, q) if $p \in \{0, n+1\}$ and $n|q$, with the only non-trivial differentials $d_n : E_n^{0,kn} \to E_n^{n+1,(k-1)n}$ are isomorphisms. Let $y \in H^n(K(\mathbb{Z}, n), \mathbb{Q})$ be the fundamental class, let $x = d_n y$. By the product rule, $d_n(y^m) = mxy^{m-1}$ is a generator of $E_n^{n+1,(m-1)n}$, where by assumption y^m is a generator of $H^{nm}(K(\mathbb{Z}, n), \mathbb{Q})$. If there was a non-trivial $H^k(K(\mathbb{Z}, n+1), \mathbb{Q})$ with $k > n-1$, then the corresponding class in $E_2^{k,0}$ could not be the image of a differential, contradicting $E_\infty^{p,q} = 0$ for $(p, q) \neq (0, 0)$. $\square$

Note that $K(\mathbb{Z}, n) \to K(\mathbb{Q}, n)$ is a rational $\pi_\bullet$-isomorphism, hence by the $\mathrm{mod}\ \mathcal{C}^{tor}$ Whitehead theorem, it also induces an isomorphism on $H_\bullet(-, \mathbb{Q})$, and hence on $H^\bullet(-, \mathbb{Q})$, for $n > 1$. This is also true for $n = 1$, which one can see by noting that a model for $K(\mathbb{Q}, 1)$ is given by the mapping telescope of $S^1 \xrightarrow{(\cdot)^2} S^1 \xrightarrow{(\cdot)^3} S^1 \to \dots$

Theorem 4.19. *We have*

$$\pi_k(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n \text{ or } n \text{ even and } k = 2n-1, \\ 0 & \text{else.} \end{cases}$$

Proof. We start with n odd. We can further assume $n \geq 3$. Then the n-th Postnikov section $S^n \to K(\mathbb{Z}, n)$ induces an isomorphism on $H^\bullet(-, \mathbb{C})$. Hence by duality also on $H_\bullet(-, \mathbb{Q})$ and by the $\mathrm{mod}\ \mathcal{C}^{tor}$ Whitehead theorem is an isomorphism on $\pi_*(-) \otimes \mathbb{Q}$. Hence $\pi_k(S^n) \otimes \mathbb{Q} \cong \pi_k(K(\mathbb{Z}, n)) \otimes \mathbb{Q}$ gives the desired result.

For even n, the map $S^n \to K(\mathbb{Z}, n)$ is not an isomorphism on $H^\bullet(-, \mathbb{Q})$, since $H^\bullet(K(\mathbb{Z}, n), \mathbb{Q})$ is polynomial, while $H^\bullet(S^n, \mathbb{Q})$ is exterior. We try to build a space out of $K(\mathbb{Z}, m)$-spaces, whose rational cohomology is exterior: Let $K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ classify the cap-square of the fundamental class. Using that $[K(\mathbb{Z}, n), K(\mathbb{Z}, 2n)] \cong H^2(K(\mathbb{Z}, n), \mathbb{Z})$, we obtain a fibre sequence $F \to K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ and a map $f : S^n \to F$, since the composite $S^n \to K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ is nullhomotopic, as $\pi_n(K(\mathbb{Z},2n)) = 0$. We claim that f induces an isomorphism on $H^\bullet(-,\mathbb{Q})$. Consider the Serre spectral sequence for the fibre sequence $\Omega K(\mathbb{Z},2n) \cong K(\mathbb{Z},2n-1) \to F \to K(\mathbb{Z},n)$. The differential $d_{2n} : H^{2n-1}(K(\mathbb{Z},2n-1),\mathbb{Q}) \to H^{2n}(K(\mathbb{Z},n),\mathbb{Q})$ must be onto, since we know by construction that the squaring map $H^n(F,\mathbb{Q}) \to H^{2n}(F,\mathbb{Q})$ is zero. Being surjective for a finite dimensional vector space means it is an isomorphism. By the product rule, each $d_{2n} : E_{2n}^{kn,2n-1} \to E_{2n}^{(k+2)n,0}$ with $k \geq 0$ is an isomorphism. Hence $H^\bullet(F,\mathbb{Q}) \cong \bigwedge(x_n)$ and $S^n \to F$ induces an isomorphism on $H^\bullet(-,\mathbb{Q})$ and therefore on $\pi_*(-)\otimes\mathbb{Q}$. Hence

$$\pi_k(S^n)\otimes\mathbb{Q} \cong \pi_k(F)\otimes\mathbb{Q} \cong \begin{cases}\mathbb{Q} & \text{if } k = n, 2n-1, \\ 0\text{else} & \end{cases}$$

if n is even, as claimed. □

Corollary 4.20.

$$\pi_k(S^n) \cong \begin{cases}\mathbb{Z}\oplus \textit{finite} & k = n,\ \textit{or } n \textit{ is even and } k = 2n-1, \\ \textit{finite} & \textit{else.}\end{cases}$$

Proof. Theorem 4.18 plus the fact that $\pi_k(S^n)$ is finitely generated. □

So what is an example of an infinite order element of $\pi_{2n-1}(S^n)$ for n even, and how does one detect those? For $n = 2$, we know that $\pi_3(S^2)$ is generated by the Hopf map $\eta : S^3 \to S^2$. For general n, let $f : S^{2n-1} \to S^n$ and choose generators $a \in H^n(S^n,\mathbb{Z}), b \in H^{2n}(S^{2n},\mathbb{Z})$. Then the mapping cone $C(f)$ satisfies $H^k(C(f),\mathbb{Z}) \cong \mathbb{Z}$ for $k = 0, n, 2n$ and 0 else. More precisely, we have isomorphisms $H^n(C(f),\mathbb{Z}) \cong H^n(S^n,\mathbb{Z})$, $\widetilde{a} \mapsto a$ and $H^{2n}(S^{2n},\mathbb{Z}) \cong H^{2n}(C(f),\mathbb{Z})$, $b \mapsto \widetilde{b}$. Then $\widetilde{a} \smile \widetilde{a} = h(f)\widetilde{b}$ for a unique $h(f) \in \mathbb{Z}$.

Definition 4.21. The number $h(f)$ is called the Hopf invariant of f.

Lemma.

- $h(-)$ *defines a group homomorphism.*
- $h(f) = 0$ *if n is odd.*
- *If n is even, $h([\iota_n,\iota_n]) = \pm 2$, where ι_n is the composite of an attaching map $S^{2n-1} \to S^n \vee S^n$ for the $2n$-cell in $S^n \times S^n$ followed by the fold map $S^n \vee S^n \to S^n$.*

Proof. Exercise. □

Corollary. *If n is even, then $h(-) : \pi_{2n-1}(S^n)/\pi_{2n-1}(S^n)^{tor} \to \mathbb{Z}$, is injective.*

For small n, we have: $h(\eta : S^3 \to S^2) = \pm 1$, since $C(\eta) \cong \mathbb{CP}^2$ and $H^\bullet(\mathbb{CP}^2\mathbb{Z}) = \mathbb{Z}[x]/x^3$. Similarly, $h(S^7 \to S^4) = \pm 1 = h(S^{15} \to S^8)$.

Theorem (Adams 1960, Hopf invariant 1 problem)**.** *The numbers $n = 2, 4, 8$ are the only dimensionas with elements of Hopf invariant* 1.

$\pi_{2n-1}(S^n)$ is the last "unstable" (in the sense of the Freudenthal suspension theorem) homotopy group of codimension $n-1$, and surjects onto the stable, finite group $\pi_{2n}(S^{n+1})$. Hence, $\Sigma : \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$ must have a nontrivial kernel. In fact, $\Sigma[\iota_n,\iota_n] = 0$. E.g. $\mathbb{Z} \cong \pi_3(S^2) \twoheadrightarrow \pi_4(S^3) \cong \mathbb{Z}/2$ or $\mathbb{Z}\oplus\mathbb{Z}/12 \cong \pi_7(S^4) \twoheadrightarrow \pi_8(S^5) \cong \mathbb{Z}/24$.

Using our computations, we can deduce an even stronger Hurewicz theorem in the rational case:

Theorem 4.22. *Let X be simply-connected, and $\pi_i(X) \otimes \mathbb{Q} \cong 0$ for $i \leq n-1$. Then the Hurewicz map $\pi_i(X) \otimes \mathbb{Q} \to H_i(X, \mathbb{Q})$ is an isomorphism for $1 \leq qi \leq 2n-2$ and a surjection for $i = 2n-1$.*

Corollary 4.23. *Let X be a pointed space. For $k \in \mathbb{N}$, consider the colimit*

$$\pi_k^{st}(X) \otimes \mathbb{Q} = \operatorname{colim}_n \pi_{k+n}(\Sigma^n X) \otimes \mathbb{Q}$$

along the suspension maps. We obtain a Hurewicz map $\pi_k^{st}(X) \otimes \mathbb{Q} \to H_k(X, \mathbb{Q})$ via

$$\begin{tikzcd}
\pi_k(X) \otimes \mathbb{Q} \arrow[r, "h"] \arrow[d] & H_k(X, \mathbb{Q}) \arrow[d, "\cong"] \\
\pi_{k+1}(\Sigma X) \otimes \mathbb{Q} \arrow[r, "h"] & H_{k+1}(\Sigma X, \mathbb{Q})
\end{tikzcd}$$

This Hurewicz map is an isomorphism.

Proof. By the rational Hurewicz theorem, the map $\pi_{k+1}(\Sigma^n X) \otimes \mathbb{Q} \to H_{k+n}(\Sigma^n X, \mathbb{Q})$ is an isomorphism up to degree $k + n \leq 2n - 2$, since $\Sigma^n X$ is $(n-1)$-connected. Hence for fixed k, eventually all maps in the colimit system are isomorphisms. □

In particular, $\pi_\bullet^{st}(-) \otimes \mathbb{Q}$ is a homology theory, i.e. it has long exact sequences for cofibre sequences, even though the unstable $\pi_\bullet$ do not. We will see in an exercise that in fact $\pi_\bullet^{st}$ defines a homology theory, called stable homotopy. Its coefficients are the stable homotopy groups of spheres.

Theorem 4.24. *Let X be 1-connected, $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i = 1, \ldots, n-1$. Then the Hurewicz map $\pi_i(X) \otimes \mathbb{Q} \to H_i(X, \mathbb{Q})$ is an isomorphism for $0 \leq i \leq 2n-2$ and surjective for $i \leq 2n-1$.*

Proof. Let $\tau_{\geq i} X$ be the $(i-1)$-connected cover of X, i.e. the fibre of $X \to \tau_{\leq i-1} X$. We consider the diagram

$$\begin{tikzcd}
\pi_i(\tau_{\geq i} X) \otimes \mathbb{Q} \arrow[r, "\cong"] \arrow[d, "\cong"] & H_i(\tau_{\geq i} X, \mathbb{Q}) \arrow[d] \\
\pi_i(X) \otimes \mathbb{Q} \arrow[r] & H_i(X, \mathbb{Q})
\end{tikzcd}$$

Hence, $\pi_i(X) \otimes \mathbb{Q} \to H_i(X, \mathbb{Q})$ is an isomorphism (surjective) if and only if $H_i(\tau_{\geq i} X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$ is an isomorphism (surjective). For this, we use

Lemma 4.25. *Let X be 1-connected, ratinally $(n-1)$-connected. Then $H_i(\tau_{\geq n+1} X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$ is an isomorphism for $n < i \leq 2n-2$ and surjective for $i = 2n-1$.*

Iterated application of this lemma yields the result. □

Proof. (of the lemma). Let A be an abelian group. Then $H_\bullet(K(A, n), \mathbb{Q})$ is concentrated in multiples of n, and $H_n(K(A, n), \mathbb{Q}) \cong A \otimes \mathbb{Q}$. We consider the fibre sequence $\tau_{\geq n+1} X \to X \to \tau_{\leq n} X$ and its rotation $\Omega \tau_{\leq n} X \to \tau_{\geq n-1} X \to X$. There is a diagram,, where the rows are fibre sequences

$$\begin{tikzcd}
\Omega \tau_{\leq qn} X \arrow[r] \arrow[d, equal] & \tau_{\geq n+1} X \arrow[r] \arrow[d] & X \arrow[d] \\
\Omega \tau_{\leq n} X \arrow[r] & * \arrow[r] & \tau_{\leq n} X \\
K(\pi_n(X), n-1) \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & K(\pi_n(X), n) \arrow[u]
\end{tikzcd}$$

The arrows from the third to the second row are isomorphisms on $\pi_*(-) \otimes \mathbb{Q}$, hence by the Hurewicz theorem also on $H_*(-, \mathbb{Q})$. Up to total degree $2n-1$, the rational Serre spectral sequence for the top row has potentially non-trivial entries only in degrees $(0,0), (i,0)$ for $n \leq i$, in $(0, n-1), (n, n-1)$ and in $(0, 2n-2)$. By naturality, the differential $d_n : H_n(X, \mathbb{Q}) \to H_{n-1}(\Omega\tau_{\leq n}X, \mathbb{Q})$ must also be an isomorphism, and $d_n : E^n_{n,n-1} \to E^n_{0,2n-2}$ is surjective. Hence, in this range, the $(n+1)$-st page has entries in at most the positions $(0,0)$, $(i,0)$ for $i > n$ and $(n, n-1)$. For degree reasons, there are no further differentials involving $H_i(X, \mathbb{Q})$ for $i < 2n$. By the edge homomorphism, the claim follows. □

5 Cohomology Operations and the Cohomology of $K(\mathbb{F}_2, n)$

Definition 5.1. A *cohomology operation* is a natural transformation $\varphi : H^k(-, A) \to H^l(-, B)$ for some abelian groups A, B and $k, l \in \mathbb{Z}$. A *stable cohomology operation* of degree n is a collection of cohomology operations $\{\varphi^{(k)} : H^k(-, A) \to H^{k+n}(-, B)\}$ which commute with the suspension isomorphism :

$$\begin{tikzcd} H^k(X, A) \arrow[r, "\varphi^{(k)}"] \arrow[d, "\cong"] & H^{k+n}(X, B) \arrow[d, "\cong"] \\ H^{k+1}(\Sigma X, A) \arrow[r, "\varphi^{(k+1)}"] & H^{k+1+n}(\Sigma X, B) \end{tikzcd}$$

Example. Let R be a ring, $n \in \mathbb{N}$. Then $x \mapsto x^{\smile n}$ is a cohomology operation. Note: This is typically not an additive operation (unless $n = p^k$ is a power of a prime and $\mathrm{char}(R) = p$).

If $f : A \to B$ is a group homomorphism, then f_* defines a stable cohomology operation of degree 0. If $0 \to A \to B \to C \to 0$ is short exact, we have a short exact sequence of cochain complexees

$$0 \to C^\bullet(X, A) \to C^\bullet(X, B) \to C^\bullet(X, C) \to 0$$

and hence the boundary maps $\delta : H^k(X, C) \to H^{k+1}(X, A)$ (called "Bockstein homomorphism") form a stable cohomology operation of degree 1.

Lemma 5.2. *Every stable operation $\{\varphi^{(k)}\}$ is additive.*

Proof. Let Y be a pointed $(m-1)$-connected space, $m \geq 2$. Then $Y \vee Y \to Y \times Y$ induces an isomorphism on $H_k(-, \mathbb{Z})$ for $k = 0, \ldots, 2m-1$ by the Künneth theorem. We obtain a diagram (if $k + n, k < 2m - 1$)

$$\begin{tikzcd} H^k(Y, A) \times H^k(Y, A) \arrow[r, "\cong"] \arrow[d, "\varphi^{(k)} \times \varphi^{(k)}"] & H^k(Y \vee Y, A) \arrow[d, "\varphi^{(k)}"] & H^k(Y \times Y, A) \arrow[l, "\cong"'] \arrow[r, "\Delta^*"] \arrow[d, "\varphi^{(k)}"] & H^k(Y, A) \arrow[d, "\varphi^{(k)}"] \\ H^{k+n}(Y, B) \times H^{k+n}(Y, B) \arrow[r, "\cong"] & H^{k+n}(Y \vee Y, B) & H^{k+n}(Y \times Y, B) \arrow[l, "\cong"'] \arrow[r, "\Delta^*"] & H^{k+n}(Y, B) \end{tikzcd}$$

where the composition along the top or bottom row is just the addition of the respective cohomology groups. By naturatlity of $\varphi^{(k)}$, this diagram commutes.

Now let X be arbitrary and $k \in \mathbb{Z}$. Then $\varphi^{(m+k)} : H^{m+k}(\Sigma^m X, A) \to H^{m+k+n}(\Sigma^m X, B)$ is additive whenever $m + k < 2m$ and $m + k + n < 2m$, which we can achieve by choosing m large enough. By stability, $\varphi^{(k)}$ agrees with this map up to suspension isomorphism, hence it is also additive. $\square$

Note: For a ring R and an abelian group A, the set of cohomology operators $H^k(-, A) \to H^*(-, B)$ form a graded ring by pointwise sum and cup-product. It is graded commutative if R is commutative.

For A an abelian group, the set of stable cohomology operations $\{H^k(-, A) \to H^{k+*}(-, A) : k \in \mathbb{Z}\}$ form a graded ring by composition. This is generally not commutative.

Proposition 5.3. *We have bijections*

a) $\{\varphi : H^k(-, A) \to H^l(-, B) \mid \varphi \text{ coh. op.}\} \xrightarrow{\cong} H^l(K(A, k), B), \quad \varphi \mapsto \varphi(c_k)$,
where c_k is the fundamental class in $H^k(K(A, k), A)$.

b) $\{\text{stable operations } H^\bullet(-, A) \to H^{\bullet+n}(-, B)\} \xrightarrow{\cong} \lim_{k \in \mathbb{N}} H^{k+n}(K(A, k), B)$,
where the limit is taken along the maps

$$H^{k+1+n}(K(A, k+1), B) \to H^{k+1+n}(\Sigma K(A, k), B) \cong H^{k+n}(K(A, k), B)$$

with the map $\Sigma K(A,k) \to K(A,k+1)$ adjoint to the weak homotopy equivalence $K(A,k) \to \Omega K(A,k+1)$ that induces id_A on π_k.

Remark: By the Freudenthal suspension theorem, the map $\Sigma K(A,k) \to K(A,k+1)$ is an isomorphism on π_* up to degree $2k-1$, hence by the Hurewicz theorem also on $H^\bullet(-,B)$ in that range. Hence for fixed n the connecting maps stabilise.

Proof. a) This is a consequence of the natural isomorphism $[X, K(A,k)] \to H^k(X,A),\ f \mapsto f^*(c_k)$ and the Yoneda lemma.

b) By definition, stable operations are the limit over k along the maps

$$\{\text{operations } H^{k+1}(-,A) \to H^{k+1+n}(-,B)\} \to \{\text{operations } H^k(-,A) \to H^k(-,B)\}$$

given by $\varphi \mapsto \sigma^{-1} \circ \varphi \circ \sigma$, where σ is the suspension isomorphism. Using part a), it hence suffices to observe that $H^{k+1}(K(A,k+1),B) \to H^{k+1}(\Sigma K(A,k),A)$ maps c_{k+1} to $\sigma(c_k)$ which follows from $K(A,k) \to \Omega K(A,k+1)$ inducing the identity on π_k and the natural identifications

$$H^{k+1}(K(A,k+1),A) \cong \mathrm{Hom}(H_{k+1}(K(A,k+1),\mathbb{Z}),A) \cong \mathrm{Hom}(\pi_{k+1}(K(A,k+1)),A).$$

□

Corollary 5.4. *All operations $H^k(-,A) \to H^l(-,B)$ with $0 < l < k$ are trivial.*

Proof. $K(A,k)$ is $(k-1)$-connected, hence $H^l(K(A,k),B) = 0$ for $0 < l < k$. □

Corollary 5.5. *a) Let k be odd, $l > 0$. Then*

$$\{\textit{operators } H^k(-,\mathbb{Q}) \to H^l(-,\mathbb{Q})\} = \begin{cases} Q & \textit{if } k = l, \\ 0 & \textit{else.} \end{cases}$$

b) For k even,

$$\{\textit{operators } H^k(-,\mathbb{Q}) \to H^l(-,\mathbb{Q})\} = \begin{cases} \mathbb{Q}\{(\cdot)^{\smile n}\} & \textit{if } l = nk, \\ 0 & \textit{else.} \end{cases}$$

c) The graded ring of stable operations $H^\bullet(-,\mathbb{Q}) \to H^{\bullet+*}(-,\mathbb{Q})$ is a copy of $\mathbb{Q}$ concentrated in degree 0.

Proof. This follows from proposition 5.3 and earlier computations

$$H^*(K(\mathbb{Q},k),\mathbb{Q}) = \begin{cases} \bigwedge(x_k) & \text{if } k \text{ odd}, \\ \mathbb{Q}[x_k] & \text{if } k \text{ even.} \end{cases}$$

□

We now turn to operations on cohomology with $\mathbb{F}_2$ coefficients.

Theorem 5.6 (Steenrod)**.** *There are unique cohomology operations $\mathrm{Sq}^i : H^n(X,\mathbb{F}_2) \to H^{n+i}(X,\mathbb{F}_2)$ for every $n, i \geq 0$, which have the following properties:*

- $\mathrm{Sq}^0 = \mathrm{id}$.

- $\mathrm{Sq}^i : H^i(X, \mathbb{F}_2) \to H^{2i}(X, \mathbb{F}_2)$ *is the cup square.*
- *If* $i > \deg(x)$*, then* $\mathrm{Sq}^i(x) = 0$.
- *Cartan formula* $\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y)$

Moreover, these operations are stable, Sq^1 *is the Bockstein homomorphism for the exact sequence* $0 \to \mathbb{F}_2 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2 \to 0$*, and they satisfy the Adem relations* $\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{n=0}^{\lfloor i/2 \rfloor} \binom{j-n-1}{i-2n} \mathrm{Sq}^{i+j-n}\,\mathrm{Sq}^n$ *for* $0 < i < 2j$.

This was discussed in Topology II, but we explain a way to consturct the Sq^i using the Serre spectral sequence. Let $S^\infty \cong EC_2 \to BC_2 \cong \mathbb{RP}^\infty$ be the universal cover. The group C_2 acts on $K(\mathbb{F}_2, n)^{\times 2} = K(\mathbb{F}_2, n) \times K(\mathbb{F}_2, n)$ by permuting the factors, and likewise on $K(\mathbb{F}_2, n)^{\wedge 2} = K(\mathbb{F}_2, n)^{\times 2}/K(\mathbb{F}_2, n) \vee K(\mathbb{F}_2, n)$. We can form the quotients $EC_2 \times_{C_2} K(\mathbb{F}_2, n)^{\times 2}$ and $EC_{2+} \wedge_{C_2} K(\mathbb{F}_2, n)^{\wedge 2}$.

Proposition 5.7. *We have a bijection*

$$H^{2n}(EC_{2+} \wedge_{C_2} K(\mathbb{F}_2, n)^{\wedge 2}, \mathbb{F}_2) \to H^{2n}(K(\mathbb{F}_2, n)^{\wedge 2}), \mathbb{F}_2) \cong \mathbb{F}_2\{\iota_n \smile \iota_n\}.$$

Conceptually, the extension of the cup square $K(\mathbb{F}_2, n)^{\wedge 2} \to K(\mathbb{F}_2, 2n)$ to a map $EC_{2+} \wedge_{C_2} K(\mathbb{F}_2, n)^{\wedge 2} \to K(\mathbb{F}_2, 2n)$ comes from the fact that the cochains $C^\bullet(X, \mathbb{F}_2)$ (or the spectrum $H(F_2)$) carries a so-called E_∞-multiplication ("commutative up to all higher homotopies") and there are maps $E\Sigma_{m+} \wedge_{\Sigma_m} K(\mathbb{F}_2, n)^{\wedge m} \to K(\mathbb{F}_2, mn)$.

We can also construct the classes in an ad hoc way using our methods. This is easiest done with a relative form of the Serre spectral sequence.

Proposition 5.8. *Let* $f : E \to B$ *be a Serre fibration and* $E' \subseteq E$ *be a subspace such that* $f' = f|_{E'} : E' \to B$ *is a Serre fibration. We write* $F = f^{-1}(b)$ *and* $F' = (f')^{-1}(b)$ *for the fibres. Then there exists a spectral sequence of the form*

$$E_2^{p,q} = H^p(B; H^q(F, F'; A)) \Rightarrow H^{p+q}(E, E'; A).$$

As before, $H^q(F, F'; A)$ *is to be interpreted as a local system in general.*

Proof. (Sketch) One obtains this Serre spectral sequence from the standard one via the quotient double complex $C_{\bullet,\bullet}(f, A)/C_{\bullet,\bullet}(f', A)$. □

We now apply this to the Serre fibration $EC_2 \times_{C_2} K(F_2, n)^{\times 2} \to BC_2$ with fibre $K(\mathbb{F}_2, n)^{\times 2}$, and its subfibration $EC_2 \times_{C_2} (K(F_2, n) \vee K(F_2, n)) \to BC_2$. By the Künneth theorem, the groups

$$H^q(K(\mathbb{F}_2, n)^{\times 2}, K(\mathbb{F}_2, n)^{\vee 2}, \mathbb{F}_2) \cong \widetilde{H}^q(K(\mathbb{F}_2, n)^{\wedge 2}, \mathbb{F}_2)$$

are 0 for $q < 2n$ and isomorphic to $\mathbb{F}_2$ spanned by $\iota_n \smile \iota_n$ in degree $2n$. Note that any group action on $\mathbb{F}_2$ is necessarily trivial, hence the local system $H^{2n}(K(\mathbb{F}_2, n)^{\times 2}, K(\mathbb{F}_2, n)^{\vee 2}, \mathbb{F}_2)$ must be constant. Hence, $E_2^{0,2n} = H^0(SC_2, H^{2n}(\ldots)) \cong \mathbb{F}_2$. By our calculation, $E_2^{p,q} = 0$ for $q < 2n$. Hence there cannot be any differential out of $E_2^{0,2n}$ and we obtain a unique class $\alpha_n \in H^{2n}(EC_{2+} \wedge_{C_2} K(\mathbb{F}_2, n)^{\vee 2}, \mathbb{F}_2)$ refining the cup square in $H^{2n}(K(\mathbb{F}_2, n)^{\vee 2}, \mathbb{F}_2)$.

Remark: The same proof works for all m:

$$H^{nm}(E\Sigma_{m+} \wedge_{\Sigma_m} K(\mathbb{F}_2, n)^{\wedge m}, \mathbb{F}_2) \cong H^{nm}(K(\mathbb{F}_2, n)^\wedge m, \mathbb{F}_2) \cong \mathbb{F}_2\{\iota_n \smile \ldots \smile \iota_n\}.$$

We proceed by pulling back α_n under the map

$$\begin{aligned} H^{2n}(EC_{2+} \wedge_{C_2} K(\mathbb{F}_2, n)^{\wedge 2}, \mathbb{F}_2) &\xrightarrow{\Delta^*} H^{2n}(BC_{2+} \wedge K(\mathbb{F}_2, n), \mathbb{F}_2) \\ &\cong \bigoplus_{i=0}^{n} H^i(BC_2, \mathbb{F}_2) \otimes \widetilde{H}^{2n-i}(K(\mathbb{F}_2, n), \mathbb{F}_2) \end{aligned}$$

Let $u \in H^1(BC_2, \mathbb{F}_2)$ be a generator. Then $\mathrm{Sq}^i(\iota_n) \in H^{n+i}(K(\mathbb{F}_2, n), \mathbb{F}_2)$ is uniquely defined by the expression $\Delta^*(\alpha_n) = \sum_{i=0}^{n} u^{n-i} \cdot \mathrm{Sq}^i(\iota_n)$. By the Yoneda lemma (Proposition 5.3a), $\mathrm{Sq}^i(\iota_n)$ corresponds to a cohomology operation $\mathrm{Sq}^i : H^n(-, \mathbb{F}_2) \to H^{n+i}(-, \mathbb{F}_2)$. We omit the proof that these satisfy the properties from the theorem.

Let $\mathcal{A}$ denote the Steenrod algebra, defined as the free graded algebra on the classes Sq^i (with $|\mathrm{Sq}^i| = i$) divided by the Adem relations and the relation $\mathrm{Sq}^0 = 1$. We obtain a map from $\mathcal{A}$ to the ring of stable cohomology operations on $H^\bullet(-, \mathbb{F}_2)$. We will later show that this is an isomorphism. Note: The cohomology ring $H^\bullet(X, A)$ is naturally a module over $\mathcal{A}$, satisfying the Cartan formula, the condition $\mathrm{Sq}^i(x) = 0$ for $i > \deg(x)$ and $\mathrm{Sq}^{|x|} = x^{\smile 2}$ (an "unstable module").

For now, we record some applications. We say that an element $x \in \mathcal{A}_n$ is decomposable if it can be written as $x = x_1 y_1 + \ldots + x_k y_k$ for homogeneous $x_i, y_i \in \mathcal{A}$ of positive degree.

Lemma 5.9. *The class Sq^m is decomposible if and only if m is not a power of 2.*

Proof. First assume that m is not a power of 2. Let i be the smallest power of 2 not appearing in the binary expansion of $m-1$. We consider the Adem relation $\mathrm{Sq}^i \mathrm{Sq}^{m-i} = \sum_{n=0}^{\lfloor i/2 \rfloor} \binom{m-i-n-1}{i-2n} \mathrm{Sq}^{m-n} \mathrm{Sq}^n$. The coefficient of Sq^m, i.e. the case $n = 0$, is given by $\binom{m-i-1}{i}$. This number is odd, since i appears in the binary expansion of $m - i - 1$ (see Lucas' theorem). Hence

$$\mathrm{Sq}^m = \mathrm{Sq}^i \mathrm{Sq}^{m-i} + \sum_{n=1}^{\lfloor i/2 \rfloor} \binom{n-i-n-1}{i-2n} \mathrm{Sq}^{m-n} \mathrm{Sq}^n .$$

is decomposable.

Now let $n = 2^k$. It suffices to give an A-module V and a homogeneous element $v \in V$ such that $\mathrm{Sq}^n v \neq 0$ and $\mathrm{Sq}^i v = 0$ for $0 < i < n$. We claim that $V = H^\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) = \mathbb{F}_2[u]$ and $v = u^{2^k}$ works. For $k = 0$, we have $\mathrm{Sq}^1(u) = u^2$ and there is no degree between 0 and 1. We proceed by induction. Hence by assumption we have $\mathrm{Sq}^0(u^{2^{k-1}} = u^{2^{k-1}}$, $\mathrm{Sq}^{2^{k-1}}(u^{2^{k-1}}) = u^{2^k}$ and $\mathrm{Sq}^i(u^{2^{k-1}} = 0$ otherwise. For $l = 1, \ldots, 2^k - 1$, we have

$$\mathrm{Sq}^l(u^{2^k}) = \mathrm{Sq}^l(u^{2^{k-1}} u^{2^{k-1}}) = \sum_{i=0}^{l} \mathrm{Sq}^i(u^{2^{k-1}}) \mathrm{Sq}^{l-i}(u^{2^{k-1}})$$

is non-trivial only if $i, l \in \{0, 2^{k-1}\}$. This is the case only for $l = 2^{k-1}$ and $i = 0$ or $i = 2^{k-1}$, and these two terms cancel each other out. Hence $\mathrm{Sq}^l(u^{2^k}) = 0$ for $0 < l < n$ and $\mathrm{Sq}^{2^k}(u^{2^k}) = (u^{2^k})^2 = u^{2^{k+1}} \neq 0$. □

Remark: More conceptually: The total squaring operator $\mathrm{Sq} = \mathrm{Sq}^0 + \mathrm{Sq}^1 + \ldots : H^\bullet(X, \mathbb{F}_2) \to H^\bullet(X, \mathbb{F}_2)$ is a ring homomorphism by the Cartan formula. Hence $\mathrm{Sq}(u^{2^k}) = \mathrm{Sq}(u)^{2^k} = (u + u^2)^{2^k} = u^{2^k} + u^{2^{k+1}}$ since the squaring map is a ring homomorphism in characteristic 2.

Example 5.10. $\mathrm{Sq}^3 = \mathrm{Sq}^1 \mathrm{Sq}^2$, $\mathrm{Sq}^5 = \mathrm{Sq}^1 \mathrm{Sq}^4$, in general $\mathrm{Sq}^{2n+1} = \mathrm{Sq}^1 \mathrm{Sq}^{2n}$. Also

$$\mathrm{Sq}^6 = \mathrm{Sq}^2 \mathrm{Sq}^4 + \mathrm{Sq}^5 \mathrm{Sq}^1 = \mathrm{Sq}^2 \mathrm{Sq}^4 + \mathrm{Sq}^1 \mathrm{Sq}^4 \mathrm{Sq}^1 .$$

Corollary 5.11. *If there exists an element in $\pi_{2n-1}(S^n)$ of Hopf invariant 1, then n is a power of 2.*

Proof. Let $f : S^{2n-1} \to S^n$ have Hopf invariant 1. Then, for a generator $x \in H^n(C(f), \mathbb{Z}) = \mathbb{Z}$, the square $x^{\smile 2}$ generates $H^{2n}(C(f), \mathbb{Z}) \cong \mathbb{Z}$. Reducing mod2, we conlcude that the cup-square of a generator $\overline{x}$ of $H^n(C(f), \mathbb{F}_2) \cong \mathbb{F}_2$ generates $H^{2n}(C(f), \mathbb{F}_2) \cong \mathbb{F}_2$. Hence $\mathrm{Sq}^n(\bar{x}) = \bar{x}^{\smile 2} \neq 0$. But for $n < k < 2n$, $H^k(C(f), \mathbb{F}_2) = 0$, hence $\mathrm{Sq}^1(\bar{x}), \dots, \mathrm{Sq}^{n-1}(\bar{x})$ are trivial. By the previous lemma, n must therefore be a power of 2. □

Example 5.10. We use Adem relations to show that $f = \eta \circ \Sigma\eta : S^4 \to S^2$ is non-trivial in $\pi_4(S^2)$. If f was trivial, we could extend $\eta : S^3 \to S^2$ over the cone of $\Sigma\eta$, which is given by $\Sigma\mathbb{CP}^2$. Let $g : \Sigma\mathbb{CP}^2 \to S^2$ be this extension. The mapping cone $C(g)$ has one cell in each dimension $0, 2, 4, 6$. Let $x_2, x_4, x_6 \in H^\bullet(C(g), \mathbb{F}_2)$ denote the generators in the respective degrees. We obtain the following picture on cohomology from the maps $\mathbb{CP}^2 = C(\eta) \hookrightarrow C(g)$ and $C(g) \to \Sigma^2\mathbb{CP}^2$:

$$\begin{tikzcd}
\widetilde{H}^\bullet(\mathbb{CP}^2,\mathbb{F}_2) & \widetilde{H}^\bullet(C(g),\mathbb{F}_2) & \widetilde{H}^\bullet(\Sigma^2\mathbb{CP}^2,\mathbb{F}_2) \\
 & \bullet\, x_6 & \bullet \arrow[l] \\
\bullet & \bullet\, x_4 \arrow[l] \arrow[u, "\mathrm{Sq}^2"] & \bullet \arrow[l] \arrow[u, "\mathrm{Sq}^2"] \\
\bullet \arrow[u, "\mathrm{Sq}^2"] & \bullet\, x_2 \arrow[l] \arrow[u, "\mathrm{Sq}^2"] &
\end{tikzcd}$$

By naturality, we have $\mathrm{Sq}^2 \mathrm{Sq}^2 x_2 = x_6 \neq 0$, but $\mathrm{Sq}^2 \mathrm{Sq}^2 = \mathrm{Sq}^3 \mathrm{Sq}^1$ by the Adem relation, and $\mathrm{Sq}^1 x_2 = 0$, contradiction. Note that this argument even shows that $\Sigma^k(\eta \circ \Sigma\eta)$ is nontrivial for all k.

Now we turn to the computation of $H^\bullet(K(\mathbb{F}_2, n), \mathbb{F}_2)$ for all n.

Definition 5.11. For $I = (i_n, i_{n-1}, \dots, i_0)$, denote by $\mathrm{Sq}^I \in \mathcal{A}$ the composite $\mathrm{Sq}^{i_n} \mathrm{Sq}^{i_{n-1}} \cdots \mathrm{Sq}^{i_0}$. We call I admissible if $i_k \geq 2i_{k-1}$ for all k, and $i_0 \geq 1$. Here, the empty sequence is admissible, and $\mathrm{Sq}^\emptyset = 1$. We write $|I| = i_n + i_{n-1} + \ldots + i_0 = |\mathrm{Sq}^I|$ for the total degree and the excess $e(I)$ of an admissible sequence I is

$$e(I) = (i_n - 2i_{n-1}) + \ldots + (i_1 - 2i_0) + i_0 = 2i_n - |I|.$$

Theorem 5.12 (Cartan-Serre)**.** *For $n \geq 1$ we have*

$$H^\bullet(K(\mathbb{F}_2, n), \mathbb{F}_2) \cong \mathbb{F}_2[\{\mathrm{Sq}^I \iota_n \mid I \text{ admissible, } e(I) < n\}].$$

For $n \geq 2$ we have

$$H^\bullet(K(\mathbb{Z}, n), \mathbb{F}_2) \cong \mathbb{F}_2[\{\mathrm{Sq}^I \iota_n \mid I \text{ admissible, } i_0 \geq 2, e(I) < n\}].$$

Example. $H^\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) \cong H^\bullet(K(\mathbb{F}_2, 1), \mathbb{F}_2) \cong \mathbb{F}_2[\iota_1]$, since every non-empty admissible sequence I has $e(I) \geq 1$. Further,

$$H^\bullet(K(\mathbb{F}_2, 2), \mathbb{F}_2) \cong \mathbb{F}_2[\{\mathrm{Sq}^{2^n} \mathrm{Sq}^{2^{n-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 \iota_2 \mid n \geq 1\}]$$

and $H^\bullet(K(\mathbb{Z}, 2), \mathbb{F}_2) \cong \mathbb{F}_2[\iota_2]$ since any non-empty admissible sequence with $i_0 \geq 2$ has $e(I) \geq 2$.

Looking at the spectral sequence for $K(\mathbb{F}_2, n) \to * \to K(\mathbb{F}_2, n+1)$ we have a differential d_{n+1} which sends ι_n to ι_{n+1}. We want to understand how these behave in relation to Sq^k.

Let $F \to E \xrightarrow{p} B$ be a fibre sequence with F path-connected and B 1-connected and $b_0 \in B$. The differential $d_n : E_n^{0,n-1} \to E_n^{n,0}$ is defined on a subgroup of $E_2^{0,n-1} \cong H^{n-1}(F, A)$ with values in a quotient of $E_2^{n,0} \cong H^n(B, A)$.

Definition 5.13. We call this operation the *transgression*, and $x \in H^{n-1}(F, A)$ *transgressive*, if it lies in $E_n^{0,n-1}$ and hence its transgression is defined. We say that x *transgresses* to $y \in H^n(B, A)$, if y is a representative of $d_n(x)$ in the quotient.

Theorem 5.14. *In the diagram*

$$H^{n-1}(F, A) \xrightarrow{\delta} H^n(E, F, A) \xleftarrow[p^*]{} H^n(B, b_0, A)$$

$x \in H^{n-1}(F, A)$ *is transgressive if and only if* $\delta(x) \in \operatorname{im}(p^*)$. *Moreover, the kernel of* $H^n(B, A) \cong H^n(B, b_0, A) \to E_n^{n,0}$ *agrees with the kernel of* p^**, and the transgression can be identified with*

$$E_n^{0,n-1} \cong \delta^{-1}(\operatorname{im}(p^*) \xrightarrow{\delta} \operatorname{im}(p^*) \xleftarrow[\cong]{p^*} H^n(B, b_0, A)/\ker(p^*) \cong E_n^{n,0}$$

Proof. (Sketch). Let $C_{\bullet,\bullet}(p, \mathbb{Z})$ denote the double complex of singular (p, q)-simplices. Then applying $\operatorname{Hom}(-, A)$ to the quotient $C_{\bullet,\bullet}(p, \mathbb{Z})/C_{\bullet,\bullet}(F \to \{b_0\}, \mathbb{Z})$ gives rise to a reduced version of the cohomological Serre spectral sequence of the form

$$\widetilde{E}_2^{p,q} = H^p(B, b_0, H^q(F, A)) \Rightarrow H^{p+q}(E, F, A)$$

and a map of spectral sequences $\widetilde{E}_r \to E_r$. Note that $\widetilde{E}_2^{0,q} = 0$ and $\widetilde{E}_2^{p,q} \cong E_2^{p,q}$ for $p > 0$. So the 2nd pages only differ in the first column. Applying our characterization of the edge homomorphism to the reduced spectral sequence, we conclude that

$$E_n^{n,0} \cong \widetilde{E}_n^{n,0} \cong \operatorname{im}(p^* : H^n(B, b_0, A) \to H^n(E, F, A))$$

and that the map

$$H^n(B, b_0, A) \cong \widetilde{E}_2^{n,0} \to \widetilde{E}_n^{n,0} \cong \widetilde{E}_\infty^{n,0} \hookrightarrow H^n(E, F, A)$$

agrees with p^*. Let $\delta : H^{n-1}(F, A) \to H^n(E, F, A)$ be the boundary map, and let $\alpha \in H^{n-1}(F, A)$. Then the first non-zero differential on α in the absolute spectral sequence determines whicch filtration entry detects $\delta\alpha \in H^n(E, F)$ in the relative spectral sequence. Hence α is transgressive if and only if $\delta\alpha$ is of filtration n, which is isomorphic to the image of p^*, and its transgression is the coset $(p^*)^{-1}(\delta\alpha)$. □

Theorem 5.15 (Transgression theorem)**.** *The subset of transgressive classes in* $H^\bullet(F, \mathbb{F}_2)$ *is closed under the application of each* Sq^i*. If* $x \in H^{n-1}(F, \mathbb{F}_2)$ *transgresses to* $y \in H^n(B, \mathbb{F}_2)$*, then* $\operatorname{Sq}^i x$ *transgresses to* $\operatorname{Sq}^i y$*.*

Proof. If $\delta(x) = p^*(y)$, then

$$\delta(\operatorname{Sq}^i x) = \operatorname{Sq}^i(\delta x) = \operatorname{Sq}^i(p^* y) = p^*(\operatorname{Sq}^i y).$$

Note that δ commutes with Sq^i since it can be written as $H^{n-1}(F, \mathbb{F}_2) \cong \widetilde{H}^n(\Sigma F, \mathbb{F}_2) \to \widetilde{H}^n(C(i), \mathbb{F}_2) \cong H^n(E, F, \mathbb{F}_2)$, where i is the inclusion $F \to E$, and the Sq^i are stable. □

Lemma 5.16. *Let* X *be a space,* $x \in H^n(X, \mathbb{F}_2)$*,* I *admissible.*

a) If $e(I) > n$*, then* $\operatorname{Sq}^I(x) = 0$*.*

b) If $e(I) = n$*, then* $\operatorname{Sq}^I(x) = (\operatorname{Sq}^{I'}(x))^2$ *for some admissible* I' *with* $e(I') \leq n$*.*

In particular, each $\operatorname{Sq}^I(x)$ *with* $e(I) = n$ *can be written as* $(\operatorname{Sq}^J(x))^{2^k}$ *with* $e(J) < n$*.*

Proof. a) Since $i_n = i_{n-1} + \ldots + i_0 + e(I)$, we have

$$|\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_0} x| = |x| + i_{n-1} + \ldots + i_0 = |x| + i_n - e(I) < i_n.$$

Hence $\operatorname{Sq}^I(x) = \operatorname{Sq}^{i_n}(\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_0} x) = 0$.
b) Similarly, if $e(I) = n$, then $|\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_0} x| = i_n$, hence $\operatorname{Sq}^I(x) = \operatorname{Sq}^{i_n}(\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_0} x) = (\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_0} x)^{\smile 2}$ and $e(I' = (i_{n-1}, \ldots, i_0) \leq e(I) = n$ □

Theorem 5.17. *Let $F \to E \to B$ be a fibre sequence, with $E \simeq *$, and B 1-connected. Assume that there exist transgressive elements $x_i \in H^\bullet(F, F_2)$ such that the square-free monomials of the x_i form a basis of $H^\bullet(F, \mathbb{F}_2)$. Let $y_i \in H^\bullet(B, \mathbb{F}_2)$ denote an element representing the regression of x_i. Then $H^\bullet(B, \mathbb{F}_2)$ is a polynomial ring on the classes y_i.*

Proof. We define a spectral sequence $\overline{E}_r^{p,q}$ as follows: $\overline{E}_r^{p,q} = \bigoplus x_{i_1} \cdots x_{i_k} \mathbb{F}_2[\{y_j \mid |y_j| \geq r\}]_q$, where the direct sum runs over all square-free monomials in the x_i of degree at least $r-1$ of total degree p, and

$$d_r((x_{i_1} \cdots x_{i_k}) \cdot P) = \sum_{j=1}^{k} d_r(x_{i_j}) \prod_{l \neq j} x_{i_l} \cdot P$$

where $d_r(x_i) = y_i$ if $|x_i| = r-1$ and 0 otherwise. We note the following: We have $d_r((x_{i_1} \cdots x_{i_k})P) = 0$ if and only if $|x_{i_l}| > r-1$ for all $l = 1, \ldots, k$. Further, $\operatorname{im}(d_r)$ is contained in the subgroup generated by expressions of the form $(x_{i_1} \cdots x_{i_k})Q$, where Q is a monomial containing at least one y_i with $|y_i| = r$. If $(x_{i_1} \cdots x_{i_k})Q$ is such a monomial, and $d_r((x_{i_1} \cdots x_{i_k})Q = 0$, then $(x_{i_1} \cdots x_{i_k})Q \in \operatorname{im}(d_i)$: By the above, we must have $|x_{i_l}| \geq r$ for all l. We write $Q = Q' \cdot y_i$ with $|y_i| = r$. Then $i_l \neg i$ for all l, hence $x_{i_1} \cdots x_{i_k} x_i$ is square-free and $d_r(x_{i_1} \cdots x_{i_k} x_i Q) = x_{i_1} \cdots x_{i_k} y_i Q' = x_{i_1} \cdots x_{i_k} Q$. One checks easily that d_r is a differential. Hence $\overline{E}$ is indeed a spectral sequence.

Moreover, we obtain a map of spectral sequences $\varphi : \overline{E} \to E$, $x_{i_1} \cdots x_{i_k} P \mapsto x_{i_1} \cdots x_{i_k} P$, using that the differentials d_r on E_r on elements of the form $x_{i_1} \cdots x_{i_k}$ are determined by the differentials on the x_i and the Leibniz rule. This map of spectral sequences is an isomorphism on $E_r^{0,q}$ for all r, and on E_∞. We now show that it is an isomorphism on $E_2^{p,0}$ by induction on p. Assume we know it is an isomorphism on $E_2^{p',0}$ for all $p' < n$. Then it is also an isomorphism on $E_2^{p,q}$ for $p < n$, and on $E_i^{n-i,i-1}$ for all $2 \leq i \leq n$, since all outgoing differentials don't leave that range until $d_i : E_i^{n-i,i-1} \to E_i^{n,0}$. Every $\alpha \in E_2^{n,0}$ must be in the image of one of these d_i since $E_\infty^{n,0} = 0$. It follows that $\overline{E}_2^{n,0} \to E_2^{n,0}$ is surjective.

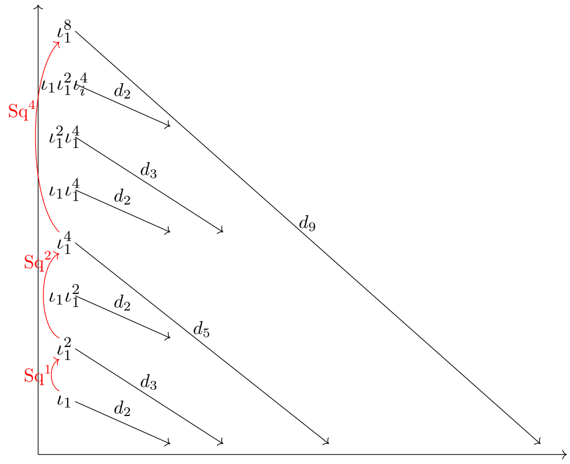
Now assume that $\varphi : \overline{E}_2^{n,0} \to E_2^{n,0}$ was not injective, so let $0 \neq \alpha \in \overline{E}_2^{n,0}$ be in the kernel of φ. Again, we can write $[\alpha] = d_i(\beta)$ for some i and $\beta \in \overline{E}_i^{n-i,i-1}$, hence $d_i(\varphi(\beta)) = \varphi(d_i\beta) = \varphi(a) = 0$. Hence $\varphi(\beta) \in E_2^{n-i,i-1}$ is a permanent cycle, as d_i is the last possible non-trivial differential. But $\varphi(\beta)$ is not in the image of d_i, since $\beta \in \overline{E}^{n-i,i-1}$ is not and all differentials landing in $E_i^{n-i,i-1}$ are determined by those into $\overline{E}_i^{n-i,i-1}$ by the induction assumption. This finishes the proof. □

Proof. (of Theorem 5.12). For $K(\mathbb{F}_2, n)$, we start the induction with $K(\mathbb{F}_2, 1) \cong \mathbb{RP}^\infty$, for which we knot the statement to hold. For $K(\mathbb{F}_2, n)$, we analogously start with $K(\mathbb{Z}, 2) \cong \mathbb{CP}^\infty$. In Theorem 5.17, applied to the induction step and the Serre spectral sequence for $K(\mathbb{F}_2, n) \to * \to K(\mathbb{F}_2, n+1)$, we set the x_i to be all the $(\operatorname{Sq}^I \iota_n)^{2^k}$ with $e(I) < n$, I admissible. Note that square-free monomials of these form a basis for $H^\bullet(K(\mathbb{F}_2, n), \mathbb{F}_2)$, as powers of the form $(\operatorname{Sq}^I \iota_n)^m$ can be uniquely decomposed via their binary expansion. For the integral case take those with $i_0 \geq 2$. Note that ι_n transgresses to ι_{n+1}, since the d_{n+1} differential on ι_n is the only possible non-trivial one. By Lemma 5.16, $\operatorname{Sq}^I \iota_n$ also transgresses, to $\operatorname{Sq}^I \iota_{n+1}$, and $(\operatorname{Sq}^I \iota_n)^{2^k}$ transgresses to

$$\operatorname{Sq}^{2^{k-1}(|I|+n)} \ldots \operatorname{Sq}^{|I|+n}(\operatorname{Sq}^I \iota_{n+1}) = \operatorname{Sq}^J \iota_{n+1}$$

with $J = (2^{k-1}(|I| + n))(2^{k-2}(|I| + n)) \cdots (|I| + n), I)$ if $k > 0$. As we saw, every J with $e(I) = n$ arises in this way from an admissible sequence of smaller excess, thus finishing the proof. $\square$

Example. $\mathbb{RP}^\infty \to * \to K(\mathbb{F}_2, 2)$. From the transgression theorem, we can infer the first non-trivial differentials out of the 0-th column, which are determined by the smallest exponent in the binary expansion of ι_1^k.



Corollary 5.18. *The map $\psi : \mathcal{A} \to \{$ring of stable operations on $H^*(-, \mathbb{F}_2)\}$ is an isomorphism.*

Proof. We saw that the ring of stable operations can be described in degree k as the limit

$$\lim_{n \in \mathbb{N}} H^{k+n}(K(\mathbb{F}_2, n), \mathbb{F}_2).$$

We note that in degree $< n$, all Sq^I for I admissible have excess $e(I) < n$, since $e(I) = 2i_n - |I| \leq 2|I| - |I| = |I|$. Moreover, every product of the form $\mathrm{Sq}^I c\iota \cdots \mathrm{Sq}^J \iota_n$ lies in degree $\geq 2n$ inside $H^\bullet(K(\mathbb{F}_2, n), \mathbb{F}_2)$. It follows that up to degree $2n - 1$, $H^\bullet(K(\mathbb{F}_2, n), \mathbb{F}_2)$ has a basis consisting of all the $\mathrm{Sq}^I \iota_n$ with I admissible and $|I| < n$. Hence, in the limit, the ring of stable operators has a basis given by all Sq^I with I admissible, since the element $(\iota_1, \iota_2, \iota_3, \ldots)$ is the image of $1 \in \mathcal{A}$. In particular, ψ is surjective.

For injectivity, it suffices to show that every element in $\mathcal{A}$ can be written as a sum of elements of the form Sq^I for I admissible, since ψ is injective on their span, given that their images are linearly independent. This can be achieved inductively using Adem relations. Cinsider $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_r}$, not necessarily admissible. Define $m(I) = \sum_{s=1}^r i_s s$. Ijf I is not admissible, there exists $j \in \{1, \ldots, r\}$ such that $i_j < 2i_{j+1}$. We can perform the Adem relation

$$\mathrm{Sq}^{i_j} \mathrm{Sq}^{i_{j+1}} = \sum_{n=0}^{\lfloor i_j/2 \rfloor} \binom{i_{j+1} - n - 1}{i_j - 2n} \mathrm{Sq}^{i_j + i_{j+1} - n} \mathrm{Sq}^n$$

to replace Sq^I by a sum of terms $\mathrm{Sq}^{I'}$ with $I' = (i_1, \ldots, i_j - 1, i_j + i_{j+1} - n, n, i_{j+2}, \ldots, i_r)$. Furthermore,

$$m(I') = m(I) + j(i_j + i_{j+1} - n) + (j+1)n - ji_j - (j+1)i_j = m(I) + n - i_{j+1} < m(I)$$

since $n \leq i_j/2$ and $i_j < 2i_{j+1}$. Since $m(-)$ cannot go below 0, this process must terminate. $\square$

Exercise. (not relevant for exam). Use our computation of $H^\bullet(K(\mathbb{F}_2, n), \mathbb{F}_2)$ and $H^\bullet(K(\mathbb{Z}, n), \mathbb{F}_2)$ and our understanding of transgressions to compute $\pi_6(S^3, *)$ as follows:

a) We know that $\pi_6(S^3, *)$ is finite and that 3-torsion is a copy of $\mathbb{Z}/3$ and that there is no p-torsion for $p > 3$.

b) To understand the 2-torsion, proceed inductively.

   1) First compute $H^\bullet(\tau_{\geq 4} S^3, \mathbb{F}_2)$ up to sufficiently high degrees via the fibre sequence $\tau_{\geq 4} S^3 \to S^3, K(\mathbb{Z}, 3)$.

   2) Compute $H^\bullet(\tau_{\geq 5} S^3, \mathbb{F}_2)$ (sufficiently high) via the fibre sequence $\tau_{\geq 5} S^3 \to \tau_{\geq 4} S^3 \to K(\mathbb{F}_2, 2)$

   3) Compute $H^\bullet(\tau_{\geq 6} S^3, \mathbb{F}_2)$ in degrees 6 and 7. Use (or prove) the following fact to understand $H_6(\tau_{\geq 6} S^3, \mathbb{Z}) \cong \pi_6(S^6, *)$: If X is a space, then every $\mathbb{Z}$-summand in $H_n(X, \mathbb{Z})$ contributes an $\mathbb{F}_2$-summand in $H^n(X, \mathbb{F}_2)$, every $\mathbb{Z}/2$-summand in $H_n(X, \mathbb{Z})$ contributes one $\mathbb{F}_2$-summand in $H^n(X, \mathbb{F}_2)$ and in in $H^{n+1}(X, \mathbb{F}_2)$ related by a Sq^1, and every $\mathbb{Z}/2^k$-summand for $k > 1$ in $H_n(X, \mathbb{Z})$ contribuets one $\mathbb{F}_2$-summand in $H^n(X, \mathbb{F}_2)$ and one in $H^{n+1}(X, \mathbb{F}_n)$, with no Sq^1 between them.

6 Vector Bundles and Characteristic Classes

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 6.1. An n-dimensional $\mathbb{F}$-vector bundle ξ over a base space B consists of a map of spaces $p : E(\xi) = E \to B$ and for each $b \in B$, the structure of an n-dimensional vector space over $\mathbb{F}$ on $E_b = p^{-1}(b)$. Those must satisfy the local triviality condition: for every $b \in B$ there exists a local neighbourhood $U \subseteq B$ of b and a homeomorphism $h : U \times \mathbb{F}^n \xrightarrow{\cong} p^{-1}(V)$ such that for each $b \in U$, the map $x \mapsto h(b, x)$ is a vector space isomorphism from $\mathbb{F}^n \cong \{b\} \times \mathbb{F}^n \to E_b$. In particular, the fibres are mapped such that the following triangle commutes:

$$\begin{tikzcd}
U \times \mathbb{F}^n \arrow[rr, "h"] \arrow[dr, "\mathrm{pr}_1"'] & & p^{-1}(U) \arrow[dl, "p|_{p^{-1}(U)}"] \\
& U &
\end{tikzcd}$$

Example 6.2. Let B be any space, $n \in \mathbb{N}$. Then $B \times \mathbb{F}^n \to B$ is a vector bundle, with vector space structure on $\mathrm{pr}_B^{-1}(b) = \{b\} \times \mathbb{F}^n$ carried over from the identification $\mathbb{F}^n \cong \{b\} \times \mathbb{F}^n,\ v \mapsto (b, v)$. This bundle is called the trivial bundle.

A non-trivial example is the following: Let $B = \mathbb{RP}^1$ be the 1-dimensional projective space and

$$E = \{(l, v) \mid l \in \mathbb{RP}^1, v \in l\} \subseteq \mathbb{RP}^1 \times \mathbb{R}^2$$

with projection $p : E \to \mathbb{RP}^1,\ (l, v) \mapsto l$, where l here really denotes the full line. Then the bijection $E_l = p^{-1}(\{l\}) = \{l\} \times l \cong l$ gives E_l a vector space structure.

To see that this defines a 1-dimensional vector bundle over $\mathbb{RP}^1$, we need to check the local triviality condition. Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projections, and let $U_i \subseteq \mathbb{RP}^1$ be the set of lines L such that $\pi_i|_L : L \to \mathbb{R}$ is an isomorphism. Then each U_i is open and $\mathbb{RP}^1 = U_1 \cup U_2$. Now $h : p^{-1}(U_i) \to U_i \times B,\ (L, v) \mapsto (L, \pi_i(v))$ defines a homeomorphism. Each π_i is linear, hence h is linear on each fibre as required. This is the Möbius bundle.

The second example generalises.

Example 6.3. Let $N, n \geq 1$. The Grassmanian $\mathrm{Gr}_n(\mathbb{F}^N)$ is the set of n-dimensional subspaces of $\mathbb{F}^N$. Let $\mathrm{Fr}_n(\mathbb{F}^N) \subseteq (\mathbb{F}^N)^n$ be the subspace of linearly independent sequences $(v_1, \ldots, v_n)$. We obtain a surjective map

$$1 : \mathrm{Fr}_n(\mathbb{F}^N) \to \mathrm{Gr}_n(\mathbb{F}^N), \quad (v_1, \ldots, v_n) \mapsto \langle v_1, \ldots, v_n \rangle_{\mathbb{F}}.$$

We give $\mathrm{Gr}_n(\mathbb{F}^N)$ the quotient topology for this map.

Let $\gamma_{\mathbb{F}}^{n,N} := \{(V, v) \in \mathrm{Gr}_n(\mathbb{F}^N) \times \mathbb{F}^n \mid v \in V\}$ and $p : \gamma_{\mathbb{F}}^{n,N} \to \mathrm{Gr}_n(\mathbb{F}^n)$ the projection $(V, v) \mapsto V$. The fibre $p^{-1}(V)$ identifies with $\{V\} \times V \cong V$ and hence forms a vector space. This is locally trivial: For $V \in \mathrm{Gr}_n(\mathbb{F}^N)$, consider the orthogonal projection $\pi_V : \mathbb{F}^N \to V$. The set $U := \{W \in \mathrm{Gr}_n(\mathbb{F}^N) \mid \pi_V|_W \text{ iso}\}$ is open, since $q^{-1}(U)$ is open in $(\mathbb{F}^N)^n$. Again, the map $p^{-1}(U) \to U \times V,\ (W, v) \mapsto (W, \pi_V(v))$ is a homeomorphism, yielding a local trivialization. The vector bundle $\gamma_{\mathbb{F}}^{n,N}$ is called the tautological vector bundle on $\mathrm{Gr}_n(\mathbb{F}^N)$.

Example. $\mathrm{Gr}_1(\mathbb{F}^n)$ is by definition given by the projective space $\mathbb{FP}^{n-1}$ from the previous example.

We next discuss operations on vector bundles.

Definition 6.4. A subbundle of a vector bundle $p : E \to B$ is a subspace $E' \subseteq E$ such that each $E' \cap E_b$ is a subspace of E_b and $p|_{E'} : E' \to B$ is locally trivial. A morphism of vector bundles

$p_1 : E_1 \to B,\ p_2 : E_2 \to B$ is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

such that each $\varphi|_{(E_1)_b} : (E_1)_b \to (E_2)_b$ is $\mathbb{F}$-linear. Two vector bundles are isomorphic if there exist mutually inverse morphisms $f : E_1 \to E_2$ and $g : E_2 \to E_1$.

If $p : E \to B$ is a vector bundle, and $f : X \to B$ a continuous map, we define the pullback bundle

$$f^*E = X \times_B E = \{(x, e) \in X \times E \mid f(x) = p(e)\}$$

and $f^*p : f^*E \to X$ the natural projection. We use the map $(f^*E)_x \to E_{f(x)},\ (x, e) \mapsto e$ to give $(f^*E)_x$ the structure of an $\mathbb{F}$-vector space. If $\varphi_U : U \times \mathbb{F}^n \xrightarrow{\cong} p^{-1}(U)$ is a local trivialization of p, then setting $V = f^{-1}(U)$ we obtain a local trivialization of f^*p via $V \times \mathbb{F}^n \to (f^*E)^{-1}(V),\ (x, v) \mapsto (x, \varphi_V(f(x), e))$. Hence f^*p indeed defines a vector bundle.

Example. By definition, each $\gamma_{\mathbb{F}}^{n,N}$ is a subbundle of the trivial bundle $\mathrm{Gr}_n(\mathbb{F}^N) \times \mathbb{F}^N \to \mathrm{Gr}_n(\mathbb{F}^N)$.

Roughly speaking, any natural continuous operation on vector spaces can be extended to vector bundles. We focus on the following examples

1. (Whitney) sum of bundles.
   If $p : E \to B$ is an n-dimensional $\mathbb{F}$-vector bundle and $p' : E' \to B$ an n'-dimensional vector bundle, we define $p \oplus p' : E \oplus E' \to B$ with $E \oplus E' = E \times_B E'$ the "fibre direct sum" with projection $(p \oplus p')(e, e') = p(e) = p'(e')$. We have $(E \oplus E')_b = E_b \times E'_b = E_b \oplus E'_b$, which inherits a $(n+n')$-dimensional vector space structure. We omit the proof that this is indeed locally trivial.

2. Realification and complexification.
   If $p : E \to B$ is an n-dimensional $\mathbb{C}$-vector bundle, we can also consider it as a $2n$-dimensional real vector bundle by neglecting structure fibrewise. We write $E_{\mathbb{R}}$ to emphasize the real structure.

   If $p : E \to B$ is an n-dimensional $\mathbb{R}$-vector bundle, we can use the natural identification $\mathbb{C} \otimes_{\mathbb{R}} V \cong V \oplus V,\ (1, v) \mapsto (v, 0),\ (i, v) \mapsto (0, v)$ to obtain from the $2n$-dimensional $\mathbb{R}$-vector bundle $E \oplus E$ an n-dimensional $\mathbb{C}$-vector bundle. We write $E \otimes_{\mathbb{R}} \mathbb{C}$ for this complex bundle.

3. Euclidean bundles.
   Recall that an Euclidean vector space is a finite dimensional real vector space together with a positive definite quadratic function $\mu : K \to \mathbb{R}$, i.e. $\mu(v) > 0$ for $v \neq 0$ and $v \cdot w := \frac{1}{2}(\mu(v + w) - \mu(v) - \mu(w))$ is bilinear. An Euclidean vector bundle is an $\mathbb{R}$-vector bundle $p : E \to B$ together with a continuous function $\mu : E \to \mathbb{R}$ which restricts to a positive definite quadratic form on each fibre. In this case, μ is called an Euclidean metric on E.

   For example, the trivial bundle $B \times \mathbb{R}^n \to B$ carries the Euclidean metric $\mu(k, x) = \sum_{i=1}^n x_i^2$. One can show that if B is paracompact, then every vector bundle over B can be given an Euclidean metric. Further, if μ and μ' are Euclidean metrics on the same bundle $p : E \to B$, then there exists a bundle automorphism φ of p such that $\mu' = \mu \circ p$. Similarly, one defines hermitian metrics on $\mathbb{C}$-vector bundles.

4. Orthogonal complement bundles.
   Let $p : E \to B$ be an Euclidean $\mathbb{R}$-vector bundle with subbundle $p|_{\widetilde{E}} : \widetilde{E} \to B,\ \widetilde{E} \subseteq E$. Then the orthogonal complement bundle $\widetilde{E}^{\perp}$ is defined to be the subspace $\widetilde{E}^{\perp} = \{e \mid e \in (\widetilde{E}_{p(e)})^{\perp}\}$ using

the scalar product induced by the metric. This is locally trivial: Let $h : U \times \mathbb{R}^n \to p^{-1}(U)$ be a local trivialization of p. Transporting the metric μ on $E \supseteq p^{-1}(U)$ over along h, we obtain an Euclidean metric on $U \times \mathbb{R}^n$. By the previous comment, we can assume up to isomorphism, that this is the standard metric.

Replacing U by a smaller neighbourhood if necessary, we can further assume that $p|_{\widetilde{E}}^{-1}(U) \cong U \times \mathbb{R}^k$ can be trivialized. Composing the two equivalences, we obtain an isometric embedding $U \times \mathbb{R}^k \hookrightarrow U \times \mathbb{R}^n$, $(b, v) \mapsto (b, \psi(b, v))$. By reordering the entries of $\mathbb{R}^n$ if necessary and replacing U yet again by a smaller neighbourhood, we can assume that $\psi(b, e_1), \ldots, \psi(b, e_k)$ intersect trivially with $\{b\} \times \mathbb{R}^0 \times \mathbb{R}^{n-k}$ for all $b \in B$. Hence, the tuple $\psi(b, e_1), \ldots, \psi(b, e_k), e_{k+1}, \ldots, e_n$ form a basis of $\mathbb{R}^n$ for all $b \in B$. Applying the Gram-Schmidt process (which is continuous in the matrix), we obtain an orthonormal basis $\overline{\psi}(b, e_1), \ldots, \overline{\psi}(b, e_k), \overline{\psi}(b)_1, \ldots, \overline{\psi}(b)_{n-k}$. Then the function

$$U \times \mathbb{R}^{n-k} \hookrightarrow U \times \mathbb{R}^n, \quad (b, x) \mapsto \sum_{x_i} \overline{\psi}(b)_i$$

defines a homeomorphism to the orthogonal complement of the image of the embedding $U \times \mathbb{R}^k \to U \times \mathbb{R}^n$ above. Translating back to h, this provides a local trivialization of $\widetilde{E}^{\perp}$.

Example. Consider the Möbius bundle $\gamma_{\mathbb{R}}^{1,2} \to \mathbb{RP}^1 \cong S^1$ as a subbundle of the trivial bunde $\mathbb{RP}^1 \times \mathbb{R}^2 \to \mathbb{RP}^1$. The orthogonal complement bundle $(\gamma_{\mathbb{R}}^{1,2})^{\perp}$ is isomorphic to $\gamma_R^{1,2}$ itself via $\gamma_R^{1,2} \to (\gamma_{\mathbb{R}}^{1,2})^{\perp}$, $(L, v) \mapsto (L, Av)$ where $A : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by $\frac{\pi}{2}$. We will soon see that $\gamma_{\mathbb{R}}^{1,2}$ is non-trivial. However, it becomes trivial after pulling back along the degree map $f : S^1 \to \mathbb{RP}^1 \cong \S^1, (x) \mapsto \langle x \rangle$: $f^*\gamma_{\mathbb{R}}^{1,2}$ is given by the space of pairs $\{(x, v) \mid v \in \langle x \rangle\} \subseteq S^1 \times \mathbb{R}^2$. The map $S^1 \times \mathbb{R}^2$, $(x, \lambda) \mapsto (x, \lambda x)$ is a bundle isomorphism.

Definition 6.5. Let X be a topological space, $n \in \mathbb{N}$. We let $\mathrm{Vect}_{\mathbb{F}}^n(X)$ denote the set of equivalence classes of n-dimensional $\mathbb{F}$-vector bundles on X. This is a contravariant functor in X, via the pullback of bundles.

We want to study this functor. First we show the following

Proposition 6.6. *Let X be paracompact and ξ a vector bundle on $X \times I$. Then the pullbacks $\iota_0^*\xi$ and $\iota_1^*\xi$ are isomorphic vector bundles over X.*

For this, we first show the following lemma.

Lemma. *There exists an open covering $\{U_j\}_j$ of X, such that ξ is trivializable over each $U_j \times I$.*

Proof. First, we note that if $U \subseteq X$ is open, and if ξ is trivializable over $U \times [a, b]$ and over $U \times [b, c]$, then it is trivializable over $U \times [a, c]$. To see this, let $h_1 : p^{-1}(U \times [a, b]) \xrightarrow{\cong} U \times [a, b] \times \mathbb{F}^n$ and $h_2 : p^{-1}(U \times [b, c]) \xrightarrow{\cong} U \times [b, c] \times \mathbb{F}^n$ be trivializations. Then

$$h_2 : p^{-1}(U \times [b, c]) \xrightarrow{h_2} U \times [b, c] \times \mathbb{F}^n \xrightarrow{\mathrm{id}_{[b,c]} \times \varphi} U \times [b, c] \times \mathbb{F}_n, \quad \text{with}$$

$$\varphi : U \times \mathbb{F}^n \cong U \times \{b\} \times \mathbb{F}^n \xrightarrow{h_2^{-1}} p^{-1}(U \times \{b\}) \xrightarrow{h_1} U \times \{b\} \times \mathbb{F}^n \cong U \times \mathbb{F}^n.$$

is another trivialization which agrees with h_1 on $U \times \{b\}$. Hence, the two glue together, giving a trivialization on all of $U \times [a, c]$.

Now by compactness of I, for every $x \in X$, we can find open neighbourhoods $U_{x,1}, \ldots, U_{x,k}$ and partitions $0 = t_0 < t_1 < \ldots < t_k = 1$ such that ξ is trivializable over each $U_{x,i} \times [t_{i-1}, t_i]$. Thus ξ is trivializable on all of $U_x \times I$, with $U_x = U_{x,1} \cap \ldots \cap U_{x,k}$ by the above. $\square$

Proof. (of proposition 6.6) Let $\{U_j\}_j$ as in the lemma. Then, as X is paracompact, there is a countable cover $\{V_i\}_{i\in\mathbb{N}}$ of X and a partition of unity $\{\varphi_i : X \to [0,1]\}_i$ with $\mathrm{supp}(\varphi_i) \subseteq V_i$, and each $V_i = \bigcup_\alpha U_{i,\alpha}$, each $U_{i,\alpha}$ open and $U_{i,\alpha} \subseteq U_j$ for some j. Hence ξ is trivializable over each $V_i \times I$. Let $\psi_i = \varphi_1 + \ldots + \varphi_i$ and let $p_i : E_i \to X$ be the pullback of ξ along $X \times X \times I$, $x \mapsto (x, \psi_i(x))$. For intuition, check the example below. We define a homeomorphism $h_i : E_i \cong E_{i-1}$ as follows: Outside $p^{-1}(V_i)$, set $h_i = \mathrm{id}$, and choose a trivialization $\widetilde{h}_i : p^{-1}(V_i \times I) \cong V_i \times I \times \mathbb{F}^n$, and set $h_i(x, \psi_i(x), v) = (x, \psi_{i-1}(x), v)$ where $(x, \psi_i(x), v) \in V_i \times I \times \mathbb{F}^n$. Then the infinite composite $\ldots \circ h_2 \circ h_1$ is an isomorphism from the pullback $\iota_0^*\xi$ to the pullback $\iota_1^*\xi$, as desired. □

Example. Let $X = V_1 \cup V_2$. Then a trivialization φ on $V_1 \times I$ gives isomorphism $\iota_0\xi \cong (id, \varphi_1)^*\xi$ that is constant outside V_1.

Corollary 6.7. *If X is paracompact, $f_1, f_2 : X \to Y$ homotopic maps and ξ a vector bundle over Y. Then $f_1^*\xi \cong f_2^*\xi$.*

Proof. Let $H : X \times I \to Y$ be a homotopy. Then $f_1 = H \otimes \iota_0,\ f_2 = H \otimes \iota_1$ and hence

$$f_1^*\xi = \iota_0^* H^*\xi \cong \iota_1^* H^*\xi = f_2^*\xi,$$

where the isomorphism comes from proposition 6.6, and $H^*\xi$ is the bundle over $X \times I$. □

Next we aim to show that there exist universal bundles at least over paracompact spaces.

Definition 6.8. For $n \in \mathbb{N}$, the infinite Grassmanian manifold is $\mathrm{Gr}_n^{\mathbb{F}} = \mathrm{Gr}_n(\mathbb{F}^\infty)$, where $\mathbb{F}^\infty$ is the $\mathbb{F}$-vector space of sequences $(x_1, x_2, \ldots)$ with almost all $x_i = 0$. Note that $\mathrm{Gr}_n^{\mathbb{F}}$ is not a manifold. It comes with the weak topology, with respect to the filtration by the $\mathrm{Gr}_n(\mathbb{F}^N)$. Similarly, we define $\gamma_{\mathbb{F}}^n = \{(V, v) \mid v \in V\} \subseteq \mathrm{Gr}_n^{\mathbb{F}} \times \mathbb{F}^\infty$ with $(\gamma_{\mathbb{F}}^n)_V \cong \{V\} \times V \cong V$ a vector space.

Lemma 6.9. *The projection $p : \gamma_{\mathbb{F}}^n \to \mathrm{Gr}_n^{\mathbb{F}},\ (V, v) \mapsto V$ is an n-dimensional $\mathbb{F}$-vector bundle.*

Proof. Similar the the finite-dimensional case, using that $\mathrm{Gr}_n^{\mathbb{F}} \times \mathbb{F}^\infty$ comes with the weak topology. □

Theorem 6.10. *If X is paracompact, then the natural map*

$$[X, \mathrm{Gr}_n^{\mathbb{F}}] \to \mathrm{Vect}_{\mathbb{F}}^n(X), \quad [f : X \to \mathrm{Gr}_n^{\mathbb{F}}] \mapsto [f * \gamma_{\mathbb{F}}^n]$$

is a bijection. That is, the functor $\mathrm{Vect}_{\mathbb{F}}^n(-)$ is represented in $\mathrm{Ho}(\mathrm{Sp}_p^{parac})$ by the pair $(\mathrm{Gr}_n^{\mathbb{F}}, \gamma_{\mathbb{F}}^n)$.

Proof. Surjectivity: Let $p : E \to X$ be an n-dimensional $\mathbb{F}$-vector bundle. We construct a continuous map $\widetilde{f} : E \to \mathbb{F}^\infty$, which is linear and injective on each fibre. Then $f : X \to \mathrm{Gr}_n(\mathbb{F}^\infty) = \mathrm{Gr}_N^{\mathbb{F}}$, $x \mapsto \widetilde{f}(E_x)$ is continuous (can be checked after trivialisation) and $E \to f^*\gamma_{\mathbb{F}}^n \subseteq X \times \mathrm{Gr}_n(\mathbb{F}^\infty) \times \mathbb{F}^\infty$, $e \mapsto (p(e), f(p(e)), \widetilde{f}(e))$ defines a bundle isomorphism from E to $f^*\gamma_{\mathbb{F}}^n$.

To construct $\widetilde{f}$, we choose a trivializing cover $\{U_i\}_i$ as before, and trivializations $h : p^{-1}(V_i) \to U_i \times \mathbb{F}^n$. Then $\widetilde{f}_i : p^{-1}(V_i) \xrightarrow{h_i} U_i \times \mathbb{F}^n \xrightarrow{\mathrm{pr}} \mathbb{F}^n$ defines such a map for the restricted bundle over each U_i with target $\mathbb{F}^n$ instead of $\mathbb{F}^\infty$. We choose a partition of unity $\{\varphi_i\}_i$ subordinate to this cover, and define

$$\widetilde{f} = (\varphi_1 \circ p \circ \widetilde{f}_1, \varphi_2 \circ p \circ \widetilde{f}_2, \ldots) \subseteq (\mathbb{F}^n)^\infty \cong \mathbb{F}^\infty$$

which has the desired properties.

Injectivity: Let $f_1, f_2 : X \to \mathrm{Gr}_n^{\mathbb{F}}$ such that $f_1^*\gamma_{\mathbb{F}}^n \cong f_2^*\gamma_{\mathbb{F}}^n$. Write E_i for the total space of $f_i^*\gamma_{\mathbb{F}}^n$. As in the first part, we obtain maps $\widetilde{f}_i : E_i \to \mathbb{F}^\infty$. We can precompose $\widetilde{f}_2$ with the homeomorphism $E_1 \to E_2$ to get two maps $\widetilde{f}_1, \widetilde{g} : E_1 \to \mathbb{F}^\infty$, each linear and injective on the fibres.

It suffices to show that $\widetilde{f_1}$ and $\widetilde{g}$ are homotopic through maps that are linear and injective on fibres, since this gives a homotopy $f_1 \xrightarrow{\sim} f_2$. To produce the homotopy between $\widetilde{f_1}$ and $\widetilde{g}$, we first note that there is a linear injective homotopy $\{h_t : \mathbb{F}^\infty \to \mathbb{F}^\infty\}$ from $\mathrm{id}_{\mathbb{F}^\infty}$ to the map $\tau(x_1, x_2, \ldots) = (x_1, 0, x_2, 0, \ldots)$, for example the direct path. After postcomposing with this homotopy, we can assume that $\widetilde{g}$ is purely in odd degrees and similarly, that $\widehat{f_1}$ is purely in even degrees. Then the direct path homotopy $\{t\widetilde{f_1} + (1-t)\widetilde{g}\}$ has the desired properties. □

Corollary 6.11. *If X is compact and $p : E \to X$ is an n-dimensional $\mathbb{F}$-vector bundle, then there exists $m \in \mathbb{N}$ and $f : X \to \mathrm{Gr}_n(\mathbb{F}^m)$, and an isomorphism $f^*\gamma_{\mathbb{F}}^{n,m} \cong E$.*

Proof. By using compactness, the classifying map $X \to \mathrm{Gr}_n^{\mathbb{F}}$ must factor through some finite stage $\mathrm{Gr}_n(\mathbb{F}^m)$, since $\mathrm{Gr}_n^{\mathbb{F}}$ carries the weak topology. □

Corollary 6.12. *Let X be compact, and $p : E \to X$ an n-dimensional $\mathbb{F}$-vector bundle. Then there is $p' : E' \to X$ such that $E \oplus E'$ is trivializible.*

Proof. Since pullback of bundles preserves direct sums and trivialisable bundles, it suffices to show the statement for the bundles $\gamma_{\mathbb{F}}^{n,m}$ by the previous corollary. But $\gamma_{\mathbb{F}}^{n,m}$ is by definition a subbundle of the trivial bundle $\mathrm{Gr}_n(\mathbb{F}^m) \times \mathbb{F}^m$, with complement given by the orthogonal complement bundle for the standard Euclidean metric. □

Example. For $n = 1$, this gives isomorphisms $\mathrm{Vect}^1_{\mathbb{R}}(X) \cong [X, \mathbb{RP}^\infty] \cong H^1(X, \mathbb{F}_2)$ and $\mathrm{Vect}^1_{\mathbb{C}}(X) \cong [X, \mathbb{CP}^\infty] \cong H^1(X, \mathbb{Z})$.

It follows that $\gamma^1_{\mathbb{R}}$ is the unique non-trivial line bundles on $\mathbb{RP}^\infty$, since $H^1(\mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2$. Since $H^1(\mathbb{RP}^\infty, \mathbb{F}_2) \xrightarrow{\cong} H^1(\mathbb{RP}^m, \mathbb{F}_2)$ for all $m \geq 1$, we see that $\gamma_{\mathbb{R}}^{1,m+1}$ is also non-trivial for all $m \geq 1$, in particular the Möbius bundle is non-trivial. Similarly, the line bundles $\gamma_{\mathbb{C}}^{1,m+1}$ are generators of $\mathrm{Vect}^1_{\mathbb{C}}(\mathbb{CP}^m)$ for $m \geq 1$. In particular, $\mathrm{Vect}^1_{\mathbb{R}}(X)$ and $\mathrm{Vect}^1_{\mathbb{C}}(X)$ have natural abelian group structures. One can show that this is given by the tensor product of line bundles. In fact, natural operations $\mathrm{Vect}^1_{\mathbb{R}}(-) \times \mathrm{Vect}^1_{\mathbb{R}}(-) \to \mathrm{Vect}^1_{\mathbb{R}}(-)$ correspond, by the Yoneda lemma, to elements in

$$H^1(\mathbb{RP}^\infty \times \mathbb{RP}^\infty, \mathbb{F}_2) \cong H^1(\mathbb{RP}^\infty \times *, \mathbb{F}_2) \oplus H^1(* \times \mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2.$$

Only two of these are invariant under the C_2-action, which is the case for the one classifying the tensor product, since it is symmetric. Since the tensor product is non-trivial in general, it must correspond to the diagonal element $(1,1) \in \mathbb{F}_2 \oplus \mathbb{F}_2$. Hence, every real line bundle ξ on X has an associated cohomology class $\omega_1(\xi) \in H^1(X, \mathbb{F}_2)$, and this is a full invariant of ξ up to isomorphism. This is an example of a characteristic class, the thoery of which we now develop systematically.

Definition 6.13. A characteristic class is a natural transormation of the form

$$\mathrm{Vect}^n_{\mathbb{F}}(-) \to H^m(-, A)$$

for $n, m \in \mathbb{N}$, A abelian group.

At least when we restrict to paracompact spaces, as before characteristic classes correspond to elements in $H^m(\mathrm{Gr}_n^{\mathbb{F}}, A)$. We will mainly focus on the case $\mathbb{F} = \mathbb{R}$ and $A = \mathbb{F}_2$. Our main goal is to show

Theorem 6.14. *For every real vector bundle $\xi : E \to B$, there exist characteristic classes $\omega_i(\xi) \in H^i(B, \mathbb{F}_2)$, $i \in \mathbb{N}$, called the* Stiefel-Whitney classes*, satisfying the following properties:*

(1) $\omega_0(\xi) = 1$, *and* $\omega_i(\xi) = 0$ *for* $i > \dim \xi$.

(2) *(Naturality) If $f : B' \to B$, then $\omega_i(f^*\xi) = f^*\omega_i(\xi)$.*

(3) *(Whitney-product formula) If ξ and η are real vector bundles over the same base space B, then $\omega_k(\xi \oplus \eta) = \sum_{i=0}^{k} \omega_i(\xi) \smile \omega_{k-i}(\eta)$.*

(4) *We have $\omega_1(\gamma_{\mathbb{R}}^{1,2}) \neq 0$ in $H^1(\mathbb{RP}^1, \mathbb{F}_2)$.*

Moreover, when restricted to paracompact spaces

We first assume the theorem and record some elementary properties: If ε is a trivial bundle, then $\omega_i(\varepsilon) = 0$ for $i > 0$, since ε can be pulled back from a point and $H^i(*, \mathbb{F}_2) = 0$ for $i > 0$. Further, we have $\omega_i(\xi \oplus \varepsilon) = \omega_i(\xi)$ for all i by the Whitney product formula (the ω_i are "stable").

We set $H^{\Pi}(B, \mathbb{F}_2) = \prod_{n\in\mathbb{N}} H^n(B, \mathbb{F}_2)$ with ring structure extended from the cup-product, i.e.

$$(x_i)_i \cdot (y_j)_j = \left(\sum_{i=0}^{k} x_i \smile y_{k-i} \right)_k .$$

Then the total Stiefel-Whitney class $\omega(\xi) \in H^{\Pi}(B, \mathbb{F}_2)$ is defined as $\omega(\xi) = \omega_0(\xi) + \omega_1(\xi) + \omega_2(\xi) + \ldots$. Then $\omega(\xi \oplus \eta) = \omega(\xi)\omega(\eta)$ in this situation. Hence, if $\xi \oplus \varepsilon^m$ is a trivial bundle, we have $\omega(\xi)\omega(\eta) = \omega(\xi \oplus \eta) = \omega(\varepsilon^m) = 1$ and hence $\omega(\eta) = \omega(\xi)^{-1}$.

Recall $\mathrm{Vect}^1_{\mathbb{R}}(\mathbb{RP}^1) = H^1(\mathbb{RP}^1, \mathbb{F}_2) \cong \mathbb{F}_2$, hence axiom 4 forces $\omega_1(\gamma_{\mathbb{R}}^{1,2})$ to be the unique non-trivial element of $H^1(\mathbb{RP}^1, \mathbb{F}_2)$. Since $\gamma_{\mathbb{R}}^{1,2}$ is the pullback of $\gamma^1_{\mathbb{R}}$ along $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$ and $H^1(\mathbb{RP}^\infty, \mathbb{F}_2) \to H^1(\mathbb{RP}^1, \mathbb{F}_2)$ is an isomorphism, we have that $\omega_1(\gamma^1_{\mathbb{R}}) \in H^1(\mathbb{RP}^\infty, \mathbb{F}_2)$ is the non-trivial element u. Hence, $\omega(\gamma^1_{\mathbb{R}}) = 1 + u \in H^{\Pi}(\mathbb{RP}^\infty, \mathbb{F}_2)$. Its inverse is given by $1 + u + u^2 + \ldots$. This contains infinitely many non-trivial terms and can hence not be equal to $\omega(\eta)$ for some vector bundle η on $\mathbb{RP}^\infty$. Hence $\gamma^1_{\mathbb{R}}$ does not embed into a trivial bundle, in contrast to bundles over compact spaces.

We now turn to the construction of the ω_i. We start with Thom classes and Thom isomorphisms: Let $p : E \to B$ be a real vector bundles. We set $E_0 \subseteq E$ to be all elements which are not the 0-element in their respective fibre. Then $p|_{E_0} : E_0 \to B$ is a fibre bundle with fibres homeomorphic to $\mathbb{R}^n \setminus \{0\}$. We can hence apply the relative Serre spectral sequence of the form

$$H^p(B, H^q(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}, \mathbb{F}_2)) \Rightarrow H^{p+q}(E, E_0, \mathbb{F}_2).$$

Now, $H^q(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}, \mathbb{F}_2) = \mathbb{F}_2$ if $q = n$ and 0 else. Moreover, the local system is necessarily trivial, since any automorphism of $\mathbb{F}_2$ is the identity. Hence

Theorem 6.15. *There is a natural isomorphism $\Phi : H^\bullet(B, \mathbb{F}_2) \xrightarrow{\cong} H^{\bullet+n}(E, E_0, \mathbb{F}_2)$ called the* Thom isomorphism. *The image $u = \Phi(1) \in H^n(E, E_0, \mathbb{F}_2)$ is called the* Thom class *of the bundle. Then the map Φ is given by the cup-product $H^\bullet(B, \mathbb{F}_2) \xrightarrow{\cong} H^\bullet(E, \mathbb{F}_2) \xrightarrow{\smile u} H^{\bullet+n}(E, E_0, \mathbb{F}_2)$.* □

Note that u is uniquely determined by the fact that it restricts to a generator in $H^n(F_b, (F_b)_0, \mathbb{F}_2)$ on each of the fibres F_b.

Remark. This crucially uses F_2-coefficients. Over $\mathbb{Z}$, the local system $H^1(F_-, (F_-)_0, \mathbb{Z})$ is trivial iff the bundle is orientable, meaning one can choose orientations on each fibre F_b which are locally compatible in the sense that the local trivializations all send them to the same orientation of $\mathbb{R}^n$. Since every complex vector space has a canonical real orientation, every complex vector bundle is orientable and hence has a Thom-isomorphism with $\mathbb{Z}$-coefficients.

The Thom-isomorphism is sometimes phrased differently: If $p : E \to B$ is euclidean, we can form the disc bundle $D(E)$ and the sphere bundle $S(E)$ by fibrewise restricting to the vectors of length ≤ 1 and

$= 1$ respectively. Then $D(E) \hookrightarrow E$ and $S(E) \hookrightarrow E_0$ are both homotopy equivalences. Homotopy inverses are given by the maps $E \to B \xrightarrow{\text{0-section}} D(E)$ and $E_0 \to S(E), v \mapsto \frac{v}{|v|}$. Hence, there is an isomorphism $H^\bullet(E, E_0, \mathbb{F}_2) \xrightarrow{\cong} H^\bullet(D(E), S(E), \mathbb{F}_2)$. The pair $(D(E), S(E))$ has the advantage that it satisfies excision, and so we can further identify $H^\bullet(D(E), S(E), \mathbb{F}_2) \cong \widetilde{H}^\bullet(D(E)/S(E), \mathbb{F}_2)$. The space $D(E)/S(E) =: \mathrm{Th}(p)$ is called the *Thom* space. Note that we have a commuting square

$$\begin{tikzcd} S(E) \arrow[r] \arrow[d, equal] & B \\ S(E) \arrow[r, hook] & D(E) \arrow[u, "\cong"'] \end{tikzcd}$$

Hence, $\mathrm{Th}(p)$ is homotopy equivalent to the mapping cone of the sphere bundle $S(E) \to B$. If B is compact, then $\mathrm{Th}(p)$ is homeomorphic to the one-point compactification of E (exercises). If $p = \varepsilon^n$ is trivial, then $\mathrm{Th}(\varepsilon)^n = S^n \wedge B_+$ is the unreduced n-fold suspension, and the Thom isomorphism is the usual suspension isomorphism.

In general, $\mathrm{Th}(p)$ is a twisted suspension of B_+ and the Thom isomorphism says that this twist is invisible to $H^\bullet(-, \mathbb{F}_2)$, at least as a functor to graded $\mathbb{F}_2$-vector space.

Definition 6.16. Let $\xi : E \to B$ be a vector bundle. We define

$$\omega_i(\xi) = \Phi^{-1}(\mathrm{Sq}^i\, u) \in H^i(B, \mathbb{F}_2),$$

where Φ is the Thom isomorphism. In other words, $\omega_i(\xi)$ is characterized by the equation

$$\omega_i(\xi) \smile u = \mathrm{Sq}^i\, u \in H^{n+i}(E, E_0, \mathbb{F}_2).$$

Slogam: "SW classes measure the failure of Φ to commute with Steenrod operations."

Remark. By definition, the Stiefel-Whitney classes only depend on the underlying sphere bundle of the vector bundle.

We verify the axioms of theorem 6.16: We have $\omega_0(\xi) = \Phi^{-1}\,\mathrm{Sq}^0\, u = \Phi^{-1} u = 1 \in H^0(B, \mathbb{F}_2)$. Moreover, u is in degree $n = \dim(\xi)$, hence $\mathrm{Sq}^i\, u = 0$ for $i > n$. So $\omega_i(\xi) = 0$ for $i > n$.

For naturality, if $f : B' \to B$ is a map, we obtain a canonical bundle map $g : f^*E \to E$, which induces a linear isomorphism on fibres $F'_{b'} = (f^*E)_{b'} \xrightarrow{\cong} E_{f(b')} = F_{f(b')}$. It follows that $g^*u \in H^n(f^*E, (f^*E)_0, \mathbb{F}_2)$ is the Thom class for f^*E, since its restriction to the fibre can be computed using

$$\begin{tikzcd} H^n(f^*E, (f^*E)_0, \mathbb{F}_2) \arrow[d] & H^n(E, E_0, \mathbb{F}_2) \arrow[l] \arrow[d] \\ H^n(F'_{b'}, (F'_{b'})_0, \mathbb{F}_2) & H^n(F_{f(b')}, (F_{f(b')})_0, \mathbb{F}_2 \arrow[l, "\cong"'] \end{tikzcd}$$

Since $u \in H^n(E, E_0, \mathbb{F}_2)$ restricts to a generator, so does g^*u. Hence we obtain $g^*u = u'$, and we see that

$$f^*\omega_i(\xi) \smile u' = f^*\omega_i(\xi) \smile g^*u \stackrel{*}{=} g^*(\omega_i(\xi) \smile u) = g^*(\mathrm{Sq}^i\, u) = \mathrm{Sq}^i(g^*u) = \mathrm{Sq}^i\, u'.$$

The equality $*$ follows from the naturality of the action.

To prove the Whitney product formula, we first discuss the effect of the external cross product on SW-classes. Given $p : E \to B$ of dimension m and $p' : E' \to B'$ of dimension n, the product $p \times p' : E \times E' \to B \times B'$ again forms a vector bundle of dimension $m+n$, giving the fibres $(E \times E')_{(b,b')} = E_b \times E_{b'}$ the product vector space structure. The local triviality follows by taking products of local trivializations.

The previously defined internal direct sum $p \oplus p'$ in the case where $B = B'$ is obtained as the pullback $p \oplus p' = \Delta^*(p \times p')$ along the diagonal $\Delta : B \to B \times B$.

We consider the cross product $H^m(E, E_0, \mathbb{F}_2) \times H^n(E', E'_0, \mathbb{F}_2) \times H^{m+n}(E \times E', E_0 \times E' \cup E \times E'_0, \mathbb{F}_2)$ and observe that $E_0 \times E' \cup E \times E'_0 = (E \times E')_0$, since $(V \times W) \setminus \{0\} = (V \setminus \{0\} \times W \cup V \times (W \setminus \{0\})$ for vector spaces V, W. We claim that $u \times u'$ is a Thom class for $p \times p'$. For this, it again suffices to show that it restricts to a generator on each fibre $H^{m+n}((E \times E')_{(b,b')}, ((E \times E')_{(b,b')})_0, \mathbb{F}_2)$. By naturality of the cross product, this restriction is the cross product of the restriction of u to $H^m(E_b, (E_b)_0, \mathbb{F}_2)$ and of u' to $H^n(E'_b, (E'_b)_0, \mathbb{F}_2)$. We know that this is a generator (for example, identify $H^n(\mathbb{R}^m, \mathbb{R}^m \setminus 0, \mathbb{F}_2)$ with $H^m(D^m, S^{m-1}, \mathbb{F}_2) \cong \widetilde{H}^m(S^m)$ and likewise for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0, \mathbb{F}_2)$, and note that by the Künneth isomorphism, $\widetilde{H}^m(S^m, \mathbb{F}_2) \otimes \widetilde{H}^n(S^n, \mathbb{F}_2) \cong \widetilde{H}^{m+n}(S^{m+n}, \mathbb{F}_2)$).

Hence, given $a \in H^\bullet(B)$, $b \in H^\bullet(B')$, we obtain

$$(a \times b) \smile (u \times u') = (a \smile u) \times (b \smile u').$$

This shows that the Thom isomorphism satisfies $\Phi(a \times b) = \Phi(a) \times \Phi(b)$. Hence

$$\begin{aligned} \omega_i(p \times p') &= \Phi^{-1}(\mathrm{Sq}^i(u \times u')) = \Phi^{-1} \sum_{j=0}^{i} \mathrm{Sq}^j u \times \mathrm{Sq}^{i-j} u') \\ &= \sum_{j=0}^{i} \Phi^{-1} \mathrm{Sq}^j u \times \Phi^{-1} \mathrm{Sq}^{i-j} u' = \sum_{j=0}^{i} \omega_j(p) \times \omega_{i-j}(p'). \end{aligned}$$

As mentioned above, we obtain $p \oplus p'$ (in case $B = B'$) as $\Delta^*(p \times p')$. By naturality,

$$\omega_i(p \oplus p') = \omega_i(\Delta^*(p \times p')) = \Delta^* \omega_i(p \times p') = \Delta^* \sum_{j=0}^{i} \omega_i(p) \times \omega_{i-j}(p') = \sum_{j=0}^{i} \omega_j(p) \smile \omega_{i-j}(p').$$

It remains to show that $\omega_1(\gamma_{\mathbb{R}}^{1,2}) \neq 0$, where $\gamma_{\mathbb{R}}^{1,2}$ is the Möbius bundle over $\mathbb{RP}^1$, equipped with the euclidean metric from its embedding into $\mathbb{RP}^1 \times \mathbb{R}^2$. Then the disc bundle $D(\gamma_{\mathbb{R}}^{1,2})$ is a closed Möbius strip, and the sphere bundle $S(\gamma_{\mathbb{R}}^{1,2})$ is its boundary. Hence

$$H^\bullet(\gamma_{\mathbb{R}}^{1,2}, (\gamma_{\mathbb{R}}^{1,2})_0, \mathbb{F}_2 \cong \widetilde{H}^\bullet(D(\gamma_{\mathbb{R}}^{1,2})/S(\gamma_{\mathbb{R}}^{1,2}), \mathbb{F}_2) \cong \widetilde{H}^\bullet(\mathbb{RP}^2, \mathbb{F}_2).$$

The Thom class $u \in \widetilde{H}^1(\mathbb{RP}^2, \mathbb{F}_2)$ must be the generator. We have $\mathrm{Sq}^1 u = u \smile u \neq 0$ in $\widetilde{H}^2(\mathbb{RP}^2, \mathbb{F}_2)$, hence $\omega_1(\gamma_{\mathbb{R}}^{1,2}) \neq 0$. □

Note that we can use the sum and cup product to construct more characteristic classes out of the ω_i. For example, $\omega_1^3 + \omega_1\omega_2$ is a characteristic class of degree 3 defined for all vector bundles. Our next goal is to show that over paracompact spaces, all characteristic classes are of this form. Using the identification of characteristic classes for bundles over paracompact spaces with $H^\bullet(\mathrm{Gr}_n^{\mathbb{R}}, \mathbb{F}_2)$, we can think of the ω_i as elements in $H^\bullet(\mathrm{Gr}_{\mathbb{R}}^n, \mathbb{F}_2)$ and abbreviate $\omega_i(\eta_{\mathbb{R}}^n)$ to ω_i. We obtain a map $\alpha_n : \mathbb{F}_2[\omega_1, \ldots, \omega_n] \to H^\bullet(\mathrm{Gr}_n^{\mathbb{R}}, \mathbb{F}_2)$. Our goal is to show

Theorem 6.17. *The map α is an isomorphism for all $n \in \mathbb{N}$.*

Proof. Since $\omega_1(\gamma_{\mathbb{R}}^1) \in H^1(\mathbb{RP}^\infty, \mathbb{F}_2)$ is the generator, we already know that α_1 is an isomorphism. We first show that α_n is always injective. For this, we consider the map $\varphi_n : (\mathbb{RP}^\infty)^{\times n} \to \mathrm{Gr}_n^{\mathbb{R}}$ corresponding to the product bundle $\gamma_{\mathbb{R}}^1 \times \cdots \gamma_{\mathbb{R}}^1$ under the bijection $\mathrm{Vect}_{\mathbb{R}}^n((\mathbb{RP}^\infty)^{\times n}) \cong [(\mathbb{RP}^\infty)^n, \mathrm{Gr}_n^{\mathbb{R}}]$. We obtain a map

$$\varphi_n^* : H^*(\mathrm{Gr}_n, \mathbb{R}) \to H^*((\mathbb{RP}^\infty)^{\times n}, \mathbb{F}_n) \cong (H^\bullet(\mathbb{RP}^\infty, \mathbb{F}_2))^{\otimes n} \cong \mathbb{F}_2[u_1, \ldots, u_n]$$

where u_i is the image of $u \in H^1(\mathbb{RP}^\infty, \mathbb{F}_2)$ under $\mathrm{pr}_i^* : H^1(\mathbb{RP}^\infty, \mathbb{F}_2) \to H^1((\mathbb{RP}^\infty)^{\times n}, \mathbb{F}_2)$. We study the composite $\varphi_n^* \circ \alpha_n : \mathbb{F}_2[\omega_1, \dots, \omega_n] \to \mathbb{F}_2[u_1, \dots, u_n]$, $\omega_i \mapsto \omega_i((\gamma_R^1)^{\times n})$. By the Whitney product formula, we have $\omega((\gamma_{\mathbb{R}}^1)^{\times n}) = \omega(\gamma_{\mathbb{R}}^1)^{\times n} = (1+u_1)\cdots(1+u_n)$. Hence

$$\omega_i((\gamma_R^1)^{\times n}) = \sum_{0<j_1<\ldots<j_i\leq n} u_{j_1}\cdots u_{j_i} = e_i(u_1, \dots, u_n),$$

where e_i is called the i-th elementary symmetric polynomial in n variables. It is called symmetric because the Σ_n-action on $\mathbb{F}_2[u_1, \dots, u_n]$ permuting the u_i's leaves each e_i invariant.

Proposition 6.18. *The e_i are algebraically independent, i.e. there are no polynomial relations between them $\varphi_n^* \circ \alpha_n$ is injective.*

Proof. Induction on n, the case $n = 1$ is clear. Assume $f(e_1, \dots, e_n) = 0$ for a polynomial $0 \neq f \in \mathbb{F}_2[x_1, \dots, x_n]$ which we can choose with $\deg_n f$ minimal. Write $f = f_0 + f_1 x_n + \ldots + f_d x_n^d$ for polynomials $f_i \in \mathbb{F}_2[x_1, \dots, x_{n-1}]$ and some d. If $f_0 = 0$, we can write $f = x_n \widetilde{f}$ and $\widetilde{f}(e_1, \dots, e_n) = 0$ in contradiction to the choice of f. Thus $f_0 \neq 0$. We apply the ring map $\mathbb{F}_2[u_1, \dots, u_n] \to \mathbb{F}_2[u_1, \dots, u_{n-1}]$ which sends u_n to 0. This sends $e_i(u_1, \dots, u_n)$ to $e_i(u_1, \dots, u_{n-1})$ if $i \leq n$ and $e_n(u_1, \dots, u_n)$ to 0. Hence $0 = f_0(e_1, \dots, e_{n-1})$, contradicting the induction hypothesis. □

Remark 6.19. One can in fact show that the e_i are polynomial generators of the subring $\mathbb{F}_2[u_1, \dots, u_n]^{\Sigma_n}$ of symmetric polynomials.

Returning to the proof of theorem 6.17, we know now that α_n is injective. To show surjectivity, we use an inductive argument built on the following: We give $\gamma_{\mathbb{F}}^n$ a Euclidean metric from its embedding in $\mathrm{Gr}_n^{\mathbb{F}} \times \mathbb{F}^\infty$. Hence we can consider the sphere bundle $S(\gamma_{\mathbb{F}}^n)$ by restricting to vectors of length 1 in each fibre.

Proposition 6.20. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. There is a homotopy equivalence $S(\gamma_{\mathbb{F}}^n) \simeq \mathrm{Gr}_{n-1}^{\mathbb{F}}$ for every $n \geq 1$.*

Proof. $S(\gamma_{\mathbb{F}}^n)$ consists of pairs (V, v) of an n-dimensional subspace $V \subseteq \mathbb{F}^\infty$ and a unit vector $v \in V$. We obtain a map

$$\Phi : S(\gamma_{\mathbb{F}}^n) \to \mathrm{Gr}_{n-1}^{\mathbb{F}}, \quad (V, v) \mapsto \langle v\rangle^\perp = \{x \in V \mid x \perp v\}.$$

In the other direction, we define

$$\Psi : \mathrm{Gr}_{n-1}^{\mathbb{F}} \to S(\gamma_{\mathbb{F}}^n), \quad W \mapsto g_*(\mathbb{F} \oplus W, (1, 0)),$$

where g is the linear isometric isomorphism $\mathbb{F} \oplus \mathbb{F}^\infty \to \mathbb{F}^\infty$ that shifts coordinates by one to the right. The composite $\Phi \circ \Psi$ sends $W \subseteq \mathbb{F}^\infty$ to $0 \oplus W \subseteq \mathbb{F} \oplus \mathbb{F}^\infty \cong \mathbb{F}^\infty$. The straight line homotopy from $\mathrm{id}_{F\oplus\mathbb{F}^\infty}$ to $\mathbb{F}^\infty \hookrightarrow \mathbb{F} \oplus \mathbb{F}^\infty$ shows that this is homotopic to the identity on $\mathrm{Gr}_{n-1}^{\mathbb{F}}$. The composite $\Psi \circ \Phi$ sends (V, v) to $\mathbb{F} \oplus \langle v\rangle, (1, 0))$. Using a composition of straight line homotopies to the odd/even parts again shows that this is homotopic to the identity on $S(\gamma_{\mathbb{F}}^n)$. □

Alternatively, $S(\gamma_{\mathbb{F}}^n) \to \mathrm{Gr}_{n-1}^{\mathbb{F}}$, $(V, v) \mapsto \langle v\rangle^\perp$ defines a fibre bundle with fibre over $W \in \mathrm{Gr}_n^{\mathbb{F}}$ given by $S(W^\perp) = \{v \in \mathbb{F}^\infty \mid |v| = 1 \wedge v \perp W\}$. This is homeomorphic to S^∞ and hence contractible. □

We further note that the map $f : \mathrm{Gr}_{n-1}^{\mathbb{F}} \xrightarrow{\Psi} S(\gamma_n^{\mathbb{F}}) \to \mathrm{Gr}_n^{\mathbb{F}}$ classifies the n-dimensional bundle $\gamma_{\mathbb{F}}^{n-1} \oplus \mathbb{F}$, since the fiber of $f^*\gamma_n^{\mathbb{F}}$ over $W \in \mathrm{Gr}_{n-1}^{\mathbb{F}}$ is given by $g(\mathbb{F}) \oplus g(W)$. The summand $g(\mathbb{F})$ is 1-dimensional and independent of W, the second summand gives a bundle isomorphism to $\gamma_{n-1}^{\mathbb{F}}$, again using that g is homotopic to the identity.

In problem 1, exercise sheet 2, we saw that for every fibre sequence $S^n \to E \to B$, $n \neq 0$, B simply connected, there is a long exact sequence of the form

$$\cdots \to H_{p-n}(B, A) \to H_p(E, A) \to H_p(B, A) \to H_{p-n-1}(B, A) \to \cdots$$

called the Gysin-sequence. We now discuss a cohomological version of this, for more general B. Let $S^n \to Y \to X$, $n \neq 0$, be a fibre sequence with X path-connected and R a commutative ring, and assume that the local system $H^n(F_-, R)$ on X is isomorphic to the constant one. Then the cohomological Serre spectral sequence becomes (after choosing an isomorphism $H^n(S^n, R) \cong R$)

$$H^p(X, R)(\delta_{q0} + \delta_{qn}) \cong H^p(X, H^q(S^n, R)) \cong H^p(X, H^q(F_-, R) \Rightarrow H^{p,q}(Y, R)$$

This uses that the local system $H^0(F_-, R)$ is always constant, since any continuous map of path-connected spaces induces the identity on $R \cong H^0(-, R)$. Let $y \in E_2^{0,n} = H^0(X, R)$ be a generator and $e \in H^{n+1}(X, R)$ be the image of y under the differential d_{n+1}. e is called the *Euler class* of the spherical fibre sequence, it depends on the chosen isomorphism $H^n(F_-, R) \cong \mathrm{const}(R)$. Since $H^m(X, R) \to E_2^{m,n}$, $x \mapsto yx$ is an isomorphism, it follows that d_{n+1} is determined by the Leibniz rule $d_{n+1}(yx) = e \smile x$. By our description of the edge homomorphism, it follows that

$$\ker(p^* : H^\bullet(X, R) \to H^\bullet(Y, R)) = \mathrm{im}(e \smile - : H^{\bullet-n-1}(X, R) \to H^\bullet(X, R)).$$

Furthermore, the image of the map $H^\bullet(Y, R) \twoheadrightarrow E_\infty^{\bullet-n,n} \hookrightarrow E_2^{\bullet-n,n} = H^{\bullet-n}, R)$ is given by the kernel of $e \smile -$. Hence we have

Corollary 6.21. *Let $S^n \to Y \xrightarrow{q} X$ be a fibre sequence, $n \neq 0$, X path-connected. Let R be a commutative ring with a choice of trivialization $H^n(F_-, R) \cong R$. Then there is a long exact sequence of the form*

$$\cdots \to H^m(Y, R) \to H^{m-n}(X, R) \xrightarrow{e\smile} H^{m+1}(X, R) \xrightarrow{q^*} H^{m+1}(Y, R) \to \cdots$$

where $e \in H^{n+1}(X, R)$ is the Euler class, depending on the choice of trivialization.

Remark 6.22. A trivialization $H^n(F_-, R) = R$ is called an R-orientation of the spherical fibre sequence. Every spherical fibre sequence has a unique $\mathbb{F}_2$-orientation, since $\mathbb{F}_2$ has no non-trivial automorphisms.

We further have

Lemma 6.23. *If $p : E \to B$ is an $(n+1)$-dimensional euclidean $\mathbb{R}$-vector bundle, B path-connected, with associated n-dimensional sphere bundle $q : S(E) \to B$, then $e = \omega_{n+1}(p) \in H^{n+1}(B, \mathbb{F}_2)$.*

Proof. We first note that $\omega_{n+1}(p) = f^*(u) \in H^{n+1}(B, \mathbb{F}_2)$, where $f : (B, \emptyset) \to (E, E_0)$ is the zero-section. In other words, f^* is the composite

$$H^\bullet(E, E_0, \mathbb{F}_2) \xrightarrow{i^*} H^\bullet(E, \mathbb{F}_2) \underset{p^*}{\overset{\cong}{\longleftarrow}} H^\bullet(B, \mathbb{F}_2).$$

Indeed, we have $\mathrm{Sq}^{n+1} u = u \smile u$ and $i^*(u) \smile u = u \smile u$ by naturality of the relative cup product.

Now let $b \in B$. We write $S(E_b)$ for F. The relevant differential $d_{n+1} : H^n(S(E_b, \mathbb{F}_2) \to H^{n+1}(B, \mathbb{F}_2)$ is a transgression, hence we know it can be computed as follows:

$$H^n(S(E_b), \mathbb{F}_2) \xrightarrow{\partial} H^{n+1}(S(E), S(E_b), \mathbb{F}_2) \hookleftarrow H^{n+1}(B, \mathbb{F}_2).$$

We first note that the diagram

$$
\begin{tikzcd}
H^n(S(E_b),\mathbb{F}_2) \arrow[rr, "\partial"] \arrow[d, "\cong"', "\partial"] & & H^{n+1}(S(E),S(E_b),\mathbb{F}_2) \\
H^{n+1}(D(E_b),S(E_b),\mathbb{F}_2) & & \\
H^{n+1}(D(E),S(E_b)k\mathbb{F}_2) \arrow[u] \arrow[rruu] & & H^{n+1}(D(E),S(E),\mathbb{F}_2) \arrow[ll] \arrow[uu]
\end{tikzcd}
$$

implies that $\partial y = j^*(u)$, where $j : (S(E),S(E_b)) \to (D(E),S(E))$ is the inclusion. Furthermore, we have a commuting diagram

$$
\begin{tikzcd}
H^{n+1}(S(E),S(E_b),\mathbb{F}_2) & H^{n+1}(B,b,\mathbb{F}_2) \arrow[l, "q^*"'] \arrow[r, "\cong"] \arrow[d, "p^*"] & H^{n+1}(B,\mathbb{F}_2) \arrow[d, "\cong"', "p^*"] \\
H^{n+1}(D(E),S(E),\mathbb{F}_2) \arrow[r] \arrow[rru, bend left, "f^*"] & H^{n+1}(D(E),S(E_b),\mathbb{F}_2) \arrow[r] \arrow[lu] & H^{n+1}(D(E),\mathbb{F}_2)
\end{tikzcd}
$$

which implies $p^*(f^*u) = j^*u = \partial(y)$, and hence $f^*u = d_{n+1}y = e$. □

We now show that $\alpha_n : \mathbb{F}_2[u_1,\dots,u_n] \to H^\bullet(\mathrm{Gr}_n^{\mathbb{R}},\mathbb{F}_2)$ is an isomorphism for all n, by induction on n. The induction start is ok, and assuming the claim for n, we obtain a Gysin long exact sequence

$$\cdots \to H^\bullet(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2) \xrightarrow{\omega_{n+1}} H^{\bullet+n+1}(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2) \to H^{\bullet+n+1}(\mathrm{Gr}_n^{\mathbb{R}};\mathbb{F}_2) \to \cdots$$

We know that $H^\bullet(\mathrm{Gr}_n^{\mathbb{R}},\mathbb{F}_2)$ is generated by $\omega_1,\dots,\omega_n$, and that each ω_i lifts to a class of $H^\bullet(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2)$ of the same name. Since $H^\bullet(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2) \to H^\bullet(\mathrm{Gr}_n^{\mathbb{R}},\mathbb{F}_2)$ is a ring map, it must hence be surjective. This implies that the Gysin sequence splits up into short exact sequences. We get a comparison map

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_2[\omega_1,\dots,\omega_{n+1}]_\bullet \arrow[r, "\omega_{n+1}"] \arrow[d, "\alpha_{n+1}"] & \mathbb{F}_2[\omega_1,\dots,\omega_{n+1}]_{\bullet+n+1} \arrow[r, "\omega_{n+1}=0"] \arrow[d, "\alpha_{n+1}"] & \mathbb{F}_2[\omega_1,\dots,\omega_n]_{\bullet+n+1} \arrow[r] \arrow[d, "\cong"', "\alpha_n"] & 0 \\
0 \arrow[r] & H^\bullet(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2) \arrow[r, "\omega_{n+1}"] & H^{\bullet+n+1}(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2) \arrow[r] & H^{\bullet+n+1}(\mathrm{Gr}_n^{\mathbb{R}},\mathbb{F}_2) \arrow[r] & 0
\end{tikzcd}
$$

Since all graded maps in this diagram are concentrated in non-negative degrees, it follows by induction on the degree of $H^\bullet(\mathrm{Gr}_{n+1}^{\mathbb{R}},\mathbb{F}_2)$ that α_{n+1} must be an isomorphism. □

Corollary 6.24. *The map $\varphi_n^* : H^\bullet(\mathrm{Gr}_n^{\mathbb{R}},\mathbb{F}_2) \to H^\bullet((\mathbb{RP}^\infty)^{\times n},\mathbb{F}_2)$ is injective.*

Proof. We already saw that $\varphi_n^* \circ \alpha_n$ is injective, and we now know that α_n is an isomorphism. □

Corollary 6.25. *If two characteristic classes $\beta_1,\beta_2 : \mathrm{Vect}_{\mathbb{R}}^n(-) \to H^m(-,\mathbb{F}_2)$ over paracompact spaces agree on all bundles that decompose into sums of line bundles, then $\beta_1 = \beta_2$.*

Proof. The bundle $\varphi_n^*\gamma_{\mathbb{R}}^n$ over $(\mathbb{RP}^\infty)^\times$ is by definition a sum of line bundles, namely

$$\varphi_n^*\gamma_{\mathbb{R}}^n = \pi_1^*\gamma_{\mathbb{R}}^1 \oplus \ldots \oplus \pi_n^*\gamma_{\mathbb{R}}^1,$$

where π_i is the i-th projection. So the assumption guarantees that $\pi_n^*\beta_1(\gamma_{\mathbb{R}}^n) = \beta_2(\varphi_n^*\gamma_{\mathbb{R}}^n) = \varphi_n^*\beta_2(\gamma_{\mathbb{R}}^n)$. Since φ_n^* is injective, it follows that $\beta_1(\gamma_{\mathbb{R}}^n) = \beta_2(\gamma_{\mathbb{R}}^n)$, and hence by universality $\beta_1 = \beta_2$. □

Corollary 6.26. *Stiefel-Whitney classes are uniquely determined by the axioms (1) to (4) over paracompact spaces.*

Proof. Assume ω_i and ω_i' both satisfy axioms (1) through (4), with associated total classes ω and ω'. Then we already saw that

$$\omega(\gamma^1_{\mathbb{R}}) = 1 + u = \omega'(\gamma^1_{\mathbb{R}}) \in H^{\prod}(\mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2[[u]].$$

Hence by universality and naturality, ω and ω' must agree for all line bundles over paracompact spaces. If $\xi = \xi_1 \oplus \ldots \oplus \xi_n$ and all ξ_i are line bundles, axiom (4) implies $\omega(\xi) = \prod_{i=1}^n \omega(\xi_i) = \omega'(\xi)$. By the above corollary, $\omega = \omega'$. □

The complex case. Now we want to compute characteristic classes on paracompact spaces of the form $\mathrm{Vect}^n_{\mathbb{C}}(-) \to H^m(-,\mathbb{Z})$ or in other words the cohomology $H^\bullet(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z})$. We want to use the same inductive process through the identification $S(\gamma^n_{\mathbb{C}}) \cong \mathrm{Gr}^{\mathbb{C}}_{n-1}$ and the Gysin sequence. For this we need to understand the local system $H^\bullet(S(E_-), \mathbb{Z})$ when $p : E \to B$ is a complex vector bundle. We fix a generator $\beta \in H^\bullet(D(\mathbb{C}), S(\mathbb{C}), \mathbb{Z})$ throughout this discussion. Then the external relative cross power $\beta_{\mathbb{C}^n} := \beta^{\times n}$ forms a generator of $H^{2n}(D(\mathbb{C}^n), S(\mathbb{C}^n), \mathbb{Z})$. If V is a hermitian n-dimensional $\mathbb{C}$-vector space, we choose an isometry $\varphi : V \to \mathbb{C}^n$ and obtain $\varphi^*\beta_{\mathbb{C}^n} \in H^{2n}(D(V), S(V), \mathbb{Z})$.

Given another choice $\psi : V \to \mathbb{C}^n$, the composite $\psi \circ \varphi^{-1}$ is an element of $U(n)$ acting by a degree 1 map on $S(\mathbb{C}^n)$. It follows that $\psi^*\beta_{\mathbb{C}^n} = \varphi^*\beta_{\mathbb{C}^n}$. Hence we obtain a canonical generator $\beta_V \in H^{2n}(D(V), S(V), \mathbb{Z})$ independent of the choice of identification $V \cong \mathbb{C}^n$. Now, let $p : E \to B$ be a hermitian n-dimensional $\mathbb{C}$-vector bundle with sphere bundle $q : S(E) \to B$. For $b \in B$, we obtain an isomorphism

$$\delta_b : H^{2n-1}(S(E_b), \mathbb{Z}) \underset{\cong}{\xrightarrow{\delta}} H^{2n}(D(E_b), S(E_b), \mathbb{Z}) \cong \mathbb{Z},$$

where the last map is given by $\beta_{E_b} \mapsfrom 1$.

Lemma 6.27. *This defines an isomorphism from the local system $H^{2n-1}(S(E_-), \mathbb{Z})$ to the constant one on $\mathbb{Z}$, i.e. a $\mathbb{Z}$-orientation of $q : S(E) \to B$.*

Proof. Given a path $\omega : I \to B$ form b to b', we obtain isomorphisms

$$H^{2n-1}(S(E_b), \mathbb{Z}) \xleftarrow{\cong} H^{2n-1}(S(\omega^*E), \mathbb{Z}) \xrightarrow{*} H^{2n-1}(S(E_{b'}, \mathbb{Z}).$$

Now ω^*E is a vector bundle over I, hence isomorphic to a trivialisable bundle. This reduces the lemma to the case of trivial bundles, for which it is trivial. □

Theorem 6.28. *$H^\bullet(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z})$ is a polynomial ring on classes $c_1, \ldots, c_n$, with $|c_i| = 2$, uniquely determined by the properties*

(1) c_n equals the Euler class for $S(\gamma^n_{\mathbb{C}})$ under the $\mathbb{Z}$-orientation just constructed.

(2) For $i < n$, the map $H^\bullet(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z}) \to H^\bullet(\mathrm{Gr}^{\mathbb{C}}_{n-1}, \mathbb{Z})$ sends c_i to c_i.

The associated characteristic classes are called the Chern classes for complex vector bundles.

Proof. By induction. The case $n = 0$ is trivial. For the induction step $n-1 \to n$, we have to set $c_n := e \in H^{2n}(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z})$ for $S(\gamma^n_{\mathbb{C}})$. The Gysin sequence shows that $H^j(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z}) \to H^j(\mathrm{Gr}^{\mathbb{C}}_{n-1}, \mathbb{Z})$ is an isomorphism for $j < 2n-1$. Hence the elements $c_1, \ldots, c_{n-1} \in H^\bullet(\mathrm{Gr}^{\mathbb{C}}_{n-1}, \mathbb{Z})$ lift uniquely to elements $c_1, \ldots, c_{n-1} \in H^\bullet(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z})$. We have hence defined all the c_i. It remains to show that $H^\bullet(\mathrm{Gr}^{\mathbb{C}}_n, \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$, which is the same argument as in the real case. □

Remark 6.29. Like the Stiefel-Whitney classes, the Chern classes are uniquely characterised by analogous axioms.

7 Applications to Smooth Manifolds: non-immersions and cobordism

We will show that smooth manifolds come with canonical vector bundles and Stiefel-Whitney classes are invariants, and that the homotopy groups of $\mathrm{Th}(\gamma^n_{\mathbb{R}})$ have geometric interpretation in terms of cobordism classes of smooth manifolds. First, we recall the following notions:

Definition 7.1. Let $U \subseteq \mathbb{R}^m$ be open. A function $f : U \to Y \subseteq \mathbb{R}^n$ is called smooth if it is differentiable infinitely often. If $X \subseteq \mathbb{R}^m$ is any subset, $f : X \to Y$ is called smooth if for every $x \in X$ there exists an open $U \subseteq \mathbb{R}^m$ and a smooth function $\tilde{f} : U \to \mathbb{R}^n$ such that $\tilde{f}|_{U\cap X} = f|_{U\cap X}$.

Lemma 7.2. *The composite of smooth maps is smooth.*

Definition 7.3. A map $f : X \to Y$ is a diffeomorphism if it is smooth, bijective, and f^{-1} is also smooth. $M \subseteq \mathbb{R}^m$ is a smooth manifold of dimension n, if for all $x \in M$ there exists $U \subseteq M$ open which is diffeomorphic to an open subset of $\mathbb{R}^n$.

Example 7.4. Every open subset of $\mathbb{R}^n$ is a smooth manifold, S^n is a smooth manifold.

Remark 7.5. One can also define abstract smooth manifolds as second countable paracompact spaces M equipped with additional data in a variety of ways, for example via an equivalence class of a smooth atlas. One way is to specify for every open subset U of M a subset $F(U) \subseteq C^0(U, \mathbb{R})$, the space of continuous functions $U \to \mathbb{R}$, such that F is a sheaf and for every $x \in M$ there is an open $U \ni x$ such that $(U, F|_U)$ is isomorphic to $(\mathbb{R}^n, C^\infty(-, \mathbb{R}))$. A morphism $f : M \to N$ is then smooth if $g \circ f \in F_M(f^{-1}(U))$ for all $U \subseteq N$ open and $g \in F_N(U)$. If $M \subseteq \mathbb{R}^m$ is a smooth manifold in the previous sense, then $F(U) = \{f : U \to \mathbb{R} \text{ smooth }\}$ defines a smooth structure on M in the abstract sense, and the concepts of smooth functions coincide. Hence, submanifolds of Euclidean space form a full subcategory of abstract manifolds. These categories are equivalenct via the Whitney embedding theorem:

Theorem 7.6 (Whitney embedding)**.** *Every abstract n-dimensional smooth manifold is diffeomorphic to a submanifold of $\mathbb{R}^{2n}$.*

Let M be an n-dimensional manifold, $x \in M$. We define the tangent space $T_xM = Df_0(\mathbb{R}^n) \subseteq \mathbb{R}^m$. where f is a choice of parametrization such that $f(0) = x$ and df_0 is its derivative at 0.

Lemma 7.7. *T_*M is independent of the choice of local parametrisation.*

Definition 7.8. Let $M \subseteq \mathbb{R}^m$ be a smooth manifold of dimension n. We define the tangent bundle τ_M with the total space

$$TM := \{(x, v) \mid v \in T_xM\} \subseteq M \times \mathbb{R}^m.$$

Lemma 7.9. *This defines an n-dimensional $\mathbb{R}$-vector bundle over M by projecting to the first coordinate and using the vector space structure on each T_xM. Moreover, every smooth map $f : M \to N$ between manifolds induces a bundle map $Tf : TM \to TN$ defined via $Tf(x, v) = (f(x), d\tilde{f}_x(v))$, where $\tilde{f}$ is a local extension of f around x to a smooth map on a subset of $\mathbb{R}^m$.*

Definition 7.10. Let $M \subseteq \mathbb{R}^m$ be an n-dimensional smooth manifold. Then its normal bundle $\nu_{M,\mathbb{R}^m}$ is defined as the orthogonal complement of the tangent bundle τ_M, viewed as a smooth subbundle of the trivial bundle $M \times \mathbb{R}^m$, that is $\nu_{M,\mathbb{R}^m} = \{(x, v) \mid v \in (T_xM)^\perp\}$. The normal bundles is defined in the following more general situation: A smooth map $i : M \to N \subseteq \mathbb{R}^m$ is called an immersion, if $di_x : T_xM \to T_{i(x)}N$ is injective for all $x \in M$. Then the normal bundle is defined as

$$\{(x, v) \mid v \in T_{i(x)}N,\ v\perp di_x(T_xM)\} \subseteq M \times \mathbb{R}^n.$$

Thus $i^*\tau_N \cong \tau_M \oplus \nu_i$

Remark. Immersions are local, but not necessarily global embeddings. (For example a smooth, non-injective map $S^i \to I^2$. Similarly, $\mathbb{RP}^2$ or the Klein bottle can be immersed into $\mathbb{R}^3$.)

Example 7.11. We have $T_xS^n = \langle x\rangle^\perp$, which is an easy calculation. Hence the normal bundle is isomorphic to the trivial bundle via

$$S^n \times \mathbb{R} \to \nu_{S^n}, \quad (x, \lambda) \mapsto (x, \lambda x).$$

Thus $\tau_{S^n} \oplus \varepsilon = \varepsilon^{n+1}$ and $\omega(\tau_{S^n}) = 1$. Nevertheless, one can show that τ_{S^n} is trivialisable if and only if $n = 1, 3, 7$. This is closely related to the Hopf invariant 1 problem.

We next want to discuss $\mathbb{RP}^\infty$, which we first have to turn into a smooth manifold. We can do so in two ways:

(a) We can declare a function $f : U \to \mathbb{R}^n$ for an open $U \subseteq \mathbb{RP}^n$ to be smooth if and only if $p^{-1}(U) \to U \xrightarrow{f} \mathbb{R}^n$ is smooth, where $p : S^n \to \mathbb{RP}^n$.

(b) Identify $\mathbb{RP}^n$ with a subspace of $\mathbb{R}^{(n+1)\times(n+1)}$ via the map $A : L = \langle x\rangle \mapsto \frac{1}{|x|^2}(x_ix_j)_{ij}$ or in other words, the map that sends a line in $\mathbb{R}^{n+1}$ to the orthogonal projection onto that line. This map is clearly injective.

The composite $\widetilde{g} : S^n \to \mathbb{RP}^n \xrightarrow{A} M_{n+1}(\mathbb{R})$ is smooth and locally has a smooth inverse, since if A_L is the orthogonal projection to L, pick a unit vector e_i which is not in $L^\perp$. Then $A_{L'} \mapsto A_{L'}(e_i)/|A_L(e_i)|$ is a local smooth inverse. Hence, the image of A is a smooth manifold and p is a local diffeomorphism. In particular, $dp_x : T_xS^n = \langle x\rangle \to T_x(\mathbb{RP}^n)$ is an isomorphism for all $x \in S^n$.

Since any line $L \in \mathbb{RP}^n$ has exactly two preimages x and $-x$ in S^n, and these satisfy $T_xS^n = T_{-x}S^n$, this identifies $T_L\mathbb{RP}^n$ with $L^\perp$. But this is misleading, since the two isomorphisms dp_x and dp_{-x} satisfy $dp_x = -dp_{-x}$. To encode this, we instead identify $T\mathbb{RP}^n$ with the set of all pairs

$$\{(x, v), (-x, -v) \mid v \in \langle x\rangle^\perp\},$$

which is now by construction independent of the choice of generator x for L under the antipode map. The fibre over a line L identifies with the set of homomorphisms $\gamma : L \to L^\perp$. Hence we obtain the following

Corollary 7.12. *The tangent bundle is isomorphic to the hom-bundle* $\mathrm{Hom}(\gamma_{\mathbb{R}}^{1,n}, (\gamma_{\mathbb{R}}^{1,n})^\perp)$.