Étale cohomology

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1 Motivation and basic definitions

1.1 Introduction and motivation

Problem: For varieties X over an algebraically closed field k (and hopefully more general schemes) define a cohomology theory $H^*(X)$ with properties similar to $H^*_{\text{sing}}(X(\mathbb{C})_{\text{ord. top. space}})$. Hopefully, there exists a Lefschitz fixed point formula

$$\#(\text{fixed points of } f \text{ with multiplicity}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(f^*|H^i(X)). \tag{L}$$

The aim of Grothendieck was to apply this to a program proposed by Weil of studying the congruence zeta function of X by applying (L) to $f = F_X$ given by $[x_0, \dots, x_n] \mapsto [x_0^q, \dots, x_n^q]$, yielding

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F_X^*|H^i(X)).$$

Counterexamples $H^*_{dR}(X) = \mathbb{H}^*(X_{\text{Zar}}, \mathcal{O}_X \to \Omega^1_X \to \cdots)$ (de Rham cohomology) is ok if the characteristic of k is zero but not in $\operatorname{char} p$ where it is unsuitable for Weil's program. Similarly, $H^*(X_{\text{Zar}}, \mathbb{Z})$ does not work: $\underline{\mathbb{Z}}(X) \to \underline{\mathbb{Z}}(V)$ is surjective when X is irreducible, implying vanishing higher sheaf cohomology.

Restrictions on the ring of coefficients: If X is a supersingular elliptic curve over $\overline{\mathbb{F}}_q$ then $H^1(X)$ ought to be two-dimensional, but $\operatorname{End}(X) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is non-split precisely over $\mathbb{Q}_p$ and $\mathbb{R}$, in which case it cannot act on a two-dimensional vector space. This excludes $\mathbb{Q}_p$ and $\mathbb{R}$ as the field of definition and hence also $\mathbb{Q}$ and $\mathbb{Z}$.

Etale cohomology with coefficients $\mathbb{Z}/l^n\mathbb{Z}$, l a prime invertible in k. Then

$$H^*(X, \mathbb{Q}_l) := \big(\varprojlim H^*(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z})\big) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Deligné used this to show the Riemann hypothesis for congurence zeta function.

Other theories include Crystilline cohomology with coefficients in $W(\overline{F}_q)$. Scholze has a way of working with $\mathbb{Z}_p$ directly, using the pro-étale site, and a proposal to work with $\mathbb{C}$ coefficients. But it is not clear how to do this.

Hence we will mostly study finite coefficients. If one works over $\mathbb{C}$, the exact exponential sequence $0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ is important. and we want at least the exactness of

$$0 \to \mu_{l^n} \to \mathcal{O}_X^\times \xrightarrow{f \mapsto f^{l^n}} \mathcal{O}_X^\times \to 0. \tag{$*$}$$

Note that $\mu_{l^n} \cong \mathbb{Z}/l^n\mathbb{Z}$ non-canonically if $k = \bar{k}$ and l is invertible in k. Unfortunately, but not unexpectedly, this is not exact on X_{Zar}. If this were exact, one could hope to get some information from it provided that $H^1(C, \mathcal{O}_C^\times) \cong \mathbb{Z} \times \operatorname{Jac}_C(k)$. The idea of Grothendieck was to enforce the exactness of $(*)$ by considering $V \to F(V)$ for étale morphisms $V \to X$ instead of only Zariski open subsets. Then, when $f \in \mathcal{O}_V^\times(V)$ one has an l^n-th root of f on $U = \{(x, \varphi) \mid x \in V, \varphi^{l^n} = f(x)\}$.

1.2 Flat morphisms

Definition 1. M is a *flat* A-module if $T \mapsto M \otimes_A T$ is exact or, equivalently, if $\mathrm{Tor}_p^A(M, T) = 0$ for all T and $p > 0$. An A-algebra B is flat if it is flat as an A-module.

Definition 2. For a morphism $f : X \to Y$ of schemes, f is called *flat* if it satisfies the folllowing equivalent conditions:

a) For all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-algebra.

b) For affine open subsets $U \subseteq X, V \subseteq Y$ s.t. $f(U) \subseteq V$, $\mathcal{O}_X(U)$ is flat as an $\mathcal{O}_Y(V)$-algebra.

c) There are affine open subsets $U_i \subseteq X, V_i \subseteq Y$ s.t. $f(U_i) \subseteq V_i$, $\mathcal{O}_X(U_i)$ is a flat $\mathcal{O}_Y(V_i)$-algebra and $X = \bigcup_{i \in I} U_i$.

Remark 1. a) See stacksproject 01U2

b) Other literature: SGA1: Etale fundamental group, SGA4_1: Topoi, Grothendieck topology, SGA4_2: Etale topology, SGA4_3: Proper and smooth base change, SGA$4\frac{1}{2}$: various stuff and Arcata – Introduction to etale cohomology by Delinge, SGA5: l-adic cohomology
Milne: Etale cohomology, Kiehl-Freitag: Etale cohomology and Weil conjectures
Matsumura: Commutative Algebra, Matsumura: Commutative Ring Theory

Let A be a ring, X quasi-compact and separated $\mathrm{Spec}\, A$-scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $H^*(X, \mathcal{M})$ can be calculated using $\check{H}(\mathcal{U}, -)$ for affine coverings. Hence, by the exactness of $- \otimes_A \widetilde{A}$, this gives

Proposition 1. *a) Let $\widetilde{A}$ be a flat A-algebra, then $H^*(\widetilde{X}, \widetilde{M}) \cong H^*(X, M) \otimes_A \widetilde{A}$, where $\widetilde{X} = X \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, \widetilde{A} \xrightarrow{p} X$ and $\widetilde{M} = p^* M$.*

b) Let $f : X \to Y$ be a quasi-compact separated morphism and $g : \widetilde{Y} \to Y$ a flat morphism, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $g^* R^* f_* \mathcal{M} \cong R^* \widetilde{f}_* \widetilde{g}^* \mathcal{M}$ where $\widetilde{X} = X \times_Y \widetilde{Y}$.

Remark 2. Base change results for etale cohomology are similar. We have b) if f is proper or if f is of finite type and g is smooth, and the sheaves are of torsion.

Definition 3. f is called *faithfully flat* if it is flat and surjective on points. $\widetilde{A}$ is a faithfully flat A-algebra if it is flat and $R \otimes_A \widetilde{A} = 0$ implies $T = 0$.

Definition 4. [1] Let $f : X \to Y$ be a morphism of schemes. A descent datum (of quasi-coherent sheaves of modules) for f is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ with an isomorphism $\mu : p_1^* \mathcal{M} \cong p_2^* \mathcal{M}$, where

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "p_{12}, p_{13}", "p_{23}"'] \arrow[rr, bend right, "q_1, q_2, q_3"'] & X \times_Y X \arrow[r, "p_1, p_2"] & X \end{tikzcd}$$

[1] see tag 023A or SGA1,VI for fibred categories: descend data for X-schemes to Y-schemes and ample line bundles

are the different projections, and the diagram

$$\begin{tikzcd}
q_1^*\mathcal{M} \arrow[r, equal] \arrow[dr, equal] & p_{12}^*p_1^*\mathcal{M} \arrow[rr, "p_{12}^*\mu", "\cong"'] & & p_{12}^*p_2^*\mathcal{M} \arrow[r, equal] & q_2^*\mathcal{M} \arrow[dl, equal] \\
& p_{13}^*p_1^*\mathcal{M} \arrow[d, "\cong", "p_{13}^*\mu"'] & & p_{23}^*p_1^*\mathcal{M} \arrow[d, "\cong"', "p_{23}^*\mu"] & \\
& p_{13}^*p_2^*\mathcal{M} \arrow[dr, equal] & & p_{23}^*p_2^*\mathcal{M} \arrow[dl, equal] & \\
& & q_3^*\mathcal{M} & &
\end{tikzcd}$$

must commute. A morphism of descent data is a morphism $\varphi : \mathcal{M} \to \widetilde{\mathcal{M}}$ compatible with μ and $\widetilde{\mu}$, i.e. $(p_2^*\varphi)\mu = \widetilde{\mu}(p_1^*\varphi)$

Remark 3. We have a functor

$$\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}, \quad \mathcal{N} \mapsto (f^*\mathcal{N}, \text{the canonical iso } p_1^*f^*\mathcal{N} \cong p_2^*f^*\mathcal{N}).$$

One would like this to be an equivalence of categories. It has a right adjoint

$$(\mathcal{R}\mathcal{M})(U) = \{m \in \mathcal{M}(f^{-1}U) \mid \mu p_1^* m = p_2^* m\}$$

Proposition 2 (stacks loc.cit., SGA1.VII.1, Milne)**.** *If f is faithfully flat and quasi-compact, the above functor* $\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}$ *is an equivalence of categories.*

Proof. If f has a section, the inverse image along that section is an inverse functor. In general, base change with $f : X \to Y$ reduces to this situation, provided that f is separated, which is a situation one can reduce to. □

Corollary 1. *If f is faithfully flat, $\mathcal{O}_Y(V) = \{\lambda \in \mathcal{O}_X(f^{-1}U) \mid p_1^*\lambda = p_2^*\lambda\}$.*

Remark 4. Both quasi-compactness and quasi-coherence in proposition 2 are needed. Consider $Y = \operatorname{Spec} R$, R a PID with $\operatorname{Spec} R$ infinite,

$$X = \coprod_{m \in \mathrm{mSpec}} \operatorname{Spec} R_m, \qquad N_1 = \coprod_{m \in \mathrm{mSpec}\, R} R/m \to N_2 = \prod_{m \in mSpecR} R/m,$$

then it is easy to see that this inclusion does not split, bit it splits canonically after applying $- \otimes_R R_m$, giving rise to a morphism of descent data which does not descend to a morphism $N_2 \to N_1$.

Definition 5. A morphism $i : X \to Y$ in a category $\mathcal{A}$ is an effective monomorphism if for all objects T,

$$\mathrm{Hom}_{\mathcal{A}}(T, X) \xrightarrow{\varphi \mapsto i\varphi} \{f \in \mathrm{Hom}_{\mathcal{A}}(T, Y) \mid \sigma f = \widetilde{\sigma} f \text{ for all } \sigma, \sigma' : Y \to S \text{ s.t. } \sigma i = \widetilde{\sigma} i\}$$

is bijective. $p : X \to Y$ is an effective epimorphism if it is an effective monomorphism in $\mathcal{A}^{\mathrm{op}}$, i.e.

$$\mathrm{Hom}_{\mathcal{A}}(Y, T) \xrightarrow[\cong]{\varphi \mapsto \varphi p} \{f \in \mathrm{Hom}_{\mathcal{A}}(X, T) \mid f\sigma = f\widetilde{\sigma} \text{ for all } \sigma, \widetilde{\sigma} : S \to X \text{ s.t. } p\sigma = p\widetilde{\sigma}\}.$$

Remark 5. If $X \times_Y X$ exists, f being an effective epimorphism is equivalent to it being a coequalizer of $X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$.

Proposition 3 (SGA1.VIII.4 or stacks 023Q)**.** *Every fpqc (quasi-compact faithfully flat) morphism of schemes is an effective epimorphism, i.e.*

$$\mathrm{Hom}(Y,T) \to \mathrm{Hom}(X,T) \rightrightarrows \mathrm{Hom}(X \times_Y X, T)$$

is an exact sequence of sets.

Remark 6. This implies that for every scheme T, the functor $X \mapsto T(X) := \mathrm{Hom}(X,T)$ satisfies the sheaf condition in the following sense:

$$T(Y) \xrightarrow{\tau \mapsto \tau f} \{t \in T(X) \mid tp_1 = tp_2\}.$$

That this should be interpreted as a kind of sheaf axiom becomes obvious if we have a covering $Y = \bigcup_{i=1}^n U_i$, $X = \coprod_{i=1}^n U_i \xrightarrow{f} Y$. Then $X \times_Y X = \coprod_{i,j=1}^n (U_i \cap U_j)$ with $tp_1|_{U_i \cap U_j}$ identified with $t|_{U_i}|_{U_i \cap U_j}$.

Proposition 4 (01UA)**.** *Every flat morphism (locally) of finite presentation is open.*

1.3 Grothendieck Topologies

As Deligne did in Arcata, we prefer the definition of Grothendieck topology by sieves.

Definition 1. Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}(\mathcal{C})$. A *sieve* (or $\mathcal{C}$-sieve) over X is a class $\mathcal{S}$ of morphisms with target X, such that $(U \to X) \in \mathcal{S}$ implies $(V \to U \to X) \in \mathcal{S}$ for every morphism $V \to U$ in $\mathcal{C}$. The empty class of morphisms is called the *empty sieve*, and the class of all morphisms with target X is called the *all sieve* (over X). For a morphism $f : Y \to X$ in $\mathbb{C}$, $f^*\mathcal{S} = \{v : U \to Y \mid fu \in \mathcal{S}\}$.

Remark 1. a) Obviously, $f^*\mathcal{S}$ is a sieve over Y if $\mathcal{S}$ is a sieve over X.

b) The fact that we work with categories where $\mathrm{Ob}\,\mathcal{C}$ is a proper class creates set-theoretic difficulties. Our way of dealing with this is to mostly ignore them.

c) The intersection of any class of sieves over X is a sieve over X.Thus, for every class $(f_i)_{i \in I}$ of morphisms with target X, there is a smallest sieve over X containing all f_i, namely $\{\xi : U \to X \mid \xi = f\eta$ for $\eta : U \to Y_i$ for some $\eta\}$. This is called the sieve generated by the f_i.

Example 1. a) X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$ turned into a category by its half ordering by $\subseteq$. If $X = \bigcup_{i \in I} U_i$ is an open covering, then the sieve generated by the (unique morphisms from) U_i is the sieve of all $V \in \mathbb{O}_X$ s.t. $V \subseteq U_i$ for at least one i.

b) If X is a complex space (e.g. $X = \mathbb{C} \setminus \{0\}$) with its complex topology, and $U \subseteq X$ open and $f \in \mathcal{O}_X(U)$, then $\mathcal{S} = \{V \subseteq U \mid \exists \varphi \in \mathcal{O}_X(V) \text{ s.t. } \varphi^2 = f|_V\}$ is a $\mathbb{O}_X$-sieve over U.

Remark. Thus, a morphism is in a sieve iff it is small enough "to pass through the sieve".

Definition 2. A *Grothendieck topology* $\mathbb{T}$ on a category $\mathcal{C}$ associates to every object X of $\mathcal{C}$ a class $\mathbb{T}_X$ of sieves over X, called the *covering sieves* of X. The following conditions must be verified:

(GTTriv) The all sieve over X covers X.

(GTTrans) If $\mathcal{S} \in \mathbb{T}_X$ and $f : Y \to X$, then $f^*\mathcal{S} \in \mathbb{T}_Y$.

(GTLoc) If $\mathcal{T} \in \mathbb{T}_X$ and $\mathcal{S}$ any sieve over X such that $f^*\mathcal{S} \in \mathbb{T}_Y$ for all $f : Y \to X$ in $\mathcal{T}$, then $\mathcal{S} \in \mathbb{T}_X$.

We will often write $\mathcal{S} \mathrel{/\!=} X$ for $S \in \mathbb{T}_X$ if there are no ambiguities (or $\mathcal{S} \mathrel{/\!=_{\mathbb{T}}} X$ it there are).

Remark 1. Pretopologies are specified by specifying a class of admissible coverings $\mathcal{U} = (f_i : Y_i \to X)_{i\in I}$. Various assumptions must be satisfied, like that $(U_i \times_X Y \to Y)_{i\in I}$ still form an admissible covering of Y (including the existence of the fibre product). By putting $\mathbb{T}_X = \{$admissible coverings $\mathcal{S}$ of X with all $f_i \in \mathcal{S}\}$ one gets a Grothendieck topology. Equivalent pretopologies define the same $\mathbb{T}_X$. If the category has fibre products, one gets a pretopology from a Grothendieck topology $\mathbb{T}_X$ by calling a covering admissible iff the f_i generate a sieve in $\mathbb{T}_X$. This is the largest pretopology in its equivalence class.

Example 2. X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$, and $\mathcal{S} \mathrel{/\!\!=} U$ iff $U = \bigcup_{V\in\mathcal{S}} V$. Other Grothendieck topologies can be introduced as well.

a) $X = [0,1]_{\mathbb{R}}$, put $\mathcal{S} \mathrel{/\!\!=} U$ iff there are countable many $(U_i)_{i\in\mathbb{N}}$ such that $U \setminus \bigcup_{i\in\mathbb{N}} U_i$ is a set of Lebesgue measure 0, or $\mathcal{S} = U = \emptyset$.

b) Rigid analytic geometry (Tate style) or real algebraic geometry (Delfs-Knebusch) enforce quasi-compactness of certain open subsets of X, making it harder to be a covering.

c) X a Noetherian scheme, $d \in \mathbb{N}$. $\mathcal{S} \mathrel{/\!\!=} \mathcal{U}$ iff $\operatorname{codim}(U \setminus \bigcup_{V\in\mathcal{S}} V) \geq d$, making it easier to be a covering.

Remark 2. You can think of (GTLoc) as the condition that being a covering is a local property.

Fact 1. a) Every sieve $\mathcal{T}$ containing a covering sieve $\mathcal{S}$ is itself covering.

b) The intersection of finitely many covering sieves is covering.

Proof. a) If $(f : U \to X) \in \mathcal{S}$, then $f^*\mathcal{T}$ is the all-sieve on U which covers U by (GTTrans). By (GTLoc), $\mathcal{T}$ covers X.

b) It is sufficient to show that $\mathcal{T} := \mathcal{S}_1 \cap \mathcal{S}_2$ covers X, where both $\mathcal{S}_i \mathrel{/\!\!=} X$. If $(f : U \to X) \in \mathcal{S}_1$, then $f^*\mathcal{T} = f^*\mathcal{S}_2 \mathrel{/\!\!=} U$ by (GTTrans) and since $\mathcal{S}_2 \mathrel{/\!\!=} X$. Again by (GTLoc), $T \mathrel{/\!\!=} X$. $\square$

Proposition 1. *Let S be a scheme, P a Zariski-local property of S-schemes and $\underline{\mathrm{Sch}}_S^P$ be the full subcategory of the category $\underline{\mathrm{Sch}}_S$ of S-schemes, with class of objects being the S-schemes with property P, and let $\mathcal{C}$ be a class of morphisms in $\underline{\mathrm{Sch}}_S^P$. The following assumptions must be satisfied:*

(A) $\mathcal{C}$ is closed under composition, base-change and finite coproducts.

(B) If U is a quasi-compact S-scheme with $P(U)$ and $U = \bigcup_{i=1}^n U_i$ is a finite affine open covering, then the morphism $\coprod_{i=1}^n U_i \to U$ belongs to $\mathcal{C}$.

If X is an S-scheme with $P(X)$ then the following conditions to a sieve $\mathcal{S}$ over X are equivalent:

(C1) There are open coverings $X = \bigcup_{i\in I} U_i$ and morphisms $V_i \to U_i$ for all $i \in I$ such that $(V_i \to U_i \to X) \in \mathcal{S}$ and V_i is covered (in the ordinary sense) by its Zariski-open subsets W such that $(W \to V_i \to U_i) \in \mathcal{C}$

(C2) The same conditions, but the U_i and V_i must be affine.

In addition, we obtain a Grothendieck topology $\mathbb{T}$ on $\underline{\mathrm{Sch}}_S^P$ by associating to X the class $\mathbb{T}_X$ of all sieves with these equivalent properties.

Remark 3. a) In (A), the stability under base change includes the condition that $X_Y\widetilde{X}$ has P when $X, Y, \widetilde{X}$ have this property and $(X \to Y) \in \mathcal{C}$.

b) It the elements of $\mathcal{C}$ are open maps, then the conditions (C1) and (C2) can be modified by simply requiring that $(V_i \to U_i) \in \mathcal{C}$ without changing anything else, i.e. $X = \bigcup_{i\in I} U_i$ and $(V_i \to U_i) \in \mathcal{C} \cap \mathcal{S}$.

Example 3. a) P the trivial property and $\mathcal{C}$ the class of all fpqc morphisms. We get the fpqc topology on $\underline{\mathrm{Sch}}_S$.

$\widetilde{\mathrm{a}}$) Let S be Noetherian, P : local Noetherianness and $\mathcal{C}$ the class of fpqc morhpisms. This will NOT work as (A) is violated: For instance, with $S = X = \operatorname{Spec}\mathbb{Q}$, the fibre product $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ is non-noetherian: The ideal $I = (x \otimes y - y \otimes x \mid x, y \in \mathbb{C})$ is not finitely generated as $\Omega_{\mathbb{C}/\mathbb{Q}} \cong I/I^2$. This is a $\mathbb{C}$-vector space of dimension equal to the continuum (the transcendence degree of $\mathbb{C}/\mathbb{Q}$).

b) Let $\mathcal{C}$ be the class of all fppf (faithfully flat of finite presentation) morphisms and the trivial property (or local Noetherianness) for P. Then fibre products don't cause any trouble, since then $\widetilde{X} \times_X Y$ is of finite type over $\widetilde{X}$ and local Noetherianness is preserved. One gets the fppf-topology on (locally noetherian) S-schemes. In this case. quasi-finiteness can be added to "of finite presentation" without modifying the topology: (stacks 056X)

c) The class $\mathcal{C}$ of all surjective morphisms which are Zariski-local isomorphisms, with P = trivial, or local Noetherianness, or regularity, ... and one gets the Zariski topology on $\underline{\mathrm{Sch}}_S$.

Proof. (of proposition 1) It is clear that (C2) implies (C1). Assume conversely that $X = \bigcup_{i\in I} U_i$ and $(p_i : V_i \to U_i) \in \mathcal{C}$ such that V_i is covered by the open $W \subseteq V_i$ such that $(W \to V_i \to X) \in \mathcal{S} \cap \mathcal{C}$. (We call such W $\mathcal{S}$-small.) Let $U_i = \bigcup_{j\in J_i} U_{ij}$ be an open affine covering and $V_{ij} = p_i^{-1} U_{ij} = V_i \times_{U_i} U_{ij}$. Thus $(V_{ij} \to U_{ij}) \in \mathcal{C}$ by (A). If $W \subseteq V_i$ is $\mathcal{S}$-small, the same holds for $W \cap V_{ij}$, showing that V_{ij} is covered by its $\mathcal{S}$-small open subsets. Thus we may assume that the U_i are affine and the V_i quasi-compact. By an application of (B), we may also assume that the V_i are affine. Then (C2) holds.

It remains to show the properties of a Grothendieck topology. For (GTTriv) this is trivial (U_i any affine covering and $V_i = U_i$). Also, (GTTrans) is easy. If $f : \widetilde{X} \to X$ is a morphism one puts $\widetilde{U}_i = f^{-1} U_i$, $\widetilde{V}_i = \widetilde{U}_i \times_{U_i} V_i$ and $(\widetilde{V}_i \to \widetilde{U}_i) \in \mathcal{C}$ by (A). Also, if $W \subseteq V$ is $\mathcal{S}$-small, then its inverse image in $\widetilde{V}_i$ is $f^*\mathcal{S}$-small, and these inverse images cover $\widetilde{V}_i$. For (GTLoc), let $\mathcal{S} \,/\!\!= X$ and $\mathcal{T}$ any sieve such that $f^*\mathcal{T} \,/\!\!= Y$ for all $(f : Y \to X) \in \mathcal{S}$. We must show $\mathcal{T} \,/\!\!= X$.

<u>Case 1:</u> One can choose $V_i = U_i \xrightarrow{\mathrm{id}} U_i$ in the condition (C1) for $\mathcal{S} \,/\!\!= X$. Then the restriction $\mathcal{T}|_{U_i} := (U_i \hookrightarrow X)^*\mathcal{T}$ covers U_i. Thus there are an open covering $U_i = \bigcup_{j\in J_i} U_{ij}$ and $V_{ij} \to U - ij$ as in (C1) for $\mathcal{T}|_{U_i}$, and then $X = \bigcup_{i\in I} \bigcup_{j\in J_i} U_{ij}$, together with the morphisms $V_{ij} \to U_{ij}$, does the same for X.

<u>Case 2:</u> X is affine, and there is a morphism $(p : V \to X) \in (\mathcal{S} \cap \mathcal{C})$ with V affine, s.t. p generates $\mathcal{S}$. Then $p^*\mathcal{T} \,/\!\!= V$. Write $V = \bigcup_{i=1}^n U_i$ and morphisms $(V_i \to U_i) \in \mathcal{C}$ such that the $\mathcal{S}$-small open susets of V_i cover V_i. Then one can satisfy (C2) for $\mathcal{T}$ by $U' = X$, $V' = \coprod_{i=1}^n V_i \to \coprod_{i=1}^n U_i \to V \to X = U'$, where the arrows are in $\mathcal{C}$ by (A), (B), and assumption, respectively.

<u>Case 3:</u> General case: If $V_i \to U_i$ are as in (C2) for $\mathcal{S}$, then the pullback of $\mathcal{T}$ to any $\mathcal{S}$-small open subset W of V_i covers W. By case 1, the pullback of $\mathcal{T}$ to V_i covers V_i. By case 2, $\mathcal{T}|_{U_i} \,/\!\!= U_i$. By case 1 again, $\mathcal{T} \,/\!\!= X$. □

Definition 3. A presheaf on a category $\mathcal{C}$ (with values in sets, (abelian) groups, rings) is a contravariant functor from $\mathcal{C}$ to <u>Set</u> (or groups, rings, ...). If a Grothendieck topology $\mathbb{T}$ on $\mathcal{C}$ is given, then a presheaf $\mathcal{F}$ is called ($\mathbb{T}$-)separated, if

$$F(X) \to \prod_{(p:U\to X)\in\mathcal{S}} F(U), \qquad f \mapsto (F(p)f)_p \tag{$*$}$$

is injective. We call a separated presheaf F a sheaf if the image of $(*)$ is $\varprojlim_{(p:U\to X)\in\mathcal{S}} F(U)$. In other words, the image of $(*)$ must be the family of all $(f_p)_p$ such that $F(q')f_p = F(p')f_q$ in $F(W)$ whenever

$$\begin{array}{ccc} W & \xrightarrow{p'} & V \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ U & \xrightarrow{p} & X \end{array}$$

is a commutative diagram in $\mathcal{C}$, with $p, q \in \mathcal{S}$.

Proposition 2. *In the situation of proposition 1, a presheaf G is a sheaf (resp. separated) if and only if for every object X of $\underline{\mathrm{Sch}}_S^P$ the presheaf $U \mapsto G(U)$ on X equipped with its Zariski topology is a sheaf (resp. separated), and for every morphism $p : U \to V$ in $\mathcal{C}$ the sequence*

$$G(V) \xrightarrow{p^*} G(U) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} G(U \times_V U)$$

is exact in the sense that the first morphism is the equalizer of the second two (resp, if p^* is injective

Proof. Let $S \,/\!\!= X$, we must show that $G(X) \to \varprojlim G$ is bijective (resp. injective), and for the proof of bijectiveness, we may assume injective.
Case 1: S is already covering for X_{Zar}: Trivial.
Case 2: There is a morphism $p : U \to X$ in $\mathcal{C}$ such that the S-small open subsets W of U cover U (as sets). If $g_1, g_2 \in G(X)$ have the same image in $\varprojlim_S G$, then $p^*g_1|_W = p_2^*g_2|_W$ when $W \subseteq U$ is S-small. By our first assumption on G, $p^*g_1 = p^*g_2$. As p^* is injective by our second assumption, $g_1 = g_2$. Let $\gamma \in \varprojlim_S G$. By our first assumption on G, there is $g_U \in G(U)$ such that $g_U|_W = \gamma_W$ whenever $W \subseteq U$ is S-small. Let $W, \widetilde{W} \subseteq U$ be S-small, then for $p_1, p_2 : U \times_X U \to U$ we have

$$p_1^* g_U|_{W \times_X \widetilde{W}} = p_1^* \gamma_W|_{W \times_X \widetilde{W}} = \gamma_{W \times_X \widetilde{W}} = p_2^* \gamma_{\widetilde{W}}|_{W \times_X \widetilde{W}} = p_2^* g_U|_{W \times_X \widetilde{W}}.$$

As these $W \times_X \widetilde{X}$ cover $U \times_X U$ as a set, $p_1^* g_U = p_2^* g_U$. By our assumption there is a unique $g \in G(X)$ such that $p^*g = g_U$. We must show that the image of g in $\varprojlim_S G$ is γ. Let $\widetilde{S} \subseteq S$ be the subsieve of S generated by the S-small $W \subseteq U$. Then $\widetilde{S} \,/\!\!= X$, and the image of g in $\varprojlim_{\widetilde{S}} G$ equals $\gamma|_{\widetilde{S}}$ by construction. For $(\nu : V \to X) \in S$, this implies that $G(\nu)g = \gamma_V$ as they have the same image in $\varprojlim_{\nu^* \widetilde{S}} G$, and $\nu^* \mathcal{S} \,/\!\!= V$. Thus the claim about g is shown.
Case 3: General case. Let $V_i \to U_i$ be as in the definition of a Grothendieck topology. If g_1, g_2 have the same image in $\varprojlim_S G$ then $g_1|_{U_i} = g_2|_{U_i}$ by case 2, hence $g_1 = g_2$ by the first assumption. Let $\gamma \in \varprojlim_S G$, by case 2 there is $\gamma_i \in G(U_i)$ such that the image of γ_i in $\varprojlim_{S|_{U_i}} G$ equals the restriction of γ. Then $\gamma_i|_{U_i \cap U_j} = \gamma_j|_{U_i \cap U_j}$ as their images in $\varprojlim_{S|_{U_i \cap U_j}} G$ are both equal to the restriction of γ to $S|_{U_i \cap U_j} \,/\!\!= U_i \cap U_j$. By our first assumption, there is $g \in G(X)$ such that $g|_{U_i} = g_i$. In a similar way as in the end of case 2, one sees that the image of g in $\varprojlim_S G$ equals γ. □

Corollary 1. *If X is any S-scheme then*

$$U \to X(U) := \mathrm{Hom}_{\underline{\mathrm{Sch}}_S}(U, X)$$

is an fpqc-sheaf on $\underline{\mathrm{Sch}}_S$.

Exercise: If $F \in \mathrm{QCoh}(S)$, then $(\upsilon : U \to S) \mapsto \upsilon^* F$ is an fpqc sheaf, and $H^*(S_{Zar}, F) \cong H^*(S_{fpqc}, F)$

1.4 Étale morphisms

Proposition 1. *Let $f : X \to Y$ be a morphism locally of finite type between Noetherian schemes, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

a) $\Omega_{X/Y,x} = 0$.

b) There is an open neighbourhood U of x in X such that $\Delta_{X/Y} : U \to X \times_Y X$ is an open embedding.

c) We have $\mathfrak{m}_x = (f^*\mathfrak{m}_y)\mathcal{O}_{X,x}$, and $k(x)$ is a separable finite field extension of $k(y)$.

If f is separated, such that $\Delta_{X/Y}$ is a closed embedding defined by the quasi-coherent sheaf of ideals $J \subseteq \mathcal{O}_{X\times_Y X}$, then the above is also equivalent to

d) $J_x = 0$.

Remark. The Noetherianness assumption can be dropped with little effort.

Proof. (Sketch) As a), b), and c), as well as the claim in d) are local in X, we may assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine. Then the equivalence of b) with d) is obvious as J is locally finitely generated: If d) holds, there is an open neighbourhood U of x in X such that $J|_U$ vanishes. The equivalence of a) with d) then comes from a well-known fact (Remark 1 below) about Kähler differentials. By Nakayama's lemma $(\Omega_{X/Y})_x = 0$ if and only if $0 = (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{X,x}} k(x) \cong (\Omega_{f^{-1}\{y\}/k(y)})_x$, by the compatability of Kähler differentials with base change. The $k(y)$-algebra $(k(y) \otimes_A B)_{\mathfrak{m}_x}$ has vanishing Kähler differentials over $k(y)$ iff this local $k(y)$-algebra is a finite separable field extension $l/k(y)$, i.e. $\mathfrak{m}_x = (f^*\mathfrak{m}_y)\mathcal{O}_x$ (othersie B_x has nilpotent elements) and $k(x) = l$ is separable over $k(y)$. $\square$

Remark 1. a) If f is separated and J as in (d), then $\Omega_{X/I} \cong \Delta^*_{X/Y} J \cong \Delta^*_{X/Y}(J/J^2)$.
b) If A and B are as in the proof, $\Omega_{B/A} \cong I/I^2$, $I = \ker(B \otimes_A B \to B)$.
c) $\mathcal{D}er_{B/A}(B, M) \cong \operatorname{Hom}_B(I/I^2, M)$, given by $d \mapsto \varphi(a \otimes b) = ad(b)$ and $d(b) = \varphi(1 \otimes b - b \otimes 1)$.

Definition 1. a) A morphism $f : X \to Y$ locally of finite type between locally Noetherian schemes is *unramified* at $x \in X$ iff it satisfies the equivalent definitions of proposition 1.
b) It is called *étale at x* if it is flat and unramified at x.
c) It is called étale iff it is étale at all $x \in X$.
d) It is called an *étale covering* if it is étale and finite.

Remark. See 00U0 for the definitions the non-Noetherian case, which are essentially the same. By 00U9 locally every étale morphism comes by base-change from a Noetherian morphism. See also EGA IV.17.

Fact 1 (00U2)**.** a) The class of étale morphisms is stable under composition and base change.
b) If $g \circ f$ is étale and g unramified, then f is étale.
c) If f is étale and a closed embedding, then f is an open embedding.

Proof. a) The stability of flatness under base change is assumed to bek nown here, and for unramifiedness this follows from $\Omega_{\widetilde{X}/\widetilde{Y}} \cong \Omega_{X/Y}$ for every Cartesian square

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

For treatment of compositions, let the morphisms always be $f : X \to Y, g : Y \to S$. Again for flatness this is well-known. Unramifiedness of $g \circ f$ follows from the exact sequence

$$f^*\Omega_{Y/S} \xrightarrow{f^*} \Omega_{X/S} \to \Omega_{X/Y} \to 0 \tag{F1}$$

b)
c) This follows form Proposition 1.2.4. even when f is flat of finite presentation, X, Y arbitary. □

Fact 2. A flat morphism $X \to Y$ is étale at $x \in X$ if and only if this holds for $f^{-1}(y)/x$ at x. The same holds for unramified morphisms.

Example 1. a) $X \to \operatorname{Spec} k$ is étale at $x \in X$ iff $\mathcal{O}_{X,x}$ is a finite separable field extension of k.
b) Every open or closed embedding is unramified.
c) Every open embedding is étale.

Lemma 1. *If A is an algebra over a field K, the following conditions are equivalent:*

a) A/K is étale,

b) $A \cong \bigoplus_{i=1}^{n} L_i$, each L_i/K separable,

c) The trace form $B_{A/K}(a, b) := \operatorname{Tr}_{A/K}(ab)$ is a perfect pairing on $A \times A$.

Proof. Omitted. □

Remark 2. If L/K is a finitely generated field extension, then $\Omega_{X/Y} \cong 0$ iff L/K is finite and separable.

Proposition 2. *Let X be locally Noetherian, $\mathcal{A}$ a coherent locally free $\mathcal{O}_X$-algebra. Then $\underline{\operatorname{Spec}}\mathcal{A} \to X$ is étale over x if and only if the trace bilinear form $B_{\mathcal{A}_x/\mathcal{O}_{X,x}}$, $B(a, b) = \operatorname{Tr}_{\mathcal{A}_x/\mathcal{O}_{X,x}}(ab)$ is non-degenerate. In particular, $\underline{\operatorname{Spec}}\mathcal{A}$ is an étale covering if the trace bilinear form is non-degenerate everywhere.*

Proof. Flatness is automatic by our assumptions. The assertion then follows with little work from fact 2 and lemma 1. □

Corollary 1. *In the situation of the proposition, $p : \underline{\operatorname{Spec}}\mathcal{A} \to X$ is an étale covering if and only if there is an open subset $U \subseteq X$ with $\operatorname{codim}(Y, X) \geq 2$ for every irreducible component Y of $X \setminus U$, and $p^{-1}(U) \to U$ is an étale covering.*

Proof. Without losing generality $X = \operatorname{Spec} R$ is affine and $\mathcal{A}$ is defined by the free R-algebra A. Using a base of the R-module A and a matrix representation of $B_{A/R}$,

$$\{x \in X \mid \operatorname{Spec} A \to X \text{ is not étale over } x\} = V(d)$$

where $d \in A$ is the determinant of that matrix representation of $B_{A/R}$. By Krull's principal ideal theorem all irreducible components of this closed subset have codimension at most 1. □

Proposition 3. *If $f : X \to Y$ is an étale morphism of locally Noetherian S-schemes, then $f^*\Omega_{Y/S} \to \Omega_{X/S}$ is an isomorphism.*

Proof. Surjectivity follows from the cotangent sequence (F1) using only that f is unramified. For the isomorphism claim consider

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/Y}"] \arrow[rr, bend left, "\Delta_{X/S}"] \arrow[dr, "f"'] & X \times_Y X \arrow[r, "j"] \arrow[d] & X \times_S X \arrow[d, "p"] \\ & Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

It is sufficient to give the proof when all schemes are affine and therefore separated. Then all diagonals are closed embeddings and given by coherent sheaves of ideals, e.g. $\Delta_{X/S}$ by $J_{X/S}$. The square being cartesian implies that j is a closed embedding with sheaf of ideals $J_j = p^* J_{Y/S}$ (this uses that p is flat) As $\Delta_{X/Y}$ is an open embedding,

$$\Omega_{X/S} = \Delta^*_{X/S} J_{X/S} = \Delta^*_{X/Y} j^* J_{X/S} \cong \Delta^*_{X/Y} j^* J_j \cong \Delta^*_{X/Y} j^* p^* J_{Y/S} = f^* \Delta^*_{Y/S} J_{Y/S} = f^* \Omega_{Y/S}$$

□

Proposition 4. *If $f : X \to Y$ is a morphism of locally finite type between locally Noetherian schemes, and if f is étale at $x \in X$, then X is regular at x iff Y is at $y = f(x)$.*

Proof. From the étaleness of f one gets $\mathfrak{m}_x^l/\mathfrak{m}_x^{l+1} \cong \mathfrak{m}_y^l/\mathfrak{m}_y^{l+1} \otimes_{k(y)} k(x)$ and the dimensions of the local rings are equal to the smallest d such that the dimension of these vector spaces are $O(l^{d-1})$ as $l \to \infty$. It follows that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} =: d$ and therefore X is regular if and only if $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = d$ if and only if $\dim_{k(y)} \mathfrak{m}_y/\mathfrak{m}_y^2 = d$ if and only if Y is regular. □

Proposition 5 (Arcata Def 1.1.)**.** *Let S be an R-algebra of finite type, where R is Noetherian. Then the following are equivalent.*

(A1) If A is a Noetherian R-algebra, $I \subseteq A$ a nilpotent ideal, then in any diagram of solid arrows

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd} \tag{L}$$

there is a unique dotted arrow (a ring homomorphism) making the diagram commute.

(A2) The same condition, but with the sharper assumption $I^2 = 0$.

(A3) The condition (A2) with the sharper assumption that A is a local ring.

(B) S is an étale R-algebra (i.e. S is flat over R and $\Omega_{S/R} = 0$).

(C1) There is a representation $S = R[x_1, \dots, x_n]/(f_1, \dots, f_n)_T$, where $T = R[x_1, \dots, x_n]$, such that the Jacobian determinant $\det(\frac{\partial f_i}{\partial x_j})_{ij}$ maps to a unit in S.

(C2) If $S \cong R[x_1, \dots, x_n]/J$, $J \subseteq T = R[x_1, \dots, x_n]$ is any representation of S as an R-algebra, then there are $g, f_1, \dots, f_n \in T$ such that $V(g) \cap V(f_1, \dots, f_n) = \emptyset$, $J_g = \langle f_1, \dots, f_n \rangle_{T_g}$ and the Jacobian determinant as in (C1) maps to a unit in S.

Proof. (A1)⇒(A2)⇒(A3) is trivial. (A2)⇒(A1) is an induction on the smallest k such that $I^{2^k} = 0$. (A3)⇒(A2): By assumption our lifting problem has local solutions $S \to A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. As S is a finitely presented A-algebra, these come from $S \to A_{\alpha_\mathfrak{p}}$, $\alpha_p \in R$. There are finitely many $\alpha_i = \alpha_{\mathfrak{p}_i}$ such that $\langle \alpha_1, \dots, \alpha_n \rangle_A = A$, and the compositions $S \to A_{\alpha_i} \to A_{\alpha_i\alpha_j}$ and $S \to A_{\alpha_j} \to A_{\alpha_i\alpha_j}$ coincide because this is so after composition with any morphism $A_{\alpha_i\alpha_j} \to A_\mathfrak{q}$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus V(\alpha_i\alpha_j)$, and the map from $A_{\alpha_i\alpha_j}$ to the product of these $A_\mathfrak{q}$ is injective. It is then well-known that there is a unique ring morphism $S \to A$ making all triangles $S \to A \to A_{\alpha_i}$ commute, and it is easy to see that this is the only solution to (L).

(A)⇒(B): One way to show flatness is to consider any presentation $S \cong T/J$, $T = R[x_1, \dots, x_n]$. By induction on n we get commutative diagrams

$$\begin{tikzcd} S \arrow[r, "\lambda_n"] \arrow[dr, dashed, "\lambda_{n+1}"] & T/J^n \\ R \arrow[u] \arrow[r] & T/J^{n+1} \arrow[u] \end{tikzcd}$$

splitting the surjective morphism $\widehat{T} \to S$ where $\widehat{T}$ is the completion of T with respect to J. As our rings are Noetherian, $\widehat{T}$ is a flat T-module, hence a flat R-module, and so is its direct summand S. $\Omega_{S/R} = 0$ follows from

Fact 3. In the situation of (L) assume $I^2 = 0$. Then $\mathcal{D}er(S/R, I)$ acts simply transitively on the set of dottet arrows $\alpha : S \to A$ making (L) commute, provided that such a solution α exists. The action of $\delta \in \mathcal{D}er(S/R, I)$ on α is $\widetilde{\alpha}(s) = \alpha(s) + \delta(s)$.

As by our assumption the set of solutions to (L) is not empty, we have $\mathcal{D}er(S/R, I) = 0$ for all such I. This can be applied to $A = S \oplus M$, $I = M$. Then $I^2 = 0$ and $\mathcal{D}er(S/R, M) = 0$ for any S-module M. Hence $\Omega_{S/R} = 0$.

(B)⇒(C2): Let $T = R[x_1, \dots, x_n]$, $S = T/J$ as in (C2). By the short exact sequence

$$J/J^2 \to \Omega_{T/R} \otimes_T S \to \Omega_{S/R} \to 0$$

and (B), the map $J/J^2 \to \Omega_{T/R} \otimes_T S \cong \bigoplus_{i=1}^n SdX_i$ (sending $f + J^2$ to $((\frac{\partial f}{\partial x_i} + J)dx_i)_i$) must be surjective. Because of this it is possible to choose the f_i in (C2) such that the Jacobian determinant becomes a unit in S (e.g. s.t. the image of f_j is $(\delta_{ij}dX_i)_i$). Let $J' = \langle f_1, \dots, f_n \rangle_T$, $X = \operatorname{Spec} S$, $X' = \operatorname{Spec} S'$, where $S' = T/J'$, $Y = \operatorname{Spec} R$. Repeating the above argument with J' implies $(\Omega_{S'/R})_x = 0$ for all x in the closed subscheme $X \subseteq X'$. Thus $X' \to S$ is unramified at the image of X', and therefore $U \to Y$ is unramified, where $U \subseteq X'$ is some open neighbourhood of the image of X. By Fact 1, $X \to U$ is étale, hence $X \to X'$ is étale and by Fact 1 $X \to X'$ is an open embedding. It is thus possible to choose an element $g \in T$ whose image in $\mathcal{O}_{X'}(X') = T/J'$ equals 1 on the clopen subset $X \subseteq X'$ and 0 on its complement. It is then easy to see that g does what we want.

(C2)⇒(C1): We start with any presentation $\pi : R[x_1, \dots, x_n]/J \xrightarrow{\cong} S$ and apply (C2). With the notations from (C2), $\pi' : R[x_1, \dots, x_{n+1}]/J' \xrightarrow{\cong} S$, where J' is the ideal generated by J and $1 - gX_{n+1}$ and π' sending X_i to $\pi(X_i)$ when $i \leq n$ and $\pi'(x_{n+1})$ is some inverse image of g in S. This presentation does what we want.

(C1)⇒(A2): With nations as in (C1), if A is an R-algebra, then the set of solid arrows $S \to A/I$ making (L) commute is (by $\alpha \mapsto (\alpha(\text{image of } x_i \text{ in } S))_i$) equivalent to the set of solutions of $f_i(x_1, \dots, x_n) = 0$ in A/I. The set of dashed arrows $S \to A$ corresponds in the same way to the set of solutions of $f_i(x_1, \dots, x_n) = 0$ in A. It is well-known (Hensel's lemma) that the solution set in A maps injectively to the solutions in A/I when the Jacobian is a unit in A/I, which it is as it is in $S^\times$. □

Remark 3. This also holds in the non-Noetherian situation, when S/R is of finite presentation.

Proposition 6. *If $X_0 \to X$ is a closed embedding defined by a nilpotent sheaf of ideals, then the functor*

$$\text{Étale } X\text{-schemes} \to \text{Étale } X_0\text{-schemes}, \qquad Y \to Y_0 := X_0 \times_X Y$$

is an equivalence of categories.

Proof. The assertion that this functor defines a bijection on morphisms (i.e. is fully faithful) is easily reduced to the situation where X, Y are affine, in which case it is an immediate consequence of Proposition 5(A). It remains to show essential surjectivity.

Let $X = \operatorname{Spec} R$, $X_0 = \operatorname{Spec} R/I$. If $Y_0 \to X_0$ is an étale morphism with affine Y_0, by Proposition 5(C1) one can choose a representation $Y_0 = \operatorname{Spec} S_0$, $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ There are $\varphi_i \in R_0[x_1, \ldots, R_n]$ such that $\varphi_i \mod I = f_i$ and the Jacobian of the φ_i is a unit in $S = R[x_1, \ldots, x_n]/(\varphi_1, \ldots, \varphi_n)$ because this is so modulo the nilpotent ideal IS. For general étale X_0-schemes one chooses a covering by affine open subsets and by full faithfulness the gluing data module I lift to gluing data for the lifts of these affine étale X_0-schemes. The case of general X is dealt with in the same way, lifting $\pi_0^{-1}(U)$, $\pi_0 : Y_0 \to X_0$ étale, for affine open subsets $U \subseteq X$, and using full faithfulness of the functor to get gluing data for these lifts. □

Remark 4. Such $X_0 \to X$ are examples of universal homeomorphisms, i.e. morphisms $X_0 \to X$ such that $Y_0 = X_0 \times_X Y \to Y$ is a homeomorphism for any X-scheme Y. This condition can be checked by verifying universal injectivity, universal surjectivity, followed by universal closedness or universal openness.

Since for every pair of morphisms $\alpha : A \to S, \beta : B \to S$ of schemes (or locally ringed spaces) the canonical map

$$A \times_S B \to [A] \times_{[S]} [B] = \{(a, b) \mid a \in A, b \in B, \alpha(a) = \beta(b)\}$$

is surjective, any surjective morphism is automatically universally surjective. If $X_0 \to X$ is injective one can show that is is universally injective if and only if for all $x_0 \in X_0$ with image x in X, $k(x_0)/k(x)$ is an algebraic and purely inseparable field extension.

For morphisms of finite type between Noetherian schemes, universal closedness is equivalent to properness. But such morphisms are quasi-finite if they are injective, and if they are also proper they are finite by an easy special case of Zariski's Main Theorem.

Proposition 7. *Proposition 6 also holds when $X_0 \to X$ is a universal homeomorphism (i.e. finite, bijective, $k(x_0)/k(x)$ always algebraic and purely inseparable) of finite type between locally Noetherian schemes.*

Example 2. This can be applied to Frobenius type morphisms, e.g. $F_X = \mathrm{id}_X$, $F_X^*(\varphi) = \varphi^p$ in $\mathcal{O}_X(U)$ if $\operatorname{char}(X) = p$. Another example would be the relative Frobenius $F_{X/\mathbb{F}_q}$ on $X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q$ sending (when X is quasi-projective) all coordinates to their q-th power.

Lemma 2. *Let $f : X \to Y$, $g : Y \to S$ be morphisms locally of finite type between locally Noetherian schemes with f étale, and let $x \in X$. Then $g \circ f$ is étale at x if and only if g is étale at $y = f(x)$.*

Proof. Since f is étale, hence flat, and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ local, $\mathcal{O}_{X,x}$ is a faithfully flat $\mathcal{O}_{Y,y}$-algebra. The if-part is the fact that étaleness is stable under composition. For the "only if"-part, use the fact that $\operatorname{Tor}_q^{\mathcal{O}_{S,s}}(\mathcal{O}_{Y,y}, T) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong \operatorname{Tor}_q^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, T) = 0$ (T any $\mathcal{O}_{S,s}$-module) when $q > 0$ (as gf is flat) and deriving $\operatorname{Tor}_q^{\mathcal{O}_{S,s}}(\mathcal{O}_{Y,y}, T) = 0$ as $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is faithfully flat.

That $\mathfrak{m}_{S,s}\mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$ can also be checked after $- \otimes \mathcal{O}_{X,x}$, as f is étale, $\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ and the desired equality again follows from the fact that gf is étale at x (hence $\mathfrak{m}_{S,s}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$). Trivially, separability of $k(y)/k(s)$ follows from $k(s) \subseteq k(y) \subseteq k(x)$ and $k(x)/k(s)$ separable. □

1.5 The étale topology

Definition 1. Let X be a scheme.

a) Let Et/X be the category of étale X-schemes. The étale topology on that category is the Grothendieck topology for which $S \mathrel{/\!=} U$ if and only if there are étale morphisms (of finite presentation) $U_i \to U$ belonging to S whose images cover U. This site (=category + Grothendieck topology) is called the small étale site $X_{\text{ét}}$

b) The étale topology of all (or all Noetherian) X-schemes is defined in the same way, dropping from a) the condition that $U \to X$ must be étale. This is the big étale site $X_{\text{Ét}}$.

Remark 1. Let $(U_i \to U)_i$ be a family of étale morphisms such that their images cover U and each U_i is covered by its open subsets $W \subseteq U_i$ which are S-small. Then the sieve generated by these $W \to U$ is covering in the sense of definition 1 and contained in S, hence $S \mathrel{/\!=} X$. Therefore technical modifications as in Proposition 1.3.1 are not necessary in this case. The proof that one has a Grothendieck topology is simplified by étale morphisms being open.

Definition 2. A morphism $f : X \to Y$ is called weakly étale if it is flat and $\Delta_{X/Y} : X \to X \times_Y X$ is also flat.

Example 1. Every étale morphism is weakly étale as it is flat and $\Delta_{X/Y}$ is an open embedding.

Theorem 1 (Bhatt,Scholze)**.** *If A is a ring and B a weakly étale A-algebra, there is a faithfully flat weakly étale B-algebra $\widetilde{B}$ such that $\widetilde{B}/A$ is a direct limit of étale A-algebras.*

Remark 2. a) The proétale topology is defined by Proposition 1.3.1 using the class of weakly étale morphisms. One can, for instance, use this to study $H^*(X, \mathbb{Z}_p)$ directly rather than indirectly as $\varprojlim H^*(X, \mathbb{Z}/p^k\mathbb{Z})$. The proof of the crucial results for Weil 1/2 still depend on the SGA 4 results on proper and smooth base change and Poincaré duality.

b) In between the ßetale and the fppf topology there is the syntonic topology where the covering sieves are generated by flat morphisms that are local complete intersections.

c) One could sharpen the condition for $S \mathrel{/\!=} U$ in Definition 1 requiring that for every $x \in U$ there must be $i \in I$ and $\xi \in U_i$ mapping to x under $U_i \to U$ such that $k(\xi)/k(x)$ is trivial. (Then $\operatorname{Spec}\mathbb{Z}$ is covered by $\operatorname{Spec}\mathbb{Z}[i]$ and $\operatorname{Spec}\mathbb{Z}[\frac{1}{5}]$.)

1.6 The Étale Fundamental Group

Definition 1. Let FET_X be the category of finite étale morphisms $\pi : \widetilde{X} \to X$.

Definition 2. A geometric point of a scheme X is a morphism $\mathbf{x} : \operatorname{Spec} K \to X$, where K is an algebraically closed field. The image under $\mathbf{x}$ of the only point of $\operatorname{Spec} K$ is called the support of $\mathbf{x}$, i.e. $\mathbf{x}$ is supported at x if $\mathbf{x}(\operatorname{Spec} K) = x$.

Remark 1. The conidition that K is algebraically closed is sometimes relaxed to being separably closed. We follow 03P0 where K is required to be algebraically closed, which also seems to be mostly followed in SGA. Relaxing algebraically closed to separably closed leads to an essentially equivalent condition but it is a bit more awkward to study lifts of geometric points under finite non-étale surjective morphisms

$Y \to X$.

The category FET_X has cartesion products, equalizers and coproducts.

Definition 3. a) For a geometric point $\mathbf{x} : \operatorname{Spec} K \to X$, let $\mathrm{Fib}_{\mathbf{x}} : \mathrm{FET}_X \to$ (finite Sets) be given by

$$(\pi : \widetilde{X} \to X) \mapsto \{\widetilde{\mathbf{x}} : \operatorname{Spec} K \to \widetilde{X} \mid \mathbf{x} = \pi \widetilde{\mathbf{x}}\}.$$

b) Let $\Pi_1^{et}(X)$ be the category with objects are geometric points of X and morphisms $\mathbf{x} \to \widetilde{\mathbf{x}}$ are functor-isomorphisms $\mathrm{Fib}_{\mathbf{x}} \to \mathrm{Fib}_{\widetilde{\mathbf{x}}}$.

c) For a geometric point $\mathbf{x}$, let $\Pi_1^{et}(X, \mathbf{x})$ be the group of automorphisms of $\mathbf{x}$ in the groupoid $\Pi_1^{et}(X)$.

Remark. a) If one uses the separably closed definition for geometric points, one gets an equivalent category $\Pi_1'(X)$. This is because for every $\mathbf{x} : \operatorname{Spec} K \to X$, K separably closed, one has an algebraic closure $i : K \to \overline{K}$, and $\overline{\mathbf{x}} : \operatorname{Spec} \overline{K} \to \operatorname{Spec} K \to X$. If x is the support of $\overline{\mathbf{x}}$, then

$$\begin{aligned}\mathrm{Fib}_{\overline{\mathbf{x}}}(Y) &\cong \{(y, \lambda) \mid y \in Y \text{ with image } x,\ \lambda \text{ an extension of } k(x) \xrightarrow{\overline{\mathbf{x}}^*} \overline{K} \text{ to } k(y) \to \overline{K}\} \\ &= \{(y, \lambda) \mid y \in Y \text{ with image } x,\ \lambda \text{ an extension of } k(x) \xrightarrow{\overline{\mathbf{x}}^*} \overline{K} \text{ to } k(y) \to K\},\end{aligned}$$

since any λ in $\overline{K}$ has image in K, as $k(y)/k(x)$ is separable. This gives an isomorphism from $\overline{\mathbf{x}} \in \Pi_1^{et}(X)$ to $\mathbf{x} \in \Pi_1'(X)$. Since $\Pi_1^{et}(X)$ is a full subcategory of $\Pi_1'(X)$ by definition, they are equivalent.
b) Note that an equivalent definition of a geometric point is to define is as a triple $(K, x, \mathbf{x})$ where K is an algebraically closed field, $x \in X$ and $\mathbf{x} : k(x) \to K$ a homomorphism.
c) One also has an equivalent subcategory $\Pi_1''(X) \subseteq \Pi_1^{et}(X)$ where objects are geometric points $\mathbf{x} : \operatorname{Spec} K \to X$ such that K is algebraic over the image of $k(x) \to K$. If $\mathbf{x} : \operatorname{Spec} K \to X$ is a geometric point in the sense of definition 2, and $\widetilde{K} \subseteq K$ is the algebraic closure of $\mathbf{x}^*(k(x))$, then there is a unique morphism $\widehat{\mathbf{x}} : \operatorname{Spec} \widetilde{K} \to X$ whose composition with $\operatorname{Spec} K \to \operatorname{Spec} \widetilde{K}$ equals $\mathbf{x}$, and a canonical isomorphism $\mathbf{x} \cong \widehat{\mathbf{x}}$ in $\Pi_1^{et}(X)$ (for similar reasons as in a).

Remark 2. One introduces a Krull topology on the set of morphisms $\mathbf{x} \to \widetilde{\mathbf{x}}$ in Π_1^{et}: A neighbourhood base of a morphism $\gamma : \mathbf{x} \to \widetilde{\mathbf{x}}$ is $\{\Omega_v \mid v \text{ any object of } \mathrm{FET}_X\}$ where

$$\Omega_v = \{\widetilde{\gamma} : \mathbf{x} \to \widetilde{\mathbf{x}} \mid \gamma = \widetilde{\gamma} \text{ on } \mathrm{Fib}_{\mathbf{x}}(V) \to \mathrm{Fib}_{\widetilde{\mathbf{x}}}(V)\}.$$

It is easy to see that $\Pi_1^{et}(X, \mathbf{x})$ is complete with this topology.

Example 1. Let $X = \operatorname{Spec} K$ where K is a field. Then, étale X-schemes are automatically finite (essentially by Hilbert's Nullstellensatz) and up to isomorphism of the form $\operatorname{Spec} A$ where A is a finite-dimensional étale K-algebra. Let $\overline{K}$ be an algebraic closure of K, $K^s \subseteq \overline{K}$ the separable closure of K in $\overline{K}$, $G = \operatorname{Aut}(\overline{K}/K) \cong \operatorname{Gal}(K^s/K)$ equipped with the Krull topology. Let $\mathbf{x}$ denote the geometric point of X given by $\operatorname{Spec} \overline{K} \to \operatorname{Spec} K$.

If Y is an object of FET_X, then $\mathrm{Fib}_{\mathbf{x}}(Y)$ is in canonical bijection with the set of pairs (y, λ) where y is any point of Y and $\lambda : k(y) \to \overline{K}$ any ring homomorphism extending $K \to \overline{K}$. If $\theta \in G$, then θ acts on this set by $(y, \lambda) \mapsto (y, \theta\lambda)$.

One gets a functor $\mathrm{FET}_X \to$ (finite sets with continuous action by G) where the continuity condition is imposed for the Krull topology on G and the discrete topology on the fintie set. This functor is an equivalence of categories with inverse functor sending a finite G-set F to

$$\operatorname{Spec}(\{f : F \to \overline{K} \mid \theta(f(x)) = f(\theta x)\}).$$

It follows that $\Pi_1^{et}(X, \mathbf{x}) \cong G$, canonically.

Remark. Note that for an étale $\operatorname{Spec} K$-scheme the morphism $X \to \operatorname{Spec} K$ is finite if X is quasi-compact.

Theorem 1 (SGA1.V)**.** *Let X be a locally connected locally Noetherian scheme.*

a) We have an equivalence of categories

$$\mathrm{FET}_X \to \mathcal{C}, \quad (\pi : Y \to X) \mapsto (\mathbf{x} \to \mathrm{Fib}_{\mathbf{x}} Y),$$

where $\mathcal{C}$ is the category of functors F from $\Pi_1^{et}(X)$ to the category of finite sets such that

$$F(\mathbf{x}) \times \mathrm{Hom}_{\Pi_1^{et}}(\mathbf{x}, \mathbf{y}) \to F(\mathbf{y}), \quad (f, \gamma) \mapsto F(\gamma)f$$

is continuous, where $F(\mathbf{x})$ and $F(\mathbf{y})$ carry the discrete topology and $\mathrm{Hom}_{\Pi_1^{et}}(\mathbf{x}, \mathbf{y})$ the Krull topology.

b) If, in addition, X is connected, then $\Pi_1^{et}(X)$ is connected (in the sense that it has only one isomorphism class of objects). Thus, if $\mathbf{x}$ is a geometrix point of X, then

$$\mathrm{FET}_X \to (\textit{fin. sets with cont. } \Pi_1^{et}(X, \mathbf{x})\textit{-action}), \qquad Y \mapsto \mathrm{Fib}_{\mathbf{x}} Y$$

is an equivalence of categories.

Remark 3. If X is a $\mathbb{Q}$-scheme, then, an alternative approach to an algebraically defined fundamental group would consider the Tannakian category of locally free coherent $\mathcal{O}_X$-modules with a connecten $\nabla : \mathcal{E}^\vee \to \mathcal{O} \otimes \Omega^1_{X/S}$ of vanishing curvature. This would play a similar role for H^*_{dR} compared with the role played by Π_1^{et} for $H^\bullet(X_{et})$.

Definition 4. A prinicpal G-covering (G a finite group) of X is an object Y of FET_X with a G-action such that the following equivalent conditions hold:

a) $G \times Y = \coprod_{g \in G} Y \mapsto Y \times_X Y$, $(g, y) \mapsto (y, gy)$ is an isomorphism and $Y \to X$ is flat.

b) The sieve on X_{et} or X_{Et} of all X-schemes U such that $U \times_X U \cong G \times U$ is the category of U-schemes with a G-action over X.

Fact 1. Let G be abelian. If X is connected and $\mathbf{x}$ any geometric point, then $\mathrm{Hom}(\Pi_1^{et}(X, \mathbf{x}), G)$ is in canonical bijection with the set of isomorphim classes of principal G-coverings.

Proposition 1 (Kummer theory for Π_1^{et})**.** *Let X be connected, $\zeta \in \mu_n^*(X)$ (i.e. a morphism $X \to \operatorname{Spec} R$, $R = (\mathbb{Z}[T]/(T^n - 1))[(T^d - 1)^{-1} \mid 1 < d < n, d|n]$). In particular, $n \in \mathcal{O}_X(X)^\times$.*

a) If $\mathcal{L}$ is a line bundle on X nad $\lambda \in (\mathcal{L}^{\otimes n})^*(X)$, then the functor

$$(\upsilon : Y \to X) \to (\textit{Sets}), \qquad Y \mapsto \{l \in (\upsilon^* \mathcal{L})(Y) \mid l^n = \nu^* \lambda\}$$

is representable by an object of FET_X which is $\mathbb{Z}/n\mathbb{Z}$-principal for the action $k \mod n : l \mapsto \zeta^k l$, and every $\mathbb{Z}/n\mathbb{Z}$-principal cover can be obtained in this way, giving us an equivalence of groupoids.

b) Thus we have an exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}_X(X)^* \xrightarrow{(\cdot)^n} \mathcal{O}_X(X)^* \to \mathrm{Hom}(\Pi_1^{et}(X, \mathbf{x}), \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Pic}(X) \xrightarrow{n\cdot} \mathrm{Pic}(X)$$

Proposition 2. *Let p be a prime and X a connected scheme over $\mathbb{F}_p$.*

a) *Let F_X denote the absolute Frobenius. If $\mathcal{T}$ is an $\mathcal{O}_X$-torsor on X_{Zar} (i.e. a sheaf of sets on X_{Zar} on which the abelian group $\mathcal{O}_X$ acts transitively) and let $\tau : F_X^*\mathcal{T} \to \mathcal{T}$ be an isomorphism. Then the functor on X-schemes*

$$(\upsilon : Y \to X) \mapsto \{t \in (\upsilon^*\mathcal{T})(Y) \mid (\upsilon^*\tau)(F_Y^* t) = t\}$$

is representable by a principal $\mathbb{Z}/p\mathbb{Z}$-cover of X, giving an equivalence of groupoids.

b) *Thus there is an exact sequence*

$$\begin{aligned} 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathcal{O}_X(X) \to \mathcal{O}_X(X) \to \operatorname{Hom}(\Pi_1^{et}(X, \mathbf{x}), \mathbb{Z}/p\mathbb{Z}) \\ \to H^1(X_{Zar}, \mathcal{O}_X) \to H^1(X_{Zar}, \mathcal{O}_X) \end{aligned}$$

Theorem 2 (Zariski-Nagata). *Let X be regular Noetherian and let $U \subseteq X$ be an open subset such that* $\operatorname{codim}(Y, X) > 1$ *if Y is any irreducible component of $X \setminus U$. Then* $\mathrm{FET}_X \to \mathrm{FET}_U$, $(\xi : \widetilde{X} \to X) \mapsto (\xi^* U \to U)$ *is an equivalence of categories. Thus* $\Pi_1^{et}(U, \mathbf{x}) \cong \Pi_1^{et}(X, \mathbf{x})$ *where $\mathbf{x}$ is any geometric point of U.*

Remark about the proof: If $\widetilde{U} \to U$ is an object of FET_U, then $\widetilde{U} = \underline{\mathrm{Spec}}\mathcal{A}$ where $\mathcal{A}$ is an étale locally free $\mathcal{O}_U$-algebra, then by "basic" commutative algebra and by corollary 1.4.1 and proposition 1.4.2 the main problem is to extend the underlying locally free $\mathcal{O}_U$-module $\mathcal{A}$ to a locally free $\mathcal{O}_X$-module. This is (relatively) trivial when $\dim X = 2$ (then any vector bundle on U extends), but is hard when $\dim X \geq 3$.

1.7 Étale neighbourhoods and stalks of étale sheaves

Definition 1. Let $x \in \operatorname{Spec} k \xrightarrow{\xi} X$ be a geometrc point of X. An étale neighbourhood of x is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\upsilon"] \arrow[dr, "\xi"'] & U \arrow[d, "p"] \\ & X \end{tikzcd}$$

where p is étale. A morphism $U \to \widetilde{U}$ of étale meighbourhoods of x is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "\widetilde{\upsilon}"] \arrow[dr, "\upsilon"'] & \widetilde{U} \\ & U \arrow[u, "\varphi"'] \end{tikzcd}$$

where φ is a morphism of X-schemes.

Remark 1. By Fact 1.4.1, the above φ is automatically étale.

Proposition 1. a) *An U-sieve S in X_{Et} or X_{et} is covering if and only if every geometric point of U has an étale neighbourhood $V \to U$ which is an element of S.*

b) *A U-sieve S in X_{et} covers U if and only if for every geometric point u of U, there is a morphism $V \to U$ in S such that u comes from some geometric point of V.*

c) *If U is Jacobson (e.g. of finite type over a field or over a PID with infinitely many primes), it is sufficient to consider geometric points supported at closed ordinary points.*

Proof. a) Necessity: Let $S \,/\!\!= U$, thus there are étale morphisms $(p_i : V_i \to U) \in S$ such that $U = \bigcup_i p_i(V_i)$. Let $\mathbf{u} : \operatorname{Spec} k \to U$ be a geometric point, given by $u \in U$ and a morphism $k(u) \to k$. There is $i \in I$ such that $u \in p_i(V_i)$. Let $v \in V_i$ with $p_i(v) = u$, then $k(v)/k(u)$ is a finite separable extension. As k is algebraically closed, the morphism $k(u) \to k$ extends to $k(v) \to k$, defining a geometric point $\mathbf{v}$ of V such that $p_i(\mathbf{v}) = \mathbf{u}$ and giving V_i the structure of an étale neighbourhood of U.

Sufficiency: Let every geometric point of U have an étale neighbourhood belonging to S (or, if U is Jacobson, assume this holds for the closed points). For every (closed) point $u \in U$, let $\overline{k(u)}$ be an algebraic closure of $k(u)$ and $\mathbf{u}$ the geometric point defined by this data. For every such u, choose an étale neighbourhood V_u of $\mathbf{u}$ such that $(p_u : V_u \to U) \in S$.Denoting by I the set of all (closed) points of U, we have $U = \bigcup_{u\in I} p_u(V_u) =: \Omega$. Indeed, certainly $I \subseteq \Omega$ as $\mathbf{u}$ lifts to a geometric point of V_u. This finishes the claim unless I is the set of closed points, in which case U was assumed to be Jacobson. In this, case, if $U \neq \Omega$, then $U \setminus \Omega$ contains a closed point, a contradiction. Hence $S \,/\!\!= U$ by definition.

b) Follows from a) in a trivial way. ☐

Definition 2. For a sheaf F (of sets, (abelian) groups, rings) on X_{et} or X_{Et}, an object U of that site and a geometric point $\mathbf{u} \in U$, define

$$F_{\mathbf{u}} = \varinjlim F(V),$$

the direct limit being taken over the category of étale neighbourhoods of $\mathbf{u}$.

Remark. In general, colimits of abelian groups may have left-derived functors like $H(G, M)$ where M is an abelian group on which the arbitrary group G acts. If $\mathcal{G}$ is the category with one object and endomorphisms G, then $\operatorname{colim}_{\mathcal{G}} M = M/\langle gm - m \mid g \in G, m \in M\rangle = H_0(G, M)$. Intuitively, by a direct limit one understands a colimit which can be obtained by "identifying things which map to the same image by applying morphisms to bigger objects of the index category", and which therefore have no higher homology.

Proposition 2. *The colimit in definition 2 is indeed a direct limit: For two arbitrary étale neighbourhoods V_1, V_2 of $\mathbf{u}$, there are an étale neighbourhood W of $\mathbf{u}$ and morphisms $p_i : W \to V_i$ of étale neighbourhoods of $\mathbf{u}$. Moreover, if $p_1, p_2 : W \to V$ are two morphisms of étale neighbourhoods of $\mathbf{u}$, there is a morphism $\omega : \Omega \to W$ such that $p_1\omega = p_2\omega$.*

Remark. It is left as an exercise to show that, as a consequence of this,

$$\operatorname{colim}_{\text{Et. nbhds of }\mathbf{u}} F = \{(V, f)\}/\sim$$

where V runs over étale neighbourhoods of U, $f \in F(V)$ and $(V, f) \sim (\widetilde{V}, \widetilde{f})$ if and only if there are morphisms $p : W \to V, \widetilde{p} : W \to \widetilde{V}$ such that $F(p)(f) = F(\widetilde{p})(\widetilde{f})$ in $F(W)$. For colimits of groups or rings, structure operations are obtained as $[(V, f)] \diamond [(\widetilde{V}, \widetilde{f})] = [W, F(p)(f) \diamond F(\widetilde{p})(\widetilde{f})]$, choosing morphisms $p, \widetilde{p}$ as required. In particular, when the target category is the category of abelian groups the colimit is exact.

Proof. (of Proposition 2). For the first point, let $W = V_1 \times_U V_2$ with the projections $p_i : W \to V_i$ and $\operatorname{Spec} k \to W$ given by (v_1, v_2) and the universal property of $V_1 \times_U V_2$ if the lift $\mathbf{v}_i$ of $\mathbf{u}$ to V is given by $v_i : \operatorname{Spec} k \to V$. For the second point, let $j : \Omega \to W = \ker(W \overset{p_1}{\underset{p_2}{\rightrightarrows}} V)$ be the equalizer. By its universal property, the morphism $w : \operatorname{Spec} k \to W$ defining the lift of $\mathbf{u}$ to W factors over $\operatorname{Spec} k \to \Omega$, giving Ω the structure of an étale neighbourhood and j is a morphism of étale neighbourhoods, provided that $\Omega \to U$ is étale, which follows from j being an open embedding: $\Omega = W \times_{V\times_U V} V$ with morphisms (p_1, p_2) and $\Delta_{V/U}$, and j is the base change of $\Delta_{V/U}$, which is an open embedding since $V \to U$ is étale, hence unramified. ☐

Fact 1. Let F be a presheaf on X_{Et} or X_{et}.

a) F is separated if and only if for every object U of that site, the restriction of F to open neighbourhoods of U is an ordinary separated presheaf, and $F(U) \hookrightarrow F(V)$, for all $V \to U$ surjective étale.

b) F is a sheaf if and only if for the same U as in a), the restriction of F to open neighbourhoods of U is an ordinary sheaf, and for every surjective étale $\pi : V \to U$,

$$\pi^* : F(U) \to \{f \in F(V) \mid \mathrm{pr}_1^* f = \mathrm{pr}_2^* f \text{ in } F(V \times_U V\}$$

is bijective.

Proof. Necessity: The first part of both conditions follows from the fact that every ordinary covering of U generates an étale covering sieve of U. For the second, one uses that V generates a covering sieve S, and

$$\varprojlim_S F = \{f \in F(V) \mid \mathrm{pr}_1^* f = \mathrm{pr}_2^* f \text{ in } F(V \times_U V)\} \tag{$*$}$$

Sufficiency: Let $S \mathrel{/\!\!=} U$, and $f_1, f_2 \in F(U)$ with the same image in $\varprojlim_S F$. As $S \mathrel{/\!\!=} U$, we have $U = \bigcup U_i$ and $(\upsilon_i : V_i \to U_i) \in S$ surjective étale. Then $\upsilon_i^*(f_1|_{U_i}) = \upsilon_i^*(f_2|_{U_i})$, hence $f_1|_{U_i} = f_2|_{U_i}$ by the second part of the condition in a), hence $f_1 = f_2$ by the first part. This finishes part a). For b), we may now assume that F is separated.

First assume that S contains a surjective étale morphism $\upsilon : V \to U$, and let $S' \subseteq S$ be the subsieve generated by υ. By $(*)$ and the separatedness of F (and since $S' \mathrel{/\!\!=} U$), if $\varphi \in \varprojlim_S F$ there is a unique $f \in F(U)$ with image $\varphi|_{S'}$ in $\varprojlim_{S'} F$. Then the image of f in $\varprojlim_S F$ is φ: If $(\pi : W \to U) \in S$, then φ_π and f have the same image in $\varprojlim_{\pi^* S'} F$, hence $\varphi_\pi = \pi^* f$ as F is separated.

Now let S be a general covering sieve. Apply the previous result to $S|_{U_i}$ to get $f_i \in F(U_i)$ such that the image of f_i in $\varprojlim_{S|_{U_i}} F$ equals $\varphi|_{S|_{U_i}}$ for a given $\varphi \in \varprojlim_S F$. Then $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ as both have the same image in $\varprojlim_{S|_{U_i \cap U_j}} F$, and F is separated. By the first condition from b), there is $f \in F(S)$ with $f|_{U_i} = f_i$. Then the image of f in $\varprojlim_S F$ is φ: Let $(\pi : W \to U) \in S$ be any element and denoty by T the sieve of all $\omega : \Omega \to W$ such that $\omega(\Omega) \subseteq \pi^{-1}(U)$, then $T \mathrel{/\!\!=} W$ and the images of φ_π and $\pi^* f$ in $\varprojlim_T F$ coincide as their components at Ω (as above) equals the component of φ at $\Omega \to U$. Thus $\omega_\pi = \pi^* f$. $\square$

Remark. The general way of sheafifying a presheaf (up to set-theoretic difficulties) is

$$(\mathrm{Sh}(F))(X) = \varinjlim_S \varprojlim_{(Y \to X) \in S} F(Y),$$

the outer limit being over the $\subseteq$-poset of covering sieves $S \mathrel{/\!\!=} X$, which are closed under intersection by Fact 1.3.1b).

Proposition 3. *Let F be a presheaf on X_{Et} or X_{et}.*

a) F is separated if and only if for every object U of the étale site under consideration, the following map is injective:

$$F(U) \to \prod_{\mathbf{u}} F_{\mathbf{u}}, \quad f \mapsto (\textit{image of } f \textit{ in } F_{\mathbf{u}})_{\mathbf{u}}$$

b) A morphism of étale sheaves is an isomorphism if and only if it induces an isomorphism on stalks at all geometric points.

c) *Let $F^{\mathrm{Sh}}(U)$ be the subset of $\prod_{\mathbf{u}} F_{\mathbf{u}}$ consisting of all $(f_{\mathbf{u}})_{\mathbf{u}} \in \prod_{\mathbf{u}} F_{\mathbf{u}}$ with the following property: For all geometric points $\mathbf{u}$ of U, there are an étale neighbourhood W of $\mathbf{u}$ in U and $\varphi \in F(W)$ such that for all geometric points $\mathbf{w}$ of W, the image of φ in F_W is $f_{\pi(\mathbf{w})}$, where $\pi : W \to U$ denotes the inclusion. Then F^{Sh} is an étale sheaf.*

d) *The canonical morphism of presheaves $F \to F^{\mathrm{Sh}}$ induces an isomorphism on stalks.*

e) *F is a sheaf if and only if $F \to F^{\mathrm{Sh}}$ is an isomorphism.*

f) *The functor $F \to \mathrm{Sh}(F) := F^{\mathrm{Sh}}$ from presheaves to sheaves is left adjoint to the inclusion functor from sheaves to presheaves.*

Proof. a) Let F be separated and $f_1, f_2 \in F(U)$ with the same image in F^{Sh}. If $\mathbf{u}$ is a geometric point of U, then f_1, f_2 have the same image in $F_{\mathbf{u}}$, hence there is an étale neighbourhood $V_{\mathbf{u}}$ of $\mathbf{u}$ in U such that the images of f_1, f_2 in $F(V_{\mathbf{u}})$ coincide. As the $V_{\mathbf{u}}$ generate a covering sieve of U and F is separated, $f_1 = f_2$.

If $F \to F^{\mathrm{Sh}}$ is injective on sections and $S /\!= U$ and $f_1, f_2 \in F(U)$ with the same image in $\varprojlim_S F$, then since every geometrc point $\mathbf{u}$ of U has an étale neighbourhood V in U belonging to S (hence $f_1|_V = f_2|_V$) and the images of f_i and $f_i|_V$ in $F_{\mathbf{u}}$ coincide, f_1 and f_2 have the same image in $F^{\mathrm{Sh}}(U)$, hence $f_1 = f_2$.

b) Let $\varphi : F \to G$ be a morphism of scheaves inducing an isomorphism on stalks. By a), φ induces injective maps on sections. Let U be an object of our site, $g \in G(U)$ and S the sieve of all U-objects V of our site such that $g|_V$ has a (unique) preimage f_V under $\varphi(V)$. From the uniqueness of f_V one has that $f \in \varprojlim_S F$. If $S /\!= U$, one has $x \in F(U)$ with image f in $\varprojlim_S F$, and $\varphi(x) = g$ as both have the same image in $\varprojlim_S G$. To see that $S /\!= U$, let $\mathbf{u} \in U$ be a geometric point. By assumption, there is $f' \in F_{\mathbf{u}}$ with $\varphi(f') = g_{\mathbf{u}}$. By the definition of stalks, there are an étale neighbourhood V of $\mathbf{u}$ in U and $f \in F(V)$ with image f' in $F_{\mathbf{u}}$, and W of $\mathbf{u}$ in V such that $\varphi(f|_W) = g|_W$. Then $(W \to U) \in S$ and W is an étale neighbourhood of $\mathbf{u}$, hence $S /\!= U$ by proposition 1.

c) We have a morphism $(F^{\mathrm{Sh}})_{\mathbf{u}} \to F_{\mathbf{u}}$ sending, for an étale neighbourhood V of $\mathbf{u}$ and $f \in (F^{\mathrm{Sh}})(V)$, the equivalence class of (V, f) tothe component of f at $\mathbf{u}$. Thus, every section of F^{Sh} is uniquely determined by its image in the stalks. By a), F^{Sh} is separated.

Let $S /\!= U$ and $f \in \varprojlim_S F^{\mathrm{Sh}}$. For every geometric point $\mathbf{u}$ of U, there is an étale neighbourhood V of $\mathbf{u}$ in U such that $(V \to U) \in S$. Let $\varphi^{(V)}_{\mathbf{u}} = (f_V)_{\mathbf{u}}$ be the component of f at the preimage of $\mathbf{u}$ in V turning V into an étale neighbourhood of $\mathbf{u}$. Then $\varphi_{\mathbf{u}} = \varphi^{(V)}_{\mathbf{u}}$ does not depend on the choice of V: By proposition 2, it is sufficient to show $\varphi^{(V)}_{\mathbf{u}} = \varphi^{(W)}_{\mathbf{u}}$ when $W \subseteq U$ are étale neighbourhoods of $\mathbf{u}$. But this follows from $f_V|_W = f|_W$. Moreover, $\varphi \in F^{\mathrm{Sh}}(U)$: If $\mathbf{u}$ is a geometric point of U, it has an étale neighbourhood V in S and the coherence condition for φ at $\mathbf{u}$ follows from the same condition satisfied by f_V at $\mathbf{u}$. Finally, the image of φ in $\varprojlim_S F^{\mathrm{Sh}}$ equals f by construction.

d) At the beginning of the proof of c) we constructed a morphism $\pi : (F^{\mathrm{Sh}})_{\mathbf{u}} \to F_{\mathbf{u}}$, and if $j : F_{\mathbf{u}} \to (F^{\mathrm{Sh}})_{\mathbf{u}}$ is the map on stalks defined by the morphism $F \to F^{\mathrm{Sh}}$, it is clear that $\pi \circ j = \mathrm{id}$. Thus $j\pi\varphi = \varphi$ when $\varphi \in (F^{\mathrm{Sh}})_{\mathbf{u}}$ for which there is $f \in F(U)$ such that φ is the image of f under $F(U) \to F^{\mathrm{Sh}}(U) \to (F^{\mathrm{Sh}})_{\mathbf{u}}$. By the coherence condition for sections of F^{Sh}, such an f can be assumed to exist after replacing U by an étale neighbourhood of $\mathbf{u}$ in U.

e) Immediate from b) and d).

f) If F is a presheaf and G a sheaf, then any morphism $\varphi : F \to G$ comes from the composition $F^{\mathrm{Sh}} \xrightarrow{\varphi^{\mathrm{Sh}}} G^{\mathrm{Sh}} \cong G$. $\square$

Remark 2. If U is Jacobson, it is sufficient to consider closed geometric points, as in proposition 1c).

Remark. a) If $\pi : U \to X$ is étale and $\mathbf{u}$ a geometric point of U, then the category of étale neighbourhoods of $\mathbf{u}$ in U is cofinal in the category of étale neighbourhoods of $\pi(\mathbf{v})$ in X. In particular, $(F|_{U_{et}})_{\mathbf{u}} \cong F_{\pi(\mathbf{u})}$.
b) An alternative formulation of the coherence condition for sections φ of F^{Sh} is: The sieve of all X-schemes $\upsilon : U \to X$ such that there is $f \in F(U)$ such that $\varphi_{\upsilon(u)} = f_{\mathbf{u}}$ for all geometric points $\mathbf{u}$ of U covers X. The equivalence with the formulation in proposition 3c) follows from proposition 1. The fact that this is a sieve follows from the general fact below, applied with $G = F$ and $H = \prod_{\mathbf{u}} F_{\mathbf{u}}$.

Fact. If $\gamma : G \to H$ is a morphism of presheaves and $h \in H(X)$, then

$$\{\upsilon : U \to X \mid \upsilon^* h \in \operatorname{im}(G(U) \to H(U))\}$$

is a sieve over X.

1.8 Henselian Rings and the Henselization

Remark. We consider Hensel's lemma in its easiest form: Let A be complete with respect to I (i.e. $A \cong \varprojlim A/I^n$), $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$ and $\overline{x} \in (A/I)^n$ such that $f_i(\overline{x}) = 0$ for all i and $\det(\frac{\partial f_i}{\partial X_j}(\overline{x})) \in (A/I)^\times$, then there is a unique $x \in A^n$ such that $\overline{x} = x \bmod I$ and $f_i(x) = 0$ for all i. In particular, this holds if $I^k = 0$ for some k.

This applies to the equation $e^2 = e$, i.e. $f(e) = 0$ where $f(T) = T^2 - T$, $f'(T) = 2T - 1$. Then $f | f'^2 - 1$, so $f'(e)$ is a unit when e is idempotent. Thus $\pi_0(\operatorname{Spec}(A/I)) \cong \pi_0(\operatorname{Spec} A)$.

Proposition 1 (Milne, Thm. I.4.3)**.** *Let A be local with maximal ideal $\mathfrak{m}$, $k = A/\mathfrak{m}$. We always write $\overline{a}$ for $a \mod \mathfrak{m}$. Let $X = \operatorname{Spec} A$ and $\xi : \operatorname{Spec} k \to X$. Then the following conditions are equivalent.*

(A1) *For $f \in A[T]$, every decomposition $\overline{f} = g_0 h_0$ in $k[T]$ with $\gcd(g_0, h_0) = 1$ lifts in an essentially unique way to $f = gh$ in $A[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$.*

(A2) *The same as (A1), but f and g_0 as well as g must be monic.*

(B1) *Every finite morphism $Y \to X$ decomposes as $Y \cong \coprod_{i=1}^n Y_i$, $Y_i \cong \operatorname{Spec} B_i$ where B_i is a local A-algebra.*

(B2) *If B is a finite A-algebra and $\overline{B} = B/\mathfrak{m}B$, then for every idempotent ε of $\overline{B}$ there is a unique idempotent e of B such that $\varepsilon = \overline{e}$.*

(B3) *Every separated quasi-finite $Y \to X$ of finite presentation decomposes as $Y \cong \coprod_{i=0}^n Y_i$, where Y_i for $1 \le i \le n$ as in (B1) and $\mathfrak{m} \in \operatorname{Spec} A = X$ is not in the image of Y_0 in X.*

(C1) *Let $\upsilon : U \to X$ be étale and $\iota : \operatorname{Spec} k \to U$ such that $\xi = \upsilon\iota$, then ι can be extended to a morphism $s : X \to U$ such that $\upsilon s = \mathrm{id}_X$.*

(C2) *If $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$ and $\alpha \in k^n$ such that $f_i(\alpha) = 0$ for $1 \le i \le n$ and $\det(\frac{\partial f_i}{\partial X_j}\alpha) \neq 0$, then there is a unique $a \in A^n$ such that $\alpha = \overline{a}$ and $f_i(a) = 0$ for all i.*

Definition 1. A is called *Henselian* if it satisfies these equivalent definitions. If, in addition, k is separably closed, then A is called *strictly Henselian*. In this series of lectures, *strictly local* will denote "local and strictly Henselian".

Remark 1. There is also a related notion of Henselian with respect to an ideal I, which is equivalent to being Henselian in the above definition when applied to local rings and $I = \mathfrak{m}$.

Fact 1. If A is a Henselian local ring then every finite local A-algebra B is Henselian by condition (B2).

Proof. (of proposition 1). We will only consider the Noetherian case. (A1)⇒(A2) is trivial.
(B1)⇔(B2): In the situation of (B1), 1 is a minimal idempotent of B_i and $B_i/\mathfrak{m}B_i$ are local, thus their spectra are connected. If (B2) holds, it can be applied to any finite A-algebra B to the idempotents $\varepsilon_1, \dots, \varepsilon_n \in \overline{B}$, giving the decomposition of the finite-dimensional k-algebra $\overline{B} \cong \bigoplus_{i=1}^n \overline{B}_i$, $\overline{B}_i$ a local k-algebra. If e_i is the lift of ε_i, then $B_i = e_i B$ gives the desired decomposition.
special case of (A2)⇒(B1): Let $Y = \operatorname{Spec} B$ and $B = A[T]/(f)$, where f is monic. If $\overline{f} = \prod_{i=1}^m \varphi_i^{e_i}$ is the prime factor decomposition of $\overline{f}$ in $k[T]$, then by (A1) this lifts to a decomposition $f = \prod_{i=1}^n f_i$ with $\overline{f_i} = \varphi_i^{e_i}$. From the fact $\langle \varphi_i^{e_i}, \varphi_j^{e_j} \rangle = k[T]$ for $i \neq j$ and Nakayama, one derives that $\langle f_i, f_j \rangle = A[T]$. Thus, $B \cong \prod_{i=1}^n B_i := \prod_{i=1}^n A[T]/(f_i)$. But $B_i/\mathfrak{m}B \cong k[T]/\varphi_i^{e_i} k[T]$ is local, hence B_i is local.
(A2)⇒(B2): Let B be a finite A-algebra and $\varepsilon \in \overline{B} = B/\mathfrak{m}B$ an idempotent. We must show the existence of an idempotent $e \in B$ such that $\varepsilon = \overline{e}$ (uniqueness being easy). Obviously there is $e \in B$ such that $\varepsilon = \overline{e}$. As B is a finite (hence integral) A-algebra, there is a monic $f \in A[T]$ such that $f(e) = 0$. Let $C = A[T]/fA[T]$, we have a homomorphism $\gamma : C \to B$ given by $P \bmod f \mapsto P(e)$. The prime factor decomposition of $\overline{f}$ gives $\overline{f} = T^a(1-T)^b \eta$ with $\eta \in k[T]$, $\eta(0) \neq 0 \neq \eta(1)$, and without loosing generality $a, b > 0$. By the Chinese Remainder Theorem there is $\tau \in A[T]$ such that $\overline{\tau} \equiv T \mod T^a(1-T)^b$ and $\overline{\tau} \equiv 0 \mod \eta$. Then $\overline{\tau}$ is an idempotent in $\overline{C} = C/\mathfrak{m}$. Since (B1)⇔(B2) holds for C because of the special case of (A2)⇒(B1) above, without loss τ is an idempotent. Thus $\widetilde{e} = \gamma(\tau)$ is an idempotent in B, and $\overline{\widetilde{e}} = \overline{\gamma(\tau)} = \overline{\tau(e)} = \overline{\tau}(\overline{e}) = \overline{\tau}(\varepsilon) = \varepsilon$ as $T(1-T) \mid (\tau - T)$ and $\varepsilon(1-\varepsilon) = 0$.
(B1)⇒(B3): By Zariski's main theorem, there is a factorization $j : Y \to \overline{Y}$, $\overline{v} : \overline{Y} \to X$ of f with j an open embedding and $\overline{v}$ finite. Let $\overline{Y} = \coprod_{i=1}^n \overline{Y}_i$ as in (B1), with the image of j being $\coprod_{i=1}^n U_i$, $U_i \subseteq \overline{Y}_i$ open, and without loosing generality, $U_i = \overline{Y}_i$, precisely when $1 \leq i \leq m$, for some integer n with $0 \leq m \leq n$. Then $Y_0 = \coprod_{i=m+1}^n U_i$, $Y_i = U_i$ for $1 \leq i \leq m$ gives the desired decomposition.
(B3)⇒(C1): Let (B3) hold. $v : U \to X$ étale, $\iota : \operatorname{Spec} k \to Y$ a lift of $\operatorname{Spec} k \to X$, which we want to extend to a section $X \to U$ of v. Replacing U by an affine open subset containing the image of ι, we may assume that U is affine. Then v is separated and (B3) can be applied with $Y = U$ and without loss $U = \coprod_{i=0}^n Y_i$ as in (B3). Let the image of ι be contained in Y_i, say $i = 1$ (as $\mathfrak{m}$ is not in the image of $Y_0 \to X$). Thus, without loss $U = Y_1$ is the spectrum of a finite local A-algebra B, then $A/\mathfrak{m} \cong B/\mathfrak{m}_B$ (as ι exists). Also B is étale, thus $\mathfrak{m}_B = \mathfrak{m}B$, and $A/\mathfrak{m} \cong B/\mathfrak{m}B$. By Nakayama, $A \to B$ is surjective, i.e. v is a closed embedding. As v is étale it must be an open embedding. As A is local and $\mathfrak{m}$ in the image of v, $v : U \to X$ is an isomorphism.
(C1)⇒(C2): The non-vanishing of the Jacobian implies thet $\mathbb{A}_X^n \supseteq Y = V(f_1, \dots, f_n) \to X$ is étale at the point under consideration, and the assertion follows by applying (C1) to an appropiate open subset of Y.
(C2)⇒(A1): Consider $f = gh$ as a system of equations for the coefficients of g and h. The Jacobian determinant modulo $\mathfrak{m}$ for this system of equations is the resultant of g_0 and h_0, which by our assumption does not vanish. The assertion follows. □

Fact 2. Every complete local ring is Henselian. ((C2) holds.)

Proposition 2. *In the notations of proposition 1, if A is Henselian, $X_0 = \operatorname{Spec} k$, then*

$$\mathrm{FET}_X \to \mathrm{FET}_{X_0}, \quad Y \mapsto Y_0 := Y \times_X X_0$$

is an equivalence of categories.

Proof. For essential surjective, we must lift an étale finite X_0-scheme Y_0 to all of X. Without loosing generality, Y_0 is connected. Then $Y_0 = \operatorname{Spec} l$ where l/k is finite separable. Let λ be a primitive element for l/k, let $P \in A[T]$ such that $\overline{P}$ equals the minimal polynomial of λ over k. Then $Y = \operatorname{Spec} A[T]/P$ is étale over X (using the Jacobian criterion of étaleness) and $Y_0 \cong \operatorname{Spec} l$.

Full faithfulness is by the following. □

Lemma 1. *Let $X_0 \to X$ be a closed embedding such that for every finite X-scheme Y, $Y_0 = Y \times_X X_0 \to X$ defines a bijection of π_0. Then*

$$\mathrm{FET}_X \to \mathrm{FET}_{X_0}, \quad Y \mapsto Y_0$$

is fully faithful.

Proof. Being finite X-schemes, objects of the categories under consideration are separated X-schemes, hence morphisms between them are separated, hence a morphism $\varphi : Y \to \widetilde{Y}$ is given by its graph, which is a closed subscheme of $Y \times_X \widetilde{Y}$, the iimage of the closed embedding $(\mathrm{id}_Y, \varphi) : Y \to Y \times_X \widetilde{Y}$. As its source and target are étale over X, that embedding is étale, hence open. Thus

$$\mathrm{Hom}_{\mathrm{FET}_X}(Y, \widetilde{Y}) \cong \{\text{clopen } U \subseteq Y \times_X \widetilde{Y} \mid U \cong Y\}.$$

Our assumption implies that clopen subsets of $Y \times_X \widetilde{Y}$ correspond bijectively to clopen subsets of $Y_0 \times_{X_0} \widetilde{Y_0} \cong Y_0 \times X\widetilde{Y_0}$. It is now sufficient to show that $U \to Y$ is an isomoprhism onto its (clopen) image if and only $U_0 \to Y_0$ is. But both morphisms are finite flat, and on each connected component of Y the degrees of U/Y and U_i/Y_i coincide and we have an isomorphism iff the degree is 1. □

Corollary 2. *If $\overline{k}$ is an algebraic closure of k and $x : \mathrm{Spec}\,\overline{k} \to \mathrm{Spec}\,k \to X$ the resulting geometric point (in the notation of Proposition 1), assuming A is Henselian, then*

$$\mathrm{Gal}(k^s/k) \cong \Pi_1^{et}(\mathrm{Spec}\,k, x) \xrightarrow{\xi} \Pi_1^{et}(X, x).$$

Corollary 3. *For a Henselian local ring A (with notation as in proposition 1), the following conditions are equivalent:*

a) A is strictly Henselian.

b) Every object of FET_X decomposes as a disjoint union of copies of X.

c) $\Pi_1^{et}(X, x) = 1$.

d) Every surjective étale morphism $U \to X$ has a section.

Proof. (Sketch). d) ⇒ b) is an induction on the degree of the finite étale A-algebra B. b) ⇒ d): By proposition 1, (B3), without loosing generality Y/X is finite. □

Definition 2. Let A be a local ring. The *Henselization* A^h of A is the initial object in the category of Henselian A-algebras, which has local[2] Henselian A-algebras B as objects and local morphisms of local A-algebras as morphisms.

If k is the residue field of A and $\eta : k \to \overline{k}$ a morphism to the separable closure, the *strict Henselization* A^{sh} of A is the initial object of the category with objects Henselian A-algebras B with maps $\eta_B : \overline{k} \to B/\mathfrak{m}_B$ making the obvious diagram commute, and morphisms the morphisms of Henselian A-algebras $\beta : B \to \widetilde{B}$ such that $\eta_B = \eta_{\widetilde{B}} \circ \beta$.

Remark. a) $\widehat{A}$ is an objecct of the category of Henselian A-algebra.
b) By the universal property, there is a unique morphism $A^h \to \widehat{A}$.

[2] B is a local ring and $A \to B$ a local ring morphism

Proposition 3. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k.*

a) A^h exists and can be constructed as the direct limit of localizations of étale A-algebras with residue field k. Moreover, every such direct limit is a Henselization of A if it is Henselian.

b) If A is Noetherian, then $\widehat{A} \cong \widehat{A^h}$, A^h is Noetherian, and $\dim A = \dim A^h$.

c) A^{sh} exists and can be constructed as the direct limit of localizations of étale A-algebras with a morphism $\overline{k} \to B/\mathfrak{m}_B$ compatible with η. Every such direct limit is a strict Henselization of A if it is strictly Henselian.

d) If A is Noetherian, then so is A^{sh}, and $\dim A = \dim A^{sh}$.

e) If k is separable closed, then $A^h \cong A^{sh}$.

Proof. (parts). Existence: Let $X = \operatorname{Spec} A$, $x \in X$ the point given by $\mathfrak{m}$ and

$$A^h = \varinjlim_{(U,u)\in\Lambda_0} \mathcal{O}_U(U) = \varinjlim_{(U,u)\in\Lambda_0^{aff}} \mathcal{O}_U(U) = \varinjlim_{(U,u)\in\Lambda_0} \mathcal{O}_{U,u}$$

where Λ_0 is the category of pairs (U, u) where $v : U \to X$ is an étale X-scheme, $u \in U$ such that $v(u) = x$ and $v^* : k(x) \to k(u)$. is an isomorphisms. Morphisms $(U, u) \to (V, v)$ in this category are morphisms of X-schemes mapping u to v. Λ_0^{aff} is the full subcategory of Λ_0 where the U is affine. Thus, A^h is a direct limit as in a), and for existence it is sufficient to show that such direct limits are Henselizations. Similarly,

$$A^{sh} = \varinjlim O_U(U) \cong \varinjlim \mathcal{O}_{U,u}$$

where the direct limit is over étale neighbourhoods in X of the geometric point given by X and η, and in the third term u is the support of the geometric point. Again it is therefore sufficient to show that A^{sh} is indeed strctly Henselian, and that direct limits as in c) are strict Henselizations if they are strict Henselian.

Henselianness of A^h: As $\varinjlim \mathcal{O}_{U,u}$ is a direct limit of local rings and local ring morphisms, A^h is indeed a local ring. Let $\mathfrak{m}^h$ be its maximal ideal. We verify condition (C1) of proposition 1. So let B be an étale A^h-algebra and let $\mathfrak{n}$ be any maximal ideal of B above $\mathfrak{a}^h$ and such that $k \to B/\mathfrak{n}$ is an isomorphism. As B is of finite presentation, there is U as in the above construction of A^h such that $B \cong A^h \otimes_{\mathcal{O}_{U,u}} \widetilde{B}$ where $\widetilde{B}$ is an étale $\mathcal{O}_{U,u}$-algebra. Replacing X by U does not change the direct limit. Hence, without loosing generality, $B \cong A^h \otimes_A \widetilde{B}$, where $\widetilde{B}$ is an étale A-algebra. If $V = \operatorname{Spec} \widetilde{B}$ and $v \in V$ the image of η, then (V, v) is an object of Λ_0, over which the required section of $\operatorname{Spec} \widetilde{B} \to X$ exist for trivial reasons. Replacing Λ_0^{aff} by the cofinal category Λ_B of pairs (U, u) with a morphism $(V, v) \to (U, u)$, we have $A^h \cong \varinjlim_{\Lambda_B} \mathcal{O}_{U,u}$ and the morphisms $\widetilde{B} \otimes_A \mathcal{O}_{U,u} \to \mathcal{O}_{U,u}$ which exist on Λ_B assemble to a morphism $B \to A^h$ giving rise to the desired section of $\operatorname{Spec} B \to \operatorname{Spec} A^h$.

Initial property: Let $A^h = \varinjlim_\Lambda A_\lambda$ be a colimit as in a). If B is a Henselian A-algebra, then proposition 1, (C1) can be applied to $B \otimes_A A_\lambda$, giving morphisms $B \otimes_A A_\lambda \to B$, which assemble to the desired morphism $A^h \to B$.

The arguments for A^{sh} are similar.

Proof of b) We have $A^h = \varinjlim_\Lambda A_\lambda$ as in a). If $\mathfrak{m}_\lambda$ is the maximal Ideal of A_λ, then $A/\mathfrak{m}^l \cong A_\lambda/\mathfrak{m}_\lambda^l$ for all l by the properties of étale morphisms. Hence $\widehat{A} \cong \widehat{A^h}$ and it follows that $\widehat{A^h}$ is Noetherian. By part a) of the next proposition, A^h is Noetherian if $\widehat{A^h}$ is flat over it, which must be shown by part b) of it. □

We omit d) as it is very difficult, e) is trivial.

Proposition. *a) If R is a ring and S a faithfully flat R-algebra which is Noetherian, then R is Noetherian. b) If R is a ring, M an R-module, then M is flat if and only if for all $(r_i)_{i=1}^n \in R^n$, $(m_i)_{i=1}^n \in M^n$ such that $\sum_{i=1}^n r_i m_i = 0$, there are $m \in \mathbb{N}$ and $(\rho_{ij})_{i=1,\dots,n;j=1,\dots,m}$ such that $\sum_{i=1}^n r_i \rho_{ij} = 0$ for all j and there are $(\mu_j)_{j=1}^m$ such that $m_i = \sum_{j=1}^m \rho_{ij}\mu_j$ for all i.*

Remark 2. If X is a scheme, $x \in X$, $\eta : k(x) \to k(x)^s$ and ξ the geometric point of X given by $k(x) \to k(x)^s \to \overline{k(x)}$, then $\mathcal{O}_{X,x}^{sh}$ (w.r.t. η) is the stack of the étale structure sheaf $U \mapsto \mathcal{O}_U(U)$ at ξ.

Fact 3. Let A be a Noetherian local ring.

a) A^h and A^{sh} are flat over A.

b) A has the property R_k iff A^h does iff A^{sh} does. The same is true for the property S_k.

c) If $\mathfrak{p} \in \operatorname{Spec}(A^h)$, then $\mathfrak{p} \in \operatorname{Ass}(A^h)$ iff $\mathfrak{p} \in \operatorname{Ass}(A)$. The same is true for $\mathfrak{p} \in \operatorname{Ass}(A^{sh}) \subseteq \operatorname{Spec}(A^{sh})$.

d) The fibres of $\operatorname{Spec} A^h \to \operatorname{Spec} A$ and $\operatorname{Spec}^{sh} \to \operatorname{Spec} A$ over $\mathfrak{p} \in \operatorname{Spec} A$ are finite, disjoint unions of spectra of separable field extensions of $k(x)$.

e) If A is universally catenary, then the same holds for A^h and A^{sh}.

f) (Strict) Henselization commutes with direct limits of local ring morphisms.

Remark. a) Except for parts a) and f) in the fact, this seems to depend on the Noetherian assumption. Notice that $\widehat{A} \to A$ might not be flat if A is not Noetherian, so Henselization is a better notion than competion.
b) Let X be locally Noetherian. X is R_k if $\mathcal{O}_{X,x}$ is regular for all $x \in X$ with $\operatorname{codim}(\overline{\{x\}}, X) = \dim \mathcal{O}_{X,x} \leq k$. X is S_k if $\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim \mathcal{O}_{X,x})$ for all $x \in X$. So S_0 always holds and one can see that X is reduced iff X is R_0 and S_1. Serre's normality criterion states that X is normal iff X is R_1 and S_2.
c) Let X be locally Noetherian. X is catenary if the underlying topological space is, and universally catenary iff every X-scheme of finite type is catenary.
d) If A is the local ring of $\operatorname{Spec}(K[X,Y]/(X^2 + X^4 - Y^2))$ at $(0,0)$, then A is a domain while $\operatorname{Spec} \widehat{A}, \operatorname{Spec} A^h$ and $\operatorname{Spec} A^{sh}$ have two minimal prime ideals.

Proposition 4 (Ratliff)**.** *For a local Noetherian ring A, the following are equivalent.*

a) A is universally catenary.

b) $A[T]$ is catenary.

c) If $\mathfrak{p} \in \operatorname{Spec} A$, all irreducible components of $\operatorname{Spec} \widehat{A/\mathfrak{p}}$ have the same dimension as $A/\mathfrak{p}$.

1.9 The Artin Approximation Property

Unless specified, all rings in this section are Noetherian.

Definition 1. Let A be a local ring with maximal ideal $\mathfrak{m}$. We say that A has the *Artin approximation property* (AAP) if the following equivalent conditions hold:

a) If $f_1, \dots, f_n \in A[X_1, \dots, X_m]$ and $x \in \widehat{A}^m$ such that $f_i(x) = 0$ for all i, then for all $l \in \mathbb{N}$ ther is $\xi \in A^m$ such that $f_i(\xi) = 0$ for all i and $x \equiv \xi \bmod \mathfrak{m}^l$.

b) If $F : A\text{-Alg}^{op} \to \text{Set}$ is a functor which commutes with direct limits and $f \in F(\widehat{A})$, $l \in \mathbb{N}$, there is $\varphi \in F(A)$ such that the images of f and φ in $F(A/\mathfrak{m}^l) \cong F(\widehat{A}/\widehat{\mathfrak{m}}^l)$ coincide.

Proof. (of equivalence). For b)⇒a), let $F(B) = \{b \in B^m \mid f_i(b) = 0 \text{ for all } i\}$. So assume a) holds. As Fcommutes with limits, there are an A-algebra B of finite type and a morphism $\beta : B \to \widehat{A}$ and $f_B \in F(B)$ such that $F(B)(f_B) = f$. Let without loss $B = A[X_1, \dots, X_m]/I$ where I is the ideal generated by the f_i. Then β is given by $x \in \widehat{A}^n$ with all $f_i(x) = 0$, $\beta([P]) = P(x)$. Let $\xi \in A^n$ be as in a) and $\widetilde{\beta} : B \to A$, $\widetilde{\beta}([P]) = P(\xi)$, and $\varphi(F(\beta)(f_B)$ does the job. □

Remark 1. Let A have the AAP. Then A is Henselian. If A is reduced (a domain), then $\widehat{A}$ is reduced (a domain): Apply AAP to the equation $X^n = 0$ ($XY = 0$). Further, if A is a domain, then A is algebraically closed in $\widehat{A}$.

Example. a) Let $\in \mathbb{Q}_p$ be transcedental over $\mathbb{Q}$, $A = \{f \in \mathbb{Q}(T) \mid f, f' \in \mathbb{Z}_p\}$ and $B = \{f \in \mathbb{Q}(T) \mid f(x) \in \mathbb{Z}_p\}$. Then $\widehat{A} \to \mathbb{Z}_p[T]/T^2$, $f \mapsto (f(x), f'(x))$ is an isomorphism. By remark 1, A does not have the AAP, nor does A^h.

b) Let K be a field of characteristic $p > 0$ such that $[K : K^p] = \infty$. Consider

$$A = \left\{ \sum_{k=0}^{\infty} f_k T^k \mid [K^p(f_0, f_1, \dots) : K^p] < \infty \right\}.$$

Then $\widehat{A}^p \subseteq A$ but $A^p \subsetneq A$ hence A fails to have the AAP (and the field of quotients of $\widehat{A}$ is a purely inseparable extension of the field of quotients of A.) Examples of this type are due to Nagata and F.K.Schmidt.

Definition 2. Let A be Noetherian.

a) A is *universally Japanese* if for all $\mathfrak{p} \in \operatorname{Spec} A$ and for all finite field extensions L of $k(\mathfrak{p})$, the integral closure of $A/\mathfrak{p}$ in L is a finitely generated A-module.

b) A is a G-ring if is is local and for all $\mathfrak{p} \in \operatorname{Spec} A$, the formal geometric fibre $\widehat{A} \otimes_A \overline{k}(\mathfrak{p})$ is regular.

c) A is excellent if A is universally catenary, all localizations $A_\mathfrak{p}$ are G-groups, and for all A-schemes X locally of finite type, the set $\{x \in X \mid \mathcal{O}_{X,x} \text{ is regular}\}$ is open in X.

Remark. a) If A is local, then A is G-ring ⇔ A^h is G-ring ⇒ A^{sh} is G-ring.
b)Rings of finite type over excellent rings are excellent.
c) Rings of finite type over universally Japanese rings are universally Japanese.
d) Excellent rings are universally Japanese.
e) Fields, $\mathbb{Z}$, and complete local rings are excellent.

Remark 2. If A is local Noetherian, then A is G-ring ⇔ A^h is G-ring ⇔ A^{sh} is G-ring (EGA IV 18.7.3 for A^h, F.V. Kuhl remark at the end of 1.1 for A^{sh} or 07QR for ⇒).

Remark 3. By a result of M. Artin from the late 60s, the Henselization of any localization of any algebra of finite type over a field is an excellent DVR. This was extended to Noetherian local Henselian G-rings by Popescu in the mid 1980s. It was shown by Ratthaus that every local, Noetherian ring with AAP is excellent.

Corollary 1. *Let A be a local G-ring.*

a) A^h is a domain $\iff$ $\widehat{A}$ is a domain.

b) In this case, A^h is the algebraic closure of A in $\widehat{A}$.

Corollary 2. *Let A be a Henselian local ring with residue field K, $S = \operatorname{Spec} A$, $S_0 = \operatorname{Spec} K$. If*

$X \to S$ is proper and $X_0 = X \times_S S_0$, then the functor

$$\mathrm{FET}_X \to \mathrm{FET}_{X_0}, \quad (Y \to X) \mapsto (Y_0)Y \times_X X_0 \cong Y \times_S S_0 \to X_0)$$

is an equivalence of categories. In particular, the inclusion $X_0 \to X$ induces bijections $\pi_0(X) \to \pi_0(X)$ and $\pi_1(X_0, x) \to \pi_1(X, x)$ for any geometric point X of X_0.

Remark 4. a) If you believe in étale homotopy there ought to be a sequence

$$\pi_2^{et}(S, s) \to \pi_1^{et}(X_0, x) \to \pi_1^{et}(X, x) \to \pi_1^{et}(S, s) \to \pi_0(X_0, x) \to \pi_0(X, x) \to \pi_0(S, s).$$

b) The case $X = S$ (or, generally, X/S having relative dimension 0) is proposition 1.8.2.
c) See SGA4, XIII2.1 for a proof in the case of dimension ≤ 1 which does not use Artin approximation.
d) The case of relative dimension ≤ 1 is sufficient for the proof of proper base change.

Remark (A). If A is Noetherian and local Henselian,

$$\{\text{idempotents in } A\} \xrightarrow{\cong} \{\text{idempotents in } K\} \xleftarrow{\cong} \{\text{idempotents in } \widehat{A}\}$$

since every finite A-algebra B is a cartesian product of Henselian local rings, and idempotents in B correspond to idempotents in $\widehat{B}$.

Remark (B). If X is a Noetherian scheme and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, then

$$U = \{x \in X \mid \mathcal{M}_x \text{ is a free } \mathcal{O}_{X,x}\text{-module}\}$$

is an open subset of X and $\mathcal{M}|_U$ is a locally free $\mathcal{O}_U$-module (i.e. a vector bundle). (If $\mathcal{O}_{X,x}$ is complete, this is the case iff $\mathcal{M}_x/\mathfrak{m}_x^k\mathcal{M}_x$ is free for all k.)

Proof. (of corollary 2).
Step 1: The assertion about π_0 is shown first, and it implies full faithfulness for the functor under consideration by lemma 1.8.1. It is sufficient to show that intersection with X_0 defines a bijection between clopen subsets of X and X_0. The two sides of this bijection are in canoncial bijection with the idempotents in $\mathcal{O}_X(X)$ and $\mathcal{O}_{X_0}(X_0)$, respectively. Let $S_n = \operatorname{Spec} A/\mathfrak{m}_A^n$, $X_n = X \times_S S_n$, then $X_n \xrightarrow{\cong} X$ as topological spaces, hence

$$\{\text{idempotents in } \mathcal{O}_{X_n}(X_n)\} \xrightarrow{\cong} \{\text{idempotents in } \mathcal{O}_{X_0}(X_0)\}.$$

Hence the same is true for the idempotents in $\varprojlim \mathcal{O}_{X_n}(X_n) = \mathcal{O}_X(X) \otimes A\widehat{A} = \widehat{\mathcal{O}_X(X)} = \widehat{B}$ for $B = \mathcal{O}_X(X)$ and the result follows from remark A.

Step 2: Essential surjectivity when A is complete.
If $Y_0 \to X_0$ is an étale covering, then it extends to étale coverings $Y_n \to X_n$ which are unique up to unique isomorphism, hence form a projective system of finite X-schemes $X_n = \operatorname{Spec} \mathcal{A}_n$ (where $\mathcal{A}_n$ is a coherent $\mathcal{O}_X$-algebra), and $\mathcal{A}_n|_{n-1} \cong \mathcal{A}_{n-1}$, canonically. By Grothendieck's existence theorem, there is a coherent algebra $\mathcal{A}$ over $\mathcal{O}_{\widehat{X}}$, $\widehat{X} = X \times_S \widehat{S} \cong X$ such that $\mathcal{A}|_{X_n} \cong \mathcal{A}_n$.

The subset $U \subseteq X$ where $\mathcal{A}$ is locally free free is open and contains X_0 by remark B. Since $X \to S$ is closed and it must not contain the closed point of S, it is empty and $U = X$. Étaleness follows from the fact that the trace bilinear form $\mathcal{A} \times \mathcal{A} \to \mathcal{O}_X$ is non-degnerate over X_0. But the set U of $x \in X$ where it is non-degenerate is open, hence $U = X$ for the same reason as before.

Remark (C). Let A be a ring, X an A-scheme of finite type, and $X_B = X \times_{\operatorname{Spec} A} \operatorname{Spec} B$, where B is any A-algebra. Then the functor

$$(A\text{-algebras}) \to (\text{Sets}), \qquad B \mapsto \{\text{isomorphism classes of } \mathrm{FET}_{X_B}\}$$

commutes with direct limits.

Step 3: By step 2 and remark C and Popescu, the functor is essentially surjective when A has the AAP.

Step 4: The general case can be reduced to this because by remark C (and related results), every étale covering of X_0 already comes from some étale covering of $\mathcal{X}_0$, where $X \cong \mathcal{X} \times_{\mathcal{T}} S$, $\mathcal{X} \to \mathcal{T}$ proper where $\mathcal{T}$ is the spectrum of the Henselization $\mathcal{A}$ of the localization of a $\mathbb{Z}$-algebra of finite type, such that A is a local $\mathcal{A}$-algebra. $\square$

Corollary 3. *Let S, S_0, X, X_0 be as in corollary 2, and let the relative dimension (i.e. the maximal dimension of the fibres of $X \to S$) be at most 1. Then $\operatorname{Pic} X \to \operatorname{Pic} X_0$ is surjective.*

Proof. Step 1: Let A be complete. It is well-known from the relation between H^1 and torsors that $\operatorname{Pic} X \cong H^1(X_{\text{Zar}}, \mathcal{O}_X^\times)$. Let $i_{n,n+1} : X_n \to X_{n+1}$ be the embedding. We have a short exact sequence of sheaves on $(X_{n+1})_{\text{Zar}}$: $0 \to J \xrightarrow{f \mapsto 1+f} \mathcal{O}_{X_{n+1}}^\times \to i_{n,n+1}\mathcal{O}_{X_n}^\times \to 0$, where the underlying topological spaces of X_n and X_{n+1} are the same. Hence

$$H^1(X_{n+1}, J) \to \underbrace{H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^\times)}_{=\operatorname{Pic}(X_{n+1})} \to \underbrace{H^1(X_{n+1}, \mathcal{O}_{X_n}^\times)}_{=\operatorname{Pic}(X_n)} \to H^2(X_{n+1}, J) = 0.$$

This shows the surjectivity of the restriction $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$. If $[\mathcal{L}_n] \in \operatorname{Pic}(X_n)$, it is therefore possible by induction on n to find line bundles $\mathcal{L}_n$ on X_n with isomorphisms $\mathcal{L}_{n+1}|_{X_n} \cong \mathcal{L}_n$. Using Grothendieck existence and showing local freeness of the result in the same way as yesterday, these $\mathcal{L}_n$ assemble to a vector bundle $\mathcal{L}$ on X, showing the desired result.

Step 2: Since $B \to \operatorname{Pic} X_B$ is a functor (A-algebras) $\to$ (sets) which commutes with direct limits, the result follows from Step 1 and Popescu's theorem if A is excellent.

Step 3: In general, there are a Henselian local ring $\mathcal{A}$ with a local morphism $\mathcal{A} \to A$ and an ($\mathcal{S} = \operatorname{Spec} \mathcal{A}$)-scheme $\mathcal{X}$ with an isomorphism $X \cong \mathcal{X} \times_{\mathcal{S}} S$ and a line bundle Λ on $\mathcal{X}_0$ and an isomorphism of $\mathcal{L}$ and the pull-back of Λ to X, where it is the Henselization of a localization of some $\mathbb{Z}$-algebra of finite type. By step 2 (where Artin's result applies to $\mathcal{A}$), there is a line bundle $\widetilde{\Lambda}$ on $\mathcal{X}$ such that $\widetilde{\Lambda}|_{\mathcal{X}_0} \cong \Lambda$ and the isomorphism class of its pullback to X gives the desired element of $\operatorname{Pic}(X)$. $\square$

Remark 5. See Arcata IV.4.1 for a proof without AAP and working when $X \to S$ is separated of relative dimension ≤ 1.

1.10 Direct and inverse limits of étale sheaves

Definition 1. Let $f : X \to Y$ be a morphism of shemes and F a presheaf on X_{et}. Let f_*F be the presheaf on Y_{et} given by $(f_*F)(U) = F(X \times_Y U)$.

Remark 1. This also works for the large étale sites, but Noetherianness may pose a technical problem.

Fact 1. If F is a sheaf on X_{et}, then f_*F is a sheaf on Y_{et}.

Proof. By Fact 1.7.1 it is sufficient to show that f_*F defines a Zariski sheaf on all objects of Y_{et} and that

$$(f_*F)(U) \to (f_*F)(V) \rightrightarrows (f_*F)(V \times_U V)$$

for every surjective étale morphism $V \to U$. The first follows from the easy observation that for every object U of Y_{et} and $W \subseteq U$ open, $(f_*F)(W) = F$(preimage of W in $U \times_Y X$) which as a functor of W is a Zariski sheaf. Similarly, the exact sequence is identified with

$$F(U \times_Y X) \to F(V \times_Y X) \rightrightarrows F((V \times_U V) \times_Y X) \cong F((V \times_Y X) \times_{U \times_Y X} (V \times_Y X))$$

and exactness of this sequence of sets follows from that of F. $\square$

Remark. If F is separated, then so is f_*F, for similar reasons.

Example 1. Let $x \in X$ and $x = \operatorname{Spec} K$ where K is a separably closed field extension of the residue field $k(x)$ of X at x. If $i : x \to X$ is the morphism given by this data, the category of sheaves on x_{et} is equivalent to the category of sets (abelian groups, rings, etc.) and we have a sheaf i_*G for every set (abelian group, ...) given by $(i_*G)(U) = \prod G$ where the product is taken over lifts $x \to U$ of $x \to X$.

Definition 2. Let $f : X \to Y$ be a morphism of schemes, G a presheaf on Y_{et} and U an object of X_{et}, let $\mathcal{C}_U$ be the category of objects V of Y_{et} with a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and morphisms are morphisms $\widetilde{V} \to V$ (which are automatically étale) such that

$$\begin{tikzcd} \widetilde{V} \arrow[rr] & & V \\ & U \arrow[ul, "\widetilde{\varphi}"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes. Let

$$(f^{\#}G)(U) = \varinjlim_{(V,\varphi)\in\mathcal{C}_U} G(V).$$

If G is a sheaf, let f^*G be the sheafification of $f^{\#}G$.

Remark 2. a) The index category $\mathcal{C}_U$ is directed in the sense that it has the property of proposition 1.7.2, the proof of which extends to this case. Therefore, the colimit over $\mathcal{C}_U$ is indeed a direct limit and $f^{\#}$ is exact on presheaves of abelian groups.
b) If $\mathbf{x} : x = \operatorname{Spec}(k) \xrightarrow{\xi} X$ is a geometric point, then $(\xi^{\#}G)(x) \cong G_{\mathbf{x}}$ essentially by definition.
c) We have an adjunction

$$\operatorname{Hom}_{\mathrm{Presh}_{X_{et}}}(f^{\#}F, G) \cong \operatorname{Hom}_{\mathrm{Presh}_{Y_{et}}}(F, f_*G)$$

making $f^{\#}$ left adjoint to f_*. The bijection sends a morphism $\tau : f^{\#}F \to G$ to the morphism $F \to F_*G$ defined on V as $F(V) \to (f^{\#}F)(V \times_Y X) \xrightarrow{\tau} G(V \times_Y X) = (f_*G)(V)$. Conversely, given $t : F \to f_*G$, define τ one the image of $\varphi \in F(V)$ under $F(V) \to f^{\#}F(U)$ as the image of $t(\varphi) \in f_*G(V)$ under $f_*G(V) \cong G(X \times_Y V) \to G(U)$. The proof that these constructions are inverse to each other is left as an exercise.
d) Since sheafification is left adjoint to the embedding from sheaves to presheaves and by fact 1, we see that $f^* : \mathrm{Sh}_{Y_{et}} \to \mathrm{Sh}_{X_{et}}$ is left adjoint to $f_* : \mathrm{Sh}_{X_{et}} \to \mathrm{Sh}_{Y_{et}}$.
e) By c) and d) and $(gf)_* \cong g_*f_*$, one also gets $(gf)^{\#} \cong f^{\#}g^{\#}$ anf $(gf)^* \cong f^*g^*$ (canonically).
f) In the situation of b), ξ^*F is the sheaf on $\mathbf{x}_{et}$ uniquely (up to isomorphism) determined by $(f^*G)(x) \cong G_{\mathbf{x}}$.
h) f^* is exact: $(f^*G)_{\mathbf{x}} \cong G_{f(\mathbf{x})}$ by e),f), and exactness of sequences of étale sheaves can be checked on stalks (see section 1.7).
i) The morphism $f^*G(U) \to \{\Gamma = (\gamma_{\mathbf{y}} \in \prod_{\mathbf{y}} G_{f(\mathbf{y})} \mid$ the sieve S of $\upsilon : \widetilde{U} \to U$ such that $\upsilon^*\Gamma$ is in the image of the morphism $(f^{\#}G)(\widetilde{U}) \to \prod_{\mathbf{y}} G_{f(\mathbf{y})}$ covers $Y\}$ is an isomorphism by f) and the description of sheafification.

Proposition 1. *Let $(\pi_{\alpha\beta} : X_\alpha \to X_\beta)_{\beta\geq\alpha}$ be an inverse system of affine morphisms between quasi-compact quasi-separated schemes, and let $X = \varprojlim X_\alpha$ with projections $\pi_\alpha : X \to X_\alpha$. For a fixed α,* *let F_α be a sheaf on $(X_\alpha)_{et}$, and $F = \pi_\alpha^* F_\alpha$, $F_\beta = \pi_{\alpha\beta}^* F_\alpha$ its pullbacks to F and F_β for $\beta > \alpha$. Then*

$$\varinjlim_{\beta \geq \alpha} F_\beta(X_\beta) \to F(X), \quad F_\beta(X_\beta) \mapsto \pi_\beta^* F_\beta(X) \cong \pi_\alpha^* F_\alpha(X)$$

is a bijection.

Remark 3. This also holds for direct/inverse limits over categories which are filtering in the sense of proposition 1.7.2, because the proof generalizes, and also because the case of direct or inverse limits over arbitrary filtering categories can be reduced to limits over posets. (see 0032) In this case, $\varinjlim_{\beta\geq\alpha}$ has to be replaced by the colimit of the comma category over α.

Remark. Morally, the reason why the above proposition holds is that every element of $F(X)$ can be written using only finitely many coefficients (sections of $\mathcal{O}_X$ on affine open subsets of X), which already come from some X_β, and sections of F_α. Moreover, all needed relations between the "coefficients" which hold on X already hold on some X_β.

Excursion on projective limits of schemes

It is easy to see that an inverse limit of $\operatorname{Spec} A_\alpha$ exists, and $\varprojlim \operatorname{Spec} A_\alpha \cong \operatorname{Spec} \varinjlim A_\alpha$. (the right-hand side enjoys the universal property of the limit, by the adjointness of Spec and $\mathcal{O}_-(-)$.) More generally, $\varprojlim \underline{\operatorname{Spec}} \mathcal{A}_\alpha \cong \underline{\operatorname{Spec}} \varinjlim \mathcal{A}_\alpha$ exists, if the A_α are quasi-coherent $\mathcal{O}_S$-algebras. Thus, limits as in Proposition 1.10.1 exist, and all projections are affine morphisms. Moreover,

$$\big(\varprojlim X_\alpha\big) \times_S \widetilde{S} \cong \varprojlim (X_\alpha \times_S \widetilde{S})$$

for inverse systems of S-schemes and morphisms. (recall 0032)

From the affinity of π_α, it follows that X is quasi-compact (resp. qcqs) if this holds for some X_α. Further, the following hold:

(α) $|\varprojlim X_\alpha| = \varprojlim |X_\alpha|$ is a homeomorphism.

(β) $\varprojlim X_\alpha \neq \emptyset$ if all X_α are quasi-compact and all $X_\alpha \neq \emptyset$.

From now on, unless mentioned otherwise, the X_α are assumed to be quasi-compact and quasi-separated, and as before, the transition morphisms are affine.

(γ) Every quasi-compact quasi-separable scheme X can be written as a limit $X = \varprojlim X_\alpha$, where the X_α are of finite type over $\mathbb{Z}$.

(δ) If F is a vector bundle over X, then $F \cong \pi_\alpha^* F_\alpha$ for some α and some vector bundle F_α on X_α, and $\mathcal{H}om_{\mathcal{O}_X}(\pi\alpha^* F, \pi_\alpha^* E_\alpha) = \varinjlim_{\beta\geq\alpha} \mathcal{H}om_{\mathcal{O}_{X_\beta}}(\pi_{\beta\alpha}^* F_\alpha, \pi_{\beta\alpha}^* E_\alpha)$.

(ε) If X is quasi-affine, then X_α is quasi-affine for some α (hence X_β is also quasi-affine for $\beta \geq \alpha$).

(η) If X is affine, then X_α is affine for some α (hence X_β is affine for $\beta \geq \alpha$).

(ζ) If $U \subseteq X$ is quasi-compact then $U = \pi_\alpha(U_\alpha)$ for some quasi-compact open $U_\alpha \subseteq X_\alpha$, and $U = \varprojlim \pi_{\beta\alpha}^{-1} U_\alpha$. If $\pi_\alpha^{-1} U_\alpha = \pi_\alpha^{-1} V_\alpha$ for $U_\alpha, V_\alpha \subseteq X_\alpha$ quasi-compact open, then there is $\beta \geq \alpha$ such that $\pi_\beta^{-1}(U_\alpha) = \pi_\beta^{-1}(V_\alpha)$.

(θ) If $Y \to X$ is of finite presentation, then $Y = \varprojlim_{\beta\geq\alpha} Y_\beta$ with $Y_\beta = X_\beta \times_{X_\alpha} Y_\alpha$, Y_α/X_α of finite presentation. Also, if $Z = \varprojlim_{\beta\geq\alpha} Z_\beta$, $Z_\beta = X_\beta \times_{X_\alpha} Z_\alpha$, Z_α/X_α of finite presentation, then $\operatorname{Hom}_X(Y, Z) \xleftarrow{\cong} \varinjlim_{\beta\geq\alpha} \operatorname{Hom}_{X_\beta}(Y_\beta, Z_\beta)$. Thus, by ($\gamma$), every finitely presented morphism between quasi-compact quasi-separated schemes is a limit of finite type morphisms between finite type $\mathbb{Z}$-schemes.

(ι) If Y/X is proper in (θ), then Y_α/X_α can be taken to be proper.

(κ) If f is étale then f_β is étale for some $\beta \geq \alpha$.

(λ) If f has one of the following properties, so does f_β for some $\beta \geq \alpha$: isomorphism, monomorphism, (open, closed) immersion, surjective, radical, (quasi)affine, (quasi)projective

For instance, consider remark C in section 1.9. Take $X \to \operatorname{Spec} A = S$ proper, $Y \to X$ an étale covering. Take $A = \varinjlim A_\alpha$, the A_α Henselizations of $\mathbb{Z}$-algebras of finite type. By (θ), without loosing generality, $X = X_\alpha \times_{S_\alpha} S$ where X_α/S_α is proper, and $Y = Y_\alpha \times_{X_\alpha} X$ for $Y_\alpha \to X_\alpha$ proper. By (λ), without loss, $Y_\alpha \to X_\alpha$ affine. By (κ), without loosing generality, $Y_\alpha \to X_\alpha$ is étale, i.e. $Y_\alpha \in \mathrm{FET}_{X_\alpha}$.

Proof. (of proposition 1). We assume that X is Noetherian to avoid non-quasi compact opens. The general case is dealt with in a similar way. One first proves the result for presheaves. In this case, the objects of $\mathcal{C}_{X,\pi_\alpha}$ can be assumed quasi-compact (as such objects are cofinal). Then the object already comes from an object in $\mathcal{C}_{X_\beta,\pi_{\beta\alpha}}$, and the assertion follows easily.

Then one shows injectivity. This follows from similar principles, using the fact that every covering sieve of X is generated by finitely many étale $U \to X$, which come from $U_{i,\alpha} \to X_\alpha$ for some α. Surjectivity then is similar, using that it is sufficient to show the conditions of being an element of $\varprojlim_S \pi_\alpha^* F_\alpha(U)$ for the projections $U_i \times U_j \to U_i, U_j$. □

If $f : X \to Y$ is a morphism and $\mathbf{y}$ a geometric point of Y, then

$$(f_* F)_{\mathbf{y}} = \varinjlim_{U \text{ etale nbhd.}} \pi^*_{U \times_Y X \to X} F(U \times_Y X) = (\pi^* F)(X_{\mathbf{y}})$$

by proposition 1 with $\pi : X_{\mathbf{y}} = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\mathbf{y}} \to X$.

Corollary 1. *If $f : X \to Y$ is a morphism, $\mathbf{y}$ a geometric point of Y, $\pi : \mathcal{O}_{X,\mathbf{y}} := X \times_Y \operatorname{Spec} \mathcal{O}_{Y,\mathbf{y}} \to X$, then $(f_* G)_{\mathbf{y}} \cong (\pi^* G)(X_{\mathbf{y}})$.*

When f is finite (say a closed immersion), then $X_{\mathbf{y}}$ can be replaced by the geometric fibre $f^{-1}(\mathbf{y}) = X_\times Y \operatorname{Spec}(k)$, where $\mathbf{y} : \operatorname{Spec} k \to Y$. This holds for the following reasons.

Fact 2. If $S = \operatorname{Spec} A$ where A is strictly Henselian and $\mathbf{s}$ is a geometric point of S given by an algebraically closed extension of the residue field of A, then $G(S) \to G_{\mathbf{s}}$ is a bijection, for all sheaves G on S_{et}.

Proof. As for every étale morphism $\widetilde{S} \to S$, every lift $\operatorname{Spec} k \to \widetilde{S}$ of $\mathbf{s}$ comes from a section $S \to \widetilde{S}$, S is cofinal in the category of étale neighbourhoods of $\mathbf{s}$. □

If $X \to Y$ is finite, $X_{\mathbf{y}} \to \operatorname{Spec} \mathcal{O}_{Y,\mathbf{y}}$ is finite, hence $X_{\mathbf{y}}$ is a classical union of spectra of stricly Henselian local finite $\mathcal{O}_{Y,\mathbf{y}}$-algebras $X_{\mathbf{y}} = \coprod_{\text{lifts } \mathbf{x} \text{ of } \mathbf{y}} \operatorname{Spec} O_{X,\mathbf{x}}$, and combining fact 1 with corollary 1 gives

Corollary 2. *If $\pi : X \to Y$ is finite, then*

$$(\pi_* F)_{\mathbf{y}} = \bigoplus_{\text{lifts } \mathbf{x} \text{ of } \mathbf{y} \text{ under } \pi} F_{\mathbf{x}}.$$

In particular, π_* is exact for étale sheaves of abelian groups.

Corollary 3. *If in addition, π is a universal homeomorphism, then π_* and π^* are equivalences of categories inverse to each other. (this also follows from proposition 1.4.7)*

2 Cohomology

2.1 Definition and Basic Properties

Proposition 1. *Let X be any scheme.*

(a) The category of sheaves of abelian groups on X_{et} is an abelian category with enough injective objects.

(b) If $j : U \to X$ is étale and F an injecctive sheaf of abelian groups on X_{et}, then j^*F is an injective sheaf of abelian groups on U_{et}.

Proof. a) One can show that this category is a Grothendieck category, from which the claim follows. Instead, we give a more down-to-earth proof: If (J_x) is a collection of injective abelian groups, summing over all (sufficiently many) geometrix points $\iota_x : x \to X$, then $(\iota_x)_* J_x$ (the skyscraper sheaf) is injective (as $(\iota_x)_*$ has an exact left adjoint ι_x^*), and so is $\prod_x (\iota_x)_* J_x$. If F is a sheaf of abelian groups we may choose embeddings $F_x \to J_x$ with J_x as before. Then $F \to \prod_x (\iota_x)_* J_x$ is a monomorphism.

b) The functor j^* has an exact left adjoint $j_!$ and thus preserves injectivity. Again, we provide another, more concrete proof, at least when j is quasi-finite: For the sheaf considered in the proof of (a), we have

$$j^* \prod_x (\iota_x)_* J_x \cong \prod_x \coprod_{\substack{\text{lifts of } x \text{ to} \\ \text{geom. points } y \text{ of } U}} (\iota_y)_* J_x,$$

where the right-hand side is injective. Then one uses lemma 1 below. □

Fact. If $L : \mathcal{A} \leftrightarrow \mathcal{B} : R$ is an adjoint pair of functors between abelian categories, then L preserves projectivity if R is exact and R preserves injectivity if L is exact.

Lemma 1. *Let $\mathcal{X}$ be a class of objects of an abelian category $\mathcal{A}$, such that for every object F there is a monomorphism $F \to X$ with $X \in \mathcal{X}$ and such that every direct summand of an element of $\mathcal{X}$ is in $\mathcal{X}$. Then $\mathcal{X}$ contains all injective objects I.*

Proof. By assumption there is a mono $I \to X$ with $X \in X$, this splits when I is injective, and then $I \in \mathcal{X}$ by assumption. □

Remark. For the proof of proposition 1b), $\mathcal{X} = \{F \mid j^*F \text{ is injective }\}$.

Proposition. *Let $j : U \to X$ be étale, then j^* has a left-adjoint $j_!$ which is exact.*

Proof. For simplicity, we only consider the case where j is quasi-compact and thus quasi-finite. One first constructs a left adjoint $j_\#$ of $j^\#$ by

$$(j_\# F)(V) = \coprod_{\substack{\text{factorizations of} \\ V \to X \text{ over } U}} F(V).$$

Then sheafification of $j_\# F$ yields $j_! F$ and $(j_! F)_x = \bigoplus F_y$, where the direct sum is taken over all factorizations y of x over $U \to X$. This shows that this is exact. □

Remark. If F has a derived functor $RF : D^+\mathcal{A} \to D^+B$, then $\mathbb{R}^p F \mathcal{X}^*$ (for a cochain complex $\mathcal{X}^*$) bounded from below) is hypercohomology in the sense of Grothendieck, and $\mathbb{R}^p F(\ldots \to 0 \to X \to 0 \to \ldots) = R^p F X$ is the classical derived functor, which is characterized by a universal property in which it is on the left.

Definition 1. We introduce the following derived functors on $\mathrm{Sh}_{\mathtt{Ab}}(X_{et})$:

(a) $H^p(X_{et}, F)$ of $F \Rightarrow F(X)$.

(b) $R^p f_*$ for any morphism $f : X \to Y$ of sheaves.

(c) $R^p \zeta_{X,*}$ of the functor $\zeta_{X,*} : \mathrm{Sh}_{\mathtt{Ab}}(X_{et}) \to \mathrm{Sh}_{\mathtt{Ab}}(X_{\mathrm{Zar}})$ restricting an étale sheaf F to open subsets of X.

Proposition 2. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms of schemes. Then we have Leray spectral sequences*

$$E_2^{p,q} = R^p g_* R^q f_* F \Rightarrow R^{p+q}(gf)_* F$$
$$E_2^{p,q} = H^p(Y_{et}, R^q f_* F) \Rightarrow H^{p+q}(X_{et}, F)$$
$$E_2^{p,q} = H^p(X_{Zar}, R^q \zeta_{X,*} F) \Rightarrow H^{p+q}(X_{et}, F)$$
$$E_2^{p,q} = R^p \zeta_{Y,*} R^q f_{et,*} F \Rightarrow R^{p+q}(\zeta_{Y,*} f_{et,*}) F = R^{p+q}(f_{Zar,*} \zeta_{X,*}) F \Leftarrow R^p f_{Zar,*} R^q \zeta_{X,*} F$$

which each correspond to commuting (up to canonical iso) diagrams of the respective derived functors in the derived setting.

Proof. These are all special cases of the Grothendieck spectral sequence for the composition of two functors. One has to check that the first functor sends injectives to acyclics: f_* and $\zeta_{X,*}$ preserve injectivity since they have an exact left adjoint. □

By corollary 1.10.2 we have

Fact 1. If $f : X \to Y$ is finite, then $R^p f_* F = 0$ when $p > 0$. Thus the Leray spectral sequences involving f collapse to

$$H^p(Y_{et}, f_* F) \cong H^p(X_{et}, F), \quad R^p(gf)_* F \cong (R^p g_*) f_* F, \quad R^p(fh)_* \cong f_* R^p h_* F$$

2.2 Relation to Galois Cohomology

Remark 1. If $\xi : \mathbf{x} = \operatorname{Spec} K \to X$ is a geometric point and $k(x)^{\mathrm{sep}}$ be the separable closure of $k(x)$ in K. If $\operatorname{Spec} K \xrightarrow{v} U \to X$ is an étale neighbourhood of ξ, supported at $y \in U$, then $v^* k(y)$ is contained in $k(x)^{\mathrm{sep}}$. Hence $G = \mathrm{Gal}(k(x)^{\mathrm{sep}}/k(x))$ acts on the category of étale neighbourhoods of $\mathbf{x}$, and hence on $F_{\mathbf{x}}$

Proposition 1. *Let k be a field, K an algebraic closure of k, $X = \operatorname{Spec} k$, $\mathbf{x} : \operatorname{Spec} K \to \operatorname{Spec} k$ defined by $k \to K$ and k^{sep} the separable closure of k in K.*

(a) The construction of Remark 1 gives an equivalence of categories

$$\mathrm{Sh}_{\mathtt{Ab}}(X_{et}) \xrightarrow{\cong} (G\text{-}modules), \quad F \mapsto F_{\mathbf{x}}.$$

(b) Under this equivalence of categories, $F(X) \cong F^G$ and $H^p(X_{et}, F) \cong H^p(G, F_{\mathbf{x}})$.

Remark. G here is a topological group and $G \times F \to F$ is assumed continuous for the discrete topology on F.

Proposition 2. *Let L/K be a Galois extension. Then*

(a) (Hilbert 90) $H^1(L/K, L^\times) := H^1(\mathrm{Gal}(L/K), L^\times) = 0$ and

(b) $H^2(L/K, L^\times) \cong \mathrm{Br}(L/K) \subseteq \mathrm{Br}(K) := \{$*iso classes of central division algebras*$\}$ *is the subgroup of division algebras* D *such that* $D \otimes_K L$ *is a matrix algebra.*

Definition 1. We say that a field E is C_k if every homogeneous polynomial of degree d in $n > d^k$ variables has a zero in $E^n \setminus \{0\}$.

Remark. We are only interested in the case $k = 1$.

Proposition 3. *(a) Every algebraic extension of a* C_1*-field is* C_1.

(b) If E *is* C_1*, then* $\mathrm{Br}(E) \cong H^1(E^{sep}/E, (E^{sep})^\times)$ *vanishes and* $H^p(E^{sep}/E, M) = 0$ *for* M *torsion when* $p > 1$.

Proof. a) Without loss let F/E be a finite extension. If $P \in F[X_1, \dots, X_n]$ is a homogeneous polynomial of degree $d < n$, and $f_1, \dots, f_D \in F$ a basis, $D = [F : E]$, then there is a polynomial $Q \in E[X_{i,j}]_{i \le n, j \le D}$ such that

$$N_{F/E}(P((\sum_j x_{ij} f_j)_i)) = Q(x_{ij}) \quad \text{for all } (x_{ij}) \in E^{nD}.$$

But Q is a homogeneous polynomial of degree $dD < nD$, hence has a zero in $E^{nD} \setminus \{0\}$.

b) The vanishing of $\mathrm{Br}(E)$ can be seen in at least two ways: If D is a central division algebra of dimension d^2 over E, then its reduced norm $N_{D/E} : D \to E$ (induced by the determinant on some splitting extension) has no non-trivial zero, but if $d > 1$, then $d^2 > d$, a contradiction to C_1. Alternatively, use Serre's "Ugly Lemma" 1.5.4 in Cassels/Frohlich, Algebraic Number Theory to derive, for F/E finite, $H^2(F/E, F^\times) = 0$ from the cyclic case. Then $H^2(E^{\mathrm{sep}}/E, (E^{\mathrm{sep}})^\times) \cong \varinjlim H^2(F/E, F^\times)$ vanishes.

For the second statement, it is sufficient to show the assertion $H^k(E^{\mathrm{sep}}/E, M) = 0$ when M is trosion and $k > 1$. This can be reduced to the vanishing of $H^2(E^{\mathrm{sep}}/E, M)$ (by dimension shifting arguments). and to M having p-primary torsion for some prime p (decompose M into p-primary parts), and to $\mathrm{Gal}(E^{\mathrm{sep}}/E)$ being a pro-p-group (Sylow theory and res/cores arguments), and to the case where M is finite (M is the limit of its finite submodules).

Let $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$, then $M^G \neq 0$ when $M \neq 0$ (orbit combinatorics), hence using induction on the cardinality of M and sequences $0 \to M' \to M \to M'' \to 0$ with $M' \cong \mathbb{Z}/p\mathbb{Z}$ without loosing generality $M \cong \mathbb{Z}/p\mathbb{Z}$. When p is different from the characteristic one uses

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{k \mapsto \zeta_p^k} (E^{\mathrm{sep}})^\times \xrightarrow{x \mapsto x^p} (E^{\mathrm{sep}})^\times \to 1$$

and

$$H^n(E^{\mathrm{sep}}/E, (E^{\mathrm{sep}})^\times) \xrightarrow{\cdot p} H^n(E^{\mathrm{sep}}/E, (E^{\mathrm{sep}})^\times) \to E^2(E^{\mathrm{sep}}/E, \mathbb{Z}/p\mathbb{Z}) \to H^2(E^{\mathrm{sep}}/E, (E^{\mathrm{sep}})^\times).$$

If p is the characteristic, then the Artin-Schreier sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to E^{\mathrm{sep}} \xrightarrow{x \mapsto x^p - x} E^{\mathrm{sep}} \to 0$$

and the vanishing of the higher Galois cohomology groups of the additive group (which even holds when E is not C_1, are used instead. $\square$

Theorem 1 (Tsen). *If* K *is algebraically closed, then* $K(T)$ *is* C_1.

Proof. Let $E = K(T)$, $A = K[T]$. It suffices to show that every $P \in A[X_1, \dots, X_n]$ which is homogeneous of degree $d < n$ has a zero in $E^n \setminus \{0\}$. Let $P = \sum p_\alpha X^\alpha$ and let D be the maximum of the degrees of the $p_\alpha \in A$. We look for solutions to $P(f_1, \dots, f_n)$ where $f_i \in K[T]_{\leq M}$. Then $P(f_1, \dots, f_n)$ can be considered a polynomial in $K[T]$ of degree $D + dM$ and the $D + dm + 1$ coefficients of this polynomial are homogeneous polynomials in the $n(M+1)$ coefficients of the f_i. As $d < n$ when M is large then the number $n(M+1)$ of free variables is larger than the number $D + dM + 1$ of equations. Now the statement follows from a well-known result from classical projective algebraic geometry (Hartshorne chapter 1). $\square$

2.3 The Relation between H^1 and Torsors

Remark 1. If $\mathcal{C}$ is a general cateogry equipped witha Grothendieck topology then a global section of a sheaf F is by definition an element of $\varprojlim_{\mathcal{C}} F \cong \mathrm{Hom}_{\mathrm{Sh}_{\mathtt{Set}}}(*, F)$. When $\mathcal{C}$ has a final object A (e.g. $A = X$ for X_{et}), then $\varprojlim_{\mathcal{C}} F \cong F(A)$ canonically. We will assume that this poses no set-theoretic problems and that the category of sheaves of abelian groups has sufficiently many injective objects, giving us derived functors $H^*(F)$ of the functor $H^0(F)$, the group of global sections.

Definition 1. Let $\mathcal{G}$ be a sheaf of groups. A $\mathcal{G}$-torsor is a sheaf $\mathcal{T}$ of sets with a left $\mathcal{G}$-action such that A and the equivalent conditions B1 and B2 hold:

(A) If $\mathcal{S}_X = \{U \to X \mid \mathcal{T}(U) \neq \emptyset\}$, then $\mathcal{S}_X /= X$. If there is a final object A, this is equivalent to $\mathcal{S}_A /= A$.

(B1) When $\mathcal{T}(X) \neq \emptyset$, then the action of $\mathcal{G}(X)$ on $\mathcal{T}(X)$ is simply transitive.

(B2) The morphism $\mathcal{G} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$, $(g, t) \mapsto (t, gt)$ is an isomorphism.

A morphism $\tau : \mathcal{T} \to \mathcal{T}'$ of $\mathcal{G}$-torsors is a morphism of sheaves of sets which is compatible with the $\mathcal{G}$-action: $\tau(gt) = g\tau(t)$ for $g \in \mathcal{G}(X), t \in \mathcal{T}(X)$. $\mathcal{T}$ is trivial or split if it has a global section or equivalently if $\mathcal{T} \equiv \mathcal{G}$ as a $\mathcal{G}$-torsor.

Lemma 1. *The category of $\mathcal{G}$-torsors is a groupoid.*

Let $0 \to F \xrightarrow{i} I \xrightarrow{\pi} \mathbb{Q} \to 0$ be a short exact sequence of sheaves of abelian groups. If $q \in H^0(Q)$ is a global section, then

$$\mathcal{T}_q(U) = \{i \in I(U) \mid \pi(i) = q\}$$

is an F-torsor (with the action of $f \in F(U)$ sending i to $f + i$).

Definition 2. An i-trivialization of a F-torsor $\mathcal{T}$ is a morphism $\lambda \mathcal{T} \to I$ of sheaves of sets such that $\lambda(f + t) = i(f) + \lambda(t)$.

Remark. Thus, an i-trivialization is a trivialization of the I-torsor $i_* F = I \overset{F}{\times} \mathcal{T} := I \times \mathcal{T}/F$ (quotient by the diagonal action of F). In particular, the torsor $\mathcal{T}_q$ constructed above has an i-trivialization given by $\lambda = \mathrm{id}_I |_{\mathcal{T}}$

Lemma 2. *The assignment $q \mapsto [\mathcal{T}_q]$ gives bijection between $H^0(Q)$ and isomorphism classes of torsors with i-trivialization. Here, a morphism $\mathcal{T} \to \mathcal{S}$ of torsors with i-trivialization is a commutative diagram*

$$\begin{tikzcd}
\mathcal{T} \arrow[d] \arrow[dr, "\lambda_{\mathcal{T}}"] & \\
\mathcal{S} \arrow[r, "\lambda_{\mathcal{S}}"'] & I
\end{tikzcd}$$

in the category of sheaves of sets where the downward arrow is a morphism of F-torsors.

Proof. Let $(\mathcal{T}, \lambda)$ be a torsor with i-trivialization. If U is such that $\mathcal{T}(U) \neq \emptyset$, then $q(U) = \pi(\lambda(t))$ does not depend on the choice of $t \in \mathcal{T}(U)$. If $\mathcal{S}$ denotes the full subcategory of $\mathcal{C}$ of objects with $\mathcal{T}(U) \neq \emptyset$, then for all objects X of $\mathcal{C}$, the sieve $\mathcal{S} \searrow X$ of morphisms of elements from $\mathcal{S}$ covers X and

$$(q_U)_{U \in \mathcal{S} \searrow X} \in \varprojlim Q(U) \xleftarrow{\cong} Q(X)$$

defining $q_x \in Q(X)$, and set $q_{\mathcal{T}} = (q_x) \in \varprojlim Q = H^0(Q)$. One then checks that $\mathcal{T} \cong \mathcal{T}_{q_{\mathcal{T}}}$. It is clear from the initial description of $\mathcal{T}_q$ that $q = q_{\mathcal{T}_q}$. Also, there is a morphism $\lambda : \mathcal{T} \to \mathcal{T}_{q_{\mathcal{T}}}$ of torsors with i-trivialization, showing that the two constructions are inverse to each other. ☐

Remark. If λ and $\widetilde{\lambda}$ are different i-trivializations of $\mathcal{T}$, then, when $\mathcal{T}(U) \neq \emptyset$, $i_U = \lambda(t) - \widetilde{\lambda}(t)$ does not depend on the choice of $t \in \mathcal{T}(U)$ giving rise to $\lambda - \widetilde{\lambda} =: l \in H^0(I)$. Thus, the lemma gives a bijection of isomorphism classes of F-torsors having some i-trivialization and $\operatorname{coker}(H^0(I) \to H^0(Q))$.

Lemma 3. *If in the above situation I is an injective object in the category of sheaves of abelian groups, then every F-torsor $\mathcal{T}$ admits some i-trivialization.*

Remark. In particular, when F is injective, every torsor has a trivialization.

Proof. Let $\operatorname{Map}(\mathcal{X}, \mathcal{Y})$ denote morphisms in the category of sheaves of sets. Then

$$\operatorname{Hom}(\mathcal{X}, \mathcal{Y})(X) := \operatorname{Map}_{\mathrm{Sh}_{\mathcal{C} \searrow X}}(\mathcal{X}|_{\mathcal{C} \searrow X}, \mathcal{Y}|_{\mathcal{C} \searrow X}),$$

where $\mathcal{C} \searrow X$ is the category of X-objects, is (up to set-theoretic difficulties) a sheaf of sets on $\mathcal{C}$. Let $\mathcal{J} = \operatorname{Hom}(\mathcal{T}, I)$. The group structure on I defines a group structure on $\mathcal{J}$ and there is a monomorphism $j : I \to |$ of sheaves of abelian groups by sending $i \in I(X)$ to the constant morphism sending every section of $\mathcal{T}$ on the X-object $v : U \to X$ to $v^* i \in I(U)$. As I is injective, there is a splitting $q : \mathcal{J} \to I$ of the monomorphism j. There is also a morphism $\tau : \mathcal{T} \to \mathcal{J}$ of sheaves of sets with $\tau(f + t) = j(i(f)) + \tau(t)$, where $\tau(t) : \mathcal{T}|_{\mathcal{C} \searrow X} \to \mathcal{J}|_{\mathcal{C} \searrow X}$ sends $\widetilde{t} \in \mathcal{T}(U)$, for anx X-object $v : U \to X$, to $v^*(t) - \widetilde{t}$. Now $\lambda := q \circ \tau$ is an i-splitting of $\mathcal{T}$. ☐

If I is injective then we have a canonical isomorphism $\operatorname{coker}(H^0(I) \to H^0(Q)) \to H^1(F)$ coming from the long exact sequence in cohomology.

Proposition 1. *The resulting bjection*

$$(\textit{isomorphism classes of } F\textit{-torsors}) \cong H^1(F)$$

does not depend on the choice of the monomorphism $i : F \to I$ with I injective.

Proof. Exercise. ☐

Corollary 1. *We have an isomorphism $H^1(X_{et}, \mathcal{O}_{X_{et}}^\times) \cong \operatorname{Pic}(X)$. The bijection is made by applying propostion 1 to the torsor*

$$\mathcal{L}^*(U) = \{\lambda \in (v^*\mathcal{L})(U) \mid \lambda : \mathcal{O}_U \to \mathcal{L} \textit{ is an isomorphism}\}.$$

(The fact that every torsor has this form is easily seen by faithfully flat descent.)

Remark. This argument also works for X_{Zar} and X_{fpqc} (up to set-theoretic issues).

Example 1. (Hilbert 90.) If $X = \operatorname{Spec} K$ then

$$H^1(K^{\mathrm{sep}}/K, (K^{\mathrm{sep}})^\times) \cong H^1((\operatorname{Spec} K)_{et}, (\mathcal{O}_{\operatorname{Spec} K})_{et}) = \operatorname{Pic}(X) = 0.$$

Remark 2. The group structure on the set of isomorphism classes of F-torsors (F abelian), which turns the bijection from proposition 1 and corollary 1 into an isomorphism of groups is $[\mathcal{T}]+[\mathcal{S}] = [\mathcal{T}\times\mathcal{S}/F]$, where the quotient is by the diagonal action of F.

2.4 Applications of Čech Cohomology

If the underlying category has the required fibre products, then one has a Čech complex

$$\check{C}^l(U/V,F) := F(\underbrace{U\times_V U\times_V \ldots \times_V U}_{(l+1)\text{ factors}}).$$

Let $d_i : \check{C}^l \to \check{C}^{l+1}$ be the pullback along the surjection omitting the i-th factor, $0 \leq i \leq l+1$, and let $d : \check{C}^l \to \check{C}^{l+1}$ be given by $d = \sum_i (-1)^i d_i$. This turns $\check{C}(U/V,F)$, for any presheaf F of abelian groups, into a cochain complex.

Remark. If $\iota : U \to \widetilde{U}$ is a morphism of V-objects, then $\iota^* : \check{C}^*(\widetilde{U}/V,F) \to \check{C}^*(U/V,F)$, and if $\widetilde{\iota}$ is another such morphism, then $h = \sum_{j=0}^{l} \eta_j^*$, where

$$\eta_j : \underbrace{U\times_V \ldots \times_V U}_{l\text{ factors}} \to \underbrace{\widetilde{U}\times_V \ldots \times_V \widetilde{U}}_{(l+1)\text{ factors}}$$

is given by $(\iota,\ldots,\iota,\widetilde{\iota},\ldots,\widetilde{\iota})$, where the "middle" ι and $\widetilde{\iota}$ both start from the j-th factor, defines a cochain homotopy between ι^* and $\widetilde{\iota}^*$.

Note that $\check{H}^l(V/V,F) = 0$ for $l > 0$. Thus, $\check{H}^l(U/V,F) = 0$ when $l > 0$ and $U \to V$ has a section.

For the étale site, if $q : Y \to X$ is a morphism, then $\check{C}(U/V, q_*F) \cong \check{C}(U\times_X Y/V\times_X Y, F)$ (canonically).

Proposition 1 (Grothendieck)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories having sufficiently many injective objects, and let $\mathcal{X}$ be a subclass of the class of objects of $\mathcal{A}$ such that*

(a) Every direct sumand of an element of $\mathcal{X}$ belongs to $\mathcal{X}$. (In particular, $\mathcal{X}$ is closed under isomorphisms.)

(b) For every object A of $\mathcal{A}$, there is a monomorphism $A \to X$ with $X \in \mathcal{X}$.

(c) If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{A}$ and $A', A \in \mathcal{X}$, then $A'' \in \mathcal{X}$, and $0 \to F(A') \to F(A) \to F(A'') \to 0$ must be exact as well.

Then all injective objects belong to $\mathcal{X}$ and $R^pFX = 0$ when $X \in \mathcal{X}$ and $p > 0$.

Proposition 2. *The assumptions of proposition 1 are satisfied in the following cases, where $\mathcal{A}$ is the category of sheaves of abelian groups on X_{et}.*

(α) $\mathcal{X} = \{F \mid \check{H}^l(U/V,F) = 0$ when $l > 0$ and U,V are affine étale X-schemes with $U \to V$ surjective$\}$ and $F = \zeta_{U,*}$

(β) $\mathcal{Y} = \{F \mid \check{H}^l(U/V,F) = 0$ for $l > 0$ and all surjective morphisms $U \to V$ in $X_{et}\}$ and $F(F) = F(U)$, any object U.

(γ) $\mathcal{Y}' = \{F \mid \check{H}^l(U/V,F) = 0$ for $l > 0$ and all surjective morphisms $U \to V$ with U,V quasi-compact and quasi-separated$\}$ and $(F) = F(U)$ for any quasi-compact, quasi-separated U.

Proof. Condition (a) is trivial in all cases. It is sufficient to prove (b) for $\mathcal{Y}$. For this, membership is stable under infinite products and for any geometric point $\mathbf{x} : x \xrightarrow{\xi} X$, $\check{C}^*(U/V, \xi_* G) \cong \check{C}^*(U \times_X \mathbf{x}/V \times_X \mathbf{x}, G)$ is acyclic in positive degrees by the above remark.

For condition (c) one first shows that $F(A) \to F(A'')$ is an epimorphism by using U/V such that the element of $A''(U)$ under consideration has a preimage in $A(V)$. One gets a short exact sequence $0 \to \check{C}^*(A') \to \check{C}^*(A) \to \check{C}^*(A'') \to 0$ in each case, showing $A'' \in \mathcal{X}$. ☐

Definition 1. An $\mathcal{O}_{X_{et}}$-module is *quasi-coherent* if $M|_{U_{\text{Zar}}}$ is quasi-coherent in the ordinary sense for every etale X-scheme U, and $\pi^* M|_{V_{\text{Zar}}} \to M|_{U_{\text{Zar}}}$ for any morphism $\pi : U \to V$.

Corollary 1. *We have an equivalence of categories* $\operatorname{Qcoh}(\mathcal{O}_{X_{et}}) \cong \operatorname{Qcoh}(\mathcal{O}_{X_{Zar}})$ *given by* $\zeta_{X,*}$ *and* $U \mapsto (\eta^* M)(U)$. *Moreover,* $R^i \zeta_{X,*} M = 0$ *when* $i > 0$ *and* M *a quasi-coherent* $\mathcal{O}_{X_{et}}$. *Thus the Leray spectral sequence degenerates to an isomorphism*

$$H^p(X_{et}, M) \xrightarrow{\cong} H^p(X_{Zar}, \zeta_{X,*} M) = H^p(X_{Zar}, M|_{X_{Zar}}).$$

Definition 2. The constant etale sheaf defined by a set (or an (abelian) group) A is the sheaf $\underline{\underline{A}}$ defined by

$$\begin{aligned} \underline{\underline{A}} &= \{A\text{-valued functions on } \Pi_0(X)\} \qquad (U \text{ locally Noetherian}) \\ &= \{\text{locally constant } A\text{-valued functions on } \Pi_0(X)\} \\ &= \text{sheaf represented by the } X\text{-scheme defined by} X \times A = \coprod_{a \in A} X \times \{a\} \end{aligned}$$

where the last definition shows that this actually defines a sheaf (even an fpqc sheaf). For an abelian group G, we write $H^p(X_{et}, G) := H^p(X_{et}, \underline{\underline{G}})$.

Corollary 2. *When* X *is quasi-compact and quasi-separable and* X *and* G *have* p*-torsion (*p *prime), then* $H^q(X_{et}, G) = 0$ *when* $q > \dim X + 1$

Proof. By the compatability of $H^p(X_{et}, \cdot)$ with limits without loss G is finite. Then without loss $G \cong \mathbb{Z}/p\mathbb{Z}$ because a general G has a filtration with such quotients. Also, if $X_0 = V(p) \subseteq X$, then $(X_0)_{et} = X_{et}$, hence without loss $p\mathcal{O}_X = 0$. Then corollary 1 and

$$0 \to \underline{\underline{\mathbb{Z}/p\mathbb{Z}}} \to \mathcal{O}_{X_{et}} \xrightarrow{AS} \mathcal{O}_{X_{et}} \to 0,$$

where the Artin-Schreier operator $AS(f) = f^p - f$, identify $R^0 \zeta_{X,*}$ and $R^1 \zeta_{X,*}$ of $\mathbb{Z}/p\mathbb{Z}$ with the kernel and cokernel of AS and show that $R^q \zeta_{X,*} \mathbb{Z}/p\mathbb{Z} = 0$ when $q > 1$. The claim then follows from the Leray spectral sequence. ☐

Remark. This is important for studying the p-torsion of etale cohomology but also shows that this p-torsion is not what we are looking for.

Proof. (of corollary 1) use proposition 2a). The needed acyclicity result for $\check{H}(U/V, M)$ is trivial when U/V has a section, hence holds for $\check{C}^*(U \times_V U/U, M) \cong \check{C}^*(U/V, M) \otimes_{\mathcal{O}_V(V)} \mathcal{O}_U(U)$ (faithfully flat base change). ☐

Corollary 3. *If* $X = \varprojlim X_\alpha$ *wit haffine transition maps and projections* π_α*, and* $(F_\alpha, \iota_{\alpha\beta})$ *is a system of sheaves* F_α *on* X_{et} *such that*

$$\begin{array}{ccc} \pi^*_{\alpha\gamma} F_\gamma & \xrightarrow{\iota_{\alpha\gamma}} & F_\alpha \\ \downarrow \cong & & \uparrow \iota_{\alpha\beta} \\ \pi^*_{\alpha\beta} \pi^*_{\beta\gamma} F_\gamma & \xrightarrow{\pi^*_{\alpha\beta} \iota_{\beta\gamma}} & \pi^*_{\alpha\beta} F_\gamma \end{array}$$

for all $\alpha \to \beta \to \gamma$. Then if X is quasi-coherent and quasi-separated,

$$\varinjlim H^*(X_\alpha, F_\alpha) \to H^*(X, \varinjlim \pi_\alpha^* F_\alpha)$$

is an isomorphism.

Proof. Proposition 2b) and proposition 1.10.1. □

Corollary 4. *Let $f : X \to Y$ be a quasi-coherent quasi-separated morphism of schemes, y a geometric point of Y, $A = \mathcal{O}_{Y_{et},y}$ and $X_y \cong X \times_Y \operatorname{Spec} A$. If $p_1 : X_Y \to X$ is the projection to the first factor, then*

$$R^q f_* F \cong H^p(X_Y, p_1^* F)$$

Proof. This holds for $q = 0$ by 1.10 and the general case follows by corollary 3. □

2.5 Constructible Sheaves

Definition 1. An object X of a category $\mathcal{A}$ is called Noetherian if it satisfies the following equivalent (assuming choice) conditions:

(a) Every sequence $X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ of subobjects of X stabilizes at some $n \in \mathbb{N}$.

(b) Every non-empty class of subobjects of X has a maximal element.

Definition 2. Let X be Noetherian. A sheaf F on X_{et} is locally constant constructible (LCC) if it satisfies the following equivalent conditions:

(a) F is representable by a finite etale group scheme.

(b) There is a surjective finite etale morphism $v : Y \to X$ such that $Y = \bigsqcup_{i=1}^n Y_i$ and $v^* F|_{Y_i} \cong \underline{\underline{F_i}}$ where each F_i is a finite abelian group.

(c) We have $S \neq X_{et}$ where S is the sieve of all objects $v : U \to X$ of X_{et} such that $v^* F \cong \underline{\underline{G_y}}$ for some finite abelian group G_y.

Definition 3. Let X be Noetherian. We call a sheaf F of abelian groups on X_{et} constructible if the following equivalent conditions hold:

(a) There is a sequence $\emptyset = U_0 \subset U_1 \subset U_1 \subset \ldots \subset U_n = X$ of open subsets such that $F|_{U_i \setminus U_{i-1}}$ is a LCC sheaf.

(b) There is a finite, jointly surjective family $\{j_k : X_k \hookrightarrow X\}$ of locally closed immersions such that all $j_k^* F$ are LCC.

(c) F is a subobjerct of $\bigoplus_{i=1}^n p_{k,*} G_k$, $p_k : X_k \to X$ finite, and G_k a constant sheaf corresponding to a finite abelian group.

(d) F is Noetherian and torsion.

Remark. This can also be done for sheaves of sets.

Proposition 1. *Every torsion sheaf on X_{et} is a direct limit of constructible sheaves.*

2.6 Cohomology of Curves

Let K be a separably closed field. By a curve over K we understand a one-dimensional scheme of finite type over K.

Proposition 1. *Let C be a smooth curve over K. Then, we have*

$$H^p(C_{et}, \mathcal{O}_{C_{et}}^\times) = \begin{cases} \mathcal{O}_{C_{et}}^\times(C_{et}) = \mathcal{O}_C(C)^\times & \text{if } p = 0, \\ \operatorname{Pic}(C) & \text{if } p = 1, \\ 0 & \text{else.} \end{cases}$$

Proof. The cases $p = 0$ and $p = 1$ hold from corollary 2.3.1. Let us proof $p > 1$ assuming K is algebraically closed. Without loss assume that C is connected, hence integral.

Let η be the generic point of C and $k = \mathcal{O}_{C,\eta}$. Then k has transcendence degree 1 over K hence is C_1 by Tsen's theorem. Let $f : Y = \operatorname{Spec} k \to C$ be the embedding. If $c \in C$ and $A = \mathcal{O}_{C_{et},\mathbf{c}}$ (where **c** is a geometric point supported at c), then $Y \times_C \operatorname{Spec} A$ is the spectrum of an algebraic field extension L of k, and L is C_1. By Corollary 2.4.4 and the vanishing of $H^p(L/k, L^*)$, we have $(R^p f_* \mathcal{O}_{Y_{et}}^\times)_{\mathbf{c}} = 0$ for $p > 0$. Hence $R^p f_* \mathcal{O}_{Y_{et}}^\times = 0$ when $p > 0$.

By divisor theory, we have an exact sequence

$$0 \to \mathcal{O}_{C_{et}}^\times \to f_* \mathcal{O}_{Y_{et}}^\times \xrightarrow{\text{div}} \bigsqcup_{\substack{c \in C \\ \text{closed}}} i_{c,*}\mathbb{Z} \to 0$$

of sheaves on C_{et}. Then, the long exact sequence is

$$\ldots \to H^{p-1}(C, \bigsqcup i_{c,*}\underline{\underline{\mathbb{Z}}}_{\mathbf{c}}) \to H^p(C, \mathcal{O}_{C_{et}}) \to H^p(C, f_* \mathcal{O}_{V_{et}}^\times) \to \ldots$$

To prove our claim, we have to show that the left and right side vanish. First,

$$H^{p-1}(C_{et}, \bigsqcup_{\substack{c \in C \\ \text{closed}}} i_{c,*}\underline{\underline{\mathbb{Z}}}_{\mathbf{c}}) \overset{2.4.3}{\cong} \bigsqcup_{\substack{c \in C \\ \text{closed}}} H^{p-1}(C_{et}, i_{c,*}\underline{\underline{\mathbb{Z}}}_{\mathbf{c}}) \overset{\text{Leray}}{\cong} \bigsqcup_{\substack{c \in C \\ \text{closed}}} H^{p-1}(c, \underline{\underline{\mathbb{Z}}}_{\mathbf{c}}) = 0.$$

For the other term, the Leray spectral sequence yields an isomorphism $H^p(C, f_* \mathcal{O}_Y^\times) \cong H^p(k^{\text{sep}}/k, (k^{\text{sep}})^\times)$ which vanishes for $p > 0$ by Tsen's theorem. □