Group Rings of Infinite Groups

by Prof. Dr. Giles Gardam

notes by Stefan Albrecht

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Literature Passman: The algebraic structure of group rings

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Lecture 1: The Kaplansky Conjectures – An Overview

Definition 1.1. Let R be a ring and G be a group. The *group ring*

$$R[G] = \left\{ \sum_{i=1}^{n} r_{g_i} g_i \mid g_i \in G, r_{g_i} \in R \right\}$$

is the ring consisting of finite formal R-linear combinations of the group elements with multiplication

$$\left(\sum r_g g\right)\left(\sum s_h h\right) = \sum r g s_h gh = \sum_k \left(\sum_{gh=k} r_g s_h\right) k.$$

In this course, we will (almost) always have $R = \mathbb{Z}$ or $R = K$ a field. In the latter case, $K[G]$ is often called the group algebra.

Example 1.2. For $G = \mathbb{Z} = \langle t \rangle$, then $R[G]$ is the ring of Laurent polynomials in t over R, usually denoted $R[t, t^{-1}]$.

Viewpoint of Noether: Representations of groups are modules over group rings.
Warning: $K[G]$ is a non-commutative ring unless G is abelian. It is (left-)Noetherian only in special settings and it is never semisimple for infinite G. (cf. Masuhke's theorem).

Although group rings tend to have bad ring-theoretic properties, they conjecturally have nice elementary properties. Note first that for $k \in K^\times$ and $g \in G$, the element $kg \in K[G]$ is a unit, such units are called *trivial*. A group is called *torsion free* if it has no nontrivial elements of finite order.

Conjecture 1.3 (Kaplansky)**.** *Let K be a field and G be a torsion free group. Then $K[G]$*

- *has no nontrivial units,*
- *has no non-zero zero divisors,*
- *has no non-trivial idempotents.*

Furthermore, for any group G (possibly with torsion), $K[G]$ is directly finite (=von Neumann-finite =Dedekind-finite), i.e. if $\alpha\beta = 1$, then $\beta\alpha = 1$.

Remark 1.4. Torsion-freeness is essential. Assume $g \in G$ has order $n \geq 2$. Then $0 = (1-g)(1+g+\ldots+g^{n-1})$

Remark 1.5. The unit conjecture is false, the others are open.

Remark 1.6. These conjectures are "local" in the sense that they only depend on the finitely generated subgroups of G.

Proposition 1.7. *For a given field K and a group G, we have*

$$\textit{unit conj.} \implies \textit{zero divisor-conj.} \implies \textit{idempotent conj.} \implies \textit{direct finite-conj.}$$

Proof. The last 2 implications are easy ring theoretic statements. The first implication requires the following theorem by Connell (which we will prove later): $K[G]$ is prime (meaning $AB = 0$ implies $A = 0$ or $B = 0$ for two-sided ideals $A, B \subseteq K[G]$) if and only if G has no non-trivial finite normal subgroups. Since G is torsion-free, $K[G]$ is prime. Now suppose $\alpha\beta = 0$ for $\alpha, \beta \neq 0$. Then there exists some $\gamma \in K[G]$ with $\beta\gamma\alpha \neq 0$: Otherwise $(K[G]\beta K[G]) \cdot (K[G]\alpha K[G]) = 0$. Now $(1 -$

$\beta\gamma\alpha)(1+\beta\gamma\alpha) = 1$ and $1+\beta\gamma\alpha$ is a non-trivial unit, since if it were trivial then $\beta\gamma\alpha = kg-1$, but $0 = (\beta\gamma\alpha)^2 = k^2g^2 - 2kg + 1$, which is absurd unless $g = 1$, in which case $\beta\gamma\alpha = k-1$ again squares to zero, hence $\beta\gamma\alpha = 0$. □

Definition 1.8. A group G is residually finite if for all $1 \neq g \in G$ there exists a homomorphism $\varphi_g : G \to Q$, Q finite, such that $\varphi_g(g) \neq 1$.

We will see later that the direct finiteness conjecture is true for $K = \mathbb{C}$. For now, we prove

Proposition 1.9. *Let G be residually finite. Then $K[G]$ is directly finite.*

Proof. A group homomorphism $\varphi : G \to Q$ induces a ring homomorphism $K[G] \to K[Q]$. Thus $K[Q]$ is a $K[G]$-module. Note that Q is a basis for the K-vector space $K[Q]$, so if Q is finite this is a finite dimensional representation of G on $V = K[Q]$.

Suppose $\alpha\beta = 1$ in $K[G]$. Let $A = \operatorname{supp}(\alpha) := \{g \in G \mid (\alpha)_g \neq 0\}$, $B = \operatorname{supp}(\beta)$. Let $C = BA$. By residual finiteness, there is a finie quotient $\varphi : G \to Q$ which is injective on C. Now the induced maps $\rho_\alpha, \rho_\beta \in \operatorname{End}(V)$ satisfy $\rho_\alpha \circ \rho_\beta = \rho_{\alpha\beta} = \operatorname{id}_V$ and thus – since V is finite-dimensional – we have $\rho_\beta \circ \rho_\alpha = \operatorname{id}_V$ as well. Write $\beta_\alpha = \sum_{c\in C} (\beta\alpha)_c c$ and thus

$$\rho_{\beta\alpha}(1_Q) = \varphi(\beta\alpha) = \sum_{c\in C} (\beta\alpha)_c \varphi(c) = 1_Q$$

forces $(\beta\alpha)_c = 1$ if $c = 1$ and 0 else. □

Lecture 2: The Unit Conjecture

There is only one known way to probe the unit conjecture for a given group G: the unique product property.

Definition 2.1. A group G has the *unique product property* (UPP, "has unique products", "has UP") if for all non-empty finite subsets $A, B \subseteq G$ there exists some $g \in G$ s.t. $g = ab$ for a unique pair $(a, b) \in A \times B$.

Example 2.2. In $(\mathbb{Z}, +)$, given finite $A, B \subseteq \mathbb{Z}$, one can take $g = \max A + \max B$. Hence $\mathbb{Z}$ has unique products.

Remark 2.3. A group with unique products is torsion-free: If $1 \neq H \leq G$, H finite, then take $A = B = H$. Each product now occurs exactly $|H|$ times.

Remark 2.4. It's difficult to produce torsion-free groups that don't have UP.

Proposition 2.5. *A group with UP satisfies the zero divisor conjecture for all fields K.*

Proof. Let $\alpha, \beta \in K[G]$ with $\alpha, \beta \neq 0$, and set $A = \operatorname{supp}(\alpha)$, $B = \operatorname{supp}(\beta)$. Write $\alpha = \sum_{a\in A} \lambda_a a$ and $\beta = \sum_{b\in B} \mu_b b$. Then if $g = a_0 b_0$, $a_0 \in A$, $b_0 \in B$ is a unique product for A, B, then we have

$$(\alpha\beta)_g = \sum_{ab=g} \lambda_a \mu_b = \lambda_{a_0} \mu_{b_0} \neq 0.$$

Hence $\alpha\beta \neq 0$ in $K[G]$. □

For the unit conjecture, we need something that is a priori stronger.

Definition 2.6. A group G has the *two unique products property* if for all finite subsets $A, B \subseteq G$ with $|A| \cdot |B| \geq 2$, there exist $g_0 \neq g_1 \in G$, such that $g_0 = a_0 b_0$ and $g_1 = a_1 b_1$ for unique pairs $(a_0, b_0), (a_1, b_1) \in A \times B$.

Proposition 2.7 (Strognowski)**.** *The two unique products property is equivalent to the unique product propterty.*

Proof. If G satisfies 2UPP, it clearly satisfies UPP (if $|A| = |B| = 1$, the product is clearly unique).

Conversely, assume that G has UP but that there exist finite sets $A, B \subseteq G$ with $|A||B| \geq 2$ with only 1 unique product. Without loss (by translating A on the left and B on the right), we may assume that $1 = 1 \cdot 1$ is the unique unique product. Now let $C = B^{-1}A$ and $D = BA^{-1}$. We claim that now there is unique product for C and D. Every element of CD can be written as $b_1^{-1} a_1 b_2 a_2^{-1}$ for some $a_i \in A, b_i \in B$. If $(a_1, b_2) \neq (1, 1)$ then by assumption there is another pair a_1', b_2' s.t. $a_1 b_2 = a_1' b_2'$ and thus $b_1^{-1} a_1 b_2 a_2^{-1} = b_1^{-1} a_1' b_2' a_2^{-1}$ is not a unique product for CD. If, on the other hand, $(a_1, b_2) = (1, 1)$, then unless $(a_2, b_1) = (1, 1)$, we find a_2', b_1' such that

$$b_1^{-1} a_1 b_2 a_2^{-1} = b_1^{-1} a_2^{-1} = (a_2 b_1)^{-1} = (a_2' b_1')^{-1} = b_1'^{-1} a_1 b_2 a_2'^{-1}$$

is not a unique product. Finally, if $a_2 = b_1 = 1$, then our element of CD is $1 = 1 \cdot 1 = b^{-1} b = a a^{-1}$ for any $a \in A, b \in B$, and A or B has an element other than 1, which gives more than one factorisation. □

Corollary 2.8. *A group with UP satisfies the unit conjecture.*

Proof. Exercise. □

Most examples of groups with UP are left-orderable.

Definition 2.9. A group G is *(left-)orderable* if it admits a total order $\prec$ that is left-invariant, i.e. if $g \prec h$, then $kg \prec kh$ for all $g, h, k \in G$.

Remark 2.10. Being left- and right-orderable are equivalent (define $g \prec' h$ iff $g^{-1} \prec h^{-1}$)) but admitting a bi-invariant total order is much stronger.

Proposition 2.11. *A left-orderable group G has unique products.*

Proof. Fix a left-order $\prec$. Given finite subsets $A, B \subseteq G$, we show that the maximum of AB is a unique product. Let $b_0 = \max B$. Then for all $a \in A$, $b \in B \setminus \{b_0\}$, we have $b \prec b_0$, so $ab \prec ab_0$. Thus the maximum of AB can only be written as ab_0 for some $a \in A$, and thus must be unique. □

Remark 2.12. It is not necessarily true that $\max(AB) = \max A \cdot \max B$.

Definition 2.13. For a left-ordered group $(G, \prec)$, the set $\mathcal{P} = \{g \in G \mid 1 \prec g\}$ is called its *positive cone*.

The positive cone clearly satisfies $\mathcal{P}^2 \subseteq \mathcal{P}$ (i.e. it's a subsemigroup) and $G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$. The converse is also true:

Lemma 2.14. *Left-orders are equivalent to choices of $\mathcal{P} \subseteq G$ satisfying $\mathcal{P}^2 \subseteq \mathcal{P}$ and $G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$.*

Proof. Exercise. □

Lemma 2.15. *A group G is left-orderable if and only if for all $g_1, \ldots, g_n \in G \setminus \{1\}$, there exists a choice of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$ such that $1 \notin S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$ (the subsemigroup generated by $g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n}$).*

Proof. If G is left-ordered, set $\varepsilon_i = 1$ iff $g_i \in \mathcal{P}$.

For the other implication, we use compactness (slogan: the inverse limit of non-empty finite sets is non-empty). Let $X = \{1, -1\}^{G \setminus \{1\}}$ be the set of functions $G \setminus \{1\} \to \{1, -1\}$, and let $A \subseteq X$ be the set of those functions that define a positive cone. This is equivalent to satisfying (simultaneously) the condition on choice of sign for all possible $g_1, \ldots, g_n \in G \setminus \{1\}$ (for $n = 3$). That is, if we denote such functions $A_{\{g_1, \ldots, g_n\}} \subseteq X$, then

$$A = \bigcap_{\substack{S \subseteq G \setminus \{1\} \\ S \text{ finite}}} A_S$$

But X is compact by Tychonoff and all the A_S are closed. Furthermore, all finite intersections of the A_S are non-empty by assumption. So $A \neq \emptyset$. $\square$

We apply the lemma to prove

Theorem 2.16 (Burns-Hale, 1972)**.** *Let G be a group. If every non-trivial finitely generated subgroup of G has a non-trivial left-orderable quotient, then G is left-orderable.*

In particular, a locally indicable group (i.e. every nontrivial finitely generated subgroup surjects onto $\mathbb{Z}$) is left-orderable.

Corollary 2.17 (Higman, 1940)**.** *Loally indicable groups satisfy the unit conjecture.* $\square$

Lecture 3: More on Orderings, and Hyperbolic Groups

Example 3.1 (of locally indicable groups)**.**

- Free groups (Niedsen-Schreier)
- Fundamental groups of closed surfaces of non-positive Euler characteristic
- Torsion-free nilpotent groups
- Torsion-free one-relator groups, i.e. groups of the form $\langle X \mid r \rangle$, $r \in F(X)$, where r is not a proper power in $F(X)$ (Brodski-Howie)

Proof. (of 2.16) Suppose G is not left-orderable and let n be minimal such that $\exists g_1, \ldots, g_n \in G \setminus \{1\}$ such that $1 \in S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$ for all choices of $\varepsilon_i \in \{-1, 1\}$. Let $H = \langle g_1, \ldots, g_n \rangle \neq 1$, so by assumption H has a non-trivial left-orderable quotient $q : H \twoheadrightarrow Q$. By relabelling, aassume $g_1, \ldots, g_t \in \ker(q)$ and $g_{t+1}, \ldots, g_n \notin \ker(q)$. As $t < n$, we can assign $\varepsilon_1, \ldots, \varepsilon_t$ such that $1 \notin S(g_1^{\varepsilon_1}, \ldots, g_t^{\varepsilon_t})$. and since Q is left-orderable, we can assign $\varepsilon_{t+1}, \ldots, \varepsilon_n$ such that $1 \notin S(q(g_{t+1})^{\varepsilon_{t+1}}, \ldots, q(g_n)^{\varepsilon_n})$. But this implies $1 \notin S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$ as every product of othese elements either only uses $g_1^{\varepsilon_1}, \ldots, g_t^{\varepsilon_t}$, hence lies in $S(g_1^{\varepsilon_1}, \ldots, g_t^{\varepsilon_t})$ or has image under q in $S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$. $\square$

Proposition 3.2. $\mathrm{Homeo}^+(\mathbb{R})$ *is left-orderable.*

Proof. Let $\{x_0, x_1, \ldots\} \subseteq \mathbb{R}$ be dense. Define the order $\prec$ on $f \in \mathrm{Homeo}^+(\mathbb{R})$ via the lexiographic order on $(f(x_0), f(x_1), \ldots)$. The map $f \mapsto (f(x_0), f(x_1), \ldots)$ is injective (because continuous functions are determined by their values on a dense set), so the order descends. $\square$

Proposition 3.3. *A countable group is left-orderable if and only if it is a subgroup of* $\mathrm{Homeo}^+(\mathbb{R})$.

Proof. Exercise. $\square$

Proposition 3.4. *Let G be a group. Suppose $N \trianglelefteq G$ such that N and G/N both have unique products. Then G has unique products.*

Proof. Let $A, B \subseteq G$ be non-empty finite subsets. Write $\varphi : G \to G/N$. Suppose $\varphi(a) \cdot \varphi(b)$ is a unique product in G/N, $a \in A$, $b \in B$. By replacing A with $a^{-1}A$ and B with Bb^{-1}, we may assume the unique product in G/N is $1 \cdot 1 = 1$. Thus $a, b \in N$. Hence the unique product of $A \cap N$ and $B \cap N$ is a unique product for A and B. $\square$

Definition 3.5. Let $A \subseteq G$ be a finite subset. An element $a \in A$ is called *extremal* (for A) if for all $s \in G \setminus \{1\}$ we have $as \notin A$ or $as^{-1} \notin A$. G is called *diffuse* if every non-empty finite subset $A \subseteq G$ has at least one extremal point.

Remark 3.6. $a \in A$ is extremal iff $a^{-1}A \cap A^{-1}a = \{1\}$

Proposition 3.7. *For any group G we have the implications*

$$\textit{left-orderable} \quad \Longrightarrow \quad \textit{diffuse} \quad \Longrightarrow \quad \textit{unique products}.$$

Proof. Suppose $(G, <)$ is a left-ordered group and let $\emptyset \neq A \subseteq G$ a finite subset. Then let $a = \max A$. For any $s \in G \setminus \{1\}$ either $s > 1$ or $s^{-1} > 1$, hence $as > a$ or $as^{-1} > a$, i.e. a is extremal.

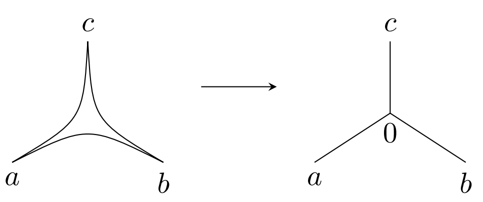
Suppose G is diffuse and let $A, B \subseteq G$ be non-empty finite subsets. Consider $C = AB$. Let $c = ab \in C$ be extremal. Suppose $c = a_1 b_1$ with $b \neq b_1$. Then $c(b_1^{-1} b_2) = a_1 b_2 \in C$ and $c(b_2^{-1} b_1) = a_2 b_1 \in C$, in contradiction to extremity. $\square$

Remark 3.8. Given a finite set $B \subseteq G$, we can easily decide if all $\emptyset \neq A \subseteq B$ have an extremal point, because if $a \in A_0 \subseteq A_1$ is extremal in A_1, then it is also extremal in A_0. Thus we can run a greedy algorithm, starting with $A = B$ and throwing out the extremal points at each step.

We can establish diffuseness geometrically, specifically for many hyperbolic groups.

Hyperbolic groups

Geodesic triangles in the hyperbolic plane $\mathbb{H}^2$ resemble tripods.



Given three points in a metric space, they embed isometrically as the vertices of a unique tripod T_Δ. The length $d(0, a)$ must be $\frac{1}{2}(d(a, b) + d(a, c) - d(b, c)) =: (b \cdot c)_a$ which we call the Gromov product. Morally, this is the distance to the incircle. Let X be a geodesic[1] metric space. For a geodesic triangle $\Delta = \Delta(a, b, c)$, define $\mathcal{X}_\Delta : \Delta \to T_\Delta$ by mapping the geodesics isometrically. Δ is called δ-thin if $p, q \in \mathcal{X}_\Delta^{-1}(t)$, then $d_X(p, q) \leq \delta$ for all $t \in T_\Delta$.

[1] $\forall x, y \in X$ $\exists$geodesic $[x, y]$, i.e. an isometric embedding $i : [0, d(x, y)] \to X$ with $i(0) = x, i(d(x, y)) = y$

Definition 3.9. X is called δ-hyperbolic if allgeodesic triangles are δ-thin. X is called (Gromov) hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

There are multiple equivalent definitions, e.g. slim triangles, but the constant δ needs to change.

Definition 3.10. A group G is called *hyperbolic* if it acts properly cocompactly by isometries on a hyperbolic space.

An alternative definition is the four-point condition.

Definition 3.11. Let $\delta \geq 0$. A metric space X is (δ)-hyperbolic if $(xy)_w \geq \min\{(xz)_w, (yz)_w\} - \delta$ for all $x, y, z, w \in X$

Remark 3.12. This definition is arguably less intuitive, but it also works for non-geodesic metric spaces such as discrete spaces.

Proposition 3.13. *Let X be a geodesic metric space. Then*

(i) X is (δ)-hyperbolic $\Rightarrow$ X is 4δ-hyperbolic.

(ii) X is δ-hyperbolic $\Rightarrow$ X is (δ)-hyperbolic.

Proof. (i) Exercise. □