Algebraic Topology
Serre spectral sequence, characteristic classes and bordism

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1 Informal introduction

One of the big goals of homotopy theory is to compute

$$[X, Y]_\bullet = \{\text{base-point preserving cont. maps } X \to Y\}/\text{homotopy}$$

for X and Y pointed CW-complexes. CW-complexes are build out of spheres, hence the building blocks are the sets $[S^n, S^k]_\bullet = \pi_n(S^k, *)$. For $n \geq 1$, there are groups, abelian if $n > 1$. What do we know about these groups?

- $\pi_n(S^k, *) = 0$ for $n < k$ by cellcular approximation.
- $\pi_n(S^n, *) \cong \mathbb{Z}$ by the Hurewicz theorem and $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$
- X is $(n-1)$-connected CW-complex: Then $\pi_n(X, *) \cong H_n(X, \mathbb{Z})$.
- $\pi_k(S^1, *) = 0$ for $k \geq 2$ by covering space theory (universal cover of S^1 is $\mathbb{R}$, which is contractible).
- $\pi_3(S^2, *) \neq 0$, since the attaching map of the 4-cell for $\mathbb{CP}^2$ is a map $\eta : S^3 \to S^2 \cong \mathbb{CP}^1$. If this was null-homotopic, then we would have $\mathbb{CP}^2 \sim S^2 \vee S^4$, which contradicts the ring structure on $H^\bullet(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}[x]/x^3$.
- $\pi_k(S^n, *) \to \pi_{k+1}(S^{n+1}, *) \to \pi_{k+2}(S^{n+2}, *) \to \cdots$ eventually stabilizes by the Freudenthal suspension theorem.

To go beyond this, we need a new tool, the Serre spectral sequence. To motivate its usefulness, consider the following strategy: There exists a map $f : S^2 \to K(\mathbb{Z}, 2)$ which induces an isomorphism $f_* : \pi_2(S^2, *) \to \pi_2(K(\mathbb{Z}, 2), *)$. We can take its homotopy fibre $H = \mathrm{hofb}_x(f)$ (2-connected cover of S^2). Then there is a fiber sequence $H \to S^2 \xrightarrow{f} K(\mathbb{Z}, 2)$ and a long exact sequence in homotopy

$$\cdots \to \pi_4(K(\mathbb{Z},2),*) \to \pi_3(H,*) \to \pi_3(S^2,*) \to \pi_3(K(\mathbb{Z},2),*) \to \pi_2(H,*) \to \pi_2(S^2,*) \to$$

$$\to \pi_2(K(\mathbb{Z},2),*) \to \pi_1(H,*) \to \pi_1(S^2,*) \to \cdots$$

from which we conclude $\pi_3(H, *) \cong \pi_3(S^2, *)$ and $\pi_1(H, *) = \pi_2(H, *) = 0$, i.e. H is 2-connected and the higher homotopy groups agree with the ones of S^2. By the Hurewicz theorem, $\pi_3(S^2, *) = H_3(H, \mathbb{Z})$. Hence we want to find a way to compute $H_*(H, *)$ from $H_*(S^2, \mathbb{Z})$ and $H_*(K(\mathbb{Z}, 2), \mathbb{Z})$.

This will also help to compute $\pi_n(S^k, *)$ in other ways (for example we will show that $\pi_n(S^k, *)$ is finite unless $n = k$ or $n = 2k - 1$ and k even). Furthermore, the Serre spectral sequence will allow us to compute the (co-)homology of spaces like $U(n), SU(n), \Omega S^n, K(\mathbb{Z}/2, n)$ etc. and (re-)prove structural theorems like Hurewicz, Freudenthal suspension, Thom isomorphisms and more.

So, given a fiber sequence $F \to Y \to X$, what could the relationship between the homology groups of F, Y and X be?

Example 1.1. Consider the easiest case $F \to X \times F \xrightarrow{\mathrm{pr}_X} X$, the trivial filtration. Then the Alexander-Whitney map induces an isomorphism

$$H_n(X \times F, \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X, H_q(F)).$$

This is the kind of result we want: It computes the homology of the total space in terms of the homology of X and F.

Example 1.2 (Hopf fibration). $S^1 \to S^3 \xrightarrow{\eta} S^2$:

n	$H_n(S^3, \mathbb{Z})$	$\bigoplus_{p+q=n} H_p(S^2, H_q(S^1, \mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}$
2	0	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0

Hence clearly the Künneth formula from the previous example is "too big" to describe the homology in this case. However, consider the "2-step"-filtration $S^1 \subseteq S^3$ which satisfies $\widetilde{H}_n(S^3/S^1, \mathbb{Z}) \cong \mathbb{Z}$ for $n = 2, 3$ and 0 otherwise. Hence $H_\bullet(S^1, \mathbb{Z}) \oplus H_\bullet(S^3/S^1, \mathbb{Z})$ agrees with the right-hand side of the table above. This does not agree with $H_*(S^3, \mathbb{Z})$, because the long exact sequence corresponding to $S^1 \to S^3 \to S^3/S^1$ does not split into nice short exact sequences. Concretely, the boundary map $\widetilde{H}_2(S^3/S^1, \mathbb{Z}) \to H_1(S^1, \mathbb{Z})$ is an isomorphism, hence these two terms do not contribute to $H_\bullet(S^3, \mathbb{Z})$.

It turns out that something similar holds for all fibre sequences $F \to Y \to X$: There exists a filtration

$$F_0 \subseteq F_1 \ldots \subseteq F_m \subseteq \ldots \subseteq C_*(Y, \mathbb{Z})$$

on $C_*(Y, \mathbb{Z})$ such that $H_{p+q}(F_p/F_{p-1}) \cong C_p^{\text{cell}}(X, H_q(F, \mathbb{Z})$. To then understand $H_\bullet(Y, \mathbb{Z})$, one needs to understand the cancellations in the associated long exact sequences. This is best encoded in a spectral sequence.

2 The Serre Spectral Sequence

Definition 2.1. A (homologically, Serre-graded) *spectral sequence* is a triple $(E^\bullet, d^\bullet, h^\bullet)$, where

- $(E^r)_{r\geq 2}$ is a sequence of $\mathbb{Z}$-bigraded abelian groups. We write $E^r_{p,q}$ for the (p,q)-graded part of E^r. E^r is called the r-th *page* of the spectral sequence.
- $(d^r : E^r \to E^r)$ is a sequence of morphisms, called *differentals*, of bidegree $(-r, r-1)$ satisfying $d^r \circ d^r = 0$.
- $h^r : H_\bullet(E^r) \to E^{r+1}$ is a sequence of bigrading-preserving isomorphisms. Here $H_\bullet(E^r)$ denotes the homology with respect to d^r, which inherits a bigrading from E^r.

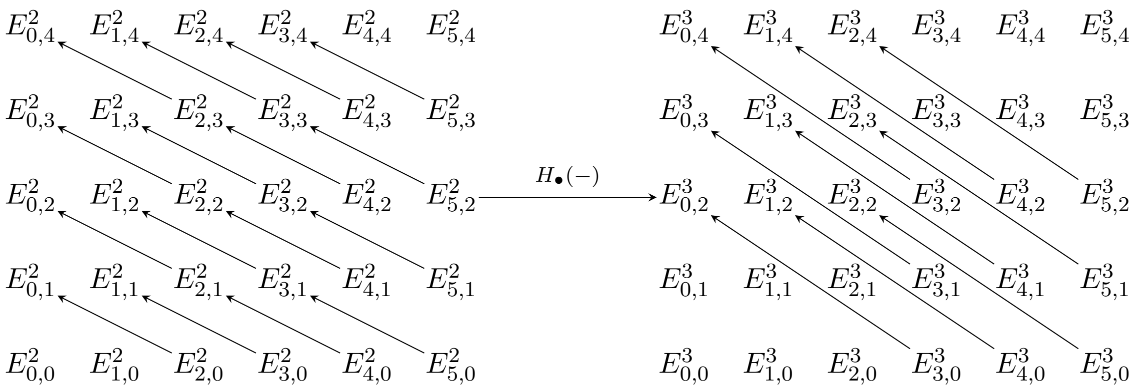

Figure 1: The second and third page of a spectral sequence

Definition 2.2. We say that a spectral sequence is *1st quadrant* if all abelian groups $E^2_{p,q}$ are trivial whenever $p < 0$ or $q < 0$.

Lemma 2.3. *For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we have $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$ for all $r \geq 2$. Moreover, for a given $(p,q) \in \mathbb{Z}^2$, the map h induces an isomorphism $E^r_{p,q} \to E^{r+1}_{p,q}$ for $r > r_0 = \max(p, q+1)$, i.e. the groups $E^r_{p,q}$ stabilize as $r \to \infty$.*

Proof. The first statement follows directly from the existence of the isomorphisms h by induction on r. For the second statement, if $r > r_0$, then the target of the differental $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial, hence every element of $E^r_{p,q}$ is a cycle. Moreover, the domain of the incoming differential $d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}$ is trivial. Hence $E^r_{p,q} \cong H_\bullet(E^r_{p,q}) \xrightarrow[h]{\cong} E^{r+1}_{p,q}$ □

Definition 2.4. For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we define the E^∞-*page* as the bigraded abelian group $E^\infty_{p,q} = E^{r_0+1}_{p,q}$ with $r_0 = \max(p, q+1)$. By the previous lemma, $E^\infty_{p,q} \cong E^r_{p,q}$ whenever $r > r_0$

By a filterted object in an abelian category $\mathcal{A}$ we mean an object $H \in \mathcal{A}$ with a sequence of inclusions

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^n \subseteq \ldots \subseteq H.$$

We will apply this to $\mathcal{A}$ the category of graded abelian groups and $H = H_*(E, \mathbb{Z})$.

Definition 2.5. A first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$ is said to *converge* to a filtered object in graded abelian groups (H, F) if there is a chosen isomorphism $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$ for all p, q and $F^p_n = H_n$ if $n \leq p$. In this case we write $E^2_{p,q} \Rightarrow H$.

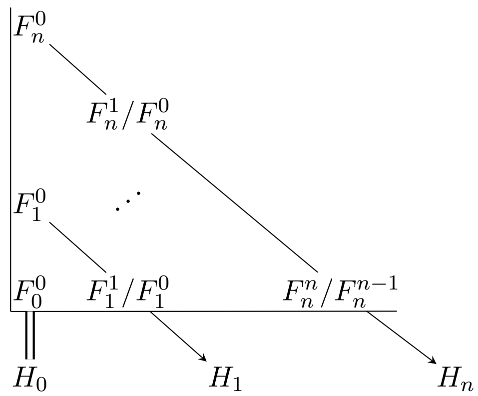

Figure 2: Visualization of E^∞ as filtrations of the H_i for a convergent spectral sequence $E^2_{p,q} \Rightarrow H$

Remark. Convergence is really a *datum* of the necessary isomorphism and not a property. Convergent spectral sequences are often simply encoded as $E^2_{p,q} \Rightarrow H$, but this suppresses not only this data, but also the higher pages, the differentials, and the filtration on H.

We now want to introduce the Serre spectral sequence for the homology of fibre sequences.

Definition 2.6. Let $f : Y \to X$ be a continuous map of topological spaces and $x \in X$ a point. The *homotopy fibre* $\mathrm{hofb}_x(f)$ of f at x is defined to be

$$\mathrm{hofb}_x(f) = P_x X \times_X Y$$

where $P_x X = \{\gamma : [0,1] \to X \mid \gamma(1) = x\}$ is the based path space of X It comes with a map $P_x X \to X$ given by $\gamma \mapsto \gamma(0)$. In words: $\mathrm{hofb}_x(f)$ is the space of pairs (γ, y) where $y \in Y$ and γ is a path in X from $f(y)$ to x. We note that $P_x X$ is contractible by the homotopy

$$H : P_x X \times [0,1] \to P_x X, \quad (\gamma, t) \mapsto s \mapsto \gamma((1-t)s + t)$$

Example 2.7. If $f : * \to X$, then $\mathrm{hofb}_x(f) = \Omega_x X$.

Definition 2.8. A *fibre sequence* of topological spaces is a sequence $F \xrightarrow{i} Y \xrightarrow{f} X$, a basepoint $x \in X$, a homotopy $h : F \to X^{[0,1]}$ from the composite $f \circ i$ to the constant map $c_x : F \to X$ such that the induced map $F \to \mathrm{hofb}_x(f),\ z \mapsto (h(z), i(z))$ is a weak homotopy equivalence.

Recall: A weak homotopy equivalence is a map inducing isomorphisms on $\pi_n(-, x)$ for all $n \in \mathbb{N}$ and all basepoints x.

Example 2.9. 1. Let $f : Y \to X$ be any continuous map, $x \in X$. Then the pair $(\mathrm{hofb}_x f \to Y \to X, H)$, where H is the homotopy from the definition of the homotopy fibre above, is a fibre sequence. Every fibre sequence is equivalent to this in the following sense: Given $(F \to Y \to X, h)$, there is a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\simeq"] \arrow[d] & \mathrm{hofb}_x(f) \arrow[d] \\
Y \arrow[r, equal] \arrow[d, "f"] & Y \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

In particular, $\Omega_x X \to x \to X$ is a fibre sequence, where $h : \Omega_x X \times [0,1] \to X$ is the evaluation map. If one instead chooses the constant homotopy, one does not obtain a fibre sequence (unless the space is contractible). This is because the induced map $\Omega_x X \to \mathrm{hofb}_x(f) = \Omega_x X$ is constant and hence usually not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces F and X, $x \in X$, the pair $(F \to F \times X \to X, \mathrm{const})$ is a fibre sequence, the *trivial fibre sequence*. To see that, note that $\mathrm{hofb}_x(\mathrm{pr}_X) = F \times P_x X$ with induced map

$$F \to F \times P_x X, \quad y \mapsto (y, \mathrm{const}),$$

which is a homotopy equivalence as $P_x X$ is contractible.

3. Let $p : E \to B$ be a fibre bundle with fibre $F = p^{-1}(b)$ for some $b \in B$. Then the sequence $F \to E \to B$ with the constant homotopy is a fibre sequence. This is a special case of the next example.

4. Recall that $p : E \to B$ is a Serre fibration if in every commutative diagram of the form

$$\begin{tikzcd} D^n \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a lift $D^n \times I \to E$ making both triangles commute. Given a Serre fibration $p : E \to B$ and $b \in B$, the sequence $F = p^{-1}(b) \to E \to B$ with the constant homotopy is a fibre sequence. (see exercises) Note: Every fibre sequence is also equivalenct to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle $S^1 \to S^3 \xrightarrow{\eta} S^2$. It arises by letting $S^1 = U(1)$ act on $S^2 \subseteq \mathbb{C}^2$ via $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$, with quotient space $\mathbb{CP}^1 \cong S^2$.

6. Example 5 generalizes to fibre bundles $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ with limit case $S^1 \to S^\infty \to \mathbb{CP}^\infty$, which is equivalent to $\Omega\mathbb{CP}^\infty \to * \to \mathbb{CP}^\infty$.

We are now ready to state the existence of the Serre spectral sequence.

Theorem 2.10 (Serre). *For every fibre sequence $(F \xrightarrow{\iota} Y \xrightarrow{p} X, h)$ with X simply-connected and abelian group A, there exists a spectral sequence of the following form*

$$E^2_{p,q} = H_p(X, H_q(F, A)) \quad \Longrightarrow \quad H_{p+q}(Y, A)$$

As noted before, this information does not include the differentials and the higher pages, as well as the filtrations on $H_\bullet(Y, A)$ and the identifications of its subquotients with the E^∞-page.

One edge case is easy to state: The map

$$H_n(F, A) = H_0(X, H_n(F, A)) = E^2_{0,n} \twoheadrightarrow E^\infty_{0,n} \hookrightarrow H_n(Y, A)$$

agrees with the factorization $H_n(F, A) \twoheadrightarrow \mathrm{im}\, \iota_* \hookrightarrow H_n(Y, A)$.

We now assume this theorem and give some sample computations.

Example 2.11. We revisit the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. S^2 is simply connected, so we get a spectral sequence. The E^2-page is $H_p(S^2, H_q(S^1, A))$, which looks like

$$\begin{tikzcd} q & & & & \\ & 0 & 0 & 0 & 0 \\ & A & 0 & A & 0 \\ & A & 0 & A \arrow[ull] & 0 \\ & & & & p \end{tikzcd}$$

There is one potentially non-trivial d^2-differential, namely $d^2 : E^2_{2,0} \to E^2_{0,1}$. All higher differentials d^r, $r > 2$, are trivial for degree reasons. Hence the E^∞-page looks as follows:

$$
\begin{array}{c|cccc}
q & & & & \\
 & 0 & 0 & 0 & 0 \\
 & \operatorname{coker}(d^2) & 0 & A & 0 \\
 & A & 0 & \ker(d^2) & 0 \\
\hline
 & & & & p
\end{array}
$$

We know that $H_n(S^3, A) = A$ for $n = 0, 3$ and 0 else. From the E^∞-page we thus get $H_0(S^3, A) = A$, $H_1(S^3, A) = \operatorname{coker}(d^2)$, $H_2(S^3, A) = \ker(d^2)$, $H_3(S^3, A) = A$. Hence d^2 must be an isomorphism.

Lemma 2.12. *There is a fibre bundle*

$$U(n-1) \overset{i}{\hookrightarrow} U(n) \to S^{2n-1},$$

where $U(n)$ denotes the topological group of unitary $n \times n$-matrices and i is the standard inclusion which adds a trivial $\mathbb{C}$-summand.

Proof. The gorup $U(n)$ acts on $\mathbb{C}^n$ by definition. This action restricts to the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Furthermore, this action is transitive, because every vector of length 1 can be extended to an orthonormal basis. Hence S^{2n-1} is in bijection with the orbit space $U(n)/\operatorname{Stab}(x)$, for any $x \in S^{2n-1}$. For $x = (0, \dots, 0, 1)$, the stabalizer equals $i(U(n-1))$. We obtain a continuous bijective map $U(n)/U(n-1) \to S^{2n-1}$, $[A] \mapsto A(0, \dots, 0, 1)^t$, which is a homeomorphism since its domain is quasi-compact and its codomain is Hausdorff. Finally, we use the fact that for a smooth, free action of a compact Lie group G on a manifold M, the map $M \to M/G$ is always a fibre bundle (in fact a G-principal bundle). $\square$

Example 2.13. We consider the case $n = 2$, i.e. the fibre sequence $S^1 \cong U(1) \hookrightarrow U(2) \to S^3$. We want to compute the homology of $U(2)$ via the Serre spectral sequence $E^2_{p,q} = H_p(S^3, H_q(S^1, \mathbb{Z}))$ All differentials on all pages have to be trivial for degree reasons. (The spectral sequence "collapses".) Hence $E^\infty = E^2$ and every antidiagonal has at most one non-trivial term, so we can read off $H_n(U(2), \mathbb{Z}) = \mathbb{Z}$ for $n = 0, 1, 3, 4$ and 0 else. In fact, one can show that $U(2) \cong S^3 \times U(1)$, so the homology could alternatively be computed with the Künneth theorem.

Example 2.14. Next we consider the fibre sequence $U(2) \hookrightarrow U(3) \to S^5$ with E^2-page $E^2_{p,q} = H_p(S^5, H_q(U(2), \mathbb{Z}))$, which looks like

$$
\begin{array}{c|cccccc}
q & & & & & & \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & Z & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
\hline
 & & & & & & p
\end{array}
$$

The first potentially non-trivial differential is $d^5 : E^2_{0,5} \to E^2_{0,4}$. At this point we cannot decide what this differential is. All higher differentials are again trivial for degree reasons, and all filtrations collapse to at most one entry. We obtain

$$H_n(U(3),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0,1,3,5,8,9, \\ \operatorname{coker}(d^5) & \text{for } n = 4, \\ \ker(d^5) & \text{for } n = 5, \\ 0 & \text{else.} \end{cases}$$

This example illustrates a typical situation, namely that one can often not fully determine all differentials but still deduce a lot. We will soon see that $d^5 = 0$ and hence $H_4(U(3),\mathbb{Z}) \cong H_5(U(3),\mathbb{Z}) \cong \mathbb{Z}$

Example 2.15. We consider $U(3) \to U(4) \to S^7$. The $E^2_{p,q} = H_p(S^7, H_q(U(3),\mathbb{Z}))$-page is

$$\begin{array}{c|ccccc}
q & & & & & \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & 0 & & & & 0 \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & ? & & \vdots & & ? \\
 & ? & \cdots & 0 & \cdots & ? \\
 & \mathbb{Z} & & \vdots & & \mathbb{Z} \\
 & 0 & & & & 0 \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & \mathbb{Z} \\
\hline
 & 0 & & & & 7 \quad p
\end{array}$$

The only possibly non-trivial differentials are $d^7 : E^2_{7,0} \to E^2_{0,6}$ and $d^7 : E^2_{7,3} \to E^2_{0,9}$ which we cannot compute at this point. Nevertheless, we can still deduce a lot, for example $H_1(U(4),\mathbb{Z}) = \mathbb{Z} = H_3(U(4),\mathbb{Z})$, $H_2(U(4),\mathbb{Z}) = 0$, $H_4(U(4),\mathbb{Z}) \cong H_4(U(3),\mathbb{Z})$,$H_5(U(4),\mathbb{Z}) \cong H_5(U(3),\mathbb{Z})$ and there is a short exact sequence $0 \to \mathbb{Z} \to H_8(U(4),\mathbb{Z}) \to \mathbb{Z} \to 0$, so $H_8(U(4),\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

In the previous examples we used the Serre spectral sequence to compute the homology of the total space of the fibre sequence. We now show that it can also be used to compute the homology of the base space or fibre.

Example 2.16. We consider the fibre sequence $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ for $n \geq 2$ and want to compute the homology of $\mathbb{CP}^n$. By path-connectedness, the E^2-page of $H_p(\mathbb{CP}^n, H_q(S^1,\mathbb{Z})) \Rightarrow H_{p+q}(S^{2n+1},\mathbb{Z})$ begins like this:

$$\begin{array}{c|cc}
q & & \\
 & 0 & 0 \\
 & \mathbb{Z} & ? \\
 & \mathbb{Z} & ? \\
\hline
 & & \quad p
\end{array}$$

Since $H_1(S^{2n+1}, \mathbb{Z}) = 0$, there must be a surjective d^2-differential $d^2 : E^2_{2,0} \to E^2_{0,1}$. But since $H_2(S^{2n+1}, \mathbb{Z}) = 0$, this differential must also be injective. Hence

$$\mathbb{Z} \cong E^2_{2,0} = H_2(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong H_2(\mathbb{CP}^n, \mathbb{Z}).$$

Furthermore, we see that $E^2_{1,0} = H_1(\mathbb{CP}^n, \mathbb{Z}) = 0$. Using $H_0(S^1, \mathbb{Z}) = H_1(S^1, \mathbb{Z})$, this implies $E^2_{1,1} = 0$ and $E_{2,1} = \mathbb{Z}$. Now we see that the 2-page looks like

$$\begin{tikzcd}[row sep=small] q & & & & \\ & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & 0 & \mathbb{Z} & ? \\ & \mathbb{Z} & 0 & \mathbb{Z} \arrow[ull] & ? \\ & & & & p \end{tikzcd}$$

By the same argument, we can deduce $d^2 : E_{4,0} \to E_{2,1}$ is an isomorphism, i.e. $H_4(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong \mathbb{Z}$, and $E_{3,0} = E_{3,1} = 0$ and so on. Since $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$, we cannot conlude that the $\mathbb{Z}$ in bidegree $(2n, 1)$ must be the image of a differental. There are two possibilities: If $d^2 : E^2_{2n+2,0} \to E^2_{2n+1}$ is the trivial map, then $E^2_{2n+2,0} = 0$ and then by induction $E^2_{p,q} = 0$ for all $p > 2n$. If, on the other hand, $d^2 : E^2_{2n+2,0} \to E^2_{2n,1}$ is non-zero, it has to be surjective: Indeed, since the cokernel is isomorphic to the lowest term of the filtration on $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$, and no $\mathbb{Z}/n\mathbb{Z}$ embeds into $\mathbb{Z}$. This then implies $H_k(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$ for all $k > 2n$. This case can be ruled out using that $\mathbb{CP}^n$ is a $2n$-dimensional CW-complex and hence $H_n(\mathbb{CP}^n, \mathbb{Z}) = 0$ for $k > 2n$. In summary, we obtain

$$H_k(\mathbb{CP}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, \ldots, 2n, \\ 0 & \text{else.} \end{cases}$$

Next we turn to an example where the Serre spectral sequence can be used to compute the homology of the fibre

Example 2.17. We consider the fibre sequence $\Omega S^3 \to * \to S^3$. On the E^2-page, we have the entries $H^p(S^3, H_q(\Omega S^3, \mathbb{Z}))$, i.e.

$$\begin{array}{c|cccc} q & & & & \\ & \vdots & \vdots & \vdots & \vdots \\ & ? & 0 & 0 & ? \\ & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \hline & & & & p \end{array}$$

The homology of the point is 0 in positive degrees, so we must have $E^\infty_{p,q} = 0$ unless $p = q = 0$. The only non-trivial differentials are $d^3 : E^3_{3,q} \to E^3_{0,q+2}$, so we conclude that these are isomorphisms. Hence $H_q(\Omega S^3, \mathbb{Z}) \cong H_{q+2}(\Omega S^3, \mathbb{Z})$. Note $E^{0,1} = 0$ since this entry cannot be killed by any differential. This implies

$$H_k(\Omega S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

In particular, ΩS^3 is an infinite-dimensional space.

We now discuss the cohomological version of the Serre spectral sequence and its multiplicative structure. This multiplication also helps in determining differentials, for example for the spectral sequences computing (co-)homology of unitary groups as above.

Definition 2.18. A *cohomologically graded spectral sequence* is a triple $(E_\bullet, d_\bullet, h_\bullet)$ where $(E_r)_r$ is a sequence of bigraded abelian groups, $(d_r : E_r \to E_r)_r$ is a sequence of differentials ($d_r \circ d_r = 0$) of bidegree $(r, 1 - r)$, and $(h_r : H_\bullet(E_r) \to E_{r+1})_r$ a sequence of bigrading-preserving isomorphisms.

As before, one defines first quadrant ($E_2^{p,q} = 0$ if $p < 0$ or $q < 0$) spectral sequences and the E_∞-page.

Rather than the filtrations $0 = F^{-1} \subseteq F^0 \subseteq \cdots \subseteq H$, one now considers filtrations

$$H = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

Definition 2.19. A cohomological first quadrant spectral sequence is said to *converge* to a filtered object (H, F) in graded abelian groups if there are isomorphisms $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$ for all p, q, and $F_p^n = 0$ for all $p > n$. Again we write $E_2^{p,q} \Longrightarrow H$.

Definition 2.20. A *(commutative) multiplicative structure* on a cohomologically graded spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ is a bigraded (commutative) ring structure on E_r, i.e. there are associative maps $E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$, such that d_r is a graded derivation, i.e.

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{p+q} x \cdot d_r(y)$$

for $x \in E_r^{p,q}$. Here (graded) commutative means $xy = (-1)^{(p+q)(p'+q')} yx$. As a result, $H_\bullet(E_r)$ is a bigraded ring and we further require that the h_r are isomorphisms of bigraded rings. Furthermore, the E_∞-page also inherits the structure of a (commutative) bigraded ring.

Definition 2.21. A filtration $\cdots \subseteq F_n \subseteq \cdots \subseteq F_1 \subseteq F_0 = H$ on a graded ring H is said to be *multiplicative* (or compatible with the multiplicative structure) if $F_s F_t \subseteq F_{s+t}$. We say that (H, F) is a *filtered graded ring.*

It follows that the associated graded object $\bigoplus F_p/F_{p+1}$ of a filtered graded (commutative) ring is a bigraded (commutative) ring.

Definition 2.22. A multiplicative first quardrant spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ is said to *converge* to a filtered graded ring (H, F) if it converges additively and the chosen isomorphism $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$ is compatible with the graded ring structure.

Theorem 2.23 (Serre)**.** *For every fibre sequence of spaces $(F \to Y \to X, h)$ with X simply connected and every abelian group A, there exists a cohomological first quadrant spectral sequence of the form*

$$E_2^{p,q} = H^p(X, H^q(F, A)) \quad \Longrightarrow \quad H^{p+q}(Y, A).$$

If A is a (commutative) ring, then the spectral sequence is multiplicative and converges multiplicatively, where on the E_2-page the multiplication is given by $(-1)^{p'q}$-times the composite

$$\begin{aligned} H^p(X, H^q(F, R)) \otimes H^{p'}(X, H^{q'}(F, R)) &\to H^{p+p'}(X, H^q(F, R) \otimes H^{q'}(F, R)) \\ &\to H^{p+p'}(X, H^{q+q'}(F, R). \end{aligned}$$

Note: If $H^\bullet(F, R)$ or $H^\bullet(X, R)$ is flat over R of finite type, then the E_2-page is isomorphic to the graded tensor product of $H^\bullet(X, R)$ and $H^\bullet(F, R)$.

Example 2.24. We reconsider the fibre sequence $U(1) \to U(2) \to S^3$ with $E_2^{p,q} = H^p(S^3, H^q(U(1), \mathbb{Z}))$, i.e.

$$\begin{array}{c|cccc} q & & & & \\ & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \hline & & & & p \end{array}$$

There cannot be non-trivial differentials. As a graded ring, the E_2-page (and hence also the E_∞-page) is isomorphic to $H^\bullet(S^3, \mathbb{Z}) \otimes H^\bullet(U(1), \mathbb{Z})$. Let $x_1 \in H^1(U(1), \mathbb{Z})$ and $x_3 \in H^3(S^3, \mathbb{Z})$ be generators.

Then $H^\bullet(U(1),\mathbb{Z}) \cong \bigwedge(x_1)$ and $H^\bullet(S^3,\mathbb{Z}) \cong \bigwedge(x_3)$, where $\bigwedge(M)$ denotes the exterior algebra on a set M, i.e. the free algebra on $x \in M$ modulo the relations $x_i x_j = -x_j x_i$ and $x_i^2 = 0$ for all $x_i, x_j \in M$. Hence, the E_2-page is isomorphic to $\bigwedge(x_1, x_3)$. The $\mathbb{Z}$ in bidegree $(3,1)$ is spanned by $x_1 x_3$. The filtration collapses degreewise and hence $H^*(U(2),\mathbb{Z})$ is exterior on classes $x_1 \in H^1(U(2),\mathbb{Z})$ and $x_3 \in H^3(U(2),\mathbb{Z})$ that are uniquely determined by the spectral sequence.

Example 2.25. We move on to the fibre sequence $U(2) \to U(3) \to S^5$. The E_2-page looks like the homological spectral sequence

$$\begin{tikzcd}
\mathbb{Z} \arrow[ddddrrrrr] & 0 & 0 & 0 & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 \\
Z & 0 & 0 & 0 & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z}
\end{tikzcd}$$

(axes: q vertical, p horizontal)

Keeping the notation from the previous example, denote generators of the first column by $1, x_1, x_3, x_1 x_3$. Again, the E_2-page is given by

$$H^\bullet(S^5,\mathbb{Z}) \otimes H^\bullet(U(2),\mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5),$$

where $x_5 \in H^5(S^5,\mathbb{Z})$ is a generator. There is one possibly non-trivial differential $d_5 : E_5^{0,4} \to E_5^{5,0}$. However, the product rule implies

$$d_5(x_1 x_3) = d_5(x_1)x_3 + (-1)^{1+0} x_1 d_5(x_3) = 0 + 0 = 0.$$

Again, the filtration collapses and hence $H^*(U(3),\mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5)$.

Example 2.26. We revisit $U(3) \to U(4) \to S^7$. The E_2-page now is

$$\begin{tikzcd}
\mathbb{Z} \arrow[ddddddrr] & & \mathbb{Z} \\
\mathbb{Z} & & \mathbb{Z} \\
0 & & 0 \\
\mathbb{Z} \arrow[ddddddrr] & & \mathbb{Z} \\
\mathbb{Z} & \vdots & \mathbb{Z} \\
\mathbb{Z} & \cdots\ 0\ \cdots & \mathbb{Z} \\
\mathbb{Z} & \vdots & \mathbb{Z} \\
0 & & 0 \\
\mathbb{Z} & & \mathbb{Z} \\
\mathbb{Z} & & \mathbb{Z} \\
0 & & 7
\end{tikzcd}$$

(axes: q vertical, p horizontal)

As before, the product rule implies that all d_7 differentials must be trivial, and $E_2 \cong \bigwedge(x_1, x_3, x_5, x_7)$. There is a non-trivial filtration on $H^8(U(4), \mathbb{Z})$ of the form

$$0 \to \mathbb{Z}(x_1x_7) \to H^8(U(4), \mathbb{Z}) \to \mathbb{Z}(x_3x_5) \to 0$$

Additively the sequence splits, but one has to be careful with the multiplicative structure. To resolve this, we need to be precise with the differentiation between the classes x_i on the E_∞-page and the corresponding classes $\overline{x_i} \in H^*(U(4), \mathbb{Z})$. Note that the choice of each $\overline{x_i}$ is unique since the filtration collapses in degrees 0 to 7. Furthermore, we record their filtrations $\overline{x_1}$ is in F_0^1, $\overline{x}_3$ is in F_0^3, $\overline{x}_5$ is in F_0^5 and $\overline{x}_7$ is in F_7^7. It follows that $\overline{x}_1\overline{x}_7$ is a generator of F_7^8, and $\overline{x}_3\overline{x}_5$ is a generator of F_0^8/F_1^8. Hence, $H^8(U(4), \mathbb{Z})$ is a free group on $\overline{x}_1\overline{x}_7$ and $\overline{x}_3\overline{x}_5$, and it follows that $H^\bullet(U(4), \mathbb{Z}) \cong \Lambda(x_1, x_3, x_5, x_7)$.

Theorem 2.27. *For all $n \in \mathbb{N}$, there is an isomorphism of graded rings*

$$H^\bullet(U(n), \mathbb{Z}) \cong \bigwedge(x_1, x_3, \ldots, x_{2n-1})$$

with x_i of degree i.

Proof. By induction on n, the start $n = 1$ is clear. Let $n \geq 2$ and assume we know the statement for $n-1$. We consider the Serre spectral sequence for the fibre sequence $U(n-1) \to U(n) \to S^{2n-1}$. By induction, its E_2-page is isomorphic to

$$E_2 \cong H^\bullet(S^{2n-1}, \mathbb{Z}) \otimes H^\bullet(U(n-1), \mathbb{Z}) = \bigwedge(x_{2n-1}) \otimes \bigwedge(x_1, x_3, \ldots, x_{2n-3}).$$

Here, x_i is a generator of $E_2^{0,i}$ for $i \leq 2n-3$ and x_{2n-1} is a generator of $E_2^{2n-1,0}$. The only possibly non-trivial differentials are d_{2n-1}. For degree reasons, d_{2n-1} vanishes on all generators $x_1, x_3, \ldots, x_{2n-1}$. By the product rule, all differentials are 0. Hence the E_∞-page is isomorphic to the E_2-page, and an exterior algebra $\bigwedge(x_1, x_3, \ldots, x_{2n-1})$.

The filtrations on $H^\bullet(U(n), \mathbb{Z})$ collapse in degrees $0, \ldots, 2n-2$, therefore we obtain unique lifts $\overline{x}_1, \ldots, \overline{x}_{2n-1} \in H^\bullet(U(n), \mathbb{Z})$. We only know from the spectral sequence that x_i^2 is of lower filtration hence a multiple of $\overline{x}_{2n-1}$, but not necessarily that $\overline{x_i}^2 = 0$. However we know from the additive structure (all subquotients are free over $\mathbb{Z}$) that $H^\bullet(U(n), \mathbb{Z})$ is torsionfree. As the multiplication is graded commutative, we hence have $\overline{x}_i^2 = 0$. We obtain a ring map $f : \bigwedge(x_1, \ldots, x_{2n-1}) \to H^\bullet(U(n), \mathbb{Z})$ by sending x_i to $\overline{x}_i$. To check that f is an isomorphism, define a grading on $\bigwedge(x_1, \ldots, x_{2n-1})$ by setting the degree of $x_1, \ldots, x_{2n-3}$ to be 0, and the degree of x_{2n-1} to be $2n-1$. This induces a filtrion by setting F_i to be the direct sum of the graded pieces of detree $2i$. Then f is filtration preserving and induces an isomorphism on associated graded pieces. The proof is finished by the following lemma. $\square$

Lemma 2.28. *Let A and B be graded abelian groups equipped with filtrations*

$$\cdots F_2 \subseteq F_1 \subseteq F_0 = A \qquad \text{and} \qquad \cdots G_2 \subseteq G_1 \subseteq G_0 = B$$

which are eventually 0 in every degree. If we havea graded filtration-preserving morphism $f : A \to B$ that induces an isomorphism on all associated graded pieces $F_i/F_{i+1} \cong G_i/G_{i+1}$, then it is an isomorphism.

Proof. This is an iterated 5-lemma argument. $\square$

Example 2.29. We revisit the fibre sequence $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and use the Serre spectral sequence to compute the ring $H^\bullet(\mathbb{CP}^n, \mathbb{Z})$. Arguing analoguously to the homological case, the E_2-page looks as follows:

$$\begin{tikzcd}
\mathbb{Z} \arrow[drr, "\cong"] & 0 & \mathbb{Z} \arrow[drr, "\cong"] & 0 & \cdots \arrow[drr, "\cong"] & 0 & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\
0 & & & & & & 2n
\end{tikzcd}$$

Let e be a generator for $E_2^{0,1}$. Then $x = d_2(e)$ is a generator of $E_2^{2,0}$ and ex is a generator for $E_2^{2,1}$. Hence $d_2(ex)$ generates $E_2^{4,0}$ By the product rule, we have $d_2(ex) = d_2(e)x + (-1)^1 ed_2(x) = x^2$. Hence x^2 generates $H^4(\mathbb{CP}^n, \mathbb{Z})$. Similarly, $d_2(ex^2) = x^3$ generates $E_2^{6,0} = H^6(\mathbb{CP}^n, \mathbb{Z})$, and so on. We obtain an isomorphism $H^\bullet(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$ and $H^\bullet(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[x]$.

Example 2.30. Next we compute the ring structure on $H^\bullet(\Omega S^2, \mathbb{Z})$. Dual to the homological case, we obtain the following E_2-page corresponding to the fibre sequence $\Omega S^3 \to * \to S^3$:

$$\begin{tikzcd}
\vdots & & & \vdots \\
\mathbb{Z} \arrow[ddrrr, "\cong"] & 0 & 0 & \mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} \arrow[ddrrr, "\cong"] & & & \mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} \arrow[ddrrr, "\cong"] & & & \mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}
\end{tikzcd}$$

Let $x \in H^2(\Omega S^3, \mathbb{Z}) = E_3^{0,2}$ be a generator. Then $e = d_3(x)$ generates $E_3^{3,0}$, and ex generates $E_3^{3,2}$. We have $d_3(x^2) = d_3(x)x + (-1)^2 xd_3(x) = 2ex$. In particular, x^2 is stwice a generator of $H^4(\Omega S^3)$, which we denote $x^2/2$. Again we repeat this: $d_3(x^3) = d_3(x)x^2 + xd_3(x^2) = 6e(x^2/2)$. Inductively, we get that x^n is $n!$ times a generator of $H^{2n}(\Omega S^3, \mathbb{Z})$. We get an isomorphism $H^\bullet(\Omega S^3, \mathbb{Z}) \cong \mathbb{Z}[x, \frac{x^2}{2}, \ldots, \frac{x^n}{n!}, \ldots] \subseteq \mathbb{Z}[x]$ or $H^\bullet(\Omega S^3, \mathbb{Z}) \cong \mathbb{Z}[x_i]/\binom{m+n}{n}x_{m+n} = x_m x_n$. This is called the divided power algebra on one generator $\Gamma(x)$. This is not finitely generated as a ring.

Remark: There is also a ring structure on the homology of $H_\bullet(\Omega S^3, \mathbb{Z})$ induced by the H-space (in fact A_∞- or E_1-space) structure by concatenation of loops. One can show that with this ring structure, $H_\bullet(\Omega S^3, \mathbb{Z})$ is a polynomial ring on a generator in degree 2. More generally, if $H_\bullet(X, \mathbb{Z})$ is free over $\mathbb{Z}$, then $H_*(\Omega\Sigma X, \mathbb{Z})$ is the tensor algebra on $H_*(X, \mathbb{Z})$ (Bott-Samelson theorem) and the map $H_\bullet(X, \mathbb{Z}) \to H_*(\Omega\Sigma X, \mathbb{Z}) = T(H_\bullet(X, \mathbb{Z}))$ is induced by the natural map $X \to \Omega\Sigma X$.

Remark: ΩS^3 vs. $\mathbb{CP}^\infty$. They have isomorphic homology groups, they both have CW-structures (up to homotopy equivalence for ΩS^3) with one cell in every even dimension, but very different homotopy groups: $\pi_n(\mathbb{CP}^\infty, *) = 0$ for $n > 2$ and $\pi_n(\Omega S^3, *) \cong \pi_{n+1}(S^3, *) \neq 0$ for all $n > 2$. Their cohomology rings are non-isomorphic (see above). The attaching maps $S^3 \to S^2$ for 4-cell for $\mathbb{CP}^\infty$ is a generator of $\pi_3(S^2, *)$, for ΩS^3 it is twice a generator. Both are loopspaces ($\mathbb{CP}^2 \simeq \Omega K(\mathbb{Z}, 3)$), but $H_*(\Omega S^3, \mathbb{Z})$ is polynomial and $H_*(\mathbb{CP}^\infty, \mathbb{Z})$ is a divided power algebra.

Example 2.31. We consider the map $S^3 \to K(\mathbb{Z}, 3)$ classifying a generator of $H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$. (equivalently, inducing an isomorphism on $\pi_3(-, *)$). Let X denote the homotopy fibre, so that $X \to S^3 \to$

$K(\mathbb{Z},3)$ is a fibre sequence. By the long exact sequence in homotopy: X is 3-connected and $\pi_n(X,*) \cong \pi_n(S^3,*)$ for $n \geq 4$. We want to understand (co-)homology of X. As we don't know the (co-)homology of $K(\mathbb{Z},3)$ yet, we take a second homotopy fibre, yielding a fibre sequence $\mathbb{CP}^\infty \cong \Omega K(\mathbb{Z},3) \to X \to S^3$. We obtain the following Serre spectral sequence: $E_2^{p,q} = H^p(S^3, H^q(\mathbb{CP}^\infty,\mathbb{Z})) \cong H^p(S^3,\mathbb{Z}) \otimes H^q(\mathbb{CP}^\infty,\mathbb{Z})$.

$$
\begin{tikzcd}[row sep=small]
q & & & & \\
\vdots & & & \vdots & \\
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & \\
0 & & & 0 & \\
\mathbb{Z} \arrow[ddrrr] & & & \mathbb{Z} & \\
0 & & & 0 & \\
\mathbb{Z} \arrow[ddrrr] & & & \mathbb{Z} & \\
0 & & & 0 & \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & \\
& & & p &
\end{tikzcd}
$$

Since X is 3-connected, $d_3 : E_2^{0,2} \to E_2^{3,0}$ must be an isomorphism. Let $x \in E_2^{0,2}$ be a generator. Then $x^i \in E_2^{0,2i}$ will be a generator, and by the product rule $d_3(x^2) = 2d_3(x)x$ is twice a generator. Inductively, $d_3(x^n)$ is n times a generator of $E_2^{3,2n-2}$. Thus the E_∞-page is

$$
\begin{array}{c|cccc}
q & & & & \\
& \vdots & & & \vdots \\
& 0 & 0 & 0 & \mathbb{Z}/4\mathbb{Z} \\
& 0 & & & 0 \\
& 0 & & & \mathbb{Z}/3\mathbb{Z} \\
& 0 & & & 0 \\
& 0 & & & \mathbb{Z}/2\mathbb{Z} \\
& 0 & & & 0 \\
& \mathbb{Z} & 0 & 0 & 0 \\
\hline
& & & & p
\end{array}
$$

and hence

$$\widetilde{H}^n(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if} n = 2k+1, \\ 0 & \text{else} \end{cases}$$

with trivial cup product. By the universal coefficient theorem we get $\widetilde{H}_n(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if } k = 2n, \\ 0 & \text{else.} \end{cases}$

Corollary 2.32. *We have $\pi_4(S^3,*) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_4(S^2,*) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. X is 3-connected, and $H_4(X,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Hurewicz theorem implies $\pi_4(X,*) = \mathbb{Z}/2\mathbb{Z}$ and we saw that $\pi_4(S^3,*) = \pi_4(X,*)$. By the Hopf fibration $(S^1 \to S^3 \to S^2)$, the same is true for S^2. □

We now know the following homotopy groups of spheres:

	π_1	π_2	π_3	π_4	π_5	π_6
S^1	$\mathbb{Z}$	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^6	0	0	0	0	0	$\mathbb{Z}$

where $\pi_5(S^4, *) = \pi_6(S^5, *) = \mathbb{Z}/2\mathbb{Z}$ follows from the Freudenthal suspension theorem.

3 Construction of the Serre spectral sequence

We focus on the cohomological version. Roughly speaking we proceed via

double complexes $\Rightarrow$ filtered complexes $\Rightarrow$ exact couple $\Rightarrow$ spectral sequence.

Definition 3.1. An exact couple is a pair of abelian groups (A, E) together with a triangle

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which is exact, i.e. exact at each of the three corners.

Define $d_1 : E \to E$ via $d_1 = j \circ k$. We have $d_1^2 = j \circ k \circ j \circ k = j \circ 0 \circ k = 0$, so d_1 is a differential. We can define $H(E) = \ker(d_1)/\operatorname{im}(d_1)$, and claim that there is a new triangle

$$\begin{tikzcd} A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd} \tag{$*$}$$

defined via $E_2 = H(E)$, $A_2 = \operatorname{im}(i) \subseteq A$, $i_2 = i|_{A_2}$. For $a \in A_2$ write $a = i(b)$ for some $b \in A$. Then $j_2(a) := [j(b)]$. This is well-defined: $d_1(j(b)) = j(k(j(b))) = 0$, and if $i(b) = i(b')$, then $b - b' \in \ker i = \operatorname{im} k$, so $j(b - b') \in \operatorname{im}(j \circ k) = \operatorname{im} d_1$. Finally, $k_2([e]) := k(e)$: Since $j(k(e)) = 0$, exactness implies $k(e) \in \operatorname{im} i$, and if $e \in \operatorname{im} d_1$, then $e \in \operatorname{im} j$ and $k(e) = 0$.

Lemma 3.2. *The triangle (∗) is again an exact couple.*

Proof. Diagram chase (omitted). □

Hence we can iterate and obtain a sequence of exact couples (A_n, E_n) with maps i_n, j_n, k_n. In particular we obtain a sequence of abelian groups E_n with differentials $d_n = j_n \circ k_n$ and isomorphisms $H(E_n) \cong E_{n+1}$. This is like a spectral sequence, except we are missing the bigrading.

For the Serre spectral sequence the two gradings play different roles: A filtration degree (x-axis), and the difference between the cohomological degree and the filtration degree.

Definition 3.3. An unrolled exact couple is a collection of pairs $(A^s, t^s)_{s\in\mathbb{Z}}$ of abelian groups with maps

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr, "i"] & & A^{s+1} \arrow[dl, "j"] \arrow[rr, "i"] & & A^s \arrow[dl, "j"] \arrow[rr, "i"] & & A^{s-1} \arrow[dl, "j"] \arrow[rr] & & \cdots \\
& \cdots & & E^s \arrow[ul, "k"] & & E^{s-1} \arrow[ul, "k"] & & &
\end{tikzcd}$$

Every unrolled exact couple gives an exact couple via $A = \bigoplus_s A^s,\ E = \bigoplus_s E^s$ combined in a single triangle. We obtain a cochain complex

$$\cdots \xrightarrow{j\circ k} E^{s-1} \xrightarrow{j\circ k} E^s \xrightarrow{j\circ k} E^{s+1} \to \cdots$$

Hence $H_*(E)$ inherits a grading, i.e. $H(E) = \bigoplus_j H^j(E)$. Generally we can write $E_r = \bigoplus_s E_r^s \cong \bigoplus_s H^s(E_{r-1})$. Assume $e \in E^s$, we can chase through its "life" in the spectral sequence: If $d_1(e) \neq 0$, then e does not define a class in $H^\bullet(E)$. If $d_1(e) = 0$, then $[e] \in H^s(E) = E_2^s$. In the unrolled picture, $d_2[e] = j_2 k_2(e)$ is computes as follows: By exactness, $k(e) \in \mathrm{im}(i)$, say $k(e) = i(b)$, and $d_2([e]) = [j(b)]$. If $d_2([e]) \neq 0$, e does not define a class in $H(E_2)$. If $d_2(e) = 0$, we can continue in this way. In general, if $k(e) = i^r(b)$ for some $r \geq 0$ and $b \in A^{s+r+1}$, then e defines a class in $E_{r+1}^s = H^s(E_r)$ and $d_{r+1}([e])$ is $j(b)$. If e survives in every step, it is called a permanent cycle. We note: If $e \in E_r^s$, then $d_r([e])$ is represented by some element of E_r^{s+r}, i.e. d_r raises the filtration degree by r.

A filtered cochain complex is a cochain complex $C^\bullet$ together with a sequence of subcomplexes

$$\ldots \subseteq F^2C^\bullet \subseteq F^1C^\bullet \subseteq F^0C^\bullet = C^\bullet$$

For convenience, we extend the filtration grading to $\mathbb{Z}$ via $F^sC^\bullet = C^\bullet$ for $s < 0$. The associated graded complex is the collection of subquotients

$$\mathrm{gr}^s C^\bullet = F^sC^\bullet / F^{s+1}C^\bullet.$$

The corresponding short exact sequences induce a long exact sequence

$$\cdots \to H^t(F^{s+1}C^\bullet) \to H^t(F^sC^\bullet) \to H^t(\mathrm{gr}^s(C^\bullet)) \to H^{t+1}(F^{s+1}C^\bullet) \to \cdots$$

Taking the direct sum over all t, we obtain

$$\begin{tikzcd}[column sep=tiny]
\cdots \arrow[rr, "i"] & & H^\bullet(F^{s+1}C^\bullet) \arrow[dl, "j"] \arrow[rr, "i"] & & H^\bullet(F^sC^\bullet) \arrow[dl, "j"] \arrow[rr, "i"] & & H^\bullet(F^{s-1}C^\bullet) \arrow[dl, "j"] \arrow[rr] & & \cdots \\
& \cdots & & H^\bullet(\mathrm{gr}^s(C^\bullet)) \arrow[ul, "k"] & & H^\bullet(\mathrm{gr}^s(C^\bullet)) \arrow[ul, "k"] & & &
\end{tikzcd}$$

in which each triangle is exact. We observe that i and j preserve the cohomological degree, but k raises it by 1. We hence obtain an unrolled exact couple with an additional cohomological degree, and an exact couple with

$$A = \bigoplus_{s,t} H^t(F^sC^\bullet), \qquad E = \bigoplus_{s,t} H^t(\mathrm{gr}^s C^\bullet)$$

and an associated spectral sequence. What does it converge to? We define a filtration on $H^\bullet(C)$ by setting $F^sH^t(C^\bullet) = \mathrm{im}(H^t(F^sC^\bullet) \to H^t(C^\bullet))$.

Theorem 3.4. *If for every t, the cohomology $H^t(F^sC^\bullet)$ is 0 for s large enough, the spectral sequence associated to the exact couple constructed above converges to $(H^\bullet C^\bullet, F^sH^\bullet(C^\bullet))$, with E_1-page $E_1^{s,t} = H^t(\mathrm{gr}^s(C^\bullet))$.*

The gradings of this spectral sequence are different to the one for the Serre spectral sequence: d_r raises the filtration degree by r and the cohomological degree by 1.

If $C^\bullet$ is concentrated in nonnegative degree, we get a first quadrant spectral sequence with the cohomological degree remaining constant along rows. For the Serre spectral sequence, we use cohomological degree minus filtration degree for the vertical axis. This regrading remains in the first quadrant because all terms with filtration degree larger than cohomological degree are trivial.

Proof. (of 3.4) In the exact couple, A_r is the direct sum over all s of

$$A_r^s := \operatorname{im}(i^{r-1} : H^\bullet(F^{s+r-1}C) \to H^\bullet(F^sC))$$

For $t \in \mathbb{Z}$ we set $n_t \in \mathbb{N}$ to be the minimum over all n such that $H^t(F^nC) = 0$. Then for $r \geq n_t + 1$, either $s > 0$, i.e. $s + r - 1 > n_t$, so $H^t(F^{s+r-1}C) = 0$, so $A_r^s = 0$, or $s \leq 0$, then $H^t(F^sC) = H^tC$, and $A_r^s = F^{s+r-1}H^tC$. So

$$A_r^t = \bigoplus_{s\leq 0} F^{s+r-1}H^t(C) = \bigoplus_{0\leq p\leq n_t} F^pH^t(C).$$

This is independent of r if $r \geq n_t+1$. Let A_∞^t be this value. The map $i_r : A_r^t \to A_r^t$ is the direct sum over the inclusions $F^{p+1}H^t(C) \to F^pH^t(C)$. In particular, it is injective, so by exactness $k_r : E_r^{t-1} \to A_r^t$ must be zero. Thus for r large, all differentials are zero and the terms E_r^t stabilize as well. Moreover, by exactness of

$$\begin{tikzcd} A_\infty \arrow[rr, "i_r"] & & A_\infty \arrow[dl, "j_r"] \\ & E_\infty \arrow[ul, "0"] & \end{tikzcd}$$

we have

$$E_\infty^t \cong \operatorname{coker}(i_r) \cong \bigoplus_{p\leq n_t} F^pH^t(C)/F^{p+1}H^t(C).$$

□

Example 3.5. Let X be a CW-complex with skeleta

$$\operatorname{sk}_0 X \subseteq \operatorname{sk}_1 X \subseteq \ldots \subseteq \operatorname{sk}_n \subseteq \ldots$$

We can filter $C^\bullet(X, A)$ by

$$F^sC^\bullet(X, A) = \ker(C^\bullet(X, A) \to C^\bullet(\operatorname{sk}_s X, A) =: C^\bullet(X, \operatorname{sk}_s X, A)$$

Then we obtain a spectral sequence with E_1-page

$$H^\bullet(C^\bullet(X, \operatorname{sk}_s X, A)/C^\bullet(X, \operatorname{sk}_{s+1} X, A) \cong H^t(\operatorname{sk}_{s+1} X, \operatorname{sk}_s X, A) \cong \widetilde{H}^t(\operatorname{sk}_{s+1} X/\operatorname{sk}_s X, A)$$

It converges to $H^\bullet(C^\bullet(X, A)) = H^\bullet(X, A)$. Note that on the E_1-page only the entries on the main diagonal are nontrivial, and these are exactly the cellular cochain complex. For degree reasons, the spectral sequence degenerates at the E_2-page, where the entries along the main diagonals are the homology of the cellular cochain complex, i.e. the cellular cohomology. Thus, this reproves that the cellular cochain complex computes cohomology.

Example 3.6. Let $p : Y \to X$ be a Serre fibration with X a CW-complex. Then we can filter Y via the preimages $p^{-1}(\operatorname{sk}_s X)$ and obtain a filtration on $C^\bullet(Y, A)$. The resulting spectral sequence "is" the Serre spectral sequence. However, some aspects (in particular, the multiplicative structure) are clearer in the construction with double complexes.

The Spectral Sequence of a Double Complex

Definition 3.7. A double complex is a bigraded abelian group $C^{\bullet,\bullet}$ equipped with two differentials

$$\delta_h : C^{p,q} \to C^{p+1.q}, \quad \text{and} \quad \delta_v : C^{p,q} \to C^{p,q+1}$$

satisfying $\delta_h^2 = 0 = \delta_v^2$ and $\delta_h\delta_v = \delta_v\delta_h$, i.e. there is an infinite commutative diagram

$$\begin{tikzcd}
C^{p-1,q+1} \arrow[r, "\delta_h"] & C^{p,q+1} \arrow[r] & C^{p+1,q+1} \arrow[r] & \cdots \\
C^{p-1,q} \arrow[r] \arrow[u, "\delta_v"] & C^{p,q} \arrow[r] \arrow[u] & C^{p+1,q} \arrow[r] \arrow[u] & \cdots \\
C^{p-1,q-1} \arrow[r] \arrow[u] & C^{p,q-1} \arrow[r] \arrow[u] & C^{p+1,q-1} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

with all rows and columns complexes. The "vertical cohomology groups" $H^q_{\delta_v}(C^{p,\bullet})$ inherit a "horizontal" differential $\delta_h : H^q_{\delta_v}(C^{p,\bullet}) \to H^q_{\delta_v}(C^{p+1,\bullet})$. We write $H^p_{\delta_h}H^q_{\delta_v}(C^{\bullet,\bullet})$ for the resulting cohomology groups.

Example 3.8. D_1, D_2 cochain complexes. Then the tensor products $D_1^p \otimes D_2^q$ form a double complex with δ_h from D_1 and δ_v from D_2.

Definition 3.9. Let $(C^{\bullet,\bullet}, \delta_h, \delta_v)$ a double complex. Its *total complex* $\mathrm{Tot}(C)$ is the cochain complex $\mathrm{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q}$ and $\delta = \delta_h + (-1)^p\delta_v$.

Note: The $(-1)^p$ is needed to guarantee $\delta^2 = 0$.

A double complex $C^{\bullet,\bullet}$ can be filtered by

$$F^s(C)^{p,q} = \begin{cases} C^{p,q} & \text{if } p \geq s, \\ 0 & \text{else.} \end{cases}$$

This induces a filtration on $\mathrm{Tot}(C)$ via $F^s\,\mathrm{Tot}(C) = \mathrm{Tot}(F^s(C))$. Then $\mathrm{gr}_s\,\mathrm{Tot}(C)^t = C^{s,t-s}$ with differential $(-1)^s\delta_v$. The sign doesn't affect cohomology, hence $H^t(\mathrm{gr}_s\,\mathrm{Tot}(C)) = H^{t-s}_{\delta_v}(C^{s,\bullet})$. We hence get a spectral sequence with E_1-page $E_1^{s,t} = H^1(\mathrm{gr}_s\,\mathrm{Tot}(C)) = H^{t-s}_{\delta_v}(C^{s,\bullet})$. Moreover, d_1 agrees with the horizontal differentials δ_h (exercise), so $E_2^{s,t} \cong H^s_{\delta_h}H^{t-s}_{\delta_v}(C^{\bullet,\bullet})$. If $C^{p,q} = 0$ if $p < 0$ or $q < 0$, then the E_1-page is concentrated in degrees $t \geq s \geq 0$ and the spectral sequence converges to the $H^\bullet(\mathrm{Tot}(C))$. It is hence customary to reindex to $(s, t-s)$ and obtain a first quadrant spectral sequence with Serre grading $H^s_{\delta_h}H^t_{\delta_v}(C) \Rightarrow H^{s+t}(\mathrm{Tot}(C))$.

Remark: Swapping the horizontal and vertical direction yields a different spectral sequence also converging to $H^\bullet(\mathrm{Tot}(C))$.

Dress's construction of the Serre spectral sequence

Let $f : E \to B$ be a Serre fibration. A singular (p,q)-simplex of f is a commutative diagram

$$\begin{tikzcd}
\Delta^p \times \Delta^q \arrow[r] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "f"] \\
\Delta^p \arrow[r] & B
\end{tikzcd}$$

Let $C_{p,q}(f)$ be the free abelian group on all singular (p,q)-simplices. There is a differential δ_h : $C_{p,q}(f) \to C_{p-1,q}(f)$ by taking the alternating sum of the faces of the p-simplex, i.e. if $d_i : \Delta^{p-1} \to \Delta^p$

is the i-th face of Δ^p, we consider

$$\begin{array}{ccccc} \Delta^{p-1}\times\Delta^q & \xrightarrow{d_i\times\mathrm{id}} & \Delta^p\times\Delta^q & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f \\ \Delta^{p-1} & \xrightarrow{d_i} & \Delta^p & \longrightarrow & B \end{array}$$

Similarly, one obtains a vertical differential $\delta_v : C_{p,q}(f) \to C_{p,q-1}(f)$ as the alternating sum of the faces of the q-simplex, obtained as

$$\begin{array}{ccccc} \Delta^{p}\times\Delta^{q-1} & \xrightarrow{\mathrm{id}\times d_i} & \Delta^p\times\Delta^q & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f \\ \Delta^{p} & \longrightarrow & \Delta^p & \longrightarrow & B \end{array}$$

By dualizing oe obtain a double complex $C^{\bullet,\bullet}(f,A) = \mathrm{Hom}(C_{\bullet,\bullet}(f),A)$ for every coefficient group A. We have $C^{p,q}(f,A) = 0$ for $p < 0$ or $q < 0$, hence we obtain a Serre-graded first quadrant spectral sequence with E_2-page $E_2^{s,t} = H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}(f,A))$ converging to $H^\bullet(\mathrm{Tot}(C^{\bullet,\bullet}(f,A)))$. This is the Serre spectral sequence. We have to show that the E_2-page and the limit agree with $H^\bullet(B,H^\bullet(F,A))$ and $H^\bullet(E,A)$, respectively.

We start with $H^\bullet(\mathrm{Tot}(C^\bullet(f,A))) \cong H^\bullet(E,A)$. For this we swap the horizontal and vertical directions and obtain another spectral sequence $H^s_{\delta_v} H^t_{\delta_h}(C(f,A)) \Rightarrow H^\bullet(\mathrm{Tot}(C(f,A)))$. Claim: The E_2-page is 0 if $s \neq 0$ and $H^0_{\delta_v} H^t_{\delta_h}(C(f,A)) \cong H^t(E,A)$.

We fix $s \geq 0$ and consider diagrams

$$\begin{array}{ccc} \Delta^t\times\Delta^s & \longrightarrow & E \\ \downarrow \mathrm{pr}_1 & & \downarrow f \\ \Delta^t & \longrightarrow & B \end{array}$$

We can rewrite this to

$$\begin{array}{ccc} \Delta^t & \longrightarrow & \mathrm{map}(\Delta^s,E) \\ \downarrow & & \downarrow \mathrm{map}(\Delta^s,f) \\ B & \xrightarrow{\mathrm{const}} & \mathrm{map}(\Delta^s,B) \end{array}$$

This in turn is equivalent to a single map

$$\Delta^t \to B \times_{\mathrm{map}(\Delta^s,B)} \mathrm{map}(\Delta^s,E) =: P,$$

i.e. a singular t-simplex of the space P. One checks that in fact $C^{s,\bullet}(f,A)$ is isomorphic to $C^\bullet(P,A)$. Now, Δ^s is contractible, hence $B \to \mathrm{map}(\Delta^s,B)$ is a homotopy equivalence. Since f is a Serre fibration, it follows that $P \to \mathrm{map}(\Delta^s,E)$ is a weak homotopy equivalence. In particular, $H^t_{\delta_h}(C^{\bullet,\bullet}(f,A)) \cong H^t(E,A)$ forall $t \geq 0$. Under these identifications, every face map of Δ^s induces the identity on these groups. Hence the complex computing $H^s_{\delta_v} H^t_{\delta_h}(E,A)$ equals

$$H^t(E,A) \xrightarrow{0} H^t(E,A) \xrightarrow{\mathrm{id}} H^t(E,A) \xrightarrow{0} H^t(E,A) \xrightarrow{\mathrm{id}} \cdots$$

thus

$$H^s_{\delta_v} H^t_{\delta_h}(C(f,A)) = \begin{cases} H^t(E,A) & s = 0 \\ 0 & s > 0. \end{cases}$$

It follows that $E_\infty = E_2$ and $H^t(\mathrm{Tot}(C(f,A))) \cong H^t(E,A)$.

It remains to compute the E_2-term.

Definition 3.10. The *fundamental grupoid* $\pi_1 X$ of a space X is the category with objects the points of X and morphisms $\mathrm{Hom}_{\pi_1 X}(x,y) = \{\text{homotopy classes of paths } \gamma \text{ from } x \text{ to } y\}$. Composition is the concatenation of paths.

$\pi_1 X$ is a groupoid, i.e. every morphism is invertible. Furthermore, $\mathrm{Hom}_{\pi_1 X}(x,x) = \pi_1(X,x)$.

Definition 3.11. A *local system* M on X is a functor $M : \pi_1 X \to \mathtt{Ab}$. We write M_x for $M(x)$

Note: If X is path-connected, $\pi_1 X$ is equivalent to the groupoid with one object $x \in X$ and automorphisms $\pi_1(X,x)$. Hence a local system is equivalent datum to an abelian group A with action by $\pi_1(X,x)$.

If X is even simply-connected, every local system is isomorphic to the constant local system for an abelian group A.

Example 3.12. Let $f : E \to B$ be a Serre fibration, A an abelian group and $q \in \mathbb{N}$. We write $F_x := f^{-1}(x)$ for the fibre of $x \in X$. We obtain a local system $x \mapsto H_q(F_x, A)$. On homotopy classes of paths $[\gamma : x \to y]$, this is defined as follows: Consider the pullback

$$\begin{tikzcd} F_\gamma \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ I \arrow[r, "\gamma"] & B \end{tikzcd}$$

which comes with weak homotopy equivalences $F_x \to F_\gamma, F_y \to F_\gamma$. Hence on homology we obtain an induced map

$$H_p(\gamma, A) : H_q(F_x, A) \xrightarrow{\simeq} H_q(F_\gamma, A) \xleftarrow{\simeq} H_q(F_y, A).$$

To show compatability with composition and homotopy invariance, consider pullbacks of f along maps $\Delta^2 \to B$.

Similarly, one obtains a local system $x \mapsto H^q(F_x, A)$.

Example 3.13. Let M be a topological manifold of dimension n, then the assignment $x \mapsto M_x = H_n(M, M \setminus \{x\}, \mathbb{Z})$ extends to a local system. For a path $\gamma : x \to y$, cover the path by contractible open sets U_i and use that $H_n(M, M \setminus \{x\}, \mathbb{Z}) \cong H_n(M, M \setminus U_i, \mathbb{Z})$. iteratively. M is orientable iff this local system is isomorphic to the constant one on $\mathbb{Z}$.

Next we define (co-)homology with coefficients in a local system M on X. We set $C_n(X,M) = \bigoplus_{\sigma:\Delta^n \to X} M_{\sigma_0}$ where $\sigma_0 \in X$ is the image of the 0th vertex of Δ^n. There is a differential

$$d : C_n(X,M) \to C_{n-1}(X,M), \quad d(\sigma, m) = (\sigma \circ d_0, (\sigma_{0,1})_*(m)) + \sum_{i=1}^{n} (-1)^i (\sigma \circ d_i, m)$$

where $\sigma_{0,1}$ is the image under σ of any path from the 0th vertex to the 1st vertex. Note that $\sigma \circ d_0$ has 0th vertex equal to the first vertex of σ, while all the other $\sigma \circ d_i$ have $(\sigma \circ d_i)_0 = \sigma_0$.

We omit the proof that this defines a chain complex.

Definition 3.14. For a local system M, we define $H_\bullet(X,M)$ as the homology of this chain complex.

If M is constant, this recovers the ordinary homology. A map of local systems $M \to N$ induces maps $H_\bullet(X,M) \to H_\bullet(X,N)$. A map of spaces $f : X' \to X$ and a local system M on X, we obtain a pullback local system f^*M and a map $H_\bullet(X', f^*M) \to H_\bullet(X,M)$, where $(f^*M)_x = M_{f(x)}$.

Let M be a local system on a space X. Then $C^\bullet(X,M)$ is defined as $C^n(X,M) = \prod_{\sigma:\Delta^n \to X} M_{\sigma_0}$

with differential
$$df(\sigma) = M(\sigma_{01})^{-1}(f(d^0\sigma)) + \sum_{i=1}^{n}(-1)^i f(d^i\sigma).$$

Definition 3.15. Cohomology with local coordinates is defined as $H^\bullet(X, M) = H^\bullet(C^\bullet(X, M))$.

Example 3.16. One can show that using local coefficients there is a version of Poincaré duality without an orientation assumption: If M is a compact topological manifold of dimension n, there is a fundamental class $[M] \in H_n(M, \mathbb{Z}^{or})$, where $\mathbb{Z}^{or}$ is the local system from example 3.13, such that there are isomorphisms $H^\bullet(M, \mathbb{Z}) \to H_{n-\bullet}(M, \mathbb{Z}^{or})$ and $H^\bullet(M, \mathbb{Z}^{or}) \to H_{n-\bullet}(M, \mathbb{Z})$

We now go back to the spectral sequence constructed out of the double complex $C^{\bullet,\bullet}(p, A)$ for a Serre fibration $p : E \to B$. We already know that it is a first quadrant spectral sequence converging to $H^\bullet(E, A)$. It remains to understand the E_2-page. We fix a map $\sigma \Delta^p \to B$ and consider a square

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta^p \arrow[r, "\sigma"] & B \end{tikzcd}$$

This is equivalent to a square

$$\begin{tikzcd} \Delta^q \arrow[r] \arrow[d] & E^{\Delta^p} \arrow[d] \\ * \arrow[r, "\sigma"] & B^{\Delta^p} \end{tikzcd}$$

This in turn is equivalent to a map $\Delta^q \to F_\sigma$, where F_σ is the fibre of the fibration $E^{\Delta^p} \to B^{\Delta^p}$ over the point σ. Thus the columns of the double complex are isomorphic to a product over all maps $\sigma : \Delta^p \to X$ of the singular cochain complex of F_σ. Again, F_σ is weakly homotopy equivalent to the fibre F_{σ_0} of the original fibration $p : E \to B$ over the 0th vertex σ_0, using that Δ^p is contractible. Hence, the vertical cohomology of the double complex are the products

$$\prod_{\sigma:\Delta^p \to B} H^q(F_\sigma, A) \cong \prod_{\sigma:\Delta^p \to B} H^q(F_{\sigma_0}, A).$$

In this decomposition, the horizontal differentials work as follows: For $i > 0$, the triangle

$$\begin{tikzcd} H^q(F_{\sigma_0}, A) & \\ H^q(F_\sigma, A) \arrow[u, "\cong"] \arrow[r, "d^i"] & H^q(F_{\sigma \circ d_i}, A) \arrow[ul, "\cong"'] \end{tikzcd}$$

commutes. For $i = 0$ we have a commuting square

$$\begin{tikzcd} H^q(F_{\sigma_0}, A) \arrow[r, "H^q(\gamma, A)", "\cong"'] & H^q(F_{(\sigma \circ d_0)_0}, A) \\ H^q(F_\sigma, A) \arrow[u, "\cong"] \arrow[r, "d_0"] & H^q(F_{\sigma \circ d_0}, A) \arrow[u, "\cong"] \end{tikzcd}$$

where γ is the image in X of any path in Δ^p from e_0 to e_1. Hence, the vertical cohomologies equipped with the horizontal differential are isomorphic to the cochain complex $C^\bullet(X, H^q(B, A))$. This shows that the E_2-page is given by $E_2^{p,q} \cong H^p(B, H^q(F_-, A))$ as claimed.

Analogously, the spectral sequence associated to the double complex $C_{\bullet,\bullet}(p, A)$ yields the homological Serre spectral sequence, with E_2-term the local system homology $H_p(B, H_q(F, A))$.

It remains to discuss multiplicative properties.

Definition 3.17. A multiplicative structure on a double complex $C^{\bullet,\bullet}$ is a collection of maps $\mu : C^{p,q} \otimes C^{p',q'} \to C^{p+p',q+q'}$ that are associative and unital (with unit 1 in degree $(0, 0)$). Moreover, the differential $\delta = \delta_h + (-1)^p\delta_v$ of $\mathrm{Tot}(C)$ satisfies the Leibniz rule, i.e. $\delta(xy) = \delta(x)y + (-1)^{p+q}x\delta(y)$.

Chasing through our constructions, we find

Proposition 3.18. *The spectral sequence associated to a multiplicative double complex is multiplicative.*

To apply this, we define a multiplicative structure on $C^{\bullet,\bullet}(p, R)$ where R is a ring, and $p : E \to B$ a Serre fibration. Recall that for cochains $\varphi \in C^p(X, R), \psi \in C^q(X, R)$ and $\sigma : \Delta^{p+q} \to X$ one defines

$$(\varphi \smile \psi)(\sigma) = \varphi(d^*_{p\text{-front}}\sigma)\psi(d^*_{q\text{-back}}\sigma).$$

Similarly, if $\varphi \in C^{p,q}(p, R), \psi \in C^{p',q'}(p, R)$ and a $(p + p', q + q')$-simplex σ represented by

$$\begin{array}{ccc} \Delta^{p+p'} \times \Delta^{q+q'} & \xrightarrow{d} & E \\ \downarrow & & \downarrow \\ \Delta^{p+p'} & \xrightarrow{\sigma} & B \end{array}$$

are given, we set

$$(\varphi \smile \psi)(\sigma) := \varphi(d^*_{(p,q)\text{-front}}\sigma) \cdot \psi(d^*_{(p',q')\text{-back}}\sigma)$$

where $d^*_{(p,q)\text{-front}}\sigma$ is the (p, q)-simplex given by

$$\begin{array}{ccccc} \Delta^p \times \Delta^q & \xrightarrow{d_{p\text{-front}} \times d_{q\text{-front}}} & \Delta^{p+p'} \times \Delta^{q+q'} & \xrightarrow{d} & E \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^p & \xrightarrow{d_{p\text{-front}}} & \Delta^{p+p'} & \xrightarrow{\sigma} & B \end{array}$$

and similarly for $d_{(p',q')\text{back}}$

Lemma 3.19. *Both the horizontal and the vertical differential on $C^{\bullet,\bullet}(p, R)$ satisfy the graded Leibniz rule with respect to this cup product. Hence $C^{\bullet,\bullet}(p, R)$ becomes a multiplicative double complex.*

Proof. Analogous to the Leibniz rule for the ordinary cup product. □

Hence, the cohomological Serre spectral sequence becomes multiplicative, and one checks that the identifaction $E_2^{p,q} \cong H^p(B, H^q(F_-, R))$ is multiplicative with respect to the multiplication on $H^\bullet(B, H^\bullet(F_-, R))$ described earlier, and that the convergence to $(H^\bullet(E, R), F_\bullet)$ is multiplicative.

Finally, we record the naturality of the Serre spectral sequence.

Definition 3.20. A morphism of cohomologically graded spectral sequences $f : (E_r, d_r, h_r) \to (E'_r, d'_r, h'_r)$ is a collection of bigraded maps $f_r : E^r \to E'_r$ that commute with differentials and satisfy $h'_r \circ f_r = H^\bullet(f_r) \circ h_r$.

Note: f is determined by f_2, but it is a condition that the higher f_r commute with the differentials. Analogously one can define morphisms of homologically graded spectral sequences.

We now consider the category Fib with objects the Serre fibrations $p : E \to B$ and morphisms the commutative squares

$$\begin{tikzcd} E \arrow[r, "g^E"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "g^B"] & B' \end{tikzcd}$$

(not necessarily pullback squares). Then $C_{\bullet,\bullet}(-, A)$ becomes a functor from Fib to double complexes by postcomposition, hence the assignment sending $p : E \to B$ to its Serre spectral sequence also becomes a functor (similarly for the cohomological version). The identification $E^2_{p,q} \cong H_p(B, H_q(F_-, A))$ is a natural isomorphism of functors $\mathrm{Fib} \to \mathtt{Ab}$. The maps $H_\bullet(g^E, A)$ and $H^\bullet(g^E, A)$ preserve the filtrations, and $E^\infty_{p,q} \cong F^p(H_{p+q}(E, A))/F^{p-1}(H_{p+q}(E, A))$ is natural, similarly for cohomology.

We give a simple application of naturality. Let $p : E \to B$ be a Serre fibration. We have surjections $E_2^{p,0} \twoheadrightarrow E_3^{p,0} \cdots \twoheadrightarrow E_\infty^{p,0}$, an inclusion $E_\infty^{p,0} = F^p H^p(E, A) \hookrightarrow H^p(E, A)$. and a map $H^p(B, A) \to H^p(B, H^0(F_-, A)) \cong E_2^{p,0}$ induced by the map of local systems $A \to H^0(F_-, A)$ from the projection $F \to *$. The composite $e(p : E \to B) : H^p(B, A) \to H^p(E, A)$ is called the edge homomorphism.

Lemma 3.21. *The morphism $e(p)$ agrees with $H^p(p, A)$.*

Proof. By naturality of the Serre spectral sequence, the edge homomorphism is also natural. We consider the square of fibrations

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "p"] & B \arrow[d, "\mathrm{id}"] \\ B \arrow[r, "\mathrm{id}"] & B \end{tikzcd}$$

and obtain a commutative square

$$\begin{tikzcd} H^\bullet(B, A) \arrow[r, "\mathrm{id}"] \arrow[d, "e(\mathrm{id})"] & H^\bullet(B, A) \arrow[d, "e(p)"] \\ H^\bullet(B, A) \arrow[r, "p^*"] & H^\bullet(E, A) \end{tikzcd}$$

It hence suffices to check that $e(\mathrm{id}) = \mathrm{id}$, which one easily checks directly. □

There is also an edge homomorphism of the form

$$H^q(E, A) \twoheadrightarrow F_0 H^q(E, A)/F_1 H^q(E, A) \cong E_\infty^{0,q} \hookrightarrow E_2^{0,q} \cong H^0(B, H^q(F_-, A)) \xrightarrow{(x \hookrightarrow B)^*} H^q(F_x, A)$$

which one can show timilarly to agree with $H^q(F_x \hookrightarrow E, A)$. There are also homological versions of these two edge homomorphisms.