Algebraic Topology
Serre spectral sequence, characteristic classes and bordism

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1 Informal introduction

One of the big goals of homotopy theory is to compute

$$[X, Y]_\bullet = \{\text{base-point preserving cont. maps } X \to Y\}/\text{homotopy}$$

for X and Y pointed CW-complexes. CW-complexes are build out of spheres, hence the building blocks are the sets $[S^n, S^k]_\bullet = \pi_n(S^k, *)$. For $n \geq 1$, there are groups, abelian if $n > 1$. What do we know about these groups?

- $\pi_n(S^k, *) = 0$ for $n < k$ by cellcular approximation.
- $\pi_n(S^n, *) \cong \mathbb{Z}$ by the Hurewicz theorem and $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$
- X is $(n-1)$-connected CW-complex: Then $\pi_n(X, *) \cong H_n(X, \mathbb{Z})$.
- $\pi_k(S^1, *) = 0$ for $k \geq 2$ by covering space theory (universal cover of S^1 is $\mathbb{R}$, which is contractible).
- $\pi_3(S^2, *) \neq 0$, since the attaching map of the 4-cell for $\mathbb{CP}^2$ is a map $\eta : S^3 \to S^2 \cong \mathbb{CP}^1$. If this was null-homotopic, then we would have $\mathbb{CP}^2 \sim S^2 \vee S^4$, which contradicts the ring structure on $H^\bullet(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}[x]/x^3$.
- $\pi_k(S^n, *) \to \pi_{k+1}(S^{n+1}, *) \to \pi_{k+2}(S^{n+2}, *) \to \cdots$ eventually stabilizes by the Freudenthal suspension theorem.

To go beyond this, we need a new tool, the Serre spectral sequence. To motivate its usefulness, consider the following strategy: There exists a map $f : S^2 \to K(\mathbb{Z}, 2)$ which induces an isomorphism $f_* : \pi_2(S^2, *) \to \pi_2(K(\mathbb{Z}, 2), *)$. We can take its homotopy fibre $H = \mathrm{hofb}_x(f)$ (2-connected cover of S^2). Then there is a fiber sequence $H \to S^2 \xrightarrow{f} K(\mathbb{Z}, 2)$ and a long exact sequence in homotopy

$$\cdots \to \pi_4(K(\mathbb{Z}, 2), *) \to \pi_3(H, *) \to \pi_3(S^2, *) \to \pi_3(K(\mathbb{Z}, 2), *) \to \pi_2(H, *) \to \pi_2(S^2, *) \to$$

$$\to \pi_2(K(\mathbb{Z}, 2), *) \to \pi_1(H, *) \to \pi_1(S^2, *) \to \cdots$$

from which we conclude $\pi_3(H, *) \cong \pi_3(S^2, *)$ and $\pi_1(H, *) = \pi_2(H, *) = 0$, i.e. H is 2-connected and the higher homotopy groups agree with the ones of S^2. By the Hurewicz theorem, $\pi_3(S^2, *) = H_3(H, \mathbb{Z})$. Hence we want to find a way to compute $H_*(H, *)$ from $H_*(S^2, \mathbb{Z})$ and $H_*(K(\mathbb{Z}, 2), \mathbb{Z})$.

This will also help to compute $\pi_n(S^k, *)$ in other ways (for example we will show that $\pi_n(S^k, *)$ is finite unless $n = k$ or $n = 2k - 1$ and k even). Furthermore, the Serre spectral sequence will allow us to compute the (co-)homology of spaces like $U(n), SU(n), \Omega S^n, K(\mathbb{Z}/2, n)$ etc. and (re-)prove structural theorems like Hurewicz, Freudenthal suspension, Thom isomorphisms and more.

So, given a fiber sequence $F \to Y \to X$, what could the relationship between the homology groups of F, Y and X be?

Example 1.1. Consider the easiest case $F \to X \times F \xrightarrow{\mathrm{pr}_X} X$, the trivial filtration. Then the Alexander-Whitney map induces an isomorphism

$$H_n(X \times F, \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X, H_q(F)).$$

This is the kind of result we want: It computes the homology of the total space in terms of the homology of X and F.

Example 1.2 (Hopf fibration)**.** $S^1 \to S^3 \xrightarrow{\eta} S^2$:

n	$H_n(S^3, \mathbb{Z})$	$\bigoplus_{p+q=n} H_p(S^2, H_q(S^1, \mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}$
2	0	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0

Hence clearly the Künneth formula from the previous example is "too big" to describe the homology in this case. However, consider the "2-step"-filtration $S^1 \subseteq S^3$ which satisfies $\widetilde{H}_n(S^3/S^1, \mathbb{Z}) \cong \mathbb{Z}$ for $n = 2, 3$ and 0 otherwise. Hence $H_\bullet(S^1, \mathbb{Z}) \oplus H_\bullet(S^3/S^1, \mathbb{Z})$ agrees with the right-hand side of the table above. This does not agree with $H_*(S^3, \mathbb{Z})$, because the long exact sequence corresponding to $S^1 \to S^3 \to S^3/S^1$ does not split into nice short exact sequences. Concretely, the boundary map $\widetilde{H}_2(S^3/S^1, \mathbb{Z}) \to H_1(S^1, \mathbb{Z})$ is an isomorphism, hence these two terms do not contribute to $H_\bullet(S^3, \mathbb{Z})$.

It turns out that something similar holds for all fibre sequences $F \to Y \to X$: There exists a filtration

$$F_0 \subseteq F_1 \ldots \subseteq F_m \subseteq \ldots \subseteq C_*(Y, \mathbb{Z})$$

on $C_*(Y, \mathbb{Z})$ such that $H_{p+q}(F_p/F_{p-1}) \cong C_p^{\text{cell}}(X, H_q(F, \mathbb{Z}))$. To then understand $H_\bullet(Y, \mathbb{Z})$, one needs to understand the cancellations in the associated long exact sequences. This is best encoded in a spectral sequence.

2 Spectral sequences

Definition 2.1. A (homologically, Serre-graded) *spectral sequence* is a triple $(E^\bullet, d^\bullet, h^\bullet)$, where

- $(E^r)_{r\geq 2}$ is a sequence of $\mathbb{Z}$-bigraded abelian groups. We write $E^r_{p,q}$ for the (p,q)-graded part of E^r. E^r is called the r-th *page* of the spectral sequence.
- $(d^r : E^r \to E^r)$ is a sequence of morphisms, called *differentals*, of bidegree $(-r, r-1)$ satisfying $d^r \circ d^r = 0$.
- $h^r : H_\bullet(E^r) \to E^{r+1}$ is a sequence of bigrading-preserving isomorphisms. Here $H_\bullet(E^r)$ denotes the homology with respect to d^r, which inherits a bigrading from E^r.

Definition 2.2. We say that a spectral sequence is *1st quadrant* if all abelian groups $E^2_{p,q}$ are trivial whenever $p < 0$ or $q < 0$.

Lemma 2.3. *For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we have $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$ for all $r \geq 2$. Moreover, for a given $(p,q) \in \mathbb{Z}^2$, the map h induces an isomorphism $E^r_{p,q} \to E^{r+1}_{p,q}$ for $r > r_0 = \max(p, q+1)$, i.e. the groups $E^r_{p,q}$ stabilize as $r \to \infty$.*

Proof. The first statement follows directly from the existence of the isomorphisms h by induction on r. For the second statement, if $r > r_0$, then the target of the differental $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial, hence every element of $E^r_{p,q}$ is a cycle. Moreover, the domain of the incoming differential $d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}$ is trivial. Hence $E^r_{p,q} \cong H_\bullet(E^r_{p,q}) \xrightarrow[h]{\cong} E^{r+1}_{p,q}$ □

Definition 2.4. For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$, we define the E^∞*-page* as the bigraded abelian group $E^\infty_{p,q} = E^{r_0+1}_{p,q}$ with $r_0 = \max(p, q+1)$. By the previous lemma, $E^\infty_{p,q} \cong E^r_{p,q}$ whenever $r > r_0$

By a filterted object in an abelian category $\mathcal{A}$ we mean an object $H \in \mathcal{A}$ with a sequence of inclusions

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^n \subseteq \ldots \subseteq H.$$

We will apply this to $\mathcal{A}$ the category of graded abelian groups and $H = H_*(E, \mathbb{Z})$.

Definition 2.5. A first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$ is said to *converge* to a filtered object in graded abelian groups (H, F) if there is a chosen isomorphism $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$ for all p, q and $F^p_n = H_n$ if $n \leq p$. In this case we write $E^2_{p,q} \Rightarrow H$.

Remark. Convergence is really a *datum* of the necessary isomorphism and not a property. Convergent spectral sequences are often simply encoded as $E^2_{p,q} \Rightarrow H$, but this suppresses not only this data, but also the higher pages, the differentials, and the filtration on H.

We now want to introduce the Serre spectral sequence for the homology of fibre sequences.

Definition 2.6. Let $f : Y \to X$ be a continuous map of topological spaces and $x \in X$ a point. The *homotopy fibre* $\mathrm{hofb}_x(f)$ of f at x is defined to be

$$\mathrm{hofb}_x(f) = P_x X \times_X Y$$

where $P_x X = \{\gamma : [0,1] \to X \mid \gamma(1) = x\}$ is the based path space of X It comes with a map $P_x X \to X$ given by $\gamma \mapsto \gamma(0)$. In words: $\mathrm{hofb}_x(f)$ is the space of pairs (γ, y) where $y \in Y$ and γ is a path in X from $f(y)$ to x. We note that $P_x X$ is contractible by the homotopy

$$H : P_x X \times [0,1] \to P_x X, \quad (\gamma, t) \mapsto s \mapsto \gamma((1-t)s + t)$$

Example 2.7. If $f : * \to X$, then $\mathrm{hofb}_x(f) = \Omega_x X$.

Definition 2.8. A *fibre sequence* of topological spaces is a sequence $F \xrightarrow{i} Y \xrightarrow{f} X$, a basepoint $x \in X$, a homotopy $h : F \to X^{[0,1]}$ from the composite $f \circ i$ to the constant map with value $c_x : F \to X$ and such that the induced map $F \to \mathrm{hofb}_x(f),\ z \mapsto (h(z), i(z))$ is a weak homotopy equivalence.

Recall: A weak homotopy equivalence is a map inducing isomorphisms on $\pi_n(-, x)$ for all $n \in \mathbb{N}$ and all basepoints x.

Example 2.9. 1. Let $f : Y \to X$ be any continuous map, $x \in X$. Then the pair $(\mathrm{hofb}_x f \to Y \to X, H)$, where H is the homotopy from the definition of the homotopy fibre above. Every fibre sequence is equivalent to this in the following sense: Given $(F \to Y \to X, h)$, there is a commutative diagram

$$\begin{tikzcd} F \arrow[r, "\simeq"] \arrow[d] & \mathrm{hofb}_x(f) \arrow[d] \\ Y \arrow[r, equal] \arrow[d, "f"] & Y \arrow[d, "f"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

In particular, $\Omega_x X \to x \to X$ is a fibre sequence where $h : \Omega_x X \times [0,1] \to X$ is the evaluation map. If one instead chooses the constant homotopy, one does not obtain a fibre sequence (unless the space is contractible). This is because the induced map $\Omega_x X \to \mathrm{hofb}_x(f) = \Omega_x X$ is constant and hence usually not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces F and X, $x \in X$, the pair $(F \to F \times X \to X, \mathrm{const})$ is a fibre sequence, the *trivial fibre sequence*. To see that, note that $\mathrm{hofb}_x(\mathrm{pr}_X) = F \times P_x X$ with induced map

$$F \to F \times P_x X, \quad y \mapsto (y, \mathrm{const}),$$

which is a homotopy equivalence as $P_x X$ is contractible.

3. Let $p : E \to B$ be a fibre bundle with fibre $F = p^{-1}(b)$ for some $b \in B$. Then the sequence $F \to E \to B$ with the constant homotopy is a fibre sequence. This is a special case of the next example.

4. Recall that $p : E \to B$ is a Serre fibration if in every commutative diagram of the form

$$\begin{tikzcd} D^n \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there exists a lift $D^n \times I \to E$ making both triangles commute. Given a Serre fibration $p : E \to B$ and $b \in B$, the sequence $F = p^{-1}(b) \to E \to B$ with the constant homotopy is a fibre sequence. (see exercises) Note: Every fibre sequence is also equivalenct to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle $S^1 \to S^3 \xrightarrow{\eta} S^2$. It arises by letting $S^1 = U(1)$ act on $S^2 \subseteq \mathbb{C}^2$ via $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$, with quotient space $\mathbb{CP}^1 \cong S^2$.

6. Example 5 generalizes to fibre bundles $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ with limit case $S^1 \to S^\infty \to \mathbb{CP}^\infty$, which is equivalent to $\Omega\mathbb{CP}^\infty \to * \to \mathbb{CP}^\infty$.