Étale cohomology

by Prof. Dr. Jens Franke

notes by Stefan Albrecht

University Bonn – winter term 2023/24

Contents

1 Motivation and basic definitions

1.1 Introduction and motivation

Problem: For varieties X over an algebraically closed field k (and hopefully more general schemes) define a cohomology theory $H^*(X)$ with properties similar to $H^*_{\text{sing}}(X(\mathbb{C})_{\text{ord. top. space}})$. Hopefully, there exists a Lefschitz fixed point formula

$$\#(\text{fixed points of } f \text{ with multiplicity}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(f^*|H^i(X)). \tag{L}$$

The aim of Grothendieck was to apply this to a program proposed by Weil of studying the congruence zeta function of X by applying (L) to $f = F_X$ given by $[x_0, \ldots, x_n] \mapsto [x_0^q, \ldots, x_n^q]$, yielding

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F_X^*|H^i(X)).$$

Counterexamples $H^*_{dR}(X) = \mathbb{H}^*(X_{\text{Zar}}, \mathcal{O}_X \to \Omega^1_X \to \cdots)$ (de Rham cohomology) is ok if the characteristic of k is zero but not in $\operatorname{char} p$ where it is unsuitable for Weil's program. Similarly, $H^*(X_{\text{Zar}}, \mathbb{Z})$ does not work: $\underline{\mathbb{Z}}(X) \to \underline{\mathbb{Z}}(V)$ is surjective when X is irreducible, implying vanishing higher sheaf cohomology.

Restrictions on the ring of coefficients: If X is a supersingular elliptic curve over $\overline{\mathbb{F}}_q$ then $H^1(X)$ ought to be two-dimensional, but $\operatorname{End}(X) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is non-split precisely over $\mathbb{Q}_p$ and $\mathbb{R}$, in which case it cannot act on a two-dimensional vector space. This excludes $\mathbb{Q}_p$ and $\mathbb{R}$ as the field of definition and hence also $\mathbb{Q}$ and $\mathbb{Z}$.

Etale cohomology with coefficients $\mathbb{Z}/l^n\mathbb{Z}$, l a prime invertible in k. Then

$$H^*(X, \mathbb{Q}_l) := \big(\varprojlim H^*(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z})\big) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Deligné used this to show the Riemann hypothesis for congurence zeta function.

Other theories include Crystilline cohomology with coefficients in $W(\overline{F}_q)$. Scholze has a way of working with $\mathbb{Z}_p$ directly, using the pro-étale site, and a proposal to work with $\mathbb{C}$ coefficients. But it is not clear how to do this.

Hence we will mostly study finite coefficients. If one works over $\mathbb{C}$, the exact exponential sequence $0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ is important. and we want at least the exactness of

$$0 \to \mu_{l^n} \to \mathcal{O}_X^\times \xrightarrow{f \mapsto f^{l^n}} \mathcal{O}_X^\times \to 0. \tag{$*$}$$

Note that $\mu_{l^n} \cong \mathbb{Z}/l^n\mathbb{Z}$ non-canonically if $k = \bar{k}$ and l is invertible in k. Unfortunately, but not unexpectedly, this is not exact on X_{Zar}. If this were exact, one could hope to get some information from it provided that $H^1(C, \mathcal{O}_C^\times) \cong \mathbb{Z} \times \operatorname{Jac}_C(k)$. The idea of Grothendieck was to enforce the exactness of $(*)$ by considering $V \to F(V)$ for étale morphisms $V \to X$ instead of only Zariski open subsets. Then, when $f \in \mathcal{O}_V^\times(V)$ one has an l^n-th root of f on $U = \{(x, \varphi) \mid x \in V, \varphi^{l^n} = f(x)\}$.

1.2 Flat morphisms

Definition 1. M is a *flat* A-module if $T \mapsto M \otimes_A T$ is exact or, equivalently, if $\mathrm{Tor}_p^A(M, T) = 0$ for all T and $p > 0$. An A-algebra B is flat if it is flat as an A-module.

Definition 2. For a morphism $f : X \to Y$ of schemes, f is called *flat* if it satisfies the folllowing equivalent conditions:

a) For all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-algebra.

b) For affine open subsets $U \subseteq X, V \subseteq Y$ s.t. $f(U) \subseteq V$, $\mathcal{O}_X(U)$ is flat as an $\mathcal{O}_Y(V)$-algebra.

c) There are affine open subsets $U_i \subseteq X, V_i \subseteq Y$ s.t. $f(U_i) \subseteq V_i$, $\mathcal{O}_X(U_i)$ is a flat $\mathcal{O}_Y(V_i)$-algebra and $X = \bigcup_{i \in I} U_i$.

Remark 1. a) See stacksproject 01U2

b) Other literature: SGA1: Etale fundamental group, SGA4_1: Topoi, Grothendieck topology, SGA4_2: Etale topology, SGA4_3: Proper and smooth base change, SGA$4\frac{1}{2}$: various stuff and Arcata – Introduction to etale cohomology by Delinge, SGA5: l-adic cohomology
Milne: Etale cohomology, Kiehl-Freitag: Etale cohomology and Weil conjectures
Matsumura: Commutative Algebra, Matsumura: Commutative Ring Theory

Let A be a ring, X quasi-compact and separated $\operatorname{Spec} A$-scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $H^*(X, \mathcal{M})$ can be calculated using $\check{H}(\mathcal{U}, -)$ for affine coverings. Hence, by the exactness of $- \otimes_A \widetilde{A}$, this gives

Proposition 1. *a) Let $\widetilde{A}$ be a flat A-algebra, then $H^*(\widetilde{X}, \widetilde{M}) \cong H^*(X, M) \otimes_A \widetilde{A}$, where $\widetilde{X} = X \times_{\operatorname{Spec} A} \operatorname{Spec} \widetilde{A} \xrightarrow{p} X$ and $\widetilde{M} = p^* M$.*

b) Let $f : X \to Y$ be a quasi-compact separated morphism and $g : \widetilde{Y} \to Y$ a flat morphism, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $g^* R^* f_* \mathcal{M} \cong R^* \widetilde{f}_* \widetilde{g}^* \mathcal{M}$ where $\widetilde{X} = X \times_Y \widetilde{Y}$.

Remark 2. Base change results for etale cohomology are similar. We have b) if f is proper or if f is of finite type and g is smooth, and the sheaves are of torsion.

Definition 3. f is called *faithfully flat* if it is flat and surjective on points. $\widetilde{A}$ is a faithfully flat A-algebra if it is flat and $R \otimes_A \widetilde{A} = 0$ implies $T = 0$.

Definition 4. [1] Let $f : X \to Y$ be a morphism of schemes. A descent datum (of quasi-coherent sheaves of modules) for f is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ with an isomorphism $\mu : p_1^* \mathcal{M} \cong p_2^* \mathcal{M}$, where

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "p_{12},p_{13}", "p_{23}"'] \arrow[rr, bend right, "q_1,q_2,q_3"'] & X \times_Y X \arrow[r, "p_1,p_2"] & X \end{tikzcd}$$

[1]see tag 023A or SGA1,VI for fibred categories: descend data for X-schemes to Y-schemes and ample line bundles

are the different projections, and the diagram

$$\begin{tikzcd}
q_1^*\mathcal{M} \arrow[rr, equal] \arrow[dr, equal] & & p_{12}^*p_1^*\mathcal{M} \arrow[rr, "p_{12}^*\mu", "\cong"'] & & p_{12}^*p_2^*\mathcal{M} \arrow[rr, equal] & & q_2^*\mathcal{M} \arrow[dl, equal] \\
& p_{13}^*p_1^*\mathcal{M} \arrow[dr, "\cong", "p_{13}^*\mu"'] & & & & p_{23}^*p_1^*\mathcal{M} \arrow[dl, "\cong"', "p_{23}^*\mu"] & \\
& & p_{13}^*p_2^*\mathcal{M} \arrow[dr, equal] & & p_{23}^*p_2^*\mathcal{M} \arrow[dl, equal] & & \\
& & & q_3^*\mathcal{M} & & &
\end{tikzcd}$$

must commute. A morphism of descent data is a morphism $\varphi : \mathcal{M} \to \widetilde{\mathcal{M}}$ compatible with μ and $\widetilde{\mu}$, i.e. $(p_2^*\varphi)\mu = \widetilde{\mu}(p_1^*\varphi)$

Remark 3. We have a functor

$$\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}, \quad \mathcal{N} \mapsto (f^*\mathcal{N}, \text{the canonical iso } p_1^*f^*\mathcal{N} \cong p_2^*f^*\mathcal{N}).$$

One would like this to be an equivalence of categories. It has a right adjoint

$$(\mathcal{R}\mathcal{M})(U) = \{m \in \mathcal{M}(f^{-1}U) \mid \mu p_1^* m = p_2^* m\}$$

Proposition 2 (stacks loc.cit., SGA1.VII.1, Milne)**.** *If f is faithfully flat and quasi-compact, the above functor $\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}$ is an equivalence of categories.*

Proof. If f has a section, the inverse image along that section is an inverse functor. In general, base change with $f : X \to Y$ reduces to this situation, provided that f is separated, which is a situation one can reduce to. □

Corollary 1. *If f is faithfully flat, $\mathcal{O}_Y(V) = \{\lambda \in \mathcal{O}_X(f^{-1}U) \mid p_1^*\lambda = p_2^*\lambda\}$.*

Remark 4. Both quasi-compactness and quasi-coherence in proposition 2 are needed. Consider $Y = \operatorname{Spec} R$, R a PID with $\operatorname{Spec} R$ infinite,

$$X = \coprod_{m \in \mathrm{mSpec}} \operatorname{Spec} R_m, \qquad N_1 = \coprod_{m \in \mathrm{mSpec}\, R} R/m \to N_2 = \prod_{m \in mSpecR} R/m,$$

then it is easy to see that this inclusion does not split, bit it splits canonically after applying $- \otimes_R R_m$, giving rise to a morphism of descent data which does not descend to a morphism $N_2 \to N_1$.

Definition 5. A morphism $i : X \to Y$ in a category $\mathcal{A}$ is an effective monomorphism if for all objects T,

$$\mathrm{Hom}_{\mathcal{A}}(T, X) \xrightarrow{\varphi \mapsto i\varphi} \{f \in \mathrm{Hom}_{\mathcal{A}}(T, Y) \mid \sigma f = \widetilde{\sigma} f \text{ for all } \sigma, \sigma' : Y \to S \text{ s.t. } \sigma i = \widetilde{\sigma} i\}$$

is bijective. $p : X \to Y$ is an effective epimorphism if it is an effective monomorphism in $\mathcal{A}^{\mathrm{op}}$, i.e.

$$\mathrm{Hom}_{\mathcal{A}}(Y, T) \xrightarrow[\cong]{\varphi \mapsto \varphi p} \{f \in \mathrm{Hom}_{\mathcal{A}}(X, T) \mid f\sigma = f\widetilde{\sigma} \text{ for all } \sigma, \widetilde{\sigma} : S \to X \text{ s.t. } p\sigma = p\widetilde{\sigma}\}.$$

Remark 5. If $X \times_Y X$ exists, f being an effective epimorphism is equivalent to it being a coequalizer of $X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$.

Proposition 3 (SGA1.VIII.4 or stacks 023Q)**.** *Every fpqc (quasi-compact faithfully flat) morphism of schemes is an effective epimorphism, i.e.*

$$\mathrm{Hom}(Y,T) \to \mathrm{Hom}(X,T) \rightrightarrows \mathrm{Hom}(X \times_Y X, T)$$

is an exact sequence of sets.

Remark 6. This implies that for every scheme T, the functor $X \mapsto T(X) := \mathrm{Hom}(X,T)$ satisfies the sheaf condition in the following sense:

$$T(Y) \xrightarrow{\tau \mapsto \tau f} \{t \in T(X) \mid tp_1 = tp_2\}.$$

That this should be interpreted as a kind of sheaf axiom becomes obvious if we have a covering $Y = \bigcup_{i=1}^n U_i$, $X = \coprod_{i=1}^n U_i \xrightarrow{f} Y$. Then $X \times_Y X = \coprod_{i,j=1}^n (U_i \cap U_j)$ with $tp_1|_{U_i \cap U_j}$ identified with $t|_{U_i}|_{U_i \cap U_j}$.

Proposition 4 (01UA)**.** *Every flat morphism (locally) of finite presentation is open.*

1.3 Grothendieck Topologies

As Deligne did in Arcata, we prefer the definition of Grothendieck topology by sieves.

Definition 1. Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}(\mathcal{C})$. A *sieve* (or $\mathcal{C}$-sieve) over X is a class $\mathcal{S}$ of morphisms with target X, such that $(U \to X) \in \mathcal{S}$ implies $(V \to U \to X) \in \mathcal{S}$ for every morphism $V \to U$ in $\mathcal{C}$. The empty class of morphisms is called the *empty sieve*, and the class of all morphisms with target X is called the *all sieve* (over X). For a morphism $f : Y \to X$ in $\mathbb{C}$, $f^*\mathcal{S} = \{v : U \to Y \mid fu \in \mathcal{S}\}$.

Remark 1. a) Obviously, $f^*\mathcal{S}$ is a sieve over Y if $\mathcal{S}$ is a sieve over X.

b) The fact that we work with categories where $\mathrm{Ob}\,\mathcal{C}$ is a proper class creates set-theoretic difficulties. Our way of dealing with this is to mostly ignore them.

c) The intersection of any class of sieves over X is a sieve over X.Thus, for every class $(f_i)_{i \in I}$ of morphisms with target X, there is a smallest sieve over X containing all f_i, namely $\{\xi : U \to X \mid \xi = f\eta \text{ for } \eta : U \to Y_i \text{ for some } \eta\}$. This is called the sieve generated by the f_i.

Example 1. a) X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$ turned into a category by its half ordering by $\subseteq$. If $X = \bigcup_{i \in I} U_i$ is an open covering, then the sieve generated by the (unique morphisms from) U_i is the sieve of all $V \in \mathbb{O}_X$ s.t. $V \subseteq U_i$ for at least one i.

b) If X is a complex space (e.g. $X = \mathbb{C} \setminus \{0\}$) with its complex topology, and $U \subseteq X$ open and $f \in \mathcal{O}_X(U)$, then $\mathcal{S} = \{V \subseteq U \mid \exists \varphi \in \mathcal{O}_X(V) \text{ s.t. } \varphi^2 = f|_V\}$ is a $\mathbb{O}_X$-sieve over U.

Remark. Thus, a morphism is in a sieve iff it is small enough "to pass through the sieve".

Definition 2. A *Grothendieck topology* $\mathbb{T}$ on a category $\mathcal{C}$ associates to every object X of $\mathcal{C}$ a class $\mathbb{T}_X$ of sieves over X, called the *covering sieves* of X. The following conditions must be verified:

(GTTriv) The all sieve over X covers X.

(GTTrans) If $\mathcal{S} \in \mathbb{T}_X$ and $f : Y \to X$, then $f^*\mathcal{S} \in \mathbb{T}_Y$.

(GTLoc) If $\mathcal{T} \in \mathbb{T}_X$ and $\mathcal{S}$ any sieve over X such that $f^*\mathcal{S} \in \mathbb{T}_Y$ for all $f : Y \to X$ in $\mathcal{T}$, then $\mathcal{S} \in \mathbb{T}_X$.

We will often write $\mathcal{S} \mathrel{/=} X$ for $S \in \mathbb{T}_X$ if there are no ambiguities (or $\mathcal{S} \mathrel{/=_{\mathbb{T}}} X$ it there are).

Remark 1. Pretopologies are specified by specifying a class of admissible coverings $\mathcal{U} = (f_i : Y_i \to X)_{i\in I}$. Various assumptions must be satisfied, like that $(U_i \times_X Y \to Y)_{i\in I}$ still form an admissible covering of Y (including the existence of the fibre product). By putting $\mathbb{T}_X = \{$admissible coverings $\mathcal{S}$ of X with all $f_i \in \mathcal{S}\}$ one gets a Grothendieck topology. Equivalent pretopologies define the same $\mathbb{T}_X$. If the category has fibre products, one gets a pretopology from a Grothendieck topology $\mathbb{T}_X$ by calling a covering admissible iff the f_i generate a sieve in $\mathbb{T}_X$. This is the largest pretopology in its equivalence class.

Example 2. X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$, and $\mathcal{S} \,/\!\!= U$ iff $U = \bigcup_{V\in\mathcal{S}} V$. Other Grothendieck topologies can be introduced as well.

a) $X = [0,1]_{\mathbb{R}}$, put $\mathcal{S} \,/\!\!= U$ iff there are countable many $(U_i)_{i\in\mathbb{N}}$ such that $U \setminus \bigcup_{i\in\mathbb{N}} U_i$ is a set of Lebesgue measure 0, or $\mathcal{S} = U = \emptyset$.

b) Rigid analytic geometry (Tate style) or real algebraic geometry (Delfs-Knebusch) enforce quasi-compactness of certain open subsets of X, making it harder to be a covering.

c) X a Noetherian scheme, $d \in \mathbb{N}$. $\mathcal{S} \,/\!\!= \mathcal{U}$ iff $\operatorname{codim}(U \setminus \bigcup_{V\in\mathcal{S}} V) \geq d$, making it easier to be a covering.

Remark 2. You can think of (GTLoc) as the condition that being a covering is a local property.

Fact 1. a) Every sieve $\mathcal{T}$ containing a covering sieve $\mathcal{S}$ is itself covering.

b) The intersection of finitely many covering sieves is covering.

Proof. a) If $(f : U \to X) \in \mathcal{S}$, then $f^*\mathcal{T}$ is the all-sieve on U which covers U by (GTTrans). By (GTLoc), $\mathcal{T}$ covers X.

b) It is sufficient to show that $\mathcal{T} := \mathcal{S}_1 \cap \mathcal{S}_2$ covers X, where both $\mathcal{S}_i \,/\!\!= X$. If $(f : U \to X) \in \mathcal{S}_1$, then $f^*\mathcal{T} = f^*\mathcal{S}_2 \,/\!\!= U$ by (GTTrans) and since $\mathcal{S}_2 \,/\!\!= X$. Again by (GTLoc), $T \,/\!\!= X$. $\square$

Proposition 1. *Let S be a scheme, P a Zariski-local property of S-schemes and $\underline{\mathrm{Sch}}_S^P$ be the full subcategory of the category $\underline{\mathrm{Sch}}_S$ of S-schemes, with class of objects being the S-schemes with property P, and let $\mathcal{C}$ be a class of morphisms in $\underline{\mathrm{Sch}}_S^P$. The following assumptions must be satisfied:*

(A) $\mathcal{C}$ is closed under composition, base-change and finite coproducts.

(B) If U is a quasi-compact S-scheme with $P(U)$ and $U = \bigcup_{i=1}^n U_i$ is a finite affine open covering, then the morphism $\coprod_{i=1}^n U_i \to U$ belongs to $\mathcal{C}$.

If X is an S-scheme with $P(X)$ then the following conditions to a sieve $\mathcal{S}$ over X are equivalent:

(C1) There are open coverings $X = \bigcup_{i\in I} U_i$ and morphisms $V_i \to U_i$ for all $i \in I$ such that $(V_i \to U_i \to X) \in \mathcal{S}$ and V_i is covered (in the ordinary sense) by its Zariski-open subsets W such that $(W \to V_i \to U_i) \in \mathcal{C}$

(C2) The same conditions, but the U_i and V_i must be affine.

In addition, we obtain a Grothendieck topology $\mathbb{T}$ on $\underline{\mathrm{Sch}}_S^P$ by associating to X the class $\mathbb{T}_X$ of all sieves with these equivalent properties.

Remark 3. a) In (A), the stability under base change includes the condition that $X_Y\widetilde{X}$ has P when $X, Y, \widetilde{X}$ have this property and $(X \to Y) \in \mathcal{C}$.

b) It the elements of $\mathcal{C}$ are open maps, then the conditions (C1) and (C2) can be modified by simply requiring that $(V_i \to U_i) \in \mathcal{C}$ without changing anything else, i.e. $X = \bigcup_{i\in I} U_i$ and $(V_i \to U_i) \in \mathcal{C} \cap \mathcal{S}$.

Example 3. a) P the trivial property and $\mathcal{C}$ the class of all fpqc morphisms. We get the fpqc topology on $\underline{\mathrm{Sch}}_S$.

$\widetilde{\text{a}}$) Let S be Noetherian, P : local Noetherianness and $\mathcal{C}$ the class of fpqc morhpisms. This will NOT work as (A) is violated: For instance, with $S = X = \operatorname{Spec}\mathbb{Q}$, the fibre product $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ is non-noetherian: The ideal $I = (x \otimes y - y \otimes x \mid x, y \in \mathbb{C})$ is not finitely generated as $\Omega_{\mathbb{C}/\mathbb{Q}} \cong I/I^2$. This is a $\mathbb{C}$-vector space of dimension equal to the continuum (the transcendence degree of $\mathbb{C}/\mathbb{Q}$).

b) Let $\mathcal{C}$ be the class of all fppf (faithfully flat of finite presentation) morphisms and the trivial property (or local Noetherianness) for P. Then fibre products don't cause any trouble, since then $\widetilde{X} \times_X Y$ is of finite type over $\widetilde{X}$ and local Noetherianness is preserved. One gets the fppf-topology on (locally noetherian) S-schemes. In this case. quasi-finiteness can be added to "of finite presentation" without modifying the topology: (stacks 056X)

c) The class $\mathcal{C}$ of all surjective morphisms which are Zariski-local isomorphisms, with P = trivial, or local Noetherianness, or regularity, ... and one gets the Zariski topology on $\underline{\mathrm{Sch}}_S$.

Proof. (of proposition 1) It is clear that (C2) implies (C1). Assume conversely that $X = \bigcup_{i\in I} U_i$ and $(p_i : V_i \to U_i) \in \mathcal{C}$ such that V_i is covered by the open $W \subseteq V_i$ such that $(W \to V_i \to X) \in \mathcal{S} \cap \mathcal{C}$. (We call such W $\mathcal{S}$-small.) Let $U_i = \bigcup_{j\in J_i} U_{ij}$ be an open affine covering and $V_{ij} = p_i^{-1} U_{ij} = V_i \times_{U_i} U_{ij}$. Thus $(V_{ij} \to U_{ij}) \in \mathcal{C}$ by (A). If $W \subseteq V_i$ is $\mathcal{S}$-small, the same holds for $W \cap V_{ij}$, showing that V_{ij} is covered by its $\mathcal{S}$-small open subsets. Thus we may assume that the U_i are affine and the V_i quasi-compact. By an application of (B), we may also assume that the V_i are affine. Then (C2) holds.

It remains to show the properties of a Grothendieck topology. For (GTTriv) this is trivial (U_i any affine covering and $V_i = U_i$). Also, (GTTrans) is easy. If $f : \widetilde{X} \to X$ is a morphism one puts $\widetilde{U}_i = f^{-1} U_i$, $\widetilde{V}_i = \widetilde{U}_i \times_{U_i} V_i$ and $(\widetilde{V}_i \to \widetilde{U}_i) \in \mathcal{C}$ by (A). Also, if $W \subseteq V$ is $\mathcal{S}$-small, then its inverse image in $\widetilde{V}_i$ is $f^*\mathcal{S}$-small, and these inverse images cover $\widetilde{V}_i$. For (GTLoc), let $\mathcal{S} \mathrel{/\!\!=} X$ and $\mathcal{T}$ any sieve such that $f^*\mathcal{T} \mathrel{/\!\!=} Y$ for all $(f : Y \to X) \in \mathcal{S}$. We must show $\mathcal{T} \mathrel{/\!\!=} X$.

<u>Case 1:</u> One can choose $V_i = U_i \xrightarrow{\mathrm{id}} U_i$ in the condition (C1) for $\mathcal{S} \mathrel{/\!\!=} X$. Then the restriction $\mathcal{T}|_{U_i} := (U_i \hookrightarrow X)^*\mathcal{T}$ covers U_i. Thus there are an open covering $U_i = \bigcup_{j\in J_i} U_{ij}$ and $V_{ij} \to U - ij$ as in (C1) for $\mathcal{T}|_{U_i}$, and then $X = \bigcup_{i\in I} \bigcup_{j\in J_i} U_{ij}$, together with the morphisms $V_{ij} \to U_{ij}$, does the same for X.

<u>Case 2:</u> X is affine, and there is a morphism $(p : V \to X) \in (\mathcal{S} \cap \mathcal{C})$ with V affine, s.t. p generates $\mathcal{S}$. Then $p^*\mathcal{T} \mathrel{/\!\!=} V$. Write $V = \bigcup_{i=1}^n U_i$ and morphisms $(V_i \to U_i) \in \mathcal{C}$ such that the $\mathcal{S}$-small open susets of V_i cover V_i. Then one can satisfy (C2) for $\mathcal{T}$ by $U' = X$, $V' = \coprod_{i=1}^n V_i \to \coprod_{i=1}^n U_i \to V \to X = U'$, where the arrows are in $\mathcal{C}$ by (A), (B), and assumption, respectively.

<u>Case 3:</u> General case: If $V_i \to U_i$ are as in (C2) for $\mathcal{S}$, then the pullback of $\mathcal{T}$ to any $\mathcal{S}$-small open subset W of V_i covers W. By case 1, the pullback of $\mathcal{T}$ to V_i covers V_i. By case 2, $\mathcal{T}|_{U_i} \mathrel{/\!\!=} U_i$. By case 1 again, $\mathcal{T} \mathrel{/\!\!=} X$. ☐

Definition 3. A presheaf on a category $\mathcal{C}$ (with values in sets, (abelian) groups, rings) is a contravariant functor from $\mathcal{C}$ to $\underline{\mathrm{Set}}$ (or groups, rings, ...). If a Grothendieck topology $\mathbb{T}$ on $\mathcal{C}$ is given, then a presheaf $\mathcal{F}$ is called ($\mathbb{T}$-)separated, if

$$F(X) \to \prod_{(p:U\to X)\in\mathcal{S}} F(U), \qquad f \mapsto (F(p)f)_p \tag{$*$}$$

is injective. We call a separated presheaf F a sheaf if the image of $(*)$ is $\varprojlim_{(p:U\to X)\in\mathcal{S}} F(U)$. In other words, the image of $(*)$ must be the family of all $(f_p)_p$ such that $F(q')f_p = F(p')f_q$ in $F(W)$ whenever

$$\begin{array}{ccc} W & \xrightarrow{p'} & V \\ \big\downarrow{\scriptstyle q'} & & \big\downarrow{\scriptstyle q} \\ U & \xrightarrow{p} & X \end{array}$$

is a commutative diagram in $\mathcal{C}$, with $p, q \in \mathcal{S}$.

Proposition 2. *In the situation of proposition 1, a presheaf G is a sheaf (resp. separated) if and only if for every object X of $\underline{\mathrm{Sch}}_S^P$ the presheaf $U \mapsto G(U)$ on X equipped with its Zariski topology is a sheaf (resp. separated), and for every morphism $p : U \to V$ in $\mathcal{C}$ the sequence*

$$G(V) \xrightarrow{p^*} G(U) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} G(U \times_V U)$$

is exact in the sense that the first morphism is the equalizer of the second two (resp, if p^* is injective

Proof. Let $S \,/\!\!= X$, we must show that $G(X) \to \varprojlim G$ is bijective (resp. injective), and for the proof of bijectiveness, we may assume injective.
Case 1: S is already covering for X_{Zar}: Trivial.
Case 2: There is a morphism $p : U \to X$ in $\mathcal{C}$ such that the S-small open subsets W of U cover U (as sets). If $g_1, g_2 \in G(X)$ have the same image in $\varprojlim_S G$, then $p^*g_1|_W = p_2^*g_2|_W$ when $W \subseteq U$ is S-small. By our first assumption on G, $p^*g_1 = p^*g_2$. As p^* is injective by our second assumption, $g_1 = g_2$. Let $\gamma \in \varprojlim_S G$. By our first assumption on G, there is $g_U \in G(U)$ such that $g_U|_W = \gamma_W$ whenever $W \subseteq U$ is S-small. Let $W, \widetilde{W} \subseteq U$ be S-small, then for $p_1, p_2 : U \times_X U \to U$ we have

$$p_1^* g_U|_{W \times_X \widetilde{W}} = p_1^* \gamma_W|_{W \times_X \widetilde{W}} = \gamma_{W \times_X \widetilde{W}} = p_2^* \gamma_{\widetilde{W}}|_{W \times_X \widetilde{W}} = p_2^* g_U|_{W \times_X \widetilde{W}}.$$

As these $W \times_X \widetilde{X}$ cover $U \times_X U$ as a set, $p_1^* g_U = p_2^* g_U$. By our assumption there is a unique $g \in G(X)$ such that $p^*g = g_U$. We must show that the image of g in $\varprojlim_S G$ is γ. Let $\widetilde{S} \subseteq S$ be the subsieve of S generated by the S-small $W \subseteq U$. Then $\widetilde{S} \,/\!\!= X$, and the image of g in $\varprojlim_{\widetilde{S}} G$ equals $\gamma|_{\widetilde{S}}$ by construction. For $(\nu : V \to X) \in S$, this implies that $G(\nu)g = \gamma_V$ as they have the same image in $\varprojlim_{\nu^* \widetilde{S}} G$, and $\nu^* \mathcal{S} \,/\!\!= V$. Thus the claim about g is shown.
Case 3: General case. Let $V_i \to U_i$ be as in the definition of a Grothendieck topology. If g_1, g_2 have the same image in $\varprojlim_S G$ then $g_1|_{U_i} = g_2|_{U_i}$ by case 2, hence $g_1 = g_2$ by the first assumption. Let $\gamma \in \varprojlim_S G$, by case 2 there is $\gamma_i \in G(U_i)$ such that the image of γ_i in $\varprojlim_{S|_{U_i}} G$ equals the restriction of γ. Then $\gamma_i|_{U_i \cap U_j} = \gamma_j|_{U_i \cap U_j}$ as their images in $\varprojlim_{S|_{U_i \cap U_j}} G$ are both equal to the restriction of γ to $S|_{U_i \cap U_j} \,/\!\!= U_i \cap U_j$. By our first assumption, there is $g \in G(X)$ such that $g|_{U_i} = g_i$. In a similar way as in the end of case 2, one sees that the image of g in $\varprojlim_S G$ equals γ. □

Corollary 1. *If X is any S-scheme then*

$$U \to X(U) := \mathrm{Hom}_{\underline{\mathrm{Sch}}_S}(U, X)$$

is an fpqc-sheaf on $\underline{\mathrm{Sch}}_S$.

Exercise: If $F \in \mathrm{QCoh}(S)$, then $(\upsilon : U \to S) \mapsto \upsilon^* F$ is an fpqc sheaf, and $H^*(S_{Zar}, F) \cong H^*(S_{fpqc}, F)$

1.4 Étale morphisms

Proposition 1. *Let $f : X \to Y$ be a morphism locally of finite type between Noetherian schemes, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

a) $\Omega_{X/Y,x} = 0$.

b) There is an open neighbourhood U of x in X such that $\Delta_{X/Y} : U \to X \times_Y X$ is an open embedding.

c) We have $\mathfrak{m}_x = (f^*\mathfrak{m}_y)\mathcal{O}_{X,x}$, and $k(x)$ is a separable finite field extension of $k(y)$.

If f is separated, such that $\Delta_{X/Y}$ is a closed embedding defined by the quasi-coherent sheaf of ideals $J \subseteq \mathcal{O}_{X\times_Y X}$, then the above is also equivalent to

d) $J_x = 0$.

Remark. The Noetherianness assumption can be dropped with little effort.

Proof. (Sketch) As a), b), and c), as well as the claim in d) are local in X, we may assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine. Then the equivalence of b) with d) is obvious as J is locally finitely generated: If d) holds, there is an open neighbourhood U of x in X such that $J|_U$ vanishes. The equivalence of a) with d) then comes from a well-known fact (Remark 1 below) about Kähler differentials. By Nakayama's lemma $(\Omega_{X/Y})_x = 0$ if and only if $0 = (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{X,x}} k(x) \cong (\Omega_{f^{-1}\{y\}/k(y)})_x$, by the compatability of Kähler differentials with base change. The $k(y)$-algebra $(k(y) \otimes_A B)_{\mathfrak{m}_x}$ has vanishing Kähler differentials over $k(y)$ iff this local $k(y)$-algebra is a finite separable field extension $l/k(y)$, i.e. $\mathfrak{m}_x = (f^*\mathfrak{m}_y)\mathcal{O}_x$ (othersie B_x has nilpotent elements) and $k(x) = l$ is separable over $k(y)$. $\square$

Remark 1. a) If f is separated and J as in (d), then $\Omega_{X/I} \cong \Delta^*_{X/Y} J \cong \Delta^*_{X/Y}(J/J^2)$.
b) If A and B are as in the proof, $\Omega_{B/A} \cong I/I^2$, $I = \ker(B \otimes_A B \to B)$.
c) $\mathcal{D}er_{B/A}(B, M) \cong \operatorname{Hom}_B(I/I^2, M)$, given by $d \mapsto \varphi(a \otimes b) = ad(b)$ and $d(b) = \varphi(1 \otimes b - b \otimes 1)$.