

# Algebraic Topology

## Serre spectral sequence, characteristic classes and bordism

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# 1 Informal introduction

One of the big goals of homotopy theory is to compute

$$[X, Y]_{\bullet} = \{\text{base-point preserving cont. maps } X \rightarrow Y\} / \text{homotopy}$$

for  $X$  and  $Y$  pointed CW-complexes. CW-complexes are build out of spheres, hence the building blocks are the sets  $[S^n, S^k]_{\bullet} = \pi_n(S^k, *)$ . For  $n \geq 1$ , there are groups, abelian if  $n > 1$ . What do we know about these groups?

- $\pi_n(S^k, *) = 0$  for  $n < k$  by cellular approximation.
- $\pi_n(S^n, *) \cong \mathbb{Z}$  by the Hurewicz theorem and  $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$
- $X$  is  $(n-1)$ -connected CW-complex: Then  $\pi_n(X, *) \cong H_n(X, \mathbb{Z})$ .
- $\pi_k(S^1, *) = 0$  for  $k \geq 2$  by covering space theory (universal cover of  $S^1$  is  $\mathbb{R}$ , which is contractible).
- $\pi_3(S^2, *) \neq 0$ , since the attaching map of the 4-cell for  $\mathbb{CP}^2$  is a map  $\eta : S^3 \rightarrow S^2 \cong \mathbb{CP}^1$ . If this was null-homotopic, then we would have  $\mathbb{CP}^2 \sim S^2 \vee S^4$ , which contradicts the ring structure on  $H^*(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}[x]/x^3$ .
- $\pi_k(S^n, *) \rightarrow \pi_{k+1}(S^{n+1}, *) \rightarrow \pi_{k+2}(S^{n+2}, *) \rightarrow \dots$  eventually stabilizes by the Freudenthal suspension theorem.

To go beyond this, we need a new tool, the Serre spectral sequence. To motivate its usefulness, consider the following strategy: There exists a map  $f : S^2 \rightarrow K(\mathbb{Z}, 2)$  which induces an isomorphism  $f_* : \pi_2(S^2, *) \rightarrow \pi_2(K(\mathbb{Z}, 2), *)$ . We can take its homotopy fibre  $H = \text{hofb}_x(f)$  (2-connected cover of  $S^2$ ). Then there is a fiber sequence  $H \rightarrow S^2 \xrightarrow{f} K(\mathbb{Z}, 2)$  and a long exact sequence in homotopy

$$\begin{aligned} \dots \rightarrow \pi_4(K(\mathbb{Z}, 2), *) \rightarrow \pi_3(H, *) \rightarrow \pi_3(S^2, *) \rightarrow \pi_3(K(\mathbb{Z}, 2), *) \rightarrow \pi_2(H, *) \rightarrow \pi_2(S^2, *) \rightarrow \\ \rightarrow \pi_2(K(\mathbb{Z}, 2), *) \rightarrow \pi_1(H, *) \rightarrow \pi_1(S^2, *) \rightarrow \dots \end{aligned}$$

from which we conclude  $\pi_3(H, *) \cong \pi_3(S^2, *)$  and  $\pi_1(H, *) = \pi_2(H, *) = 0$ , i.e.  $H$  is 2-connected and the higher homotopy groups agree with the ones of  $S^2$ . By the Hurewicz theorem,  $\pi_3(S^2, *) = H_3(H, \mathbb{Z})$ . Hence we want to find a way to compute  $H_*(H, *)$  from  $H_*(S^2, \mathbb{Z})$  and  $H_*(K(\mathbb{Z}, 2), \mathbb{Z})$ .

This will also help to compute  $\pi_n(S^k, *)$  in other ways (for example we will show that  $\pi_n(S^k, *)$  is finite unless  $n = k$  or  $n = 2k - 1$  and  $k$  even). Furthermore, the Serre spectral sequence will allow us to compute the (co-)homology of spaces like  $U(n)$ ,  $SU(n)$ ,  $\Omega S^n$ ,  $K(\mathbb{Z}/2, n)$  etc. and (re-)prove structural theorems like Hurewicz, Freudenthal suspension, Thom isomorphisms and more.

So, given a fiber sequence  $F \rightarrow Y \rightarrow X$ , what could the relationship between the homology groups of  $F$ ,  $Y$  and  $X$  be?

**Example 1.1.** Consider the easiest case  $F \rightarrow X \times F \xrightarrow{\text{pr}_X} X$ , the trivial filtration. Then the Alexander-Whitney map induces an isomorphism

$$H_n(X \times F, \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X, H_q(F)).$$

This is the kind of result we want: It computes the homology of the total space in terms of the homology of  $X$  and  $F$ .

**Example 1.2** (Hopf fibration).  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$ :

$n$	$H_n(S^3, \mathbb{Z})$	$\bigoplus_{p+q=n} H_p(S^2, H_q(S^1, \mathbb{Z}))$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}$
2	0	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0

Hence clearly the Künneth formula from the previous example is "too big" to describe the homology in this case. However, consider the "2-step"-filtration  $S^1 \subseteq S^3$  which satisfies  $\tilde{H}_n(S^3/S^1, \mathbb{Z}) \cong \mathbb{Z}$  for  $n = 2, 3$  and 0 otherwise. Hence  $H_\bullet(S^1, \mathbb{Z}) \oplus H_\bullet(S^3/S^1, \mathbb{Z})$  agrees with the right-hand side of the table above. This does not agree with  $H_*(S^3, \mathbb{Z})$ , because the long exact sequence corresponding to  $S^1 \rightarrow S^3 \rightarrow S^3/S^1$  does not split into nice short exact sequences. Concretely, the boundary map  $\tilde{H}_2(S^3/S^1, \mathbb{Z}) \rightarrow H_1(S^1, \mathbb{Z})$  is an isomorphism, hence these two terms do not contribute to  $H_\bullet(S^3, \mathbb{Z})$ .

It turns out that something similar holds for all fibre sequences  $F \rightarrow Y \rightarrow X$ : There exists a filtration

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_m \subseteq \dots \subseteq C_*(Y, \mathbb{Z})$$

on  $C_*(Y, \mathbb{Z})$  such that  $H_{p+q}(F_p/F_{p-1}) \cong C_p^{\text{cell}}(X, H_q(F, \mathbb{Z}))$ . To then understand  $H_\bullet(Y, \mathbb{Z})$ , one needs to understand the cancellations in the associated long exact sequences. This is best encoded in a spectral sequence.

## 2 The Serre Spectral Sequence

**Definition 2.1.** A (homologically, Serre-graded) *spectral sequence* is a triple  $(E^\bullet, d^\bullet, h^\bullet)$ , where

- $(E^r)_{r \geq 2}$  is a sequence of  $\mathbb{Z}$ -bigraded abelian groups. We write  $E_{p,q}^r$  for the  $(p, q)$ -graded part of  $E^r$ .  $E^r$  is called the  $r$ -th *page* of the spectral sequence.
- $(d^r : E^r \rightarrow E^r)$  is a sequence of morphisms, called *differentials*, of bidegree  $(-r, r-1)$  satisfying  $d^r \circ d^r = 0$ .
- $h^r : H_\bullet(E^r) \rightarrow E^{r+1}$  is a sequence of bigrading-preserving isomorphisms. Here  $H_\bullet(E^r)$  denotes the homology with respect to  $d^r$ , which inherits a bigrading from  $E^r$ .

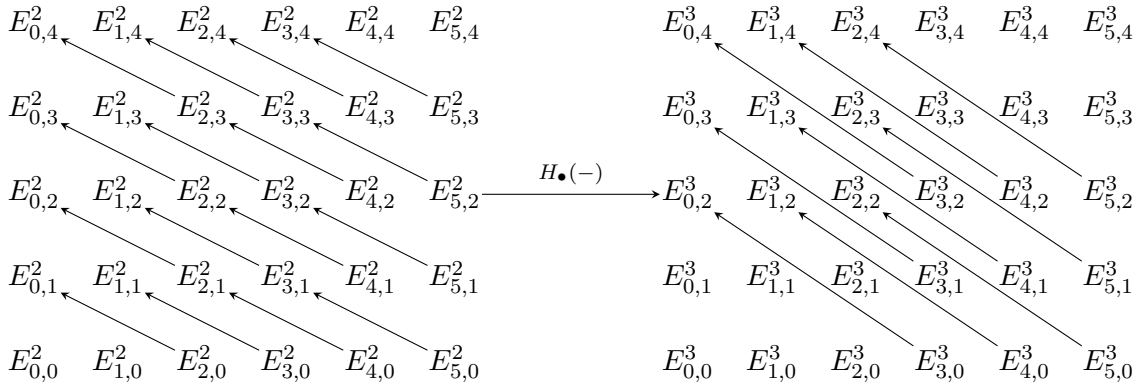


Figure 1: The second and third page of a spectral sequence

**Definition 2.2.** We say that a spectral sequence is *1st quadrant* if all abelian groups  $E_{p,q}^2$  are trivial whenever  $p < 0$  or  $q < 0$ .

**Lemma 2.3.** For a first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$ , we have  $E_{p,q}^r = 0$  if  $p < 0$  or  $q < 0$  for all  $r \geq 2$ . Moreover, for a given  $(p, q) \in \mathbb{Z}^2$ , the map  $h$  induces an isomorphism  $E_{p,q}^r \rightarrow E_{p,q}^{r+1}$  for  $r > r_0 = \max(p, q + 1)$ , i.e. the groups  $E_{p,q}^r$  stabilize as  $r \rightarrow \infty$ .

*Proof.* The first statement follows directly from the existence of the isomorphisms  $h$  by induction on  $r$ . For the second statement, if  $r > r_0$ , then the target of the differential  $d^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$  is trivial, hence every element of  $E_{p,q}^r$  is a cycle. Moreover, the domain of the incoming differential  $d^r : E_{p+r,q-r+1}^r \rightarrow E_{p,q}^r$  is trivial. Hence  $E_{p,q}^r \cong H_\bullet(E_{p,q}^r) \xrightarrow[h]{} E_{p,q}^{r+1}$   $\square$

**Definition 2.4.** For a first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$ , we define the  $E^\infty$ -page as the bigraded abelian group  $E_{p,q}^\infty = E_{p,q}^{r_0+1}$  with  $r_0 = \max(p, q + 1)$ . By the previous lemma,  $E_{p,q}^\infty \cong E_{p,q}^r$  whenever  $r > r_0$ .

By a filtered object in an abelian category  $\mathcal{A}$  we mean an object  $H \in \mathcal{A}$  with a sequence of inclusions

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \dots \subseteq F^n \subseteq \dots \subseteq H.$$

We will apply this to  $\mathcal{A}$  the category of graded abelian groups and  $H = H_*(E, \mathbb{Z})$ .

**Definition 2.5.** A first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$  is said to *converge* to a filtered object in graded abelian groups  $(H, F)$  if there is a chosen isomorphism  $E_{p,q}^\infty \cong F_{p+q}^p / F_{p+q}^{p-1}$  for all  $p, q$  and  $F_n^p = H_n$  if  $n \leq p$ . In this case we write  $E_{p,q}^2 \Rightarrow H$ .

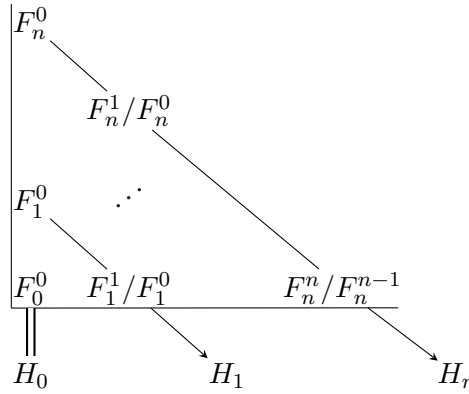


Figure 2: Visualization of  $E^\infty$  as filtrations of the  $H_i$  for a convergent spectral sequence  $E_{p,q}^2 \Rightarrow H$

**Remark.** Convergence is really a *datum* of the necessary isomorphism and not a property. Convergent spectral sequences are often simply encoded as  $E_{p,q}^2 \Rightarrow H$ , but this suppresses not only this data, but also the higher pages, the differentials, and the filtration on  $H$ .

We now want to introduce the Serre spectral sequence for the homology of fibre sequences.

**Definition 2.6.** Let  $f : Y \rightarrow X$  be a continuous map of topological spaces and  $x \in X$  a point. The *homotopy fibre*  $\text{hofb}_x(f)$  of  $f$  at  $x$  is defined to be

$$\text{hofb}_x(f) = P_x X \times_X Y$$

where  $P_x X = \{\gamma : [0, 1] \rightarrow X \mid \gamma(1) = x\}$  is the based path space of  $X$ . It comes with a map  $P_x X \rightarrow X$  given by  $\gamma \mapsto \gamma(0)$ . In words:  $\text{hofb}_x(f)$  is the space of pairs  $(\gamma, y)$  where  $y \in Y$  and  $\gamma$  is a path in  $X$  from  $f(y)$  to  $x$ . We note that  $P_x X$  is contractible by the homotopy

$$H : P_x X \times [0, 1] \rightarrow P_x X, \quad (\gamma, t) \mapsto s \mapsto \gamma((1-t)s + t)$$

**Example 2.7.** If  $f : * \rightarrow X$ , then  $\text{hofb}_x(f) = \Omega_x X$ .

**Definition 2.8.** A *fibre sequence* of topological spaces is a sequence  $F \xrightarrow{i} Y \xrightarrow{f} X$ , a basepoint  $x \in X$ , a homotopy  $h : F \rightarrow X^{[0,1]}$  from the composite  $f \circ i$  to the constant map  $c_x : F \rightarrow X$  such that the induced map  $F \rightarrow \text{hofb}_x(f)$ ,  $z \mapsto (h(z), i(z))$  is a weak homotopy equivalence.

Recall: A weak homotopy equivalence is a map inducing isomorphisms on  $\pi_n(-, x)$  for all  $n \in \mathbb{N}$  and all basepoints  $x$ .

**Example 2.9.** 1. Let  $f : Y \rightarrow X$  be any continuous map,  $x \in X$ . Then the pair  $(\text{hofb}_x f \rightarrow Y \rightarrow X, H)$ , where  $H$  is the homotopy from the definition of the homotopy fibre above, is a fibre sequence. Every fibre sequence is equivalent to this in the following sense: Given  $(F \rightarrow Y \rightarrow X, h)$ , there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\simeq} & \text{hofb}_x(f) \\ \downarrow & & \downarrow \\ Y & \xlongequal{\quad} & Y \\ \downarrow f & & \downarrow f \\ X & \xlongequal{\quad} & X \end{array}$$

In particular,  $\Omega_x X \rightarrow x \rightarrow X$  is a fibre sequence, where  $h : \Omega_x X \times [0, 1] \rightarrow X$  is the evaluation map. If one instead chooses the constant homotopy, one does not obtain a fibre sequence (unless the space is

contractible). This is because the induced map  $\Omega_x X \rightarrow \text{hofb}_x(f) = \Omega_x X$  is constant and hence usually not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces  $F$  and  $X$ ,  $x \in X$ , the pair  $(F \rightarrow F \times X \rightarrow X, \text{const})$  is a fibre sequence, the *trivial fibre sequence*. To see that, note that  $\text{hofb}_x(\text{pr}_X) = F \times P_x X$  with induced map

$$F \rightarrow F \times P_x X, \quad y \mapsto (y, \text{const}),$$

which is a homotopy equivalence as  $P_x X$  is contractible.

3. Let  $p : E \rightarrow B$  be a fibre bundle with fibre  $F = p^{-1}(b)$  for some  $b \in B$ . Then the sequence  $F \rightarrow E \rightarrow B$  with the constant homotopy is a fibre sequence. This is a special case of the next example.

4. Recall that  $p : E \rightarrow B$  is a Serre fibration if in every commutative diagram of the form

$$\begin{array}{ccc} D^n \times 0 & \longrightarrow & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ D^n \times I & \longrightarrow & B \end{array}$$

there exists a lift  $D^n \times I \rightarrow E$  making both triangles commute. Given a Serre fibration  $p : E \rightarrow B$  and  $b \in B$ , the sequence  $F = p^{-1}(b) \rightarrow E \rightarrow B$  with the constant homotopy is a fibre sequence. (see exercises) Note: Every fibre sequence is also equivalent to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$ . It arises by letting  $S^1 = U(1)$  act on  $S^2 \subseteq \mathbb{C}^2$  via  $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$ , with quotient space  $\mathbb{CP}^1 \cong S^2$ .

6. Example 5 generalizes to fibre bundles  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  with limit case  $S^1 \rightarrow S^\infty \rightarrow \mathbb{CP}^\infty$ , which is equivalent to  $\Omega \mathbb{CP}^\infty \rightarrow * \rightarrow \mathbb{CP}^\infty$ .

We are now ready to state the existence of the Serre spectral sequence.

**Theorem 2.10** (Serre). *For every fibre sequence  $(F \xrightarrow{\iota} Y \xrightarrow{p} X, h)$  with  $X$  simply-connected and abelian group  $A$ , there exists a spectral sequence of the following form*

$$E_{p,q}^2 = H_p(X, H_q(F, A)) \implies H_{p+q}(Y, A)$$

As noted before, this information does not include the differentials and the higher pages, as well as the filtrations on  $H_\bullet(Y, A)$  and the identifications of its subquotients with the  $E^\infty$ -page.

One edge case is easy to state: The map

$$H_n(F, A) = H_0(X, H_n(F, A)) = E_{0,n}^2 \twoheadrightarrow E_{0,n}^\infty \hookrightarrow H_n(Y, A)$$

agrees with the factorization  $H_n(F, A) \twoheadrightarrow \text{im } \iota_* \hookrightarrow H_n(Y, A)$ .

We now assume this theorem and give some sample computations.

**Example 2.11.** We revisit the Hopf fibration  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$ .  $S^2$  is simply connected, so we get a spectral sequence. The  $E^2$ -page is  $H_p(S^2, H_q(S^1, A))$ , which looks like

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ A & 0 & A & 0 \\ & \swarrow & & \\ A & 0 & A & 0 \end{array} \\ p \rightarrow \end{array}$$

There is one potentially non-trivial  $d^2$ -differential, namely  $d^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$ . All higher differentials  $d^r$ ,  $r > 2$ , are trivial for degree reasons. Hence the  $E^\infty$ -page looks as follows:

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ \text{coker}(d^2) & 0 & A & 0 \\ A & 0 & \ker(d^2) & 0 \end{array} \\ \rightarrow p \end{array}$$

We know that  $H_n(S^3, A) = A$  for  $n = 0, 3$  and 0 else. From the  $E^\infty$ -page we thus get  $H_0(S^3, A) = A$ ,  $H_1(S^3, A) = \text{coker}(d^2)$ ,  $H_2(S^3, A) = \ker(d^2)$ ,  $H_3(S^3, A) = A$ . Hence  $d^2$  must be an isomorphism.

**Lemma 2.12.** *There is a fibre bundle*

$$U(n-1) \xrightarrow{i} U(n) \rightarrow S^{2n-1},$$

where  $U(n)$  denotes the topological group of unitary  $n \times n$ -matrices and  $i$  is the standard inclusion which adds a trivial  $\mathbb{C}$ -summand.

*Proof.* The group  $U(n)$  acts on  $\mathbb{C}^n$  by definition. This action restricts to the unit sphere  $S^{2n-1} \subseteq \mathbb{C}^n$ . Furthermore, this action is transitive, because every vector of length 1 can be extended to an orthonormal basis. Hence  $S^{2n-1}$  is in bijection with the orbit space  $U(n)/\text{Stab}(x)$ , for any  $x \in S^{2n-1}$ . For  $x = (0, \dots, 0, 1)$ , the stabilizer equals  $i(U(n-1))$ . We obtain a continuous bijective map  $U(n)/U(n-1) \rightarrow S^{2n-1}$ ,  $[A] \mapsto A(0, \dots, 0, 1)^t$ , which is a homeomorphism since its domain is quasi-compact and its codomain is Hausdorff. Finally, we use the fact that for a smooth, free action of a compact Lie group  $G$  on a manifold  $M$ , the map  $M \rightarrow M/G$  is always a fibre bundle (in fact a  $G$ -principal bundle).  $\square$

**Example 2.13.** We consider the case  $n = 2$ , i.e. the fibre sequence  $S^1 \cong U(1) \hookrightarrow U(2) \rightarrow S^3$ . We want to compute the homology of  $U(2)$  via the Serre spectral sequence  $E_{p,q}^2 = H_p(S^3, H_q(S^1, \mathbb{Z}))$ . All differentials on all pages have to be trivial for degree reasons. (The spectral sequence "collapses".) Hence  $E^\infty = E^2$  and every antidiagonal has at most one non-trivial term, so we can read off  $H_n(U(2), \mathbb{Z}) = \mathbb{Z}$  for  $n = 0, 1, 3, 4$  and 0 else. In fact, one can show that  $U(2) \cong S^3 \times U(1)$ , so the homology could alternatively be computed with the Künneth theorem.

**Example 2.14.** Next we consider the fibre sequence  $U(2) \hookrightarrow U(3) \rightarrow S^5$  with  $E^2$ -page  $E_{p,q}^2 = H_p(S^5, H_q(U(2), \mathbb{Z}))$ , which looks like

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccccc} \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \end{array} \\ \rightarrow p \end{array}$$

The first potentially non-trivial differential is  $d^5 : E_{0,5}^2 \rightarrow E_{0,4}^2$ . At this point we cannot decide what this differential is. All higher differentials are again trivial for degree reasons, and all filtrations collapse to

at most one entry. We obtain

$$H_n(U(3), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, 3, 5, 8, 9, \\ \text{coker}(d^5) & \text{for } n = 4, \\ \ker(d^5) & \text{for } n = 5, \\ 0 & \text{else.} \end{cases}$$

This example illustrates a typical situation, namely that one can often not fully determine all differentials but still deduce a lot. We will soon see that  $d^5 = 0$  and hence  $H_4(U(3), \mathbb{Z}) \cong H_5(U(3), \mathbb{Z}) \cong \mathbb{Z}$

**Example 2.15.** We consider  $U(3) \rightarrow U(4) \rightarrow S^7$ . The  $E_{p,q}^2 = H_p(S^7, H_q(U(3), \mathbb{Z}))$ -page is

$$\begin{array}{c} \begin{array}{c} q \uparrow \\ \mathbb{Z} \qquad \qquad \mathbb{Z} \\ \mathbb{Z} \qquad \qquad \mathbb{Z} \\ 0 \qquad \qquad 0 \\ \mathbb{Z} \qquad \qquad \mathbb{Z} \\ ? \qquad \vdots \qquad ? \\ ? \quad \dots \quad 0 \quad \dots \quad ? \\ \mathbb{Z} \qquad \vdots \qquad \mathbb{Z} \\ 0 \qquad \qquad 0 \\ \mathbb{Z} \qquad \qquad \mathbb{Z} \\ \mathbb{Z} \qquad \qquad \mathbb{Z} \\ \downarrow \qquad \downarrow \\ 0 \qquad \qquad 7 \end{array} \end{array}$$

The only possibly non-trivial differentials are  $d^7 : E_{7,0}^2 \rightarrow E_{0,6}^2$  and  $d^7 : E_{7,3}^2 \rightarrow E_{0,9}^2$  which we cannot compute at this point. Nevertheless, we can still deduce a lot, for example  $H_1(U(4), \mathbb{Z}) = \mathbb{Z} = H_3(U(4), \mathbb{Z})$ ,  $H_2(U(4), \mathbb{Z}) = 0$ ,  $H_4(U(4), \mathbb{Z}) \cong H_4(U(3), \mathbb{Z})$ ,  $H_5(U(4), \mathbb{Z}) \cong H_5(U(3), \mathbb{Z})$  and there is a short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow H_8(U(4), \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0$ , so  $H_8(U(4), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

In the previous examples we used the Serre spectral sequence to compute the homology of the total space of the fibre sequence. We now show that it can also be used to compute the homology of the base space or fibre.

**Example 2.16.** We consider the fibre sequence  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  for  $n \geq 2$  and want to compute the homology of  $\mathbb{CP}^n$ . By path-connectedness, the  $E^2$ -page of  $H_p(\mathbb{CP}^n, H_q(S^1, \mathbb{Z})) \Rightarrow H_{p+q}(S^{2n+1}, \mathbb{Z})$  begins like this:

$$\begin{array}{c} \begin{array}{c} q \uparrow \\ 0 \quad 0 \\ \mathbb{Z} \quad ? \\ \mathbb{Z} \quad ? \\ \downarrow \quad \downarrow \\ \phantom{0} \quad \phantom{0} \end{array} \end{array}$$



Since  $H_1(S^{2n+1}, \mathbb{Z}) = 0$ , there must be a surjective  $d^2$ -differential  $d^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$ . But since  $H_2(S^{2n+1}, \mathbb{Z}) = 0$ , this differential must also be injective. Hence

$$\mathbb{Z} \cong E_{2,0}^2 = H_2(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong H_2(\mathbb{CP}^n, \mathbb{Z}).$$

Furthermore, we see that  $E_{1,0}^2 = H_1(\mathbb{CP}^n, \mathbb{Z}) = 0$ . Using  $H_0(S^1, \mathbb{Z}) = H_1(S^1, \mathbb{Z})$ , this implies  $E_{1,1}^2 = 0$  and  $E_{2,1} = \mathbb{Z}$ . Now we see that the 2-page looks like

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} & ? \\ \mathbb{Z} & 0 & \mathbb{Z} & ? \end{array} \\ p \rightarrow \end{array}$$

By the same argument, we can deduce  $d^2 : E_{4,0} \rightarrow E_{2,1}$  is an isomorphism, i.e.  $H_4(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong \mathbb{Z}$ , and  $E_{3,0} = E_{3,1} = 0$  and so on. Since  $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ , we cannot conclude that the  $\mathbb{Z}$  in bidegree  $(2n, 1)$  must be the image of a differential. There are two possibilities: If  $d^2 : E_{2n+2,0}^2 \rightarrow E_{2n+1}^2$  is the trivial map, then  $E_{2n+2,0}^2 = 0$  and then by induction  $E_{p,q}^2 = 0$  for all  $p > 2n$ . If, on the other hand,  $d^2 : E_{2n+2,0}^2 \rightarrow E_{2n+1}^2$  is non-zero, it has to be surjective: Indeed, since the cokernel is isomorphic to the lowest term of the filtration on  $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ , and no  $\mathbb{Z}/n\mathbb{Z}$  embeds into  $\mathbb{Z}$ . This then implies  $H_k(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$  for all  $k > 2n$ . This case can be ruled out using that  $\mathbb{CP}^n$  is a  $2n$ -dimensional CW-complex and hence  $H_n(\mathbb{CP}^n, \mathbb{Z}) = 0$  for  $k > 2n$ . In summary, we obtain

$$H_k(\mathbb{CP}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, \dots, 2n, \\ 0 & \text{else.} \end{cases}$$

Next we turn to an example where the Serre spectral sequence can be used to compute the homology of the fibre

**Example 2.17.** We consider the fibre sequence  $\Omega S^3 \rightarrow * \rightarrow S^3$ . On the  $E^2$ -page, we have the entries  $H^p(S^3, H_q(\Omega S^3, \mathbb{Z}))$ , i.e.

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ ? & 0 & 0 & ? \\ \mathbb{Z} & 0 & 0 & \mathbb{Z} \end{array} \\ p \rightarrow \end{array}$$

The homology of the point is 0 in positive degrees, so we must have  $E_{p,q}^\infty = 0$  unless  $p = q = 0$ . The only non-trivial differentials are  $d^3 : E_{3,q}^3 \rightarrow E_{0,q+2}^3$ , so we conclude that these are isomorphisms. Hence  $H_q(\Omega S^3, \mathbb{Z}) \cong H_{q+2}(S^3, \mathbb{Z})$ . Note  $E_{0,1}^{0,1} = 0$  since this entry cannot be killed by any differential. This implies

$$H_k(\Omega S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

In particular,  $\Omega S^3$  is an infinite-dimensional space.

We now discuss the cohomological version of the Serre spectral sequence and its multiplicative structure. This multiplication also helps in determining differentials, for example for the spectral sequences computing (co-)homology of unitary groups as above.

**Definition 2.18.** A cohomologically graded spectral sequence is a triple  $(E_\bullet, d_\bullet, h_\bullet)$  where  $(E_r)_r$  is a sequence of bigraded abelian groups,  $(d_r : E_r \rightarrow E_r)_r$  is a sequence of differentials ( $d_r \circ d_r = 0$ ) of bidegree  $(r, 1 - r)$ , and  $(h_r : H_\bullet(E_r) \rightarrow E_{r+1})_r$  a sequence of bigrading-preserving isomorphisms.

As before, one defines first quadrant ( $E_2^{p,q} = 0$  if  $p < 0$  or  $q < 0$ ) spectral sequences and the  $E_\infty$ -page.

Rather than the filtrations  $0 = F^{-1} \subseteq F^0 \subseteq \cdots \subseteq H$ , one now considers filtrations

$$H = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

**Definition 2.19.** A cohomological first quadrant spectral sequence is said to *converge* to a filtered object  $(H, F)$  in graded abelian groups if there are isomorphisms  $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$  for all  $p, q$ , and  $F_p^n = 0$  for all  $p > n$ . Again we write  $E_2^{p,q} \implies H$ .

**Definition 2.20.** A (commutative) multiplicative structure on a cohomologically graded spectral sequence  $(E_\bullet, d_\bullet, h_\bullet)$  is a bigraded (commutative) ring structure on  $E_r$ , i.e. there are associative maps  $E_r^{p,q} \otimes E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$ , such that  $d_r$  is a graded derivation, i.e.

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{p+q} x \cdot d_r(y)$$

for  $x \in E_r^{p,q}$ . Here (graded) commutative means  $xy = (-1)^{(p+q)(p'+q')}yx$ . As a result,  $H_\bullet(E_r)$  is a bigraded ring and we further require that the  $h_r$  are isomorphisms of bigraded rings. Furthermore, the  $E_\infty$ -page also inherits the structure of a (commutative) bigraded ring.

**Definition 2.21.** A filtration  $\cdots \subseteq F_n \subseteq \cdots \subseteq F_1 \subseteq F_0 = H$  on a graded ring  $H$  is said to be *multiplicative* (or compatible with the multiplicative structure) if  $F_s F_t \subseteq F_{s+t}$ . We say that  $(H, F)$  is a *filtered graded ring*.

It follows that the associated graded object  $\bigoplus F_p/F_{p+1}$  of a filtered graded (commutative) ring is a bigraded (commutative) ring.

**Definition 2.22.** A multiplicative first quadrant spectral sequence  $(E_\bullet, d_\bullet, h_\bullet)$  is said to *converge* to a filtered graded ring  $(H, F)$  if it converges additively and the chosen isomorphism  $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$  is compatible with the graded ring structure.

**Theorem 2.23** (Serre). *For every fibre sequence of spaces  $(F \rightarrow Y \rightarrow X, h)$  with  $X$  simply connected and every abelian group  $A$ , there exists a cohomological first quadrant spectral sequence of the form*

$$E_2^{p,q} = H^p(X, H^q(F, A)) \implies H^{p+q}(Y, A).$$

*If  $A$  is a (commutative) ring, then the spectral sequence is multiplicative and converges multiplicatively, where on the  $E_2$ -page the multiplication is given by  $(-1)^{p'q}$ -times the composite*

$$\begin{aligned} H^p(X, H^q(F, R)) \otimes H^{p'}(X, H^{q'}(F, R)) &\rightarrow H^{p+p'}(X, H^q(F, R) \otimes H^{q'}(F, R)) \\ &\rightarrow H^{p+p'}(X, H^{q+q'}(F, R)). \end{aligned}$$

Note: If  $H^\bullet(F, R)$  or  $H^\bullet(X, R)$  is flat over  $R$  of finite type, then the  $E_2$ -page is isomorphic to the graded tensor product of  $H^\bullet(X, R)$  and  $H^\bullet(F, R)$ .

**Example 2.24.** We reconsider the fibre sequence  $U(1) \rightarrow U(2) \rightarrow S^3$  with  $E_2^{p,q} = H^p(S^3, H^q(U(1), \mathbb{Z}))$ , i.e.

$$\begin{array}{c} \begin{array}{cccc} \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & \mathbb{Z} \end{array} \\ \begin{array}{c} \uparrow q \\ \xrightarrow{p} \end{array} \end{array}$$

There cannot be non-trivial differentials. As a graded ring, the  $E_2$ -page (and hence also the  $E_\infty$ -page) is isomorphic to  $H^\bullet(S^3, \mathbb{Z}) \otimes H^\bullet(U(1), \mathbb{Z})$ . Let  $x_1 \in H^1(U(1), \mathbb{Z})$  and  $x_3 \in H^3(S^3, \mathbb{Z})$  be generators.

Then  $H^\bullet(U(1), \mathbb{Z}) \cong \bigwedge(x_1)$  and  $H^\bullet(S^3, \mathbb{Z}) \cong \bigwedge(x_3)$ , where  $\bigwedge(M)$  denotes the exterior algebra on a set  $M$ , i.e. the free algebra on  $x \in M$  modulo the relations  $x_i x_j = -x_j x_i$  and  $x_i^2 = 0$  for all  $x_i, x_j \in M$ . Hence, the  $E_2$ -page is isomorphic to  $\bigwedge(x_1, x_3)$ . The  $\mathbb{Z}$  in bidegree  $(3, 1)$  is spanned by  $x_1 x_3$ . The filtration collapses degreewise and hence  $H^*(U(2), \mathbb{Z})$  is exterior on classes  $x_1 \in H^1(U(2), \mathbb{Z})$  and  $x_3 \in H^3(U(2), \mathbb{Z})$  that are uniquely determined by the spectral sequence.

**Example 2.25.** We move on to the fibre sequence  $U(2) \rightarrow U(3) \rightarrow S^5$ . The  $E_2$ -page looks like the homological spectral sequence

$$\begin{array}{cccccc}
 q \uparrow & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & & & & & & p \rightarrow
 \end{array}$$

Keeping the notation from the previous example, denote generators of the first column by  $1, x_1, x_3, x_1 x_3$ . Again, the  $E_2$ -page is given by

$$H^\bullet(S^5, \mathbb{Z}) \otimes H^\bullet(U(2), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5),$$

where  $x_5 \in H^5(S^5, \mathbb{Z})$  is a generator. There is one possibly non-trivial differential  $d_5 : E_5^{0,4} \rightarrow E_5^{5,0}$ . However, the product rule implies

$$d_5(x_1 x_3) = d_5(x_1) x_3 + (-1)^{1+0} x_1 d_5(x_3) = 0 + 0 = 0.$$

Again, the filtration collapses and hence  $H^*(U(3), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5)$ .

**Example 2.26.** We revisit  $U(3) \rightarrow U(4) \rightarrow S^7$ . The  $E_2$ -page now is

$$\begin{array}{ccccccc}
 q \uparrow & \mathbb{Z} & & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & 0 & & & & & 0 \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & \mathbb{Z} & & \vdots & & & \mathbb{Z} \\
 & \mathbb{Z} & \cdots & 0 & \cdots & & \mathbb{Z} \\
 & \mathbb{Z} & & \vdots & & & \mathbb{Z} \\
 & 0 & & & & & 0 \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & & & & & & p \rightarrow \\
 & 0 & & & & & 7
 \end{array}$$

As before, the product rule implies that all  $d_7$  differentials must be trivial, and  $E_2 \cong \bigwedge(x_1, x_3, x_5, x_7)$ . There is a non-trivial filtration on  $H^8(U(4), \mathbb{Z})$  of the form

$$0 \rightarrow \mathbb{Z}(x_1 x_7) \rightarrow H^8(U(4), \mathbb{Z}) \rightarrow \mathbb{Z}(x_3 x_5) \rightarrow 0$$

Additively the sequence splits, but one has to be careful with the multiplicative structure. To resolve this, we need to be precise with the differentiation between the classes  $x_i$  on the  $E_\infty$ -page and the corresponding classes  $\bar{x}_i \in H^*(U(4), \mathbb{Z})$ . Note that the choice of each  $\bar{x}_i$  is unique since the filtration collapses in degrees 0 to 7. Furthermore, we record their filtrations  $\bar{x}_1$  is in  $F_0^1$ ,  $\bar{x}_3$  is in  $F_0^3$ ,  $\bar{x}_5$  is in  $F_0^5$  and  $\bar{x}_7$  is in  $F_7^7$ . It follows that  $\bar{x}_1 \bar{x}_7$  is a generator of  $F_7^8$ , and  $\bar{x}_3 \bar{x}_5$  is a generator of  $F_0^8 / F_1^8$ . Hence,  $H^8(U(4), \mathbb{Z})$  is a free group on  $\bar{x}_1 \bar{x}_7$  and  $\bar{x}_3 \bar{x}_5$ , and it follows that  $H^\bullet(U(4), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5, x_7)$ .

**Theorem 2.27.** *For all  $n \in \mathbb{N}$ , there is an isomorphism of graded rings*

$$H^\bullet(U(n), \mathbb{Z}) \cong \bigwedge(x_1, x_3, \dots, x_{2n-1})$$

with  $x_i$  of degree  $i$ .

*Proof.* By induction on  $n$ , the start  $n = 1$  is clear. Let  $n \geq 2$  and assume we know the statement for  $n - 1$ . We consider the Serre spectral sequence for the fibre sequence  $U(n - 1) \rightarrow U(n) \rightarrow S^{2n-1}$ . By induction, its  $E_2$ -page is isomorphic to

$$E_2 \cong H^\bullet(S^{2n-1}, \mathbb{Z}) \otimes H^\bullet(U(n - 1), \mathbb{Z}) = \bigwedge(x_{2n-1}) \otimes \bigwedge(x_1, x_3, \dots, x_{2n-3}).$$

Here,  $x_i$  is a generator of  $E_2^{0,i}$  for  $i \leq 2n - 3$  and  $x_{2n-1}$  is a generator of  $E_2^{2n-1,0}$ . The only possibly non-trivial differentials are  $d_{2n-1}$ . For degree reasons,  $d_{2n-1}$  vanishes on all generators  $x_1, x_3, \dots, x_{2n-1}$ . By the product rule, all differentials are 0. Hence the  $E_\infty$ -page is isomorphic to the  $E_2$ -page, and an exterior algebra  $\bigwedge(x_1, x_3, \dots, x_{2n-1})$ .

The filtrations on  $H^\bullet(U(n), \mathbb{Z})$  collapse in degrees  $0, \dots, 2n - 2$ , therefore we obtain unique lifts  $\bar{x}_1, \dots, \bar{x}_{2n-1} \in H^\bullet(U(n), \mathbb{Z})$ . We only know from the spectral sequence that  $x_i^2$  is of lower filtration hence a multiple of  $\bar{x}_{2n-1}$ , but not necessarily that  $\bar{x}_i^2 = 0$ . However we know from the additive structure (all subquotients are free over  $\mathbb{Z}$ ) that  $H^\bullet(U(n), \mathbb{Z})$  is torsionfree. As the multiplication is graded commutative, we hence have  $\bar{x}_i^2 = 0$ . We obtain a ring map  $f : \bigwedge(x_1, \dots, x_{2n-1}) \rightarrow H^\bullet(U(n), \mathbb{Z})$  by sending  $x_i$  to  $\bar{x}_i$ . To check that  $f$  is an isomorphism, define a grading on  $\bigwedge(x_1, \dots, x_{2n-1})$  by setting the degree of  $x_1, \dots, x_{2n-3}$  to be 0, and the degree of  $x_{2n-1}$  to be  $2n - 1$ . This induces a filtration by setting  $F_i$  to be the direct sum of the graded pieces of degree  $2i$ . Then  $f$  is filtration preserving and induces an isomorphism on associated graded pieces. The proof is finished by the following lemma.  $\square$

**Lemma 2.28.** *Let  $A$  and  $B$  be graded abelian groups equipped with filtrations*

$$\dots F_2 \subseteq F_1 \subseteq F_0 = A \quad \text{and} \quad \dots G_2 \subseteq G_1 \subseteq G_0 = B$$

*which are eventually 0 in every degree. If we have a graded filtration-preserving morphism  $f : A \rightarrow B$  that induces an isomorphism on all associated graded pieces  $F_i/F_{i+1} \cong G_i/G_{i+1}$ , then it is an isomorphism.*

*Proof.* This is an iterated 5-lemma argument.  $\square$

**Example 2.29.** We revisit the fibre sequence  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  and use the Serre spectral sequence to compute the ring  $H^\bullet(\mathbb{CP}^n, \mathbb{Z})$ . Arguing analogously to the homological case, the  $E_2$ -page looks as

$$\begin{array}{ccccccc}
q & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\
& \searrow \wr & & \searrow \wr & & \searrow \wr & & \\
& \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\
p & | & & & & & & | \\
& 0 & & & & & & 2n
\end{array}$$

**Example 2.30.** Next we compute the ring structure on  $H^\bullet(\Omega S^2, \mathbb{Z})$ . Dual to the homological case, we obtain the following  $E_2$ -page corresponding to the fibre sequence  $\Omega S^3 \rightarrow * \rightarrow S^3$ :

$$\begin{array}{ccccccc}
& & & & & & q \\
& & & & & & \vdots \\
& & & & & & \mathbb{Z} \\
& & & 0 & 0 & & \mathbb{Z} \\
& & & \searrow & & & \\
& & & 0 & & & 0 \\
& & & \mathbb{Z} & & & \mathbb{Z} \\
& & & \searrow & & & \\
& & & 0 & & & 0 \\
& & & \mathbb{Z} & & & \mathbb{Z} \\
& & & \searrow & & & \\
& & & 0 & & & 0 \\
& & & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
& & & & & & \vdots \\
& & & & & & p
\end{array}$$

**Remark:** There is also a ring structure on the homology of  $H_*(\Omega S^3, \mathbb{Z})$  induced by the  $H$ -space (in fact  $A_\infty$ - or  $E_1$ -space) structure by concatenation of loops. One can show that with this ring structure,  $H_*(\Omega S^3, \mathbb{Z})$  is a polynomial ring on a generator in degree 2. More generally, if  $H_*(X, \mathbb{Z})$  is free over  $\mathbb{Z}$ , then  $H_*(\Omega \Sigma X, \mathbb{Z})$  is the tensor algebra on  $H_*(X, \mathbb{Z})$  (Bott-Samelson theorem) and the map  $H_*(X, \mathbb{Z}) \rightarrow H_*(\Omega \Sigma X, \mathbb{Z}) = T(H_*(X, \mathbb{Z}))$  is induced by the natural map  $X \rightarrow \Omega \Sigma X$ .

**Example 2.31.** We consider the map  $S^3 \rightarrow K(\mathbb{Z}, 3)$  classifying a generator of  $H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$ . (equivalently, inducing an isomorphism on  $\pi_3(-, *)$ ). Let  $X$  denote the homotopy fibre, so that  $X \rightarrow S^3 \rightarrow$

$$\begin{array}{cccc}
 \vdots & & & \vdots \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
 & \searrow & & \\
 0 & & & 0 \\
 \mathbb{Z} & & & \mathbb{Z} \\
 & \searrow & & \\
 0 & & & 0 \\
 \mathbb{Z} & & & \mathbb{Z} \\
 & \searrow & & \\
 0 & & & 0 \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z}
 \end{array}$$
$$\begin{array}{cccc}
q & \vdots & & \vdots \\
& 0 & 0 & 0 & \mathbb{Z}/4\mathbb{Z} \\
& 0 & & & 0 \\
& 0 & & & \mathbb{Z}/3\mathbb{Z} \\
& 0 & & & 0 \\
& 0 & & & \mathbb{Z}/2\mathbb{Z} \\
& 0 & & & 0 \\
& \mathbb{Z} & 0 & 0 & 0 \\
& & & & p
\end{array}$$
$$\tilde{H}^n(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if } n = 2k + 1, \\ 0 & \text{else} \end{cases}$$

**Corollary 2.32.** *We have  $\pi_4(S^3, *) \cong \mathbb{Z}/2\mathbb{Z}$  and  $\pi_4(S^2, *) \cong \mathbb{Z}/2\mathbb{Z}$ .*

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We now know the following homotopy groups of spheres:

	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$
$S^1$	$\mathbb{Z}$	0	0	0	0	0
$S^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
$S^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
$S^4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
$S^5$	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$S^6$	0	0	0	0	0	$\mathbb{Z}$

where  $\pi_5(S^4, *) = \pi_6(S^5, *) = \mathbb{Z}/2\mathbb{Z}$  follows from the Freudenthal suspension theorem.