Étale cohomology

by Prof. Dr. Jens Franke

notes by Stefan Albrecht

University Bonn – winter term 2023/24

Contents

1 Motivation and basic definitions

1.1 Introduction and motivation

Problem: For varieties X over an algebraically closed field k (and hopefully more general schemes) define a cohomology theory $H^*(X)$ with properties similar to $H^*_{\text{sing}}(X(\mathbb{C})_{\text{ord. top. space}})$. Hopefully, there exists a Lefschitz fixed point formula

$$\#(\text{fixed points of } f \text{ with multiplicity}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(f^*|H^i(X)). \tag{L}$$

The aim of Grothendieck was to apply this to a program proposed by Weil of studying the congruence zeta function of X by applying (L) to $f = F_X$ given by $[x_0, \dots, x_n] \mapsto [x_0^q, \dots, x_n^q]$, yielding

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F_X^*|H^i(X)).$$

Counterexamples $H^*_{dR}(X) = \mathbb{H}^*(X_{\text{Zar}}, \mathcal{O}_X \to \Omega^1_X \to \cdots)$ (de Rham cohomology) is ok if the characteristic of k is zero but not in $\operatorname{char} p$ where it is unsuitable for Weil's program. Similarly, $H^*(X_{\text{Zar}}, \mathbb{Z})$ does not work: $\underline{\mathbb{Z}}(X) \to \underline{\mathbb{Z}}(V)$ is surjective when X is irreducible, implying vanishing higher sheaf cohomology.

Restrictions on the ring of coefficients: If X is a supersingular elliptic curve over $\overline{\mathbb{F}}_q$ then $H^1(X)$ ought to be two-dimensional, but $\operatorname{End}(X) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is non-split precisely over $\mathbb{Q}_p$ and $\mathbb{R}$, in which case it cannot act on a two-dimensional vector space. This excludes $\mathbb{Q}_p$ and $\mathbb{R}$ as the field of definition and hence also $\mathbb{Q}$ and $\mathbb{Z}$.

Etale cohomology with coefficients $\mathbb{Z}/l^n\mathbb{Z}$, l a prime invertible in k. Then

$$H^*(X, \mathbb{Q}_l) := \left(\varprojlim H^*(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Deligné used this to show the Riemann hypothesis for congurence zeta function.

Other theories include Crystilline cohomology with coefficients in $W(\overline{F}_q)$. Scholze has a way of working with $\mathbb{Z}_p$ directly, using the pro-étale site, and a proposal to work with $\mathbb{C}$ coefficients. But it is not clear how to do this.

Hence we will mostly study finite coefficients. If one works over $\mathbb{C}$, the exact exponential sequence $0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ is important. and we want at least the exactness of

$$0 \to \mu_{l^n} \to \mathcal{O}_X^\times \xrightarrow{f \mapsto f^{l^n}} \mathcal{O}_X^\times \to 0. \tag{$*$}$$

Note that $\mu_{l^n} \cong \mathbb{Z}/l^n\mathbb{Z}$ non-canonically if $k = \bar{k}$ and l is invertible in k. Unfortunately, but not unexpectedly, this is not exact on X_{Zar}. If this were exact, one could hope to get some information from it provided that $H^1(C, \mathcal{O}_C^\times) \cong \mathbb{Z} \times \operatorname{Jac}_C(k)$. The idea of Grothendieck was to enforce the exactness of $(*)$ by considering $V \to F(V)$ for étale morphisms $V \to X$ instead of only Zariski open subsets. Then, when $f \in \mathcal{O}_V^\times(V)$ one has an l^n-th root of f on $U = \{(x, \varphi) \mid x \in V, \varphi^{l^n} = f(x)\}$.

1.2 Flat morphisms

Definition 1. M is a *flat* A-module if $T \mapsto M \otimes_A T$ is exact or, equivalently, if $\mathrm{Tor}_p^A(M,T) = 0$ for all T and $p > 0$. An A-algebra B is flat if it is flat as an A-module.

Definition 2. For a morphism $f : X \to Y$ of schemes, f is called *flat* if it satisfies the folllowing equivalent conditions:

a) For all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-algebra.

b) For affine open subsets $U \subseteq X, V \subseteq Y$ s.t. $f(U) \subseteq V$, $\mathcal{O}_X(U)$ is flat as an $\mathcal{O}_Y(V)$-algebra.

c) There are affine open subsets $U_i \subseteq X, V_i \subseteq Y$ s.t. $f(U_i) \subseteq V_i$, $\mathcal{O}_X(U_i)$ is a flat $\mathcal{O}_Y(V_i)$-algebra and $X = \bigcup_{i \in I} U_i$.

Remark 1. a) See stacksproject 01U2

b) Other literature: SGA1: Etale fundamental group, SGA4_1: Topoi, Grothendieck topology, SGA4_2: Etale topology, SGA4_3: Proper and smooth base change, SGA$4\frac{1}{2}$: various stuff and Arcata – Introduction to etale cohomology by Delinge, SGA5: l-adic cohomology
Milne: Etale cohomology, Kiehl-Freitag: Etale cohomology and Weil conjectures
Matsumura: Commutative Algebra, Matsumura: Commutative Ring Theory

Let A be a ring, X quasi-compact and separated $\operatorname{Spec} A$-scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $H^*(X, \mathcal{M})$ can be calculated using $\check{H}(\mathcal{U}, -)$ for affine coverings. Hence, by the exactness of $- \otimes_A \widetilde{A}$, this gives

Proposition 1. *a) Let $\widetilde{A}$ be a flat A-algebra, then $H^*(\widetilde{X}, \widetilde{M}) \cong H^*(X, M) \otimes_A \widetilde{A}$, where $\widetilde{X} = X \times_{\operatorname{Spec} A} \operatorname{Spec} \widetilde{A} \xrightarrow{p} X$ and $\widetilde{M} = p^* M$.*

b) Let $f : X \to Y$ be a quasi-compact separated morphism and $g : \widetilde{Y} \to Y$ a flat morphism, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $g^* R^* f_* \mathcal{M} \cong R^* \widetilde{f}_* \widetilde{g}^* \mathcal{M}$ where $\widetilde{X} = X \times_Y \widetilde{Y}$.

Remark 2. Base change results for etale cohomology are similar. We have b) if f is proper or if f is of finite type and g is smooth, and the sheaves are of torsion.

Definition 3. f is called *faithfully flat* if it is flat and surjective on points. $\widetilde{A}$ is a faithfully flat A-algebra if it is flat and $R \otimes_A \widetilde{A} = 0$ implies $T = 0$.

Definition 4. [1] Let $f : X \to Y$ be a morphism of schemes. A descent datum (of quasi-coherent sheaves of modules) for f is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ with an isomorphism $\mu : p_1^* \mathcal{M} \cong p_2^* \mathcal{M}$, where

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "p_{12}, p_{13}", "p_{23}"'] \arrow[rr, bend right, "q_1, q_2, q_3"'] & X \times_Y X \arrow[r, "p_1, p_2"] & X \end{tikzcd}$$

[1] see tag 023A or SGA1,VI for fibred categories: descend data for X-schemes to Y-schemes and ample line bundles

are the different projections, and the diagram

$$\begin{tikzcd}
q_1^*\mathcal{M} \arrow[rr, equal] \arrow[dr, equal] & & p_{12}^*p_1^*\mathcal{M} \arrow[rr, "p_{12}^*\mu", "\cong"'] & & p_{12}^*p_2^*\mathcal{M} \arrow[rr, equal] & & q_2^*\mathcal{M} \arrow[dl, equal] \\
& p_{13}^*p_1^*\mathcal{M} \arrow[dr, "\cong", "p_{13}^*\mu"'] & & & & p_{23}^*p_1^*\mathcal{M} \arrow[dl, "\cong"', "p_{23}^*\mu"] & \\
& & p_{13}^*p_2^*\mathcal{M} \arrow[dr, equal] & & p_{23}^*p_2^*\mathcal{M} \arrow[dl, equal] & & \\
& & & q_3^*\mathcal{M} & & &
\end{tikzcd}$$

must commute. A morphism of descent data is a morphism $\varphi : \mathcal{M} \to \widetilde{\mathcal{M}}$ compatible with μ and $\widetilde{\mu}$, i.e. $(p_2^*\varphi)\mu = \widetilde{\mu}(p_1^*\varphi)$

Remark 3. We have a functor

$$\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}, \quad \mathcal{N} \mapsto (f^*\mathcal{N}, \text{the canonical iso } p_1^*f^*\mathcal{N} \cong p_2^*f^*\mathcal{N}).$$

One would like this to be an equivalence of categories. It has a right adjoint

$$(\mathcal{R}\mathcal{M})(U) = \{m \in \mathcal{M}(f^{-1}U) \mid \mu p_1^*m = p_2^*m\}$$

Proposition 2 (stacks loc.cit., SGA1.VII.1, Milne)**.** *If f is faithfully flat and quasi-compact, the above functor* $\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}$ *is an equivalence of categories.*

Proof. If f has a section, the inverse image along that section is an inverse functor. In general, base change with $f : X \to Y$ reduces to this situation, provided that f is separated, which is a situation one can reduce to. □

Corollary 1. *If f is faithfully flat, $\mathcal{O}_Y(V) = \{\lambda \in \mathcal{O}_X(f^{-1}U) \mid p_1^*\lambda = p_2^*\lambda\}$.*

Remark 4. Both quasi-compactness and quasi-coherence in proposition 2 are needed. Consider $Y = \operatorname{Spec} R$, R a PID with $\operatorname{Spec} R$ infinite,

$$X = \coprod_{m \in \mathrm{mSpec}} \operatorname{Spec} R_m, \qquad N_1 = \coprod_{m \in \mathrm{mSpec}\, R} R/m \to N_2 = \prod_{m \in mSpecR} R/m,$$

then it is easy to see that this inclusion does not split, bit it splits canonically after applying $- \otimes_R R_m$, giving rise to a morphism of descent data which does not descend to a morphism $N_2 \to N_1$.

Definition 5. A morphism $i : X \to Y$ in a category $\mathcal{A}$ is an effective monomorphism if for all objects T,

$$\mathrm{Hom}_{\mathcal{A}}(T, X) \xrightarrow{\varphi \mapsto i\varphi} \{f \in \mathrm{Hom}_{\mathcal{A}}(T, Y) \mid \sigma f = \widetilde{\sigma} f \text{ for all } \sigma, \sigma' : Y \to S \text{ s.t. } \sigma i = \widetilde{\sigma} i\}$$

is bijective. $p : X \to Y$ is an effective epimorphism if it is an effective monomorphism in $\mathcal{A}^{\mathrm{op}}$, i.e.

$$\mathrm{Hom}_{\mathcal{A}}(Y, T) \xrightarrow[\cong]{\varphi \mapsto \varphi p} \{f \in \mathrm{Hom}_{\mathcal{A}}(X, T) \mid f\sigma = f\widetilde{\sigma} \text{ for all } \sigma, \widetilde{\sigma} : S \to X \text{ s.t. } p\sigma = p\widetilde{\sigma}\}.$$

Remark 5. If $X \times_Y X$ exists, f being an effective epimorphism is equivalent to it being a coequalizer of $X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$.

Proposition 3 (SGA1.VIII.4 or stacks 023Q). *Every fpqc (quasi-compact faithfully flat) morphism of schemes is an effective epimorphism, i.e.*

$$\mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T) \rightrightarrows \mathrm{Hom}(X \times_Y X, T)$$

is an exact sequence of sets.

Remark 6. This implies that for every scheme T, the functor $X \mapsto T(X) := \mathrm{Hom}(X, T)$ satisfies the sheaf condition in the following sense:

$$T(Y) \xrightarrow{\tau \mapsto \tau f} \{t \in T(X) \mid tp_1 = tp_2\}.$$

That this should be interpreted as a kind of sheaf axiom becomes obvious if we have a covering $Y = \bigcup_{i=1}^n U_i$, $X = \coprod_{i=1}^n U_i \xrightarrow{f} Y$. Then $X \times_Y X = \coprod_{i,j=1}^n (U_i \cap U_j)$ with $tp_1|_{U_i \cap U_j}$ identified with $t|_{U_i}|_{U_i \cap U_j}$.

Proposition 4 (01UA). *Every flat morphism (locally) of finite presentation is open.*

1.3 Grothendieck Topologies

As Deligne did in Arcata, we prefer the definition of Grothendieck topology by sieves.

Definition 1. Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}(\mathcal{C})$. A *sieve* (or $\mathcal{C}$-sieve) over X is a class $\mathcal{S}$ of morphisms with target X, such that $(U \to X) \in \mathcal{S}$ implies $(V \to U \to X) \in \mathcal{S}$ for every morphism $V \to U$ in $\mathcal{C}$. The empty class of morphisms is called the *empty sieve*, and the class of all morphisms with target X is called the *all sieve* (over X). For a morphism $f : Y \to X$ in $\mathbb{C}$, $f^*\mathcal{S} = \{v : U \to Y \mid fu \in \mathcal{S}\}$.

Remark 1. a) Obviously, $f^*\mathcal{S}$ is a sieve over Y if $\mathcal{S}$ is a sieve over X.

b) The fact that we work with categories where $\mathrm{Ob}\,\mathcal{C}$ is a proper class creates set-theoretic difficulties. Our way of dealing with this is to mostly ignore them.

c) The intersection of any class of sieves over X is a sieve over X.Thus, for every class $(f_i)_{i \in I}$ of morphisms with target X, there is a smallest sieve over X containing all f_i, namely $\{\xi : U \to X \mid \xi = f\eta \text{ for } \eta : U \to Y_i \text{ for some } \eta\}$. This is called the sieve generated by the f_i.

Example 1. a) X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$ turned into a category by its half ordering by $\subseteq$. If $X = \bigcup_{i \in I} U_i$ is an open covering, then the sieve generated by the (unique morphisms from) U_i is the sieve of all $V \in \mathbb{O}_X$ s.t. $V \subseteq U_i$ for at least one i.

b) If X is a complex space (e.g. $X = \mathbb{C} \setminus \{0\}$) with its complex topology, and $U \subseteq X$ open and $f \in \mathcal{O}_X(U)$, then $\mathcal{S} = \{V \subseteq U \mid \exists \varphi \in \mathcal{O}_X(V) \text{ s.t. } \varphi^2 = f|_V\}$ is a $\mathbb{O}_X$-sieve over U.

Remark. Thus, a morphism is in a sieve iff it is small enough "to pass through the sieve".

Definition 2. A *Grothendieck topology* $\mathbb{T}$ on a category $\mathcal{C}$ associates to every object X of $\mathcal{C}$ a class $\mathbb{T}_X$ of sieves over X, called the *covering sieves* of X. The following conditions must be verified:

(GTTriv) The all sieve over X covers X.

(GTTrans) If $\mathcal{S} \in \mathbb{T}_X$ and $f : Y \to X$, then $f^*\mathcal{S} \in \mathbb{T}_Y$.

(GTLoc) If $\mathcal{T} \in \mathbb{T}_X$ and $\mathcal{S}$ any sieve over X such that $f^*\mathcal{S} \in \mathbb{T}_Y$ for all $f : Y \to X$ in $\mathcal{T}$, then $\mathcal{S} \in \mathbb{T}_X$.

We will often write $\mathcal{S} \mathrel{/\!\!=} X$ for $S \in \mathbb{T}_X$ if there are no ambiguities (or $\mathcal{S} \mathrel{/\!\!=_{\mathbb{T}}} X$ it there are).

Remark 1. Pretopologies are specified by specifying a class of admissible coverings $\mathcal{U} = (f_i : Y_i \to X)_{i\in I}$. Various assumptions must be satisfied, like that $(U_i \times_X Y \to Y)_{i\in I}$ still form an admissible covering of Y (including the existence of the fibre product). By putting $\mathbb{T}_X = \{$admissible coverings $\mathcal{S}$ of X with all $f_i \in \mathcal{S}\}$ one gets a Grothendieck topology. Equivalent pretopologies define the same $\mathbb{T}_X$. If the category has fibre products, one gets a pretopology from a Grothendieck topology $\mathbb{T}_X$ by calling a covering admissible iff the f_i generate a sieve in $\mathbb{T}_X$. This is the largest pretopology in its equivalence class.

Example 2. X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$, and $\mathcal{S} \mathrel{/\!=} U$ iff $U = \bigcup_{V\in\mathcal{S}} V$. Other Grothendieck topologies can be introduced as well.

a) $X = [0,1]_{\mathbb{R}}$, put $\mathcal{S} \mathrel{/\!=} U$ iff there are countable many $(U_i)_{i\in\mathbb{N}}$ such that $U \setminus \bigcup_{i\in\mathbb{N}} U_i$ is a set of Lebesgue measure 0, or $\mathcal{S} = U = \emptyset$.

b) Rigid analytic geometry (Tate style) or real algebraic geometry (Delfs-Knebusch) enforce quasi-compactness of certain open subsets of X, making it harder to be a covering.

c) X a Noetherian scheme, $d \in \mathbb{N}$. $\mathcal{S} \mathrel{/\!=} \mathcal{U}$ iff $\operatorname{codim}(U \setminus \bigcup_{V\in\mathcal{S}} V) \geq d$, making it easier to be a covering.

Remark 2. You can think of (GTLoc) as the condition that being a covering is a local property.

Fact 1. a) Every sieve $\mathcal{T}$ containing a covering sieve $\mathcal{S}$ is itself covering.

b) The intersection of finitely many covering sieves is covering.

Proof. a) If $(f : U \to X) \in \mathcal{S}$, then $f^*\mathcal{T}$ is the all-sieve on U which covers U by (GTTrans). By (GTLoc), $\mathcal{T}$ covers X.

b) It is sufficient to show that $\mathcal{T} := \mathcal{S}_1 \cap \mathcal{S}_2$ covers X, where both $\mathcal{S}_i \mathrel{/\!=} X$. If $(f : U \to X) \in \mathcal{S}_1$, then $f^*\mathcal{T} = f^*\mathcal{S}_2 \mathrel{/\!=} U$ by (GTTrans) and since $\mathcal{S}_2 \mathrel{/\!=} X$. Again by (GTLoc), $T \mathrel{/\!=} X$. □

Proposition 1. *Let S be a scheme, P a Zariski-local property of S-schemes and $\underline{\mathrm{Sch}}_S^P$ be the full subcategory of the category $\underline{\mathrm{Sch}}_S$ of S-schemes, with class of objects being the S-schemes with property P, and let $\mathcal{C}$ be a class of morphisms in $\underline{\mathrm{Sch}}_S^P$. The following assumptions must be satisfied:*

(A) $\mathcal{C}$ is closed under composition, base-change and finite coproducts.

(B) If U is a quasi-compact S-scheme with $P(U)$ and $U = \bigcup_{i=1}^n U_i$ is a finite affine open covering, then the morphism $\coprod_{i=1}^n U_i \to U$ belongs to $\mathcal{C}$.

If X is an S-scheme with $P(X)$ then the following conditions to a sieve $\mathcal{S}$ over X are equivalent:

(C1) There are open coverings $X = \bigcup_{i\in I} U_i$ and morphisms $V_i \to U_i$ for all $i \in I$ such that $(V_i \to U_i \to X) \in \mathcal{S}$ and V_i is covered (in the ordinary sense) by its Zariski-open subsets W such that $(W \to V_i \to U_i) \in \mathcal{C}$

(C2) The same conditions, but the U_i and V_i must be affine.

In addition, we obtain a Grothendieck topology $\mathbb{T}$ on $\underline{\mathrm{Sch}}_S^P$ by associating to X the class $\mathbb{T}_X$ of all sieves with these equivalent properties.

Remark 3. a) In (A), the stability under base change includes the condition that $X_Y\widetilde{X}$ has P when $X, Y, \widetilde{X}$ have this property and $(X \to Y) \in \mathcal{C}$.

b) It the elements of $\mathcal{C}$ are open maps, then the conditions (C1) and (C2) can be modified by simply requiring that $(V_i \to U_i) \in \mathcal{C}$ without changing anything else, i.e. $X = \bigcup_{i\in I} U_i$ and $(V_i \to U_i) \in \mathcal{C} \cap \mathcal{S}$.

Example 3. a) P the trivial property and $\mathcal{C}$ the class of all fpqc morphisms. We get the fpqc topology on $\underline{\mathrm{Sch}}_S$.

$\widetilde{\mathrm{a}}$) Let S be Noetherian, P : local Noetherianness and $\mathcal{C}$ the class of fpqc morhpisms. This will NOT work as (A) is violated: For instance, with $S = X = \operatorname{Spec}\mathbb{Q}$, the fibre product $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ is non-noetherian: The ideal $I = (x \otimes y - y \otimes x \mid x, y \in \mathbb{C})$ is not finitely generated as $\Omega_{\mathbb{C}/\mathbb{Q}} \cong I/I^2$. This is a $\mathbb{C}$-vector space of dimension equal to the continuum (the transcendence degree of $\mathbb{C}/\mathbb{Q}$).

b) Let $\mathcal{C}$ be the class of all fppf (faithfully flat of finite presentation) morphisms and the trivial property (or local Noetherianness) for P. Then fibre products don't cause any trouble, since then $\widetilde{X} \times_X Y$ is of finite type over $\widetilde{X}$ and local Noetherianness is preserved. One gets the fppf-topology on (locally noetherian) S-schemes. In this case. quasi-finiteness can be added to "of finite presentation" without modifying the topology: (stacks 056X)

c) The class $\mathcal{C}$ of all surjective morphisms which are Zariski-local isomorphisms, with $P =$ trivial, or local Noetherianness, or regularity, ... and one gets the Zariski topology on $\underline{\mathrm{Sch}}_S$.

Proof. (of proposition 1) It is clear that (C2) implies (C1). Assume conversely that $X = \bigcup_{i\in I} U_i$ and $(p_i : V_i \to U_i) \in \mathcal{C}$ such that V_i is covered by the open $W \subseteq V_i$ such that $(W \to V_i \to X) \in \mathcal{S} \cap \mathcal{C}$. (We call such W $\mathcal{S}$-small.) Let $U_i = \bigcup_{j\in J_i} U_{ij}$ be an open affine covering and $V_{ij} = p_i^{-1} U_{ij} = V_i \times_{U_i} U_{ij}$. Thus $(V_{ij} \to U_{ij}) \in \mathcal{C}$ by (A). If $W \subseteq V_i$ is $\mathcal{S}$-small, the same holds for $W \cap V_{ij}$, showing that V_{ij} is covered by its $\mathcal{S}$-small open subsets. Thus we may assume that the U_i are affine and the V_i quasi-compact. By an application of (B), we may also assume that the V_i are affine. Then (C2) holds.

It remains to show the properties of a Grothendieck topology. For (GTTriv) this is trivial (U_i any affine covering and $V_i = U_i$). Also, (GTTrans) is easy. If $f : \widetilde{X} \to X$ is a morphism one puts $\widetilde{U}_i = f^{-1}U_i$, $\widetilde{V}_i = \widetilde{U}_i \times_{U_i} V_i$ and $(\widetilde{V}_i \to \widetilde{U}_i) \in \mathcal{C}$ by (A). Also, if $W \subseteq V$ is $\mathcal{S}$-small, then its inverse image in $\widetilde{V}_i$ is $f^*\mathcal{S}$-small, and these inverse images cover $\widetilde{V}_i$. For (GTLoc), let $\mathcal{S} \mathrel{/\!\!=} X$ and $\mathcal{T}$ any sieve such that $f^*\mathcal{T} \mathrel{/\!\!=} Y$ for all $(f : Y \to X) \in \mathcal{S}$. We must show $\mathcal{T} \mathrel{/\!\!=} X$.

<u>Case 1:</u> One can choose $V_i = U_i \xrightarrow{\mathrm{id}} U_i$ in the condition (C1) for $\mathcal{S} \mathrel{/\!\!=} X$. Then the restriction $\mathcal{T}|_{U_i} := (U_i \hookrightarrow X)^*\mathcal{T}$ covers U_i. Thus there are an open covering $U_i = \bigcup_{j\in J_i} U_{ij}$ and $V_{ij} \to U - ij$ as in (C1) for $\mathcal{T}|_{U_i}$, and then $X = \bigcup_{i\in I} \bigcup_{j\in J_i} U_{ij}$, together with the morphisms $V_{ij} \to U_{ij}$, does the same for X.

<u>Case 2:</u> X is affine, and there is a morphism $(p : V \to X) \in (\mathcal{S} \cap \mathcal{C})$ with V affine, s.t. p generates $\mathcal{S}$. Then $p^*\mathcal{T} \mathrel{/\!\!=} V$. Write $V = \bigcup_{i=1}^n U_i$ and morphisms $(V_i \to U_i) \in \mathcal{C}$ such that the $\mathcal{S}$-small open susets of V_i cover V_i. Then one can satisfy (C2) for $\mathcal{T}$ by $U' = X$, $V' = \coprod_{i=1}^n V_i \to \coprod_{i=1}^n U_i \to V \to X = U'$, where the arrows are in $\mathcal{C}$ by (A), (B), and assumption, respectively.

<u>Case 3:</u> General case: If $V_i \to U_i$ are as in (C2) for $\mathcal{S}$, then the pullback of $\mathcal{T}$ to any $\mathcal{S}$-small open subset W of V_i covers W. By case 1, the pullback of $\mathcal{T}$ to V_i covers V_i. By case 2, $\mathcal{T}|_{U_i} \mathrel{/\!\!=} U_i$. By case 1 again, $\mathcal{T} \mathrel{/\!\!=} X$. $\square$

Definition 3. A presheaf on a category $\mathcal{C}$ (with values in sets, (abelian) groups, rings) is a contravariant functor from $\mathcal{C}$ to $\underline{\mathrm{Set}}$ (or groups, rings, ...). If a Grothendieck topology $\mathbb{T}$ on $\mathcal{C}$ is given, then a presheaf $\mathcal{F}$ is called ($\mathbb{T}$-)separated, if

$$F(X) \to \prod_{(p:U\to X)\in\mathcal{S}} F(U), \qquad f \mapsto (F(p)f)_p \tag{$*$}$$

is injective. We call a separated presheaf F a sheaf if the image of $(*)$ is $\varprojlim_{(p:U\to X)\in\mathcal{S}} F(U)$. In other words, the image of $(*)$ must be the family of all $(f_p)_p$ such that $F(q')f_p = F(p')f_q$ in $F(W)$ whenever

$$\begin{tikzcd} W \arrow[r, "p'"] \arrow[d, "q'"] & V \arrow[d, "q"] \\ U \arrow[r, "p"] & X \end{tikzcd}$$

is a commutative diagram in $\mathcal{C}$, with $p, q \in \mathcal{S}$.

Proposition 2. *In the situation of proposition 1, a presheaf G is a sheaf (resp. separated) if and only if for every object X of $\underline{\mathrm{Sch}}_S^P$ the presheaf $U \mapsto G(U)$ on X equipped with its Zariski topology is a sheaf (resp. separated), and for every morphism $p : U \to V$ in $\mathcal{C}$ the sequence*

$$G(V) \xrightarrow{p^*} G(U) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} G(U \times_V U)$$

is exact in the sense that the first morphism is the equalizer of the second two (resp, if p^* is injective

Proof. Let $S \,/\!\!= X$, we must show that $G(X) \to \varprojlim G$ is bijective (resp. injective), and for the proof of bijectiveness, we may assume injective.
Case 1: S is already covering for X_{Zar}: Trivial.
Case 2: There is a morphism $p : U \to X$ in $\mathcal{C}$ such that the S-small open subsets W of U cover U (as sets). If $g_1, g_2 \in G(X)$ have the same image in $\varprojlim_S G$, then $p^*g_1|_W = p_2^*g_2|_W$ when $W \subseteq U$ is S-small. By our first assumption on G, $p^*g_1 = p^*g_2$. As p^* is injective by our second assumption, $g_1 = g_2$. Let $\gamma \in \varprojlim_S G$. By our first assumption on G, there is $g_U \in G(U)$ such that $g_U|_W = \gamma_W$ whenever $W \subseteq U$ is S-small. Let $W, \widetilde{W} \subseteq U$ be S-small, then for $p_1, p_2 : U \times_X U \to U$ we have

$$p_1^* g_U|_{W \times_X \widetilde{W}} = p_1^* \gamma_W|_{W \times_X \widetilde{W}} = \gamma_{W \times_X \widetilde{W}} = p_2^* \gamma_{\widetilde{W}}|_{W \times_X \widetilde{W}} = p_2^* g_U|_{W \times_X \widetilde{W}}.$$

As these $W \times_X \widetilde{X}$ cover $U \times_X U$ as a set, $p_1^* g_U = p_2^* g_U$. By our assumption there is a unique $g \in G(X)$ such that $p^*g = g_U$. We must show that the image of g in $\varprojlim_S G$ is γ. Let $\widetilde{S} \subseteq S$ be the subsieve of S generated by the S-small $W \subseteq U$. Then $\widetilde{S} \,/\!\!= X$, and the image of g in $\varprojlim_{\widetilde{S}} G$ equals $\gamma|_{\widetilde{S}}$ by construction. For $(\nu : V \to X) \in S$, this implies that $G(\nu)g = \gamma_V$ as they have the same image in $\varprojlim_{\nu^* \widetilde{S}} G$, and $\nu^* \mathcal{S} \,/\!\!= V$. Thus the claim about g is shown.
Case 3: General case. Let $V_i \to U_i$ be as in the definition of a Grothendieck topology. If g_1, g_2 have the same image in $\varprojlim_S G$ then $g_1|_{U_i} = g_2|_{U_i}$ by case 2, hence $g_1 = g_2$ by the first assumption. Let $\gamma \in \varprojlim_S G$, by case 2 there is $\gamma_i \in G(U_i)$ such that the image of γ_i in $\varprojlim_{S|_{U_i}} G$ equals the restriction of γ. Then $\gamma_i|_{U_i \cap U_j} = \gamma_j|_{U_i \cap U_j}$ as their images in $\varprojlim_{S|_{U_i \cap U_j}} G$ are both equal to the restriction of γ to $S|_{U_i \cap U_j} \,/\!\!= U_i \cap U_j$. By our first assumption, there is $g \in G(X)$ such that $g|_{U_i} = g_i$. In a similar way as in the end of case 2, one sees that the image of g in $\varprojlim_S G$ equals γ. □

Corollary 1. *If X is any S-scheme then*

$$U \to X(U) := \mathrm{Hom}_{\underline{\mathrm{Sch}}_S}(U, X)$$

is an fpqc-sheaf on $\underline{\mathrm{Sch}}_S$.

Exercise: If $F \in \mathrm{QCoh}(S)$, then $(\upsilon : U \to S) \mapsto \upsilon^* F$ is an fpqc sheaf, and $H^*(S_{Zar}, F) \cong H^*(S_{fpqc}, F)$

1.4 Étale morphisms

Proposition 1. *Let $f : X \to Y$ be a morphism locally of finite type between Noetherian schemes, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

a) $\Omega_{X/Y,x} = 0$.

b) There is an open neighbourhood U of x in X such that $\Delta_{X/Y} : U \to X \times_Y X$ is an open embedding.

c) We have $\mathfrak{m}_x = (f^*\mathfrak{m}_y)\mathcal{O}_{X,x}$, and $k(x)$ is a separable finite field extension of $k(y)$.

If f is separated, such that $\Delta_{X/Y}$ is a closed embedding defined by the quasi-coherent sheaf of ideals $J \subseteq \mathcal{O}_{X\times_Y X}$, then the above is also equivalent to

d) $J_x = 0$.

Remark. The Noetherianness assumption can be dropped with little effort.

Proof. (Sketch) As a), b), and c), as well as the claim in d) are local in X, we may assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine. Then the equivalence of b) with d) is obvious as J is locally finitely generated: If d) holds, there is an open neighbourhood U of x in X such that $J|_U$ vanishes. The equivalence of a) with d) then comes from a well-known fact (Remark 1 below) about Kähler differentials. By Nakayama's lemma $(\Omega_{X/Y})_x = 0$ if and only if $0 = (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{X,x}} k(x) \cong (\Omega_{f^{-1}\{y\}/k(y)})_x$, by the compatability of Kähler differentials with base change. The $k(y)$-algebra $(k(y) \otimes_A B)_{\mathfrak{m}_x}$ has vanishing Kähler differentials over $k(y)$ iff this local $k(y)$-algebra is a finite separable field extension $l/k(y)$, i.e. $\mathfrak{m}_x = (f^*\mathfrak{m}_y)\mathcal{O}_x$ (othersie B_x has nilpotent elements) and $k(x) = l$ is separable over $k(y)$. $\square$

Remark 1. a) If f is separated and J as in (d), then $\Omega_{X/I} \cong \Delta^*_{X/Y} J \cong \Delta^*_{X/Y}(J/J^2)$.
b) If A and B are as in the proof, $\Omega_{B/A} \cong I/I^2$, $I = \ker(B \otimes_A B \to B)$.
c) $\mathcal{D}er_{B/A}(B, M) \cong \operatorname{Hom}_B(I/I^2, M)$, given by $d \mapsto \varphi(a \otimes b) = ad(b)$ and $d(b) = \varphi(1 \otimes b - b \otimes 1)$.

Definition 1. a) A morphism $f : X \to Y$ locally of finite type between locally Noetherian schemes is *unramified* at $x \in X$ iff it satisfies the equivalent definitions of proposition 1.
b) It is called *étale at x* if it is flat and unramified at x.
c) It is called étale iff it is étale at all $x \in X$.
d) It is called an *étale covering* if it is étale and finite.

Remark. See 00U0 for the definitions the non-Noetherian case, which are essentially the same. By 00U9 locally every étale morphism comes by base-change from a Noetherian morphism. See also EGA IV.17.

Fact 1 (00U2)**.** a) The class of étale morphisms is stable under composition and base change.
b) If $g \circ f$ is étale and g unramified, then f is étale.
c) If f is étale and a closed embedding, then f is an open embedding.

Proof. a) The stability of flatness under base change is assumed to bek nown here, and for unramifiedness this follows from $\Omega_{\widetilde{X}/\widetilde{Y}} \cong \Omega_{X/Y}$ for every Cartesian square

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

For treatment of compositions, let the morphisms always be $f : X \to Y, g : Y \to S$. Again for flatness this is well-known. Unramifiedness of $g \circ f$ follows from the exact sequence

$$f^*\Omega_{Y/S} \xrightarrow{f^*} \Omega_{X/S} \to \Omega_{X/Y} \to 0 \tag{F1}$$

b)
c) This follows form Proposition 1.2.4. even when f is flat of finite presentation, X, Y arbitary. □

Fact 2. A flat morphism $X \to Y$ is étale at $x \in X$ if and only if this holds for $f^{-1}(y)/x$ at x. The same holds for unramified morphisms.

Example 1. a) $X \to \operatorname{Spec} k$ is étale at $x \in X$ iff $\mathcal{O}_{X,x}$ is a finite separable field extension of k.
b) Every open or closed embedding is unramified.
c) Every open embedding is étale.

Lemma 1. *If A is an algebra over a field K, the following conditions are equivalent:*

a) A/K is étale,

b) $A \cong \bigoplus_{i=1}^{n} L_i$, each L_i/K separable,

c) The trace form $B_{A/K}(a,b) := \operatorname{Tr}_{A/K}(ab)$ is a perfect pairing on $A \times A$.

Proof. Omitted. □

Remark 2. If L/K is a finitely generated field extension, then $\Omega_{X/Y} \cong 0$ iff L/K is finite and separable.

Proposition 2. *Let X be locally Noetherian, $\mathcal{A}$ a coherent locally free $\mathcal{O}_X$-algebra. Then $\underline{\operatorname{Spec}}\mathcal{A} \to X$ is étale over x if and only if the trace bilinear form $B_{\mathcal{A}_x/\mathcal{O}_{X,x}}$, $B(a,b) = \operatorname{Tr}_{\mathcal{A}_x/\mathcal{O}_{X,x}}(ab)$ is non-degenerate. In particular, $\underline{\operatorname{Spec}}\mathcal{A}$ is an étale covering if the trace bilinear form is non-degenerate everywhere.*

Proof. Flatness is automatic by our assumptions. The assertion then follows with little work from fact 2 and lemma 1. □

Corollary 1. *In the situation of the proposition, $p : \underline{\operatorname{Spec}}\mathcal{A} \to X$ is an étale covering if and only if there is an open subset $U \subseteq X$ with $\operatorname{codim}(Y, X) \geq 2$ for every irreducible component Y of $X \setminus U$, and $p^{-1}(U) \to U$ is an étale covering.*

Proof. Without losing generality $X = \operatorname{Spec} R$ is affine and $\mathcal{A}$ is defined by the free R-algebra A. Using a base of the R-module A and a matrix representation of $B_{A/R}$,

$$\{x \in X \mid \operatorname{Spec} A \to X \text{ is not étale over } x\} = V(d)$$

where $d \in A$ is the determinant of that matrix representation of $B_{A/R}$. By Krull's principal ideal theorem all irreducible components of this closed subset have codimension at most 1. □

Proposition 3. *If $f : X \to Y$ is an étale morphism of locally Noetherian S-schemes, then $f^*\Omega_{Y/S} \to \Omega_{X/S}$ is an isomorphism.*

Proof. Surjectivity follows from the cotangent sequence (F1) using only that f is unramified. For the isomorphism claim consider

$$\begin{tikzcd}
X \arrow[r, "\Delta_{X/Y}"] \arrow[rr, bend left, "\Delta_{X/S}"] \arrow[dr, "f"'] & X \times_Y X \arrow[r, "j"] \arrow[d] & X \times_S X \arrow[d, "p"] \\
& Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y
\end{tikzcd}$$

It is sufficient to give the proof when all schemes are affine and therefore separated. Then all diagonals are closed embeddings and given by coherent sheaves of ideals, e.g. $\Delta_{X/S}$ by $J_{X/S}$. The square being cartesian implies that j is a closed embedding with sheaf of ideals $J_j = p^* J_{Y/S}$ (this uses that p is flat) As $\Delta_{X/Y}$ is an open embedding,

$$\Omega_{X/S} = \Delta^*_{X/S} J_{X/S} = \Delta^*_{X/Y} j^* J_{X/S} \cong \Delta^*_{X/Y} j^* J_j \cong \Delta^*_{X/Y} j^* p^* J_{Y/S} = f^* \Delta^*_{Y/S} J_{Y/S} = f^* \Omega_{Y/S}$$

□

Proposition 4. *If $f : X \to Y$ is a morphism of locally finite type between locally Noetherian schemes, and if f is étale at $x \in X$, then X is regular at x iff Y is at $y = f(x)$.*

Proof. From the étaleness of f one gets $\mathfrak{m}_x^l / \mathfrak{m}_x^{l+1} \cong \mathfrak{m}_y^l / \mathfrak{m}_y^{l+1} \otimes_{k(y)} k(x)$ and the dimensions of the local rings are equal to the smallest d such that the dimension of these vector spaces are $O(l^{d-1})$ as $l \to \infty$. It follows that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} =: d$ and therefore X is regular if and only if $\dim_{k(x)} \mathfrak{m}_x / \mathfrak{m}_x^2 = d$ if and only if $\dim_{k(y)} \mathfrak{m}_y / \mathfrak{m}_y^2 = d$ if and only if Y is regular. □

Proposition 5 (Arcata Def 1.1.)**.** *Let S be an R-algebra of finite type, where R is Noetherian. Then the following are equivalent.*

(A1) If A is a Noetherian R-algebra, $I \subseteq A$ a nilpotent ideal, then in any diagram of solid arrows

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, dashed] & A/I \\
R \arrow[u] \arrow[r] & A \arrow[u]
\end{tikzcd} \tag{L}$$

there is a unique dotted arrow (a ring homomorphism) making the diagram commute.

(A2) The same condition, but with the sharper assumption $I^2 = 0$.

(A3) The condition (A2) with the sharper assumption that A is a local ring.

(B) S is an étale R-algebra (i.e. S is flat over R and $\Omega_{S/R} = 0$).

(C1) There is a representation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)_T$, where $T = R[x_1, \ldots, x_n]$, such that the Jacobian determinant $\det(\frac{\partial f_i}{\partial x_j})_{ij}$ maps to a unit in S.

(C2) If $S \cong R[x_1, \ldots, x_n]/J$, $J \subseteq T = R[x_1, \ldots, x_n]$ is any representation of S as an R-algebra, then there are $g, f_1, \ldots, f_n \in T$ such that $V(g) \cap V(f_1, \ldots, f_n) = \emptyset$, $J_g = \langle f_1, \ldots, f_n \rangle_{T_g}$ and the Jacobian determinant as in (C1) maps to a unit in S.

Proof. (A1)⇒(A2)⇒(A3) is trivial. (A2)⇒(A1) is an induction on the smallest k such that $I^{2^k} = 0$.
(A3)⇒(A2): By assumption our lifting problem has local solutions $S \to A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. As S is a

finitely presented A-algebra, these come from $S \to A_{\alpha_\mathfrak{p}}$, $\alpha_p \in R$. There are finitely many $\alpha_i = \alpha_{\mathfrak{p}_i}$ such that $\langle \alpha_1, \dots, \alpha_n \rangle_A = A$, and the compositions $S \to A_{\alpha_i} \to A_{\alpha_i \alpha_j}$ and $S \to A_{\alpha_j} \to A_{\alpha_i \alpha_j}$ coincide because this is so after composition with any morphism $A_{\alpha_i \alpha_j} \to A_\mathfrak{q}$ for any $\mathfrak{q} \in \operatorname{Spec} A \setminus V(\alpha_i \alpha_j)$, and the map from $A_{\alpha_i \alpha_j}$ to the product of these $A_\mathfrak{q}$ is injective. It is then well-known that there is a unique ring morphism $S \to A$ making all triangles $S \to A \to A_{\alpha_i}$ commute, and it is easy to see that this is the only solution to (L).

(A)⇒(B): One way to show flatness is to consider any presentation $S \cong T/J$, $T = R[x_1, \dots, x_n]$. By induction on n we get commutative diagrams

$$\begin{tikzcd} S \arrow[r, "\lambda_n"] \arrow[dr, dashed, "\lambda_{n+1}"] & T/J^n \\ R \arrow[u] \arrow[r] & T/J^{n+1} \arrow[u] \end{tikzcd}$$

splitting the surjective morphism $\widehat{T} \to S$ where $\widehat{T}$ is the completion of T with respect to J. As our rings are Noetherian, $\widehat{T}$ is a flat T-module, hence a flat R-module, and so is its direct summand S. $\Omega_{S/R} = 0$ follows from

Fact 3. In the situation of (L) assume $I^2 = 0$. Then $\mathcal{D}er(S/R, I)$ acts simply transitively on the set of dottet arrows $\alpha : S \to A$ making (L) commute, provided that such a solution α exists. The action of $\delta \in \mathcal{D}er(S/R, I)$ on α is $\widetilde{\alpha}(s) = \alpha(s) + \delta(s)$.

As by our assumption the set of solutions to (L) is not empty, we have $\mathcal{D}er(S/R, I) = 0$ for all such I. This can be applied to $A = S \oplus M$, $I = M$. Then $I^2 = 0$ and $\mathcal{D}er(S/R, M) = 0$ for any S-module M. Hence $\Omega_{S/R} = 0$.

(B)⇒(C2): Let $T = R[x_1, \dots, x_n]$, $S = T/J$ as in (C2). By the short exact sequence

$$J/J^2 \to \Omega_{T/R} \otimes_T S \to \Omega_{S/R} \to 0$$

and (B), the map $J/J^2 \to \Omega_{T/R} \otimes_T S \cong \bigoplus_{i=1}^n S dX_i$ (sending $f + J^2$ to $((\frac{\partial f}{\partial x_i} + J) dx_i)_i$) must be surjective. Because of this it is possible to choose the f_i in (C2) such that the Jacobian determinant becomes a unit in S (e.g. s.t. the image of f_j is $(\delta_{ij} dX_i)_i$). Let $J' = \langle f_1, \dots, f_n \rangle_T$, $X = \operatorname{Spec} S$, $X' = \operatorname{Spec} S'$, where $S' = T/J'$, $Y = \operatorname{Spec} R$. Repeating the above argument with J' implies $(\Omega_{S'/R})_x = 0$ for all x in the closed subscheme $X \subseteq X'$. Thus $X' \to S$ is unramified at the image of X', and therefore $U \to Y$ is unramified, where $U \subseteq X'$ is some open neighbourhood of the image of X. By Fact 1, $X \to U$ is étale, hence $X \to X'$ is étale and by Fact 1 $X \to X'$ is an open embedding. It is thus possible to choose an element $g \in T$ whose image in $\mathcal{O}_{X'}(X') = T/J'$ equals 1 on the clopen subset $X \subseteq X'$ and 0 on its complement. It is then easy to see that g does what we want.

(C2)⇒(C1): We start with any presentation $\pi : R[x_1, \dots, x_n]/J \xrightarrow{\cong} S$ and apply (C2). With the notations from (C2), $\pi' : R[x_1, \dots, x_{n+1}]/J' \xrightarrow{\cong} S$, where J' is the ideal generated by J and $1 - gX_{n+1}$ and π' sending X_i to $\pi(X_i)$ when $i \leq n$ and $\pi'(x_{n+1})$ is some inverse image of g in S. This presentation does what we want.

(C1)⇒(A2): With nations as in (C1), if A is an R-algebra, then the set of solid arrows $S \to A/I$ making (L) commute is (by $\alpha \mapsto (\alpha(\text{image of } x_i \text{ in } S))_i$) equivalent to the set of solutions of $f_i(x_1, \dots, x_n) = 0$ in A/I. The set of dashed arrows $S \to A$ corresponds in the same way to the set of solutions of $f_i(x_1, \dots, x_n) = 0$ in A. It is well-known (Hensel's lemma) that the solution set in A maps injectively to the solutions in A/I when the Jacobian is a unit in A/I, which it is as it is in $S^\times$. □

Remark 3. This also holds in the non-Noetherian situation, when S/R is of finite presentation.

Proposition 6. *If $X_0 \to X$ is a closed embedding defined by a nilpotent sheaf of ideals, then the functor*

$$\text{Étale } X\text{-schemes} \to \text{Étale } X_0\text{-schemes}, \qquad Y \to Y_0 := X_0 \times_X Y$$

is an equivalence of categories.

Proof. The assertion that this functor defines a bijection on morphisms (i.e. is fully faithful) is easily reduced to the situation where X, Y are affine, in which case it is an immediate consequence of Proposition 5(A). It remains to show essential surjectivity.

Let $X = \operatorname{Spec} R$, $X_0 = \operatorname{Spec} R/I$. If $Y_0 \to X_0$ is an étale morphism with affine Y_0, by Proposition 5(C1) one can choose a representation $Y_0 = \operatorname{Spec} S_0$, $S_0 = R_0[x_1, \dots, x_n]/(f_1, \dots, f_n)$ There are $\varphi_i \in R_0[x_1, \dots, R_n]$ such that $\varphi_i \mod I = f_i$ and the Jacobian of the φ_i is a unit in $S = R[x_1, \dots, x_n]/(\varphi_1, \dots, \varphi_n)$ because this is so modulo the nilpotent ideal IS. For general étale X_0-schemes one chooses a covering by affine open subsets and by full faithfulness the gluing data module I lift to gluing data for the lifts of these affine étale X_0-schemes. The case of general X is dealt with in the same way, lifting $\pi_0^{-1}(U)$, $\pi_0 : Y_0 \to X_0$ étale, for affine open subsets $U \subseteq X$, and using full faithfulness of the functor to get gluing data for these lifts. □

Remark 4. Such $X_0 \to X$ are examples of universal homeomorphisms, i.e. morphisms $X_0 \to X$ such that $Y_0 = X_0 \times_X Y \to Y$ is a homeomorphism for any X-scheme Y. This condition can be checked by verifying universal injectivity, universal surjectivity, followed by universal closedness or universal openness.

Since for every pair of morphisms $\alpha : A \to S, \beta : B \to S$ of schemes (or locally ringed spaces) the canonical map

$$A \times_S B \to [A] \times_{[S]} [B] = \{(a, b) \mid a \in A, b \in B, \alpha(a) = \beta(b)\}$$

is surjective, any surjective morphism is automatically universally surjective. If $X_0 \to X$ is injective one can show that is is universally injective if and only if for all $x_0 \in X_0$ with image x in X, $k(x_0)/k(x)$ is an algebraic and purely inseparable field extension.

For morphisms of finite type between Noetherian schemes, universal closedness is equivalent to properness. But such morphisms are quasi-finite if they are injective, and if they are also proper they are finite by an easy special case of Zariski's Main Theorem.

Proposition 7. *Proposition 6 also holds when $X_0 \to X$ is a universal homeomorphism (i.e. finite, bijective, $k(x_0)/k(x)$ always algebraic and purely inseparable) of finite type between locally Noetherian schemes.*

Example 2. This can be applied to Frobenius type morphisms, e.g. $F_X = \mathrm{id}_X$, $F_X^*(\varphi) = \varphi^p$ in $\mathcal{O}_X(U)$ if $\mathrm{char}(X) = p$. Another example would be the relative Frobenius $F_{X/\mathbb{F}_q}$ on $X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q$ sending (when X is quasi-projective) all coordinates to their q-th power.

Lemma 2. *Let $f : X \to Y$, $g : Y \to S$ be morphisms locally of finite type between locally Noetherian schemes with f étale, and let $x \in X$. Then $g \circ f$ is étale at x if and only if g is étale at $y = f(x)$.*

Proof. Since f is étale, hence flat, and $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ local, $\mathcal{O}_{X,x}$ is a faithfully flat $\mathcal{O}_{Y,y}$-algebra. The if-part is the fact that étaleness is stable under composition. For the "only if"-part, use the fact that $\operatorname{Tor}_q^{\mathcal{O}_{S,s}}(\mathcal{O}_{Y,y}, T) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong \operatorname{Tor}_q^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, T) = 0$ (T any $\mathcal{O}_{S,s}$-module) when $q > 0$ (as gf is flat) and deriving $\operatorname{Tor}_q^{\mathcal{O}_{S,s}}(\mathcal{O}_{Y,y}, T) = 0$ as $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is faithfully flat.

That $\mathfrak{m}_{S,s}\mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$ can also be checked after $- \otimes \mathcal{O}_{X,x}$, as f is étale, $\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ and the desired equality again follows from the fact that gf is étale at x (hence $\mathfrak{m}_{S,s}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$). Trivially, separability of $k(y)/k(s)$ follows from $k(s) \subseteq k(y) \subseteq k(x)$ and $k(x)/k(s)$ separable. $\square$

1.5 The étale topology

Definition 1. Let X be a scheme.

a) Let Et/X be the category of étale X-schemes. The étale topology on that category is the Grothendieck topology for which $S \models U$ if and only if there are étale morphisms (of finite presentation) $U_i \to U$ belonging to S whose images cover U. This site (=category + Grothendieck topology) is called the small étale site $X_{\text{ét}}$

b) The étale topology of all (or all Noetherian) X-schemes is defined in the same way, dropping from a) the condition that $U \to X$ must be étale. This is the big étale site $X_{\text{Ét}}$.

Remark 1. Let $(U_i \to U)_i$ be a family of étale morphisms such that their images cover U and each U_i is covered by its open subsets $W \subseteq U_i$ which are S-small. Then the sieve generated by these $W \to U$ is covering in the sense of definition 1 and contained in S, hence $S \models X$. Therefore technical modifications as in Proposition 1.3.1 are not necessary in this case. The proof that one has a Grothendieck topology is simplified by étale morphisms being open.

Definition 2. A morphism $f : X \to Y$ is called weakly étale if it is flat and $\Delta_{X/Y} : X \to X \times_Y X$ is also flat.

Example 1. Every étale morphism is weakly étale as it is flat and $\Delta_{X/Y}$ is an open embedding.

Theorem 1 (Bhatt,Scholze)**.** *If A is a ring and B a weakly étale A-algebra, there is a faithfully flat weakly étale B-algebra $\widetilde{B}$ such that $\widetilde{B}/A$ is a direct limit of étale A-algebras.*

Remark 2. a) The proétale topology is defined by Proposition 1.3.1 using the class of weakly étale morphisms. One can, for instance, use this to study $H^*(X, \mathbb{Z}_p)$ directly rather than indirectly as $\varprojlim H^*(X, \mathbb{Z}/p^k\mathbb{Z})$. The proof of the crucial results for Weil 1/2 still depend on the SGA 4 results on proper and smooth base change and Poincaré duality.

b) In between the ßetale and the fppf topology there is the syntonic topology where the covering sieves are generated by flat morphisms that are local complete intersections.

c) One could sharpen the condition for $S \models U$ in Definition 1 requiring that for every $x \in U$ there must be $i \in I$ and $\xi \in U_i$ mapping to x under $U_i \to U$ such that $k(\xi)/k(x)$ is trivial. (Then $\operatorname{Spec}\mathbb{Z}$ is covered by $\operatorname{Spec}\mathbb{Z}[i]$ and $\operatorname{Spec}\mathbb{Z}[\frac{1}{5}]$.)

1.6 The Étale Fundamental Group

Definition 1. Let FET_X be the category of finite étale morphisms $\pi : \widetilde{X} \to X$.

Definition 2. A geometric point of a scheme X is a morphism $\mathbf{x} : \operatorname{Spec} K \to X$, where K is an algebraically closed field. The image under $\mathbf{x}$ of the only point of $\operatorname{Spec} K$ is called the support of $\mathbf{x}$, i.e. $\mathbf{x}$ is supported at x if $\mathbf{x}(\operatorname{Spec} K) = x$.

Remark 1. The conidition that K is algebraically closed is sometimes relaxed to being separably closed. We follow 03P0 where K is required to be algebraically closed, which also seems to be mostly followed in SGA. Relaxing algebraically closed to separably closed leads to an essentially equivalent condition but it is a bit more awkward to study lifts of geometric points under finite non-étale surjective morphisms

$Y \to X$.

The category FET_X has cartesion products, equalizers and coproducts.

Definition 3. a) For a geometric point $\mathbf{x} : \operatorname{Spec} K \to X$, let $\mathrm{Fib}_{\mathbf{x}} : \mathrm{FET}_X \to$ (finite Sets) be given by

$$(\pi : \widetilde{X} \to X) \mapsto \{\widetilde{\mathbf{x}} : \operatorname{Spec} K \to \widetilde{X} \mid \mathbf{x} = \pi\widetilde{\mathbf{x}}\}.$$

b) Let $\Pi_1^{et}(X)$ be the category with objects are geometrix points of X and morphisms $\mathbf{x} \to \widetilde{\mathbf{x}}$ are functor-isomorphisms $\mathrm{Fib}_{\mathbf{x}} \to \mathrm{Fib}_{\widetilde{\mathbf{x}}}$.

c) For a geometric point $\mathbf{x}$, let $\Pi_1^{et}(X, \mathbf{x})$ be the group of automorphisms of $\mathbf{x}$ in the groupoid $\Pi_1^{et}(X)$.

Remark. a) If one uses the separably closed definition for geometric points, one gets an equivalent category $\Pi_1'(X)$. This is because for every $\mathbf{x} : \operatorname{Spec} K \to X$, K separably closed, one has an algebraic closure $i : K \to \overline{K}$, and $\overline{\mathbf{x}} : \operatorname{Spec} \overline{K} \to \operatorname{Spec} K \to X$. If x is the support of $\overline{\mathbf{x}}$, then

$$\begin{aligned}\mathrm{Fib}_{\overline{\mathbf{x}}}(Y) &\cong \{(y, \lambda) \mid y \in Y \text{ with image } x,\ \lambda \text{ an extension of } k(x) \xrightarrow{\overline{\mathbf{x}}^*} \overline{K} \text{ to } k(y) \to \overline{K}\} \\ &=\cong \{(y, \lambda) \mid y \in Y \text{ with image } x,\ \lambda \text{ an extension of } k(x) \xrightarrow{\overline{\mathbf{x}}^*} \overline{K} \text{ to } k(y) \to K\},\end{aligned}$$

since any λ in $\overline{K}$ has image in K, as $k(y)/k(x)$ is separable. This gives an isomorphism from $\overline{\mathbf{x}} \in \Pi_1^{et}(X)$ to $\mathbf{x} \in \Pi_1'(X)$. Since $\Pi_1^{et}(X)$ is a full subcategory of $\Pi_1'(X)$ by definition, they are equivalent.
b) Note that an equivalent definition of a geometric point is to define is as a triple $(K, x, \mathbf{x})$ where K is an algebraically closed field, $x \in X$ and $\mathbf{x} : k(x) \to K$ a homomorphism.
c) One also has an equivalent subcategory $\Pi_1''(X) \subseteq \Pi_1^{et}(X)$ where objects are geometric points $\mathbf{x} : \operatorname{Spec} K \to X$ such that K is algebraic over the image of $k(x) \to K$. If $\mathbf{x} : \operatorname{Spec} K \to X$ is a geometric point in the sense of definition 2, and $\widetilde{K} \subseteq K$ is the algebraic closure of $\mathbf{x}^*(k(x))$, then there is a unique morphism $\widehat{\mathbf{x}} : \operatorname{Spec} \widetilde{K} \to X$ whose composition with $\operatorname{Spec} K \to \operatorname{Spec} \widetilde{K}$ equals $\mathbf{x}$, and a canonical isomorphism $\mathbf{x} \cong \widehat{\mathbf{x}}$ in Π_1^{et} (for similar reasons as in a).

Remark 2. One introduces a Krull topology on the set of morphisms $\mathbf{x} \to \widetilde{\mathbf{x}}$ in Π_1^{et}: A neighbourhood base of a morphism $\gamma : \mathbf{x} \to \widetilde{\mathbf{x}}$ is $\{\Omega_v \mid v \text{ any object of } \mathrm{FET}_X\}$ where

$$\Omega_v = \{\widetilde{\gamma} : \mathbf{x} \to \widetilde{\mathbf{x}} \mid \gamma = \widetilde{\gamma} \text{ on } \mathrm{Fib}_{\mathbf{x}}(V) \to \mathrm{Fib}_{\widetilde{\mathbf{x}}}(V)\}.$$

It is easy to see that $\Pi_1^{et}(X, \mathbf{x})$ is complete with this topology.

Example 1. Let $X = \operatorname{Spec} K$ where K is a field. Then, étale X-schemes are automatically finite (essentially by Hilbert's Nullstellensatz) and up to isomorphism of the form $\operatorname{Spec} A$ where A is a finite-dimensional étale K-algebra. Let $\overline{K}$ be an algebraic closure of K, $K^s \subseteq \overline{K}$ the separable closure of K in $\overline{K}$, $G = \operatorname{Aut}(\overline{K}/K) \cong \operatorname{Gal}(K^s/K)$ equipped with the Krull topology. Let $\mathbf{x}$ denote the geometric point of X given by $\operatorname{Spec} \overline{K} \to \operatorname{Spec} K$.

If Y is an object of FET_X, then $\mathrm{Fib}_{\mathbf{x}}(Y)$ is in canonical bijection with the set of pairs (y, λ) where y is any point of Y and $\lambda : k(y) \to \overline{K}$ any ring homomorphism extending $K \to \overline{K}$. If $\theta \in G$, then θ acts on this set by $(y, \lambda) \mapsto (y, \theta\lambda)$.

One gets a functor $\mathrm{FET}_X \to$ (finite sets with continuous action by G) where the continuity condition is imposed for the Krull topology on G and the discrete topology on the fintie set. This functor is an equivalence of categories with inverse functor sending a finite G-set F to

$$\operatorname{Spec}(\{f : F \to \overline{K} \mid \theta(f(x)) = f(\theta x)\}).$$

It follows that $\Pi_1^{et}(X, \mathbf{x}) \cong G$, canonically.

Remark. Note that for an étale $\operatorname{Spec} K$-scheme the morphism $X \to \operatorname{Spec} K$ is finite if X is quasi-compact.

Theorem 1 (SGA1.V)**.** *Let X be a locally connected locally Noetherian scheme.*

a) We have an equivalence of categories

$$\mathrm{FET}_X \to \mathcal{C}, \quad (\pi : Y \to X) \mapsto (\mathbf{x} \to \mathrm{Fib}_{\mathbf{x}} Y),$$

where $\mathcal{C}$ is the category of functors F from $\Pi_1^{et}(X)$ to the category of finite sets such that

$$F(\mathbf{x}) \times \operatorname{Hom}_{\Pi_1^{et}}(\mathbf{x}, \mathbf{y}) \to F(\mathbf{y}), \quad (f, \gamma) \mapsto F(\gamma) f$$

is continuous, where $F(\mathbf{x})$ and $F(\mathbf{y})$ carry the discrete topology and $\operatorname{Hom}_{\Pi_1^{et}}(\mathbf{x}, \mathbf{y})$ the Krull topology.

b) If, in addition, X is connected, then $\Pi_1^{et}(X)$ is connected (in the sense that it has only one isomorphism class of objects). Thus, if $\mathbf{x}$ is a geometrix point of X, then

$$\mathrm{FET}_X \to (\textit{fin. sets with cont. } \Pi_1^{et}(X, \mathbf{x})\textit{-action}), \qquad Y \mapsto \mathrm{Fib}_{\mathbf{x}} Y$$

is an equivalence of categories.

Remark 3. If X is a $\mathbb{Q}$-scheme, then, an alternative approach to an algebraically defined fundamental group would consider the Tannakian category of locally free coherent $\mathcal{O}_X$-modules with a connecten $\nabla : \mathcal{E}^\vee \to \mathcal{O} \otimes \Omega^1_{X/S}$ of vanishing curvature. This would play a similar role for H^*_{dR} compared with the role played by Π_1^{et} for $H^\bullet(X_{et})$.

Definition 4. A prinicpal G-covering (G a finite group) of X is an object Y of FET_X with a G-action such that the following equivalent conditions hold:

a) $G \times Y = \coprod_{g \in G} Y \mapsto Y \times_X Y$, $(g, y) \mapsto (y, gy)$ is an isomorphism and $Y \to X$ is flat.

b) The sieve on X_{et} or X_{Et} of all X-schemes U such that $U \times_X U \cong G \times U$ is the category of U-schemes with a G-action over X.

Fact 1. Let G be abelian. If X is connected and $\mathbf{x}$ any geometric point, then $\operatorname{Hom}(\Pi_1^{et}(X, \mathbf{x}), G)$ is in canonical bijection with the set of isomorphim classes of principal G-coverings.

Proposition 1 (Kummer theory for Π_1^{et})**.** *Let X be connected, $\zeta \in \mu_n^*(X)$ (i.e. a morphism $X \to \operatorname{Spec} R$, $R = (\mathbb{Z}[T]/(T^n - 1))[(T^d - 1)^{-1} \mid 1 < d < n, d|n]$). In particular, $n \in \mathcal{O}_X(X)^\times$.*

a) If $\mathcal{L}$ is a line bundle on X nad $\lambda \in (\mathcal{L}^{\otimes n})^*(X)$, then the functor

$$(\upsilon : Y \to X) \to (\textit{Sets}), \qquad Y \mapsto \{l \in (\upsilon^* \mathcal{L})(Y) \mid l^n = \nu^* \lambda\}$$

is representable by an object of FET_X which is $\mathbb{Z}/n\mathbb{Z}$-principal for the action $k \mod n : l \mapsto \zeta^k l$, and every $\mathbb{Z}/n\mathbb{Z}$-principal cover can be obtained in this way, giving us an equivalence of groupoids.

b) Thus we have an exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}_X(X)^* \xrightarrow{(\cdot)^n} \mathcal{O}_X(X)^* \to \operatorname{Hom}(\Pi_1^{et}(X, \mathbf{x}), \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Pic}(X) \xrightarrow{n\cdot} \operatorname{Pic}(X)$$

Proposition 2. *Let p be a prime and X a connected scheme over $\mathbb{F}_p$.*

a) *Let F_X denote the absolute Frobenius. If $\mathcal{T}$ is an $\mathcal{O}_X$-torsor on X_{Zar} (i.e. a sheaf of sets on X_{Zar} on which the abelian group $\mathcal{O}_X$ acts transitively) and let $\tau : F_X^*\mathcal{T} \to \mathcal{T}$ be an isomorphism. Then the functor on X-schemes*

$$(\upsilon : Y \to X) \mapsto \{t \in (\upsilon^*\mathcal{T})(Y) \mid (\upsilon^*\tau)(F_Y^* t) = t\}$$

is representable by a principal $\mathbb{Z}/p\mathbb{Z}$-cover of X, giving an equivalence of groupoids.

b) *Thus there is an exact sequence*

$$\begin{aligned} 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathcal{O}_X(X) \to \mathcal{O}_X(X) \to \operatorname{Hom}(\Pi_1^{et}(X, \mathbf{x}), \mathbb{Z}/p\mathbb{Z}) \\ \to H^1(X_{Zar}, \mathcal{O}_X) \to H^1(X_{Zar}, \mathcal{O}_X) \end{aligned}$$

Theorem 2 (Zariski-Nagata)**.** *Let X be regular Noetherian and let $U \subseteq X$ be an open subset such that* $\operatorname{codim}(Y, X) > 1$ *if Y is any irreducible component of $X \setminus U$. Then* $\mathrm{FET}_X \to \mathrm{FET}_U$*,* $(\xi : \widetilde{X} \to X) \mapsto (\xi^* U \to U)$ *is an equivalence of categories. Thus* $\Pi_1^{et}(U, \mathbf{x}) \cong \Pi_1^{et}(X, \mathbf{x})$ *where* $\mathbf{x}$ *is any geometric point of U.*

Remark about the proof: If $\widetilde{U} \to U$ is an object of FET_U, then $\widetilde{U} = \underline{\mathrm{Spec}}\mathcal{A}$ where $\mathcal{A}$ is an étale locally free $\mathcal{O}_U$-algebra, then by "basic" commutative algebra and by corollary 1.4.1 and proposition 1.4.2 the main problem is to extend the underlying locally free $\mathcal{O}_U$-module $\mathcal{A}$ to a locally free $\mathcal{O}_X$-module. This is (relatively) trivial when $\dim X = 2$ (then any vector bundle on U extends), but is hard when $\dim X \geq 3$.