Étale cohomology

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1.1 Introduction and motivation

Problem: For varieties X over an algebraically closed field k (and hopefully more general schemes) define a cohomology theory $H^*(X)$ with properties similar to $H^*_{\text{sing}}(X(\mathbb{C})_{\text{ord. top. space}})$. Hopefully, there exists a Lefschitz fixed point formula

$$\#(\text{fixed points of } f \text{ with multiplicity}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(f^*|H^i(X)). \tag{L}$$

The aim of Grothendieck was to apply this to a program proposed by Weil of studying the congruence zeta function of X by applying (L) to $f = F_X$ given by $[x_0, \dots, x_n] \mapsto [x_0^q, \dots, x_n^q]$, yielding

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F_X^*|H^i(X)).$$

Counterexamples $H^*_{dR}(X) = \mathbb{H}^*(X_{\text{Zar}}, \mathcal{O}_X \to \Omega^1_X \to \cdots)$ (de Rham cohomology) is ok if the characteristic of k is zero but not in $\operatorname{char} p$ where it is unsuitable for Weil's program. Similarly, $H^*(X_{\text{Zar}}, \mathbb{Z})$ does not work: $\underline{\mathbb{Z}}(X) \to \underline{\mathbb{Z}}(V)$ is surjective when X is irreducible, implying vanishing higher sheaf cohomology.

Restrictions on the ring of coefficients: If X is a supersingular elliptic curve over $\overline{\mathbb{F}}_q$ then $H^1(X)$ ought to be two-dimensional, but $\operatorname{End}(X) \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is non-split precisely over $\mathbb{Q}_p$ and $\mathbb{R}$, in which case it cannot act on a two-dimensional vector space. This excludes $\mathbb{Q}_p$ and $\mathbb{R}$ as the field of definition and hence also $\mathbb{Q}$ and $\mathbb{Z}$.

Etale cohomology with coefficients $\mathbb{Z}/l^n\mathbb{Z}$, l a prime invertible in k. Then

$$H^*(X, \mathbb{Q}_l) := \big(\varprojlim H^*(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z})\big) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Deligné used this to show the Riemann hypothesis for congurence zeta function.

Other theories include Crystilline cohomology with coefficients in $W(\overline{F}_q)$. Scholze has a way of working with $\mathbb{Z}_p$ directly, using the pro-étale site, and a proposal to work with $\mathbb{C}$ coefficients. But it is not clear how to do this.

Hence we will mostly study finite coefficients. If one works over $\mathbb{C}$, the exact exponential sequence $0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ is important. and we want at least the exactness of

$$0 \to \mu_{l^n} \to \mathcal{O}_X^\times \xrightarrow{f \mapsto f^{l^n}} \mathcal{O}_X^\times \to 0. \tag{$*$}$$

Note that $\mu_{l^n} \cong \mathbb{Z}/l^n\mathbb{Z}$ non-canonically if $k = \bar{k}$ and l is invertible in k. Unfortunately, but not unexpectedly, this is not exact on X_{Zar}. If this were exact, one could hope to get some information from it provided that $H^1(C, \mathcal{O}_C^\times) \cong \mathbb{Z} \times \operatorname{Jac}_C(k)$. The idea of Grothendieck was to enforce the exactness of $(*)$ by considering $V \to F(V)$ for étale morphisms $V \to X$ instead of only Zariski open subsets. Then, when $f \in \mathcal{O}_V^\times(V)$ one has an l^n-th root of f on $U = \{(x, \varphi) \mid x \in V, \varphi^{l^n} = f(x)\}$.

1.2 Flat morphisms

Definition 1. M is a *flat* A-module if $T \mapsto M \otimes_A T$ is exact or, equivalently, if $\mathrm{Tor}_p^A(M,T) = 0$ for all T and $p > 0$. An A-algebra B is flat if it is flat as an A-module.

Definition 2. For a morphism $f : X \to Y$ of schemes, f is called *flat* if it satisfies the folllowing equivalent conditions:

a) For all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-algebra.

b) For affine open subsets $U \subseteq X, V \subseteq Y$ s.t. $f(U) \subseteq V$, $\mathcal{O}_X(U)$ is flat as an $\mathcal{O}_Y(V)$-algebra.

c) There are affine open subsets $U_i \subseteq X, V_i \subseteq Y$ s.t. $f(U_i) \subseteq V_i$, $\mathcal{O}_X(U_i)$ is a flat $\mathcal{O}_Y(V_i)$-algebra and $X = \bigcup_{i \in I} U_i$.

Remark 1. a) See stacksproject 01U2

b) Other literature: SGA1: Etale fundamental group, SGA4_1: Topoi, Grothendieck topology, SGA4_2: Etale topology, SGA4_3: Proper and smooth base change, SGA$4\frac{1}{2}$: various stuff and Arcata – Introduction to etale cohomology by Delinge, SGA5: l-adic cohomology
Milne: Etale cohomology, Kiehl-Freitag: Etale cohomology and Weil conjectures
Matsumura: Commutative Algebra, Matsumura: Commutative Ring Theory

Let A be a ring, X quasi-compact and separated $\mathrm{Spec}\, A$-scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $H^*(X,\mathcal{M})$ can be calculated using $\check{H}(\mathcal{U},-)$ for affine coverings. Hence, by the exactness of $- \otimes_A \widetilde{A}$, this gives

Proposition 1. *a) Let $\widetilde{A}$ be a flat A-algebra, then $H^*(\widetilde{X},\widetilde{M}) \cong H^*(X,M) \otimes_A \widetilde{A}$, where $\widetilde{X} = X \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, \widetilde{A} \xrightarrow{p} X$ and $\widetilde{M} = p^*M$.*

b) Let $f : X \to Y$ be a quasi-compact separated morphism and $g : \widetilde{Y} \to Y$ a flat morphism, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then $g^* R^* f_* \mathcal{M} \cong R^* \widetilde{f}_* \widetilde{g}^* \mathcal{M}$ where $\widetilde{X} = X \times_Y \widetilde{Y}$.

Remark 2. Base change results for etale cohomology are similar. We have b) if f is proper or if f is of finite type and g is smooth, and the sheaves are of torsion.

Definition 3. f is called *faithfully flat* if it is flat and surjective on points. $\widetilde{A}$ is a faithfully flat A-algebra if it is flat and $R \otimes_A \widetilde{A} = 0$ implies $T = 0$.

Definition 4. [1] Let $f : X \to Y$ be a morphism of schemes. A descent datum (of quasi-coherent sheaves of modules) for f is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ with an isomorphism $\mu : p_1^*\mathcal{M} \cong p_2^*\mathcal{M}$, where

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "p_{12},p_{13}", "p_{23}"'] \arrow[rr, bend right, "q_1,q_2,q_3"'] & X \times_Y X \arrow[r, "p_1,p_2"] & X \end{tikzcd}$$

[1] see tag 023A or SGA1,VI for fibred categories: descend data for X-schemes to Y-schemes and ample line bundles

are the different projections, and the diagram

$$\begin{tikzcd}
q_1^*\mathcal{M} \arrow[rr, equal] \arrow[dr, equal] & & p_{12}^*p_1^*\mathcal{M} \arrow[rr, "p_{12}^*\mu", "\cong"'] & & p_{12}^*p_2^*\mathcal{M} \arrow[rr, equal] & & q_2^*\mathcal{M} \arrow[dl, equal] \\
& p_{13}^*p_1^*\mathcal{M} \arrow[dr, "\cong", "p_{13}^*\mu"'] & & & & p_{23}^*p_1^*\mathcal{M} \arrow[dl, "\cong"', "p_{23}^*\mu"] & \\
& & p_{13}^*p_2^*\mathcal{M} \arrow[dr, equal] & & p_{23}^*p_2^*\mathcal{M} \arrow[dl, equal] & & \\
& & & q_3^*\mathcal{M} & & &
\end{tikzcd}$$

must commute. A morphism of descent data is a morphism $\varphi : \mathcal{M} \to \widetilde{\mathcal{M}}$ compatible with μ and $\widetilde{\mu}$, i.e. $(p_2^*\varphi)\mu = \widetilde{\mu}(p_1^*\varphi)$

Remark 3. We have a functor

$$\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}, \quad \mathcal{N} \mapsto (f^*\mathcal{N}, \text{the canonical iso } p_1^*f^*\mathcal{N} \cong p_2^*f^*\mathcal{N}).$$

One would like this to be an equivalence of categories. It has a right adjoint

$$(\mathcal{R}\mathcal{M})(U) = \{m \in \mathcal{M}(f^{-1}U) \mid \mu p_1^* m = p_2^* m\}$$

Proposition 2 (stacks loc.cit., SGA1.VII.1, Milne)**.** *If f is faithfully flat and quasi-compact, the above functor $\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}(X),f}$ is an equivalence of categories.*

Proof. If f has a section, the inverse image along that section is an inverse functor. In general, base change with $f : X \to Y$ reduces to this situation, provided that f is separated, which is a situation one can reduce to. □

Corollary 1. *If f is faithfully flat, $\mathcal{O}_Y(V) = \{\lambda \in \mathcal{O}_X(f^{-1}U) \mid p_1^*\lambda = p_2^*\lambda\}$.*

Remark 4. Both quasi-compactness and quasi-coherence in proposition 2 are needed. Consider $Y = \operatorname{Spec} R$, R a PID with $\operatorname{Spec} R$ infinite,

$$X = \coprod_{m \in \mathrm{mSpec}} \operatorname{Spec} R_m, \qquad N_1 = \coprod_{m \in \mathrm{mSpec}\, R} R/m \to N_2 = \prod_{m \in mSpecR} R/m,$$

then it is easy to see that this inclusion does not split, bit it splits canonically after applying $- \otimes_R R_m$, giving rise to a morphism of descent data which does not descend to a morphism $N_2 \to N_1$.

Definition 5. A morphism $i : X \to Y$ in a category $\mathcal{A}$ is an effective monomorphism if for all objects T,

$$\mathrm{Hom}_{\mathcal{A}}(T, X) \xrightarrow{\varphi \mapsto i\varphi} \{f \in \mathrm{Hom}_{\mathcal{A}}(T, Y) \mid \sigma f = \widetilde{\sigma} f \text{ for all } \sigma, \sigma' : Y \to S \text{ s.t. } \sigma i = \widetilde{\sigma} i\}$$

is bijective. $p : X \to Y$ is an effective epimorphism if it is an effective monomorphism in $\mathcal{A}^{\mathrm{op}}$, i.e.

$$\mathrm{Hom}_{\mathcal{A}}(Y, T) \xrightarrow[\cong]{\varphi \mapsto \varphi p} \{f \in \mathrm{Hom}_{\mathcal{A}}(X, T) \mid f\sigma = f\widetilde{\sigma} \text{ for all } \sigma, \widetilde{\sigma} : S \to X \text{ s.t. } p\sigma = p\widetilde{\sigma}\}.$$

Remark 5. If $X \times_Y X$ exists, f being an effective epimorphism is equivalent to it being a coequalizer of $X \times_Y X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$.

Proposition 3 (SGA1.VIII.4 or stacks 023Q)**.** *Every fpqc (quasi-compact faithfully flat) morphism of schemes is an effective epimorphism, i.e.*

$$\mathrm{Hom}(Y,T) \to \mathrm{Hom}(X,T) \rightrightarrows \mathrm{Hom}(X \times_Y X, T)$$

is an exact sequence of sets.

Remark 6. This implies that for every scheme T, the functor $X \mapsto T(X) := \mathrm{Hom}(X,T)$ satisfies the sheaf condition in the following sense:

$$T(Y) \xrightarrow{\tau \mapsto \tau f} \{t \in T(X) \mid tp_1 = tp_2\}.$$

That this should be interpreted as a kind of sheaf axiom becomes obvious if we have a covering $Y = \bigcup_{i=1}^n U_i$, $X = \coprod_{i=1}^n U_i \xrightarrow{f} Y$. Then $X \times_Y X = \coprod_{i,j=1}^n (U_i \cap U_j)$ with $tp_1|_{U_i \cap U_j}$ identified with $t|_{U_i}|_{U_i \cap U_j}$.

Proposition 4 (01UA)**.** *Every flat morphism (locally) of finite presentation is open.*

1.3 Grothendieck Topologies

As Deligne did in Arcata, we prefer the definition of Grothendieck topology by sieves.

Definition 1. Let $\mathcal{C}$ be a category, $X \in \mathrm{Ob}(\mathcal{C})$. A *sieve* (or $\mathcal{C}$-sieve) over X is a class $\mathcal{S}$ of morphisms with target X, such that $(U \to X) \in \mathcal{S}$ implies $(V \to U \to X) \in \mathcal{S}$ for every morphism $V \to U$ in $\mathcal{C}$. The empty class of morphisms is called the *empty sieve*, and the class of all morphisms with target X is called the *all sieve* (over X). For a morphism $f : Y \to X$ in $\mathbb{C}$, $f^*\mathcal{S} = \{v : U \to Y \mid fu \in \mathcal{S}\}$.

Remark 1. a) Obviously, $f^*\mathcal{S}$ is a sieve over Y if $\mathcal{S}$ is a sieve over X.

b) The fact that we work with categories where $\mathrm{Ob}\,\mathcal{C}$ is a proper class creates set-theoretic difficulties. Our way of dealing with this is to mostly ignore them.

c) The intersection of any class of sieves over X is a sieve over X.Thus, for every class $(f_i)_{i\in I}$ of morphisms with target X, there is a smallest sieve over X containing all f_i, namely $\{\xi : U \to X \mid \xi = f\eta$ for $\eta : U \to Y_i$ for some $\eta\}$. This is called the sieve generated by the f_i.

Example 1. a) X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$ turned into a category by its half ordering by $\subseteq$. If $X = \bigcup_{i\in I} U_i$ is an open covering, then the sieve generated by the (unique morphisms from) U_i is the sieve of all $V \in \mathbb{O}_X$ s.t. $V \subseteq U_i$ for at least one i.

b) If X is a complex space (e.g. $X = \mathbb{C} \setminus \{0\}$) with its complex topology, and $U \subseteq X$ open and $f \in \mathcal{O}_X(U)$, then $\mathcal{S} = \{V \subseteq U \mid \exists \varphi \in \mathcal{O}_X(V)$ s.t. $\varphi^2 = f|_V\}$ is a $\mathbb{O}_X$-sieve over U.

Remark. Thus, a morphism is in a sieve iff it is small enough "to pass through the sieve".

Definition 2. A *Grothendieck topology* $\mathbb{T}$ on a category $\mathcal{C}$ associates to every object X of $\mathcal{C}$ a class $\mathbb{T}_X$ of sieves over X, called the *covering sieves* of X. The following conditions must be verified:

(GTTriv) The all sieve over X covers X.

(GTTrans) If $\mathcal{S} \in \mathbb{T}_X$ and $f : Y \to X$, then $f^*\mathcal{S} \in \mathbb{T}_X$.

(GTLoc) If $\mathcal{T} \in \mathbb{T}_X$ and $\mathcal{S}$ any sieve over X such that $f^*\mathcal{S} \in \mathbb{T}_Y$ for all $f : Y \to X$ in $\mathcal{T}$, then $\mathcal{S} \in \mathbb{T}_X$.

We will often write $\mathcal{S} \mathrel{/\!\!=} X$ for $S \in \mathbb{T}_X$ if there are no ambiguities (or $\mathcal{S} \mathrel{/\!\!=}_{\mathbb{T}} X$ it there are).

Remark 1. Pretopologies are specified by specifying a class of admissible coverings $\mathcal{U} = (f_i : Y_i \to X)_{i \in I}$. Various assumptions must be satisfied, like that $(U_i \times_X Y \to Y)_{i \in I}$ still form an admissible covering of Y (including the existence of the fibre product). By putting $\mathbb{T}_X = \{$admissible coverings $\mathcal{S}$ of X with all $f_i \in \mathcal{S}\}$ one gets a Grothendieck topology. Equivalent pretopologies define the same $\mathbb{T}_X$. If the category has fibre products, one gets a pretopology from a Grothendieck topology $\mathbb{T}_X$ by calling a covering admissible iff the f_i generate a sieve in $\mathbb{T}_X$. This is the largest pretopology in its equivalence class.

Example 2. X an ordinary topological space, $\mathcal{C} = \mathbb{O}_X$, and $\mathcal{S} /\!\!= U$ iff $U = \bigcup_{V \in \mathcal{S}} V$. Other Grothendiech topologies can be introduced as well.

a) $X = [0,1]_{\mathbb{R}}$, put $\mathcal{S} /\!\!= U$ iff there are countable many $(U_i)_{i \in \mathbb{N}}$ such that $U \setminus \bigcup_{i \in \mathbb{N}} U_i$ is a set of Lebesgue measure 0, or $\mathcal{S} = U = \emptyset$.

b) Rigid analytic geometry (Tate style) or real algebraic geometry (Delfs-Knebusch) enforce quasi-compactness of certain open subsets of X, making it harder to be a covering.

c) X a Noetherian scheme, $d \in \mathbb{N}$. $\mathcal{S} /\!\!= \mathcal{U}$ iff $\operatorname{codim}(U \setminus \bigcup_{V \in \mathcal{S}} V) \geq d$, making it easier to be a covering.

Remark 2. You can think of (GTLoc) as the condition that being a covering is a local property.

Theorem 1 (Fact). *a) Every sieve containing a covering sieve is itself covering.*

b) The intersection of finitely many covering sieves is covering.