# Group Rings of Infinite Groups

by Prof. Dr. Giles Gardam

notes by Stefan Albrecht

University Bonn – winter term 2023/24

**Literature** Passman: The algebraic structure of group rings

## Contents

# 1 The Kaplansky Conjectures for Group Rings

**Definition 1.1.** Let $R$ be a ring and $G$ be a group. The *group ring*

$$R[G] = \left\{ \sum_{i=1}^{n} r_{g_i} g_i \mid g_i \in G, r_{g_i} \in R \right\}$$

is the ring consisting of finite formal $R$-linear combinations of the group elements with multiplication

$$\left(\sum r_g g\right)\left(\sum s_h h\right) = \sum r g s_h gh = \sum_k \left(\sum_{gh=k} r_g s_h\right) k.$$

In this course, we will (almost) always have $R = \mathbb{Z}$ or $R = K$ a field. In the latter case, $K[G]$ is often called the group algebra.

**Example 1.2.** For $G = \mathbb{Z} = \langle t \rangle$, then $R[G]$ is the ring of Laurent polynomials in $t$ over $R$, usually denoted $R[t, t^{-1}]$.

Viewpoint of Noether: Representations of groups are modules over group rings.
Warning: $K[G]$ is a non-commutative ring unless $G$ is abelian. It is (left-)Noetherian only in special settings and it is never semisimple for infinite $G$. (cf. Masuhke's theorem).

Although group rings tend to have bad ring-theoretic properties, they conjecturally have nice elementary properties. Note first that for $k \in K^\times$ and $g \in G$, the element $kg \in K[G]$ is a unit, such units are called *trivial*. A group is called *torsion free* if it has no nontrivial elements of finite order.

**Conjecture 1.3** (Kaplansky)**.** *Let $K$ be a field and $G$ be a torsion free group. Then $K[G]$*

- *has no nontrivial units,*
- *has no non-zero zero divisors,*
- *has no non-trivial idempotents.*

*Furthermore, for any group $G$ (possibly with torsion), $K[G]$ is directly finite (=von Neumann-finite =Dedekind-finite), i.e. if $\alpha\beta = 1$, then $\beta\alpha = 1$.*

**Remark 1.4.** Torsion-freeness is essential. Assume $g \in G$ has order $n \geq 2$. Then $0 = (1-g)(1 + g + \ldots + g^{n-1})$

**Remark 1.5.** The unit conjecture is false, the others are open.

**Remark 1.6.** These conjectures are "local" in the sense that they only depend on the finitely generated subgroups of $G$.

**Proposition 1.7.** *For a given field $K$ and a group $G$, we have*

$$\textit{unit conj.} \implies \textit{zero divisor-conj.} \implies \textit{idempotent conj.} \implies \textit{direct finite-conj.}$$

*Proof.* The last 2 implications are easy ring theoretic statements. The first implication requires the following theorem by Connell (which we will prove later): $K[G]$ is prime (meaning $AB = 0$ implies $A = 0$ or $B = 0$ for two-sided ideals $A, B \subseteq K[G]$) if and only if $G$ has no non-trivial finite normal subgroups. Since $G$ is torsion-free, $K[G]$ is prime. Now suppose $\alpha\beta = 0$ for $\alpha, \beta \neq 0$. Then there exists some $\gamma \in K[G]$ with $\beta\gamma\alpha \neq 0$: Otherwise $(K[G]\beta K[G]) \cdot (K[G]\alpha K[G]) = 0$. Now $(1 -$

$\beta\gamma\alpha)(1+\beta\gamma\alpha) = 1$ and $1+\beta\gamma\alpha$ is a non-trivial unit, since if it were trivial then $\beta\gamma\alpha = kg - 1$, but $0 = (\beta\gamma\alpha)^2 = k^2g^2 - 2kg + 1$, which is absurd unless $g = 1$, in which case $\beta\gamma\alpha = k - 1$ again squares to zero, hence $\beta\gamma\alpha = 0$. □

**Definition 1.8.** A group $G$ is residually finite if for all $1 \neq g \in G$ there exists a homomorphism $\varphi_g : G \to Q$, $Q$ finite, such that $\varphi_g(g) \neq 1$.

We will see later that the direct finiteness conjecture is true for $K = \mathbb{C}$. For now, we prove

**Proposition 1.9.** *Let $G$ be residually finite. Then $K[G]$ is directly finite.*

*Proof.* A group homomorphism $\varphi : G \to Q$ induces a ring homomorphism $K[G] \to K[Q]$. Thus $K[Q]$ is a $K[G]$-module. Note that $Q$ is a basis for the $K$-vector space $K[Q]$, so if $Q$ is finite this is a finite dimensional representation of $G$ on $V = K[Q]$.

Suppose $\alpha\beta = 1$ in $K[G]$. Let $A = \mathrm{supp}(\alpha) := \{g \in G \mid (\alpha)_g \neq 0\}$, $B = \mathrm{supp}(\beta)$. Let $C = BA$. By residual finiteness, there is a finie quotient $\varphi : G \to Q$ which is injective on $C$. Now the induced maps $\rho_\alpha, \rho_\beta \in \mathrm{End}(V)$ satisfy $\rho_\alpha \circ \rho_\beta = \rho_{\alpha\beta} = \mathrm{id}_V$ and thus – since $V$ is finite-dimensional – we have $\rho_\beta \circ \rho_\alpha = \mathrm{id}_V$ as well. Write $\beta_\alpha = \sum_{c\in C}(\beta\alpha)_c c$ and thus

$$\rho_{\beta\alpha}(1_Q) = \varphi(\beta\alpha) = \sum_{c\in C}(\beta\alpha)_c\varphi(c) = 1_Q$$

forces $(\beta\alpha)_c = 1$ if $c = 1$ and $0$ else. □

## 1.1 The Unit Conjecture

There is only one known way to probe the unit conjecture for a given group $G$: the unique product property.

**Definition 1.10.** A group $G$ has the *unique product property* (UPP, "has unique products", "has UP") if for all non-empty finite subsets $A, B \subseteq G$ there exists some $g \in G$ s.t. $g = ab$ for a unique pair $(a, b) \in A \times B$.

**Example 1.11.** In $(\mathbb{Z}, +)$, given finite $A, B \subseteq \mathbb{Z}$, one can take $g = \max A + \max B$. Hence $\mathbb{Z}$ has unique products.

**Remark 1.12.** A group with unique products is torsion-free: If $1 \neq H \leq G$, $H$ finite, then take $A = B = H$. Each product now occurs exactly $|H|$ times.

**Remark 1.13.** It's difficult to produce torsion-free groups that don't have UP.

**Proposition 1.14.** *A group with UP satisfies the zero divisor conjecture for all fields $K$.*

*Proof.* Let $\alpha, \beta \in K[G]$ with $\alpha, \beta \neq 0$, and set $A = \mathrm{supp}(\alpha)$, $B = \mathrm{supp}(\beta)$. Write $\alpha = \sum_{a\in A}\lambda_a a$ and $\beta = \sum_{b\in B}\mu_b b$. Then if $g = a_0b_0$, $a_0 \in A$, $b_0 \in B$ is a unique product for $A, B$, then we have

$$(\alpha\beta)_g = \sum_{ab=g}\lambda_a\mu_b = \lambda_{a_0}\mu_{b_0} \neq 0.$$

Hence $\alpha\beta \neq 0$ in $K[G]$. □

For the unit conjecture, we need something that is a priori stronger.

**Definition 1.15.** A group $G$ has the *two unique products property* if for all finite subsets $A, B \subseteq G$ with $|A| \cdot |B| \geq 2$, there exist $g_0 \neq g_1 \in G$, such that $g_0 = a_0 b_0$ and $g_1 = a_1 b_1$ for unique pairs $(a_0, b_0), (a_1, b_1) \in A \times B$.

**Proposition 1.16** (Strognowski)**.** *The two unique products property is equivalent to the unique product propterty.*

*Proof.* If $G$ satisfies 2UPP, it clearly satisfies UPP (if $|A| = |B| = 1$, the product is clearly unique).

Conversely, assume that $G$ has UP but that there exist finite sets $A, B \subseteq G$ with $|A||B| \geq 2$ with only 1 unique product. Without loss (by translating $A$ on the left and $B$ on the right), we may assume that $1 = 1 \cdot 1$ is the unique unique product. Now let $C = B^{-1}A$ and $D = BA^{-1}$. We claim that now there is unique product for $C$ and $D$. Every element of $CD$ can be written as $b_1^{-1} a_1 b_2 a_2^{-1}$ for some $a_i \in A, b_i \in B$. If $(a_1, b_2) \neq (1, 1)$ then by assumption there is another pair $a_1', b_2'$ s.t. $a_1 b_2 = a_1' b_2'$ and thus $b_1^{-1} a_1 b_2 a_2^{-1} = b_1^{-1} a_1' b_2' a_2^{-1}$ is not a unique product for $CD$. If, on the other hand, $(a_1, b_2) = (1, 1)$, then unless $(a_2, b_1) = (1, 1)$, we find $a_2', b_1'$ such that

$$b_1^{-1} a_1 b_2 a_2^{-1} = b_1^{-1} a_2^{-1} = (a_2 b_1)^{-1} = (a_2' b_1')^{-1} = b_1'^{-1} a_1 b_2 a_2'^{-1}$$

is not a unique product. Finally, if $a_2 = b_1 = 1$, then our element of $CD$ is $1 = 1 \cdot 1 = b^{-1} b = a a^{-1}$ for any $a \in A, b \in B$, and $A$ or $B$ has an element other than 1, which gives more than one factorisation. □

**Corollary 1.17.** *A group with UP satisfies the unit conjecture.*

*Proof.* Exercise. □

Most examples of groups with UP are left-orderable.

## 1.2 Ordered Groups and related Properties

**Definition 1.18.** A group $G$ is *(left-)orderable* if it admits a total order $\prec$ that is left-invariant, i.e. if $g \prec h$, then $kg \prec kh$ for all $g, h, k \in G$.

**Remark 1.19.** Being left- and right-orderable are equivalent (define $g \prec' h$ iff $g^{-1} \prec h^{-1}$)) but admitting a bi-invariant total order is much stronger.

**Proposition 1.20.** *A left-orderable group G has unique products.*

*Proof.* Fix a left-order $\prec$. Given finite subsets $A, B \subseteq G$, we show that the maximum of $AB$ is a unique product. Let $b_0 = \max B$. Then for all $a \in A$, $b \in B \setminus \{b_0\}$, we have $b \prec b_0$, so $ab \prec ab_0$. Thus the maximum of $AB$ can only be written as $ab_0$ for some $a \in A$, and thus must be unique. □

**Remark 1.21.** It is not necessarily true that $\max(AB) = \max A \cdot \max B$.

**Definition 1.22.** For a left-ordered group $(G, \prec)$, the set $\mathcal{P} = \{g \in G \mid 1 \prec g\}$ is called its *positive cone*.

The positive cone clearly satisfies $\mathcal{P}^2 \subseteq \mathcal{P}$ (i.e. it's a subsemigroup) and $G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$. The converse is also true:

**Lemma 1.23.** *Left-orders are equivalent to choices of $\mathcal{P} \subseteq G$ satisfying $\mathcal{P}^2 \subseteq \mathcal{P}$ and $G = \mathcal{P} \sqcup \{1\} \sqcup \mathcal{P}^{-1}$.*

*Proof.* Exercise. □

**Lemma 1.24.** *A group $G$ is left-orderable if and only if for all $g_1, \ldots, g_n \in G \setminus \{1\}$, there exists a choice of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$ such that $1 \notin S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$ (the subsemigroup generated by $g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n}$).*

*Proof.* If $G$ is left-ordered, set $\varepsilon_i = 1$ iff $g_i \in \mathcal{P}$.

For the other implication, we use compactness (slogan: the inverse limit of non-empty finite sets is non-empty). Let $X = \{1, -1\}^{G \setminus \{1\}}$ be the set of functions $G \setminus \{1\} \to \{1, -1\}$, and let $A \subseteq X$ be the set of those functions that define a positive cone. This is equivalent to satisfying (simultaneously) the condition on choice of sign for all possible $g_1, \ldots, g_n \in G \setminus \{1\}$ (for $n = 3$). That is, if we denote such functions $A_{\{g_1, \ldots, g_n\}} \subseteq X$, then

$$A = \bigcap_{\substack{S \subseteq G \setminus \{1\} \\ S \text{ finite}}} A_S$$

But $X$ is compact by Tychonoff and all the $A_S$ are closed. Furthermore, all finite intersections of the $A_S$ are non-empty by assumption. So $A \neq \emptyset$. □

We apply the lemma to prove

**Theorem 1.25** (Burns-Hale, 1972)**.** *Let $G$ be a group. If every non-trivial finitely generated subgroup of $G$ has a non-trivial left-orderable quotient, then $G$ is left-orderable.*

*In particular, a locally indicable group (i.e. every nontrivial finitely generated subgroup surjects onto $\mathbb{Z}$) is left-orderable.*

**Corollary 1.26** (Higman, 1940)**.** *Loally indicable groups satisfy the unit conjecture.* □

**Example 1.27** (of locally indicable groups)**.**

- Free groups (Niedsen-Schreier)
- Fundamental groups of closed surfaces of non-positive Euler characteristic
- Torsion-free nilpotent groups
- Torsion-free one-relator groups, i.e. groups of the form $\langle X \mid r \rangle$, $r \in F(X)$, where $r$ is not a proper power in $F(X)$ (Brodski-Howie)

*Proof.* (of 2.16) Suppose $G$ is not left-orderable and let $n$ be minimal such that $\exists g_1, \ldots, g_n \in G \setminus \{1\}$ such that $1 \in S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$ for all choices of $\varepsilon_i \in \{-1, 1\}$. Let $H = \langle g_1, \ldots, g_n \rangle \neq 1$, so by assumption $H$ has a non-trivial left-orderable quotient $q : H \twoheadrightarrow Q$. By relabelling, aassume $g_1, \ldots, g_t \in \ker(q)$ and $g_{t+1}, \ldots, g_n \notin \ker(q)$. As $t < n$, we can assign $\varepsilon_1, \ldots, \varepsilon_t$ such that $1 \notin S(g_1^{\varepsilon_1}, \ldots, g_t^{\varepsilon_t})$. and since $Q$ is left-orderable, we can assign $\varepsilon_{t+1}, \ldots, \varepsilon_n$ such that $1 \notin S(q(g_{t+1})^{\varepsilon_{t+1}}, \ldots, q(g_n)^{\varepsilon_n})$. But this implies $1 \notin S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$ as every product of othese elements either only uses $g_1^{\varepsilon_1}, \ldots, g_t^{\varepsilon_t}$, hence lies in $S(g_1^{\varepsilon_1}, \ldots, g_t^{\varepsilon_t})$ or has image under $q$ in $S(g_1^{\varepsilon_1}, \ldots, g_n^{\varepsilon_n})$. □

**Proposition 1.28.** $\mathrm{Homeo}^+(\mathbb{R})$ *is left-orderable.*

*Proof.* Let $\{x_0, x_1, \ldots\} \subseteq \mathbb{R}$ be dense. Define the order $\prec$ on $f \in \mathrm{Homeo}^+(\mathbb{R})$ via the lexiographic order on $(f(x_0), f(x_1), \ldots)$. The map $f \mapsto (f(x_0), f(x_1), \ldots)$ is injective (because continuous functions are determined by their values on a dense set), so the order descends. □

**Proposition 1.29.** *A countable group is left-orderable if and only if it is a subgroup of* $\mathrm{Homeo}^+(\mathbb{R})$.

*Proof.* Exercise. □

**Proposition 1.30.** *Let $G$ be a group. Suppose $N \trianglelefteq G$ such that $N$ and $G/N$ both have unique products. Then $G$ has unique products.*

*Proof.* Let $A, B \subseteq G$ be non-empty finite subsets. Write $\varphi : G \to G/N$. Suppose $\varphi(a) \cdot \varphi(b)$ is a unique product in $G/N$, $a \in A$, $b \in B$. By replacing $A$ with $a^{-1}A$ and $B$ with $Bb^{-1}$, we may assume the unique product in $G/N$ is $1 \cdot 1 = 1$. Thus $a, b \in N$. Hence the unique product of $A \cap N$ and $B \cap N$ is a unique product for $A$ and $B$. □

**Definition 1.31.** Let $A \subseteq G$ be a finite subset. An element $a \in A$ is called *extremal* (for $A$) if for all $s \in G \setminus \{1\}$ we have $as \notin A$ or $as^{-1} \notin A$. $G$ is called *diffuse* if every non-empty finite subset $A \subseteq G$ has at least one extremal point.

**Remark 1.32.** $a \in A$ is extremal iff $a^{-1}A \cap A^{-1}a = \{1\}$

**Proposition 1.33.** *For any group $G$ we have the implications*

$$\textit{left-orderable} \quad \Longrightarrow \quad \textit{diffuse} \quad \Longrightarrow \quad \textit{unique products}.$$

*Proof.* Suppose $(G, <)$ is a left-ordered group and let $\emptyset \neq A \subseteq G$ a finite subset. Then let $a = \max A$. For any $s \in G \setminus \{1\}$ either $s > 1$ or $s^{-1} > 1$, hence $as > a$ or $as^{-1} > a$, i.e. $a$ is extremal.

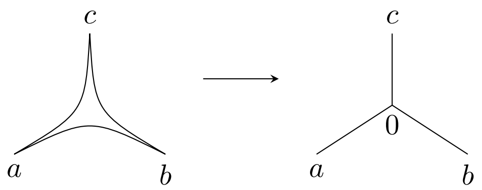
Suppose $G$ is diffuse and let $A, B \subseteq G$ be non-empty finite subsets. Consider $C = AB$. Let $c = ab \in C$ be extremal. Suppose $c = a_1b_1$ with $b \neq b_1$. Then $c(b_1^{-1}b_2) = a_1b_2 \in C$ and $c(b_2^{-1}b_1) = a_2b_1 \in C$, in contradiction to extremity. □

**Remark 1.34.** Given a finite set $B \subseteq G$, we can easily decide if all $\emptyset \neq A \subseteq B$ have an extremal point, because if $a \in A_0 \subseteq A_1$ is extremal in $A_1$, then it is also extremal in $A_0$. Thus we can run a greedy algorithm, starting with $A = B$ and throwing out the extremal points at each step.

We can establish diffuseness geometrically, specifically for many hyperbolic groups.

# 2 Hyperbolic Groups

Geodesic triangles in the hyperbolic plane $\mathbb{H}^2$ resemble tripods.




Given three points in a metric space, they embed isometrically as the vertices of a unique tripod $T_\Delta$. The length $d(0,a)$ must be $\frac{1}{2}(d(a,b)+d(a,c)-d(b,c)) =: (b \cdot c)_a$ which we call the Gromov product. Morally, this is the distance to the incircle. Let $X$ be a geodesic[1] metric space. For a geodesic triangle $\Delta = \Delta(a,b,c)$, define $\mathcal{X}_\Delta : \Delta \to T_\Delta$ by mapping the geodesics isometrically. $\Delta$ is called $\delta$-thin if $p, q \in \mathcal{X}_\Delta^{-1}(t)$, then $d_X(p,q) \leq \delta$ for all $t \in T_\Delta$.

**Definition 2.1.** $X$ is called $\delta$-hyperbolic if allgeodesic triangles are $\delta$-thin. $X$ is called (Gromov) hyperbolic if it is $\delta$-hyperbolic for some $\delta \geq 0$.

There are multiple equivalent definitions, e.g. slim triangles, but the constant $\delta$ needs to change.

**Definition 2.2.** A group $G$ is called *hyperbolic* if it acts properly cocompactly by isometries on a proper hyperbolic space.

An action of a group $G$ on a topological space $X$ is *proper*, if for all compact $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. It is *cocompact*, if there exists a compact $K \subseteq X$ such that $X = G \cdot K$. A metric space $X$ is called *proper*, if $\bar{B}_r(x)$ is compact for all $x \in X, r \geq 0$.

**Remark 2.3.** For a proper metric space, an action $G \curvearrowright X$ is proper iff for all $x \in X, r \geq 0$, the set $\{g \in G : d(gx, x) \leq r\}$ is finite. The action is cocompact iff $X = G\bar{B}_r(x)$ for some $x \in X, r > 0$.

**Example 2.4.** A tree is 0-hyperbolic.

**Corollary 2.5.** *$F_2 = \pi_1(S^1 \vee S^1)$ acts on the universal cover of $S^1 \vee S^1$, which is a locally finite graph. Hence $F_2$ is a hyperbolic group.*

**Lemma 2.6.** *Let $G \curvearrowright X$ be a proper cocompact isometric action on a proper metric space. Let $r > 0$. Then*

$$\{g \in G \mid \exists x \in X : d(gx, x) \leq r\}$$

*consists of finitely many conjugacy classes*

*Proof.* By cocompactness, $X = G\bar{B}_{r_0}(x_0)$ for some $x_0 \in X, r_0 > 0$. Suppose $g \in G$ and $x \in X$ s.t. $d(gx, x) \leq r$. There exist $h \in G$ s.t. $x \in h\bar{B}_r(x_0)$, i.e. $d(x_0, h^{-1}x) \leq r_0$. Then

$$d(g^h h^{-1}x, h^{-1}x) = d(h^{-1}gx, h^{-1}x) = d(gx, x) \leq r.$$

Thus

$$d(g^h x_0, x_0) \leq d(g^h x_0, g^h h^{-1}x) + d(g^h h^{-1}x, h^{-1}x) + d(h^{-1}x, x_0) \leq 2r_0 + r.$$

By properness, there are only finitely many possibilities for $g^h$. □

An alternative definition for hyperbolic spaces is the four-point condition.

[1]$\forall x, y \in X$ $\exists$geodesic $[x,y]$, i.e. an isometric embedding $i : [0, d(x,y)] \to X$ with $i(0) = x, i(d(x,y)) = y$

**Definition 2.7.** Let $\delta \geq 0$. A metric space $X$ is $(\delta)$-hyperbolic if $(xy)_w \geq \min\{(xz)_w, (yz)_w\} - \delta$ for all $x, y, z, w \in X$

**Remark 2.8.** This definition is arguably less intuitive, but it also works for non-geodesic metric spaces such as discrete spaces.

**Proposition 2.9.** *Let $X$ be a geodesic metric space. Then*

*(i) $X$ is $(\delta)$-hyperbolic $\Rightarrow$ $X$ is $4\delta$-hyperbolic.*

*(ii) $X$ is $\delta$-hyperbolic $\Rightarrow$ $X$ is $(\delta)$-hyperbolic.*

*Proof.* (i) Exercise.
(ii) Let $x, y, z, w \in X$. Pick $x' \in [w, x]$, $y' \in [w, y]$ and $z' \in [w, z]$ such that $d(w, x') = d(w, y') = d(w, z') = \min\{(x \cdot z)_w, (z \cdot y)_w\}$. By $\delta$-thinness of $\Delta(w, x, z)$, we have $d(x', y') \leq \delta$. Similarly, $d(y', z') \leq \delta$, hence $d(x', z') \leq 2\delta$. Thus

$$d(x, y) \leq d(x, x') + 2\delta + d(y, y') = d(w, x) + d(w, y) + 2\delta - 2\min\{(xz)_w, (yz)_w\},$$

which is equivalent to $(xy)_w \geq \min\{(xz)_w, (yz)_w\} - \delta$ □

The 4 point condition can also be phrased symetrically: We have either $(xy)_w \geq (xz)_w - \delta$ or $(xy)_w \geq (yz)_w - \delta$, that is, $d(x, y) + d(z, w) \leq d(x, z) + d(y, w) + 2\delta$ or $d(x, y) + d(z, w) \leq d(x, w) + d(y, z) + 2\delta$, together

$$d(x, y) + d(w, z) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta.$$

There are 3 ways to pair up $\{x, y, z, w\}$. Suppose $S \leq M \leq L$ are the corresponding sums of pair-distances. Then the above inequality is equivalent to $L \leq M + 2\delta$.

**Theorem 2.10** (Delzant)**.** *Let $X$ be a $\delta$-hyperbolic geodesic metric space. Suppose $G \curvearrowright X$ by isometries s.t. for all $g \in G \setminus \{1\}$ and $x \in X$, we have $d(gx, x) > 3\delta$. Then $G$ is diffuse.*

*Proof.* We claim that for all $g \in G$, $1 \neq h \in G$ and $p \in X$ we have either $d(ghp, p) > d(gp, p)$ or $d(gh^{-1}p, p) > d(gp, p)$. Then we are done because for finite $A \subseteq G$ and any $p \in X$, an element $a \in A$ achieving $\max\{d(gp, p) \mid g \in A\}$ will be extremal.

Suppose that $d(gp, p) \geq d(ghp, p), d(gh^{-1}p, p)$ for some $g, h \in G, h \neq 1, p \in X$. Consider the symmetric 4-point condition for these four points as described above. The three possible distances are

$$\begin{aligned} d(gp, p) + d(ghp, gh^{-1}p) &= d(gp, p) + d(h^2p, p) \\ d(ghp, p) + d(gp, gh^{-1}p) &= d(ghp, p) + d(hp, p) \\ d(gh^{-1}p, p) + d(gp, ghp) &= d(gh^{-1}p) + d(hp, p) \end{aligned}$$

If we assume $d(h^2p, p) \geq d(hp, p)$, then the first of these three is the largest and thus by the 4-point condition $d(gp, p) + d(h^2p, p) \leq d(gh^{\pm 1}p, p) + d(hp, p) + 2\delta \leq d(gp, p) + d(hp, p) + 2\delta$. In either case, $d(h^2p, p) \leq d(hp, p) + 2\delta$. Thus $(hp, h^{-1}p)_p \geq \frac{1}{2}d(hp, p) - \delta$. If we let $q$ be the midpoint of $[h^{-1}p, p]$, and let $q', q''$ on $[q, p]$ and $[hq, p]$ at distance $\delta$ from $q$, resp. $hq$. Then $d(q', q'') \leq \delta$ by $\delta$-thinness of $\Delta(p, hp, h^{-1}p)$. (Pick the geodesic $[p, hp]$ so that it contains $hq$.) Together,

$$d(hq, q) \leq d(q, q') + d(q, q'') + d(q'', hq) \leq 3\delta,$$

in contradiction to the assumption $d(hq, q) > 3\delta$. □

**Definition 2.11.** Let $X$ be a property of groups. Then $G$ is *virtually* $X$ if there exists a finite index subgroup $G_0 \leq G$ such that $G_0$ has $X$.

**Corollary 2.12.** *Let $G$ be a residually finite hyperbolic group. Then $G$ is virtually diffuse.*

**Remark 2.13.** It is a famous open problem whether every hyperbolic group is residually finite.

*Proof.* Let $G \curvearrowright X$ properly cocompactly by isometries on a proper $\delta$-hyperbolic space. By Lemma 2.6, there exists $1 = g_0, \ldots, g_n \in G$ s.t. for all $g \in G$, if there is $x \in X$ with $d(gx, x) \leq 3\delta$, then $g$ is conjugate to some $g_i$. By residual finiteness, we can find $\varphi : G \to Q$ finite such that $\varphi(g_1), \ldots, \varphi(g_n) \neq 1$. Then $G_0 = \ker(\varphi)$ satisfies the assumption of Delzant's theorem. □

# 3 Primality of group rings

Our aim is to give a complete proof of the following, which was used to show that the unit conjecture for $K[G]$ implies the zero-divisor conjecture $K[G]$.

**Theorem 3.1** (Connell)**.** *$K[G]$ is prime if and only if $G$ has no nontrivial finite normal subgroup.*

Recall that a ring $R$ is prime if for all $0 \neq \alpha, \beta \in R$ there exists $\gamma \in R$ such that $\alpha\gamma\beta \neq 0$. (i.e. the zero ideal is a prime ideal in the sense of noncommutative ring theory.) For the proof, we need some more group ring basics, a fair bit of group theory and an ingenious trick of Passman.

**Definition 3.2.** Let $H \leq G$. Then there is a *projection* $\pi_H : K[G] \to K[H]$ defined by $\sum_{g\in G} a_g g \mapsto \sum_{g\in H} a_g g$.

Warning: This is never a ring homomorphism unless $G = H$. But we do have

**Lemma 3.3.** *$\pi_H$ is a homomorphism of $(K[H], K[H])$-bimodules.*

*Proof.* Exercise. □

**Corollary 3.4.** *Let $H \leq G$. If $\alpha \in K[H]$ is a unit in $K[G]$, then it is a unit in $K[H]$.*

*Proof.* For $\beta \in K[G]$ with $\alpha\beta = \beta\alpha = 1$ we have $\alpha\pi_H(\beta) = \pi_H(\alpha\beta) = \pi_H(1) = 1 = \pi_H(\beta)\alpha$, so in fact $\alpha^{-1} = \beta = \pi_H(\beta) \in K[H]$. □

Recall that a left transversal for $H \leq G$ is a set $X$ containing exactly one representative $x$ of each coset of $H$, so that $G = \bigsqcup_{x\in X} xH$

**Lemma 3.5.** *Let $X$ be a left transversal for $H \leq G$. Then every element $\alpha \in K[G]$ can be written uniquely as a finite sum $\alpha = \sum_{x\in X} x\alpha_x$ with $\alpha_x \in H[X]$. In fact, $\alpha_x = \pi_H(x^{-1}\alpha)$. Thus $K[G]$ is a free right $K[H]$-module with $X$ as basis.* □

Let $M_n(R)$ denote the ring of $n \times n$-matrices over the ring $R$.

**Lemma 3.6.** *Let $[G : H] = n < \infty$. Then $K[G] \hookrightarrow M_n(K[H])$.*

*Proof.* Let $X = \{x_1, \ldots, x_n\}$ be a left transversal for $H \leq G$. Then $V = K[G]$ is a free right $K[H]$-module with basis $X$. It is also a left $K[G]$-module and since left and right multiplication commute, $K[G]$ acts by $K[H]$-linear transformations of $V \cong K[H]^n$. Since for each $\alpha \in K[G]$ and each index $1 \leq j \leq n$ we have $\alpha x_j = \sum_{i=1}^n x_i \pi_H(x_i^{-1}\alpha x_j)$, sending $\alpha$ to the matrix

$$\eta_X(\alpha) = \left(\pi_H(x_i^{-1}\alpha x_j)\right)_{i,j=1,\ldots,n}$$

defines an embedding (for choice of basis $X$). □

**Remark 3.7.** If $G$ is finite, $H = 1$, this is just the regular representation of $G$.

**Example 3.8.** Let $D_\infty = \langle r, t \mid r^2, t^r t\rangle = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ and take $X = \{1, r\}$ as traversal for $\langle t\rangle \cong \mathbb{Z}$ in $D_\infty$. Then we calculate $\alpha x_i = x_j \eta_X(\alpha)$ for $\alpha = r, t$. Since

$$r \cdot 1 = r \cdot 1,\ r \cdot r = 1 \cdot 1,\ t \cdot 1 = 1 \cdot t,\ t \cdot r = r \cdot t^{-1},$$

the map $\eta_X : K[D_\infty] \to M_2(K[t, t^{-1}])$ is given by extending

$$r \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

**Lemma 3.9** (Schur)**.** *If $[G : Z(G)] < \infty$, then $|G'| < \infty$.*

**Definition 3.10.** Let $H \leq G$ and $[G : H] = n < \infty$. Pick a left transversal $\{x_1, \ldots, x_n\}$ for $H$. For each $g \in G$ and $1 \leq j \leq n$ we have $gx_j = x_i h_j$ for some unique $i$ and $h_j \in H$. The *transfer* is defined as the map $G/G' \to H/H'$ given by $gG' \mapsto h_1 h_2 \cdots h_n H'$.

**Lemma 3.11.** *The transfer is a group homomorphism and doesn't depend on the choice of transversal.*

*Proof.* Map $K[G] \to M_n(K[H])$ by lemma 3.6, $M_n(K[H]) \to M_n(K[H/H'])$ extending $H \to H/H'$ and $M_n(K[H/H']) \to H/H'$ by taking the determinant. For given $g$ and $1 \leq j \leq n$, $gx_j = x_i h_j$, i.e. $h_j = x_i^{-1} g x_j$, so $\eta_X(g)_{ij} = h_j$ and $\eta_X(g)_{i'j} = 0$ for $i' \neq i$. Letting $\mathrm{sgn}(g) \in \{\pm 1\}$ denote the sign of the permutation $g$ induces on the set $G/H$, we have that the composition of our maps sends $g$ to $\mathrm{sgn}(g) h_1 h_2 \cdots h_n \in K[H/H']$. As $\mathrm{sgn} : G \to \{\pm 1\}$ is a homomorphism, so is the map $G \to H/H' : g \mapsto h_1 \cdots h_n$. As the image is abelian, it factors through $G/G'$ to define the transfer. It is independent of the choice of transversal $X$, since change of basis of $K[G]$ produces similar matrices. □

*Proof.* (Schur) Let $Z = Z(G)$ and $[G : Z] = n$. Consider the transfer $G/G' \to Z/Z' = Z$. By centrality of $Z$, this is just $g \mapsto g^n$. Hence if $g \in G'$, then $g^n = 1$. If $\{x_1, \ldots, x_n\}$ is a transversal, then every commutator is of the form $[x_i z_1, x_j z_2] = [x_i, x_j]$ and thus $G'$ is finitely generated. Now $[G' : G' \cap Z] \leq n$, so $G' \cap Z$ is finitely generated, has finite exponent and is abelian, hence finite. Therefore $G'$ is also finite. □

**Definition 3.12.** The *FC-center* ("finite conjugacy") of a group $G$ is

$$\Delta(G) = \{g \in G \mid |g^G| < \infty\},$$

that is, the set of elements whose conjugacy class is finite. This is alternatively the set of elements whose zentralizer $C_G(g)$ has finite index (by the orbit-stabilizer theorem for $G \curvearrowright G$ by conjugating). As $C_G(gh) \geq C_G(g) \cap C_G(h)$, we see that $\Delta(G)$ is in fact a (characteristic) subgroup of $G$. We call $G$ an *FC-group* if $G = \Delta(G)$.

**Lemma 3.13.** *An FC-group is locally finite-by-(free abelian), i.e. all finitely generated subgroups are extensions of a finite by a free abelian group.*

*Proof.* Let $H = \langle h_1, \ldots, h_n\rangle \leq G$, $G$ an FC-group. Then $C_G(H) = \bigcap_{i=1}^n C_G(h)$ is finite index in $G$ and thus $Z(H) = C_G(H) \cap H$ is finite index in $H$. Thus Schur's lemma applies and $H'$ is finite. Now $H/H'$ is a finitely generated abelian group, so of the form $T \oplus \mathbb{Z}^d$ with $T$ finite. Thus the kernel of the torsion-free abelianization $H \to H/H' \to \mathbb{Z}^d$ is finite. □