Introduction to derived categories

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Literature

Weibel. Introduction to homological algebra.
stacksproject.

Overview

Let $\mathcal{A}$ be an abelian category, set $\mathcal{C}(\mathcal{A})$ the category of complexes of $\mathcal{A}$. Given two complexes $A^\bullet, B^\bullet$, we would like to identify them if there is a quasi-isomorphism between then, i.e. $f : A^\bullet \to B^\bullet$ such that $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isos for all i. To do this, pass from $\mathcal{C}(\mathcal{A})$ to the "derived" category $\mathcal{D}(\mathcal{A})$, which we will study. We will prove that $\mathcal{D}(\mathcal{A})$ is a triangulated category and give more explicit descriptions of $\mathcal{D}(\mathcal{A})$ under certain assumptions.

Furthermore, we will study when functors F between abelian categories can be extended to a functor between the derived categories, in particular for $F = \mathrm{Hom}(A, -)$ and $F = - \otimes_R M$ in the category of R-modules. If there is time, we might finish with t-structures, stable categories or tensor-triangulated categories.

Contents

1 Complexes for abelian categories

Fix an abelian category $\mathcal{A}$. In particular, all Hom-sets are abelian groups and the composition of morphisms is bilinear, we can form direct sums and products and every morphism has a kernel and cokernel.

Example 1.1. $\mathtt{Ab}$, $R\text{−}\mathtt{Mod}$ for a ring R, $\mathtt{Sh}(X)$ for a top. space X are all abelian. For a small category $\mathcal{C}$ and any abelian category $\mathcal{D}$, then $\mathcal{D}^{\mathcal{C}}$ is abelian.

Theorem 1.2 (Mitchell's embedding theorem)**.** *Let $\mathcal{A}$ be a small abelian category. Then there is a ring R and a fully faithful exact functor $F : \mathcal{A} \to R\text{−}\mathtt{Mod}$.*

This means that one can do diagram chases in abstract small abelian categories. More generally, even if the category is not small, one can replace it by the abelian category generated by the given diagram, and can then still use the embedding theorem.

Definition 1.3. Let $\mathcal{A}$ be an additive category. A sequence of morphisms $\cdots \to X \to Y \to Z \to \cdots$ is a *complex* if the composition of any two consecutive morphisms is zero. If $\mathcal{A}$ is abelian, a complex is *exact* at Y if $\operatorname{im}(X \to Y) = \ker(Y \to Z)$. The complex is exact if it is exact at every object. A short exact sequence is an exact complex of the form $0 \to A \to B \to C \to 0$.

Lemma 1.4 (Snake lemma)**.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & U \arrow[r, "k"] & V \arrow[r, "l"] & W & \end{tikzcd}$$

be a commutative diagram with exact rows. Then there exists a unique morphism $\delta : \ker\gamma \to \operatorname{coker}\alpha$ such that the induced sequence

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$$

is exact.

2 The category of (cochain) complexes

Definition 2.1. A *morphism between two complexes* $f : A^\bullet \to B^\bullet$ is given by a collection of morphisms $f^i : A^i \to B^i$ such that

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d^i"] & B^{i+1} \end{tikzcd}$$

commutes for all i. This defines the category $\mathcal{C}(\mathcal{A})$ of complexes over an additive category $\mathcal{A}$.

A complex is called *bounded below* if $A^n = 0$ for n sufficiently small and is called *bounded above* if $A^n = 0$ for n sufficiently large, and called *bounded* if it is bounded below and above. Define the full subcategories $\mathcal{C}^+(\mathcal{A}), \mathcal{C}^-(\mathcal{A})$ and $\mathcal{C}^b(\mathcal{A})$ of bounded below, bounded above, and bounded complexes.

Remark 2.2. We can view $\mathcal{A}$ as a full subcategory of $\mathcal{C}(\mathcal{A})$ via $A \mapsto (\cdots \to 0 \to 0 \to A \to 0 \to \cdots)$.

Remark 2.3. Every functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor $\mathcal{C}(A) \to \mathcal{C}(B)$.

Lemma 2.4. *Let $\mathcal{A}$ be an abelian category.*

(i) $\mathcal{C}(\mathcal{A})$ is abelian.

(ii) $f : A^\bullet \to B^\bullet$ *is injective iff all* $f^i : A^i \to B^i$ *are injective.*

(iii) $f : A^\bullet \to B^\bullet$ *is surjective iff all* $f^i : A^i \to B^i$ *are surjective.*

(iv) A sequence $A^\bullet \to B^\bullet \to C^\bullet$ *in* $\mathcal{C}(\mathcal{A})$ *is exact iff all* $A^i \to B^i \to C^i$ *are exact.*

Definition 2.5. Let $A^\bullet \in \mathcal{C}(\mathcal{A})$, $i \in \mathbb{Z}$. Set $H^i(A^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1})$.

Remark 2.6. This is functorial: If $f : A^\bullet \to B^\bullet$ is a morphism of chain complexes, there is an induced morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ because $f^i(\ker(d^i : A^i \to A^{i+1})) \subset \ker(d^i : B^i \to B^{i+1})$ and similarly for $\operatorname{im}(d^{i-1})$. Hence we obtain a functor $H^i : \mathcal{C}(\mathcal{A}) \to \mathcal{A}$.

Definition 2.7. Let $\mathcal{A}$ be an abelian category. Let $f : A^\bullet \to B^\bullet$. f is a *quasi-isomorphism* if all $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms. A complex $A^\bullet$ is acyclic if $H^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$.

Lemma 2.8. *Given a short exact sequence* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *in* $\mathcal{C}(\mathcal{A})$*, there is an associated long exact cohomology sequence*

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{\delta} H^{i+1}(A^\bullet) \to \cdots$$

Definition 2.9 (Shifts)**.** Let $\mathcal{A}$ be an additive category and $A^\bullet \in \mathcal{C}(\mathcal{A})$. For any $k \in \mathbb{Z}$ define the k-shifted complex $A[k]^n = A^{n+k}$ and $d^n_{A[k]} = (-1)^k d_A^{n+k}$. If $f : A^\bullet \to B^\bullet$, there is an induced morphism $f[k] : A[k] \to B[k]$ given by $f[k]^n = f^{k+n}$. Hence we obtain functors $[k] : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$ such that $A[k][l]^\bullet = A[k+l]^\bullet$.

Example 2.10. Let $A \in \mathcal{A}$. Denote by $A[k]$ the complex which is zero in all degrees except in degree $-k$, where it is A.

Remark 2.11. One can naturally identify $H^{i+k}(A^\bullet) \cong H^i(A[k]^\bullet)$.

Definition 2.12 (Truncation)**.** Let $\mathcal{A}$ be an abelian category, $A^\bullet \in \mathcal{C}(\mathcal{A})$.

1. The stupid truncation $\sigma_{\geq n}$ is the subcomplex defined by

$$(\sigma_{\geq n} A^\bullet)^i = \begin{cases} 0 & \text{if } i < n, \\ A^i & \text{if } i \geq n \end{cases}$$

Note $\sigma_{\geq n} A^\bullet / \sigma_{\geq n+1} A^\bullet = A^n[-n]$.

2. The stupid truncation $\sigma_{\leq n} A^\bullet$ is defined by

$$(\sigma_{\leq n} A^\bullet)^i = \begin{cases} 0 & \text{if } i > n, \\ A^i & \text{if } i \leq n. \end{cases}$$

3. The canonical truncation $\tau_{\leq n}$ is defined via

$$(\tau_{\leq n} A^\bullet)^i = \begin{cases} A^i & \text{if } i < n, \\ \ker(d^n) & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

Note $H^i(\tau_{\leq n} A^\bullet) = H^i(A^\bullet)$ for $i \leq n$ and 0 otherwise.

4. The canonical truncation $\tau_{\geq n}$ is defined via

$$(\tau_{\geq n} A^\bullet)^i = \begin{cases} A^i & \text{if } i > n, \\ \operatorname{coker}(d^n) & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

Note $H^i(\tau_{\leq n} A^\bullet) = H^i(A^\bullet)$ for $i \geq n$ and 0 otherwise.

3 The homotopy category

Definition 3.1. A homotopy h between a pair of morphisms $f, g : A^\bullet \to B^\bullet$ is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that $f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$ for all $i \in \mathbb{Z}$. Two morphisms are homotopic if there exists a homotopy between them.

Lemma 3.2. *Let $\mathcal{A}$ be additive, $f, g : B^\bullet \to C^\bullet$ morphisms in $\mathcal{C}(\mathcal{A})$. Suppose we are given morphisms $a : A^\bullet \to B^\bullet, c : C^\bullet \to D^\bullet$ and a homotopy h between f and g. Then $\{c^{i-1} \circ h^i \circ a^i\}_i$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.* ☐

Definition 3.3 (and Corollary)**.** Let $\mathcal{K}(\mathcal{A})$ be the category of complexes of $\mathcal{A}$ with morphisms "up to homotopy", i.e. with objects the objects of $\mathcal{C}(\mathcal{A})$ and

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{C}(A)}(A^\bullet, B^\bullet)/\sim .$$

By Lemma 3.2 this is a well-defined category.

Definition 3.4. A morphism $a : A^\bullet \to B^\bullet$ is a homotopy equivalence if there is a morphism $b : B^\bullet \to A^\bullet$ such that there exist homotopies $a \circ b$ and $\mathrm{id}_{A^\bullet}$ as well as between $b \circ a$ and $\mathrm{id}_{B^\bullet}$. In this case, we say the complexes are homotpy equivalent.

That is, two complexes are homotopy equivalent if and only if they are isomorphic in $\mathcal{K}(\mathcal{A})$.

Remark 3.5. Let $\mathcal{A}$ be abelian. If $f, g : A^\bullet \to B^\bullet$ are homotopic, then $H^\bullet(f) = H^\bullet(g)$. If f is a homotopy equivalence, then f is a quasi-isomorphism.

Lemma 3.6. *$\mathcal{K}(\mathcal{A})$ is an additive category.*

Proof. (parts). $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, B^\bullet)$ is an abelian group: Check that sums of homotopic maps are again homotopic. Everything else is clear. ☐

Corollary 3.7. *The canonical functor $\mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is additive.*

Corollary 3.8. *$H^\bullet(-)$ induces a well-defined family of functors $H^i : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$.*

Lemma 3.9. *Let $\alpha : A^\bullet \to B^\bullet$ be a morphism in $\mathcal{C}(\mathcal{A})$. Then there is a bijection between the set of homotopies from α to α and $\mathrm{Hom}_{\mathcal{C}(A)}(A^\bullet, B[-1]^\bullet)$.*

Proof. Let h be a homotopy between α and α. Then $0 = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, which is exactly the condition for a morphism between $A^\bullet$ and $B[-1]^\bullet$. ☐

Proposition 3.10. *Lef $F : \mathcal{C}(\mathcal{A}) \to \mathcal{B}$ be a functor which sends homotopy classes to isomorphisms. Then F factors uniquely over $\mathcal{K}(\mathcal{A})$.*

Definition 3.11. Consider $f : K^\bullet \to L^\bullet$. The cone of f is the complex $C(f)^\bullet$ given by $C(f)^n = K^{n+1} \oplus L^n$ and

$$d^n_{C(f)} = \begin{pmatrix} -d_K^{n+1} & 0 \\ f^{n+1} & d_L^n \end{pmatrix}.$$

There are canonical morphisms $i(f) : L^\bullet \to C(f)^\bullet$ and $p(f) : C(f) \to K[1]^\bullet$ induced by the inclusion into the first and projection to the second factor, respectively.

In pictures, we have a sequence $K^\bullet \xrightarrow{f} L^\bullet \to C(f)^\bullet \to K^\bullet[1]$, which is often written as

$$\begin{tikzcd} & C(f)^\bullet \arrow[dl] & \\ K^\bullet \arrow[rr] & & L^\bullet \arrow[ul] \end{tikzcd}$$

In $\mathcal{C}(\mathcal{A})$, $\mathcal{A}$ abelian, we get a short exact sequence

$$0 \to L^\bullet \xrightarrow{i(f)} C(f)^\bullet \xrightarrow{p(f)} K^\bullet[1] \to 0.$$

Definition 3.12. (i) Fix f as above. The strict (or distinguished) triangle on f is the triple $(f, i(f), p(f))$ of morphisms in $\mathcal{C}(\mathcal{A})$.

(ii) Let $A^\bullet, B^\bullet, C^\bullet \in \mathcal{K}(\mathcal{A})$ and morphisms $u : A^\bullet \to B^\bullet, v : B^\bullet \to C^\bullet, w : C^\bullet \to A[1]^\bullet$. Then (u, v, w) form and exact triangle on (A, B, C) if it is isomorphic to a strict triangle (f', u', δ) on a morphism $f' : A'^\bullet \to B'^\bullet$ such that in $\mathcal{K}(\mathcal{A})$ there is a commutative diagram

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d, "f"] & B^\bullet \arrow[r, "v"] \arrow[d, "g"] & C^\bullet \arrow[r, "w"] \arrow[d, "h"] & A[1]^\bullet \arrow[d, "f[1]"] \\ A'^\bullet \arrow[r, "f'"] & B'^\bullet \arrow[r, "u'"] & C(f') \arrow[r, "\delta"] & A'[1]^\bullet \end{tikzcd}$$

and f, g, h are isomorphisms in $\mathcal{K}(\mathcal{A})$.

Corollary 3.13. *Given an exact triangle (u, v, w) on (A, B, C) there is an associated long exact cohomolgy sequence*

$$\cdots \xrightarrow{H^{i-1}(w)} H^i(A^\bullet) \xrightarrow{H(u)} H^i(B) \xrightarrow{H^i(v)} H^i(C) \xrightarrow{H^i(w)} H^{i+1}(A) \to \cdots$$

Proof. It is sufficient to construct this for strict triangles, but for these we have a short exact sequence of complexes which gives the long exact sequence. □

4 Triangulated categories

Definition 4.1. Let $\mathcal{D}$ be an additive category, equipped with endofunctors $[n] : \mathcal{D} \to \mathcal{D}$ for all $n \in \mathbb{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}}$. A triangle is a sextuple (X, Y, Z, f, g, h) with objects X, Y, Z and morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$. A morphism of triangles $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X', b : Y \to Y', c : Z \to Z'$ such that $b \circ f = f' \circ a, c \circ g = g' \circ b, a[1] \circ h = h' \circ c$, i.e. such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

Definition 4.2. A pre-triangulated category is a triple $(\mathcal{D}, \{[n]\}, \mathcal{T})$ where $\mathcal{D}$ is an additive category, $\{[n]\}$ is a family of auto-equivalences as above and $\mathcal{T}$ is a set of so-called distinguished triangles such that

(TR1) Any triangle isomorphic to a distinguished triangles is distinguished. Any triangle of the form $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished and for any morphism $f : X \to Y$ there exist Z, g, h such that (X, Y, Z, f, g, h) is a distinguished triangles

(TR2) The triangle (X, Y, Z, f, g, h) is distinguished if and only if $(Y, Z, X[1], g, h, -f[1])$ is distinguished.

(TR3) Given two distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') such that there are morphisms $a : X \to X'$, $b : Y \to Y'$ such that $b \circ f = f' \circ a$, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism between the two triangles.

(TR4) For each pair of morphisms $f : A \to B$, $g : B \to C$, there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "h"] \arrow[d, "g"] & C' \arrow[r] \arrow[d] & A[1] \arrow[d, equal] \\
A \arrow[r, "g\circ f"] & C \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "f[1]"] \\
 & A' \arrow[r, equal] \arrow[d] & A' \arrow[r] \arrow[d] & B[1] \\
 & B[1] \arrow[r, "h[1]"] & C'[1] &
\end{tikzcd}$$

such that the first two rows and the middle two columns form distinguished triangles.

Remark 4.3. If you just require (TR1)-(TR3), then $\mathcal{D}$ is called pre-triangulated. No example of a pre-triangulated, non-triangulated triangle is known.

Denote by $C(f)$ the object Z from the last part of (TR1), which one can think of as replacement of the cokernel Y/X. From (TR4) one gets a distinguished triangle

$$C(f) \to C(g \circ f) \to C(g) \to C(f)[1],$$

i.e. the relation "$(C/A)/(B/A) = C/B$".

(TR2) implies that for a distinguished triangle (X, Y, Z, f, g, h) any four term sequence in the long sequence

$$\cdots \to C[i-1] \to A[i] \to B[i] \to C[i] \to A[i+1] \to \cdots$$

is distinguished.

We now consider $\mathcal{D} = \mathcal{K}(\mathcal{A})$, $\mathcal{A}$ additive. Recall the definition of distinguished = exact triangles from definition 3.12

Theorem 4.4. *Let $\mathcal{A}$ be an additive category. Then $\mathcal{K}(\mathcal{A})$ is a triangulated category.*

Proof. The first and third part of (TR1) are clear. (TR2) follows from Lemma 4.5. below. The second part of (TR1) follows from (TR2) since $C(0 \to X) = X$. It is sufficient to prove (TR3) for strict triangles. So assume we have a commutative diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, "\varphi"] & B^\bullet \arrow[r] \arrow[d, "\psi"] & C(f) \arrow[r] & A^\bullet[1] \\
A'^\bullet \arrow[r, "f'"] & B'^\bullet \arrow[r] & C(f') \arrow[r] & A'^\bullet[1]
\end{tikzcd}$$

Since $\psi \circ f$ is homotopic to $f' \circ \varphi$, there are maps $h^k : A^k \to B^{k-1}$ such that $\psi^k \circ f^k - f'^k \circ \psi^k = h^{k+1} \circ d^k + d^{k-1} \circ h^k$. Define ω by

$$\omega^k = \begin{pmatrix} \varphi^{k+1} & 0 \\ h^{k+1} & \psi^k \end{pmatrix} : C(f)^k = A^{k+1} \oplus B^k \to C(f') = A'^{k+1} \oplus B'^k.$$

An explicit calculation shows that this ω satisfies the requirements of (TR3).

For (TR4), let $C' = C(f), B' = C(g \circ f), A' = C(g)$. Therefore we have to construct a distinguished triangle

$$C(f) \xrightarrow{u} C(g \circ f) \xrightarrow{v} C(g) \xrightarrow{w} C(f)[1]$$

with the maps $u : A^{k+1} \oplus B^k \to A^{k+1} \oplus C^k$, $v^k : A^{k+1} \oplus C^k \to B^{k+1} \oplus C^k$, $w : B^{k+1} \oplus C^k \to A^{k+2} \oplus B^{k+1}$ given by

$$u^k = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & g^k \end{pmatrix}, \quad v^k = \begin{pmatrix} f^{k+1} & 0 \\ 0 & \mathrm{id} \end{pmatrix}, \quad w^k = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{pmatrix}.$$

To show that this triangle is distinguished, construct an isomorphism $\varphi : C(u) \to C(g)$ such that $\varphi \circ i(u) = v$ and $p(u) = w \circ \varphi$. For that, the following works:

$$\varphi^k = \begin{pmatrix} 0 & \mathrm{id} & f^{k+1} & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix} : A^{k+2} \oplus B^{k+1} \oplus A^{k+1} \oplus C^k \to B^{k+1} \oplus C^k$$

with homotopy inverse ψ and homotopy h given by

$$\psi^k = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}, \qquad h^k = \begin{pmatrix} 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Again explicit calculations show that everything works out. □

Lemma 4.5. *For any $f : A^\bullet \to B^\bullet$ there exists a map $\varphi : A[1]^\bullet \to C(i(f))$ such that φ is an isomorphism in $\mathcal{K}(\mathcal{A})$ and the following diagram commutes.*

$$\begin{tikzcd} B^\bullet \arrow[r, "i(f)"] \arrow[d, equal] & C(f) \arrow[r, "p(f)"] \arrow[d, equal] & A^\bullet[1] \arrow[r, "-f[1]"] \arrow[d, "\varphi"] & B^\bullet[1] \arrow[d, equal] \\ B^\bullet \arrow[r, "i(f)"] & C(f) \arrow[r, "i(i(f))"] & C(i(f)) \arrow[r, "p(i(f))"] & B^\bullet[1] \end{tikzcd}$$

Proof. Note that $C(i(f))^k = B^{k+1} \oplus C(f)^k = B^{k+1} \oplus A^{k+1} \oplus B^k$. Define $\phi^k : A[1]^k \to C(i(f))^k$ by $(-f^{k+1}, \mathrm{id}_A^{k+1}, 0)^t$, which has homotopy inverse $\psi^k = (0, \mathrm{id}_{A^{k+1}}, 0)$: Check that ϕ, ψ are morphisms of complexes and indeed homotopy inverses, and that the diagram of the lemma commutes. Those are straightforward calculations. □

Proposition 4.6. *Let (A, B, C, f, g, h) be a distinguished triangle in a triangulated category. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

Proof. It suffices to prove $g \circ f = 0$ by (TR2). Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed, "\gamma"] & A[1] \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & A[1] \end{tikzcd}$$

Since the first row is a distinguished triangle by (TR1), by (TR3) γ exists and $g \circ f = \gamma \circ 0 = 0$. □

Proposition 4.7. *Let $A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$ be a distinguished triangle in a triangulated category. For and A_0 in that category, the sequences*

$$\mathrm{Hom}(A_0, A) \to \mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, C)$$
$$\mathrm{Hom}(C, A_0) \to \mathrm{Hom}(B, A_0) \to \mathrm{Hom}(A, A_0)$$

are exact.

Proof. The sequences are complexes by Proposition 4.6. Let $h : A_0 \to B$ be a map such that $g \circ h = 0$. Apply (TR1)-(TR3) to

$$\begin{tikzcd} A_0 \arrow[r, "\mathrm{id}"] & A_0 \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & A_0[1] \arrow[d, dashed, "\widetilde{m}"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1] \end{tikzcd}$$

to get $\widetilde{m}$ making the diagram commute. Set $m = \widetilde{m}[-1]$, then $h = f \circ m$ as required. The second sequence works similarly. □

Proposition 4.8. *Let $\mathcal{D}$ be a triangulated category and a morphism of distinguished triangles*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

If two of the three vertical maps are isomorphisms, then so is the third one.

Proof. Assume α, β are isomorphisms, the other cases then follow through rotation. Apply $\mathrm{Hom}(C', -)$ to the diagram to get

$$\begin{tikzcd} \mathrm{Hom}(C', A) \arrow[r] \arrow[d, "\alpha_*"] & \mathrm{Hom}(C', B) \arrow[r] \arrow[d, "\beta_*"] & \mathrm{Hom}(C', C) \arrow[r] \arrow[d, "\gamma_*"] & \mathrm{Hom}(C', A[1]) \arrow[r] \arrow[d, "\alpha[1]_*"] & \mathrm{Hom}(C', B[1]) \arrow[d, "\beta[1]_*"] \\ \mathrm{Hom}(C', A') \arrow[r] & \mathrm{Hom}(C', B') \arrow[r] & \mathrm{Hom}(C', C') \arrow[r] & \mathrm{Hom}(C', A'[1]) \arrow[r] & \mathrm{Hom}(C', B'[1]) \end{tikzcd}$$

Now $\alpha_*, \beta_*, \alpha[1]_*$ and $\beta[1]_*$ are isomorphisms, so by the five lemma, γ_* is an isomorphism as well. Hence there exists a map $\delta : C' \to C$ such that $\gamma \circ \delta = \gamma_*(\delta) = \mathrm{id}_{C'}$ and γ has a right inverse. Repeat the above argument with $\mathrm{Hom}(-, C)$ to get a left inverse as well. □

Remark 4.9. This proposition implies that the cone $C(f)$ of any morphism in a triangulated category is unique up to isomorphism. But this isomorphism is itself not unique.

Definition 4.10. Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ abelian. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called homological if for all distinguished triangles (X, Y, Z, f, g, h) the sequence $H(X) \to H(Y) \to H(Z)$ is exact in $\mathcal{A}$. An additive functor $H : \mathcal{D}^{\mathrm{op}} \to \mathcal{A}$ is cohomological if the opposite functor $H^{\mathrm{op}} : \mathcal{D} \to \mathcal{A}^{\mathrm{op}}$ is homological. For a homological functor $H : \mathcal{D} \to \mathcal{A}$, we write $H^n(X) = H(X[n])$

By (TR2) a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence

$$\cdots \to H^{-1}(Z) \to H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to \cdots$$

Example 4.11. Let $\mathcal{D}$ be a triangulated category, $A_0 \in \mathcal{D}$. Then $\mathrm{Hom}(A_0, -)$ is homological and $\mathrm{Hom}(-, A_0)$ is cohomological by Proposition 4.7.

Example 4.12. $\mathcal{D} = \mathcal{K}(\mathcal{A})$, $\mathcal{A}$ abelian. Then $H^\bullet : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$ is homological

5 Localization

Given a category $\mathcal{C}$ and S a class of morphism, the aim is to construct a category $S^{-1}\mathcal{C}$ with the same objects and such that all morphisms in S become invertible, in a "universal" way. If $\mathcal{C}$ is triangulated, we want $S^{-1}\mathcal{C}$ to be "naturally" triangulated as well.

Remark 5.1.
- We will only define $S^{-1}\mathcal{C}$ if S is multiplicative since in this case the morphisms in the localization can be described by "a calculus of fractions".
- Let R be a ring. Consider the category $\mathcal{R}$ with one object $\bullet$ and $\mathrm{End}(\bullet) = R$. If $S \subseteq R$ is a "two-sided denominator set", $S^{-1}R$ is defined and each element in $S^{-1}R$ is of the form fs^{-1} of $s^{-1}f$ for $s \in S$, $f \in R$. Our construction will coincide with this, i.e. $S^{-1}\mathcal{R}$ will have morphism set $S^{-1}R$.

Definition 5.2. Let $\mathcal{C}$ be a category and S a family of morphisms in $\mathcal{C}$. S is called multiplicative if

(S1) For all $X \in \mathcal{C}$, $\mathrm{id}_X \in S$,

(S2) If $f, g \in S$ and $g \circ f$ exists, then $g \circ f \in S$,

(S3) Any diagram of the form

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with $(Z \to Y) \in S$ can be completed to a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

such that $(W \to X) \in S$. Likewise with all arrows reversed.

(S4) If $f, g : X \to Y$, then the following are equivalent:

(a) There is $t : Y \to Y'$, $t \in S$ such that $t \circ f = t \circ g$

(b) There is $s : X' \to X$, $s \in S$ such that $f \circ s = g \circ s$

Definition 5.3. Let $\mathcal{C}$ be a category and S a multiplicative family of morphisms. Then $S^{-1}\mathcal{C}$ is the category with objects the objects of $\mathcal{C}$ and

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) = \{(X', s, f) \mid X' \in \mathcal{C}, s : X' \to X, s \in S, f : X' \to Y\}/\sim$$

where $(X', s, f) \sim (X'', t, g)$ if there exists a commutative diagram of the form

$$\begin{tikzcd} & X & \\ X' \arrow[ur, "s"] \arrow[dr, "f"'] & \widetilde{X} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X'' \arrow[ul, "t"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

and two morphisms $(X', s, f) : X \to Y$ and $(Y', t, g) : Y \to Z$ are composed as follows: By (S3) there exists X'' such that the diagram

$$\begin{tikzcd} & & X'' \arrow[dl, "t'"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

commutes. Hence set

$$(Y', t, g) \circ (X', s, f) = (X'', s \circ t', g \circ h).$$

Lemma 5.4. *Assuming all $\operatorname{Hom}_{S^{-1}\mathcal{C}}(X, Y)$ are sets, $S^{-1}\mathcal{C}$ defines a category. More precisely: The given relation is an equivalence relation, the composition rule is well-defined, and composition is associative, and respects identities.*

Proof. Stacksproject, Part 1, Chapter 4, Lemma 4.27.2. □

There is a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ defined by $X \mapsto X$ and $f \mapsto (X, \mathrm{id}_X, f)$.

Lemma 5.5. *(i) Two morphisms $f, g : X \to Y$ in $\mathcal{C}$ get identified in $S^{-1}\mathcal{C}$ if and only if $s \circ f = s \circ g$ for some $s : Y \to \widetilde{Y}$ in S.*

(ii) q preserves zero, initial and terminal objects.

(iii) If $X \times Y$ exists in $\mathcal{C}$, then $q(X \times Y) \cong q(X) \times q(Y)$ in $S^{-1}\mathcal{C}$.

(iv) If $\mathcal{C}$ is additive, then so is $S^{-1}\mathcal{C}$, and q is additive.

Proof. We only show how to define addition of morphisms for (iv). So let $(X', s, f), (X'', s', f') : X \to Y$. By (S3), we can find U to make a commutative diagram

$$\begin{tikzcd} U \arrow[r, "r'"] \arrow[d, "r"] & X'' \arrow[d, "s'"] \\ X' \arrow[r, "s"] & X \end{tikzcd}$$

Now check that $(X', s, f) \sim (U, s \circ r, f \circ r)$. Hence define the sum of the morphisms as $(U, s \circ r, f \circ r + f' \circ r')$. Check that this is well-defined. □

Proposition 5.6. *For any $s \in S$, $q(s)$ is an isomorphism. If $\mathcal{C}'$ is any other category and $F : \mathcal{C} \to \mathcal{C}'$ is a functor such that $F(s)$ is an isomorphism for all $s \in S$, then F factors uniquely through q.*

Proof. By definition, $q(s) = (X, \mathrm{id}_X, s)$ has inverse (X, s, id_X). If a functor $G : S^{-1}\mathcal{C} \to \mathcal{C}'$ with $G \circ q = F$ exists, then it must be unique, since G must agree with F on objects, and for a morphism (Z, s, f), we have $(Z, s, f) \circ q(s) = q(f)$, so applying G yields $G(Z, s, f) \circ F(s) = F(f)$. Since $F(s)$ is invertible, this uniquely determines $G(Z, s, f) = F(f) \circ F(s)^{-1}$. To show that G exists, define G in the above way and check that this is well-defined. □

6 The derived category

Definition 6.1. Let $\mathcal{D}$ be a triangulated category and $\mathcal{N}$ a family of objects in $\mathcal{D}$. Then $\mathcal{N}$ is called a null system if it satisfies

(N1) $0 \in \mathcal{N}$

(N2) $A \in \mathcal{N}$ if and only if $A[1] \in \mathcal{N}$

(N3) If $A \to B \to C \to A[1]$ is distinguished and $A, B \in \mathcal{N}$, then $C \in \mathcal{N}$.

Remark 6.2. By (N2), (N3) and rotation of triangles, if 2 of the three objects in a distinguished triangle are in $\mathcal{N}$, so is the third.

Definition 6.3. Let $\mathcal{N}$ be a null system. Define $\mathcal{S}(\mathcal{N}) = \{f : A \to B \mid C(f) \in \mathcal{N}\}$.

Proposition 6.4. *If $\mathcal{N}$ is a null system, then $\mathcal{S}(\mathcal{N})$ is multiplicative.*

Proof. (S1) Let $X \in \mathcal{D}$, consider $X \to X \to 0 \to X[1]$. Since $0 \in \mathcal{N}$, we have $\mathrm{id}_X \in \mathcal{S}(\mathcal{N})$.
(S2) Let $f : A \to B$, $g : B \to C$, $w = g \circ f$, and assume $C(f), C(g) \in \mathcal{N}$. The octahedron axiom gives a distinguished triangle $C(f) \to C(w) \to C(g) \to C(f)[1]$, and hence $C(w) \in \mathcal{N}$, i.e. $w \in \mathcal{S}(\mathcal{N})$.
(S3) Let $f : A \to B, g : C \to B$ with $g \in \mathcal{S}(\mathcal{N})$ be given. Consider $\pi = i(g) \circ f : A \to C(G)$. Then we get a commutative diagram

$$\begin{tikzcd}
D \arrow[r] & A \arrow[r, "\pi"] \arrow[d, "f"] & C(g) \arrow[r] \arrow[d, "\mathrm{id}"] & D[1] := C(\pi) \\
C \arrow[r, "g"] & B \arrow[r, "i(g)"] & C(g) \arrow[r] & C[1]
\end{tikzcd}$$

Then there exists a morphism $D \to C$ completing the diagram and forming the required completion. Similarly with reversed arrows.
(S4) Assume $f : A \to B$ such that there exists $t : B \to B'$ with $t \circ f = 0$, $t \in \mathcal{S}(\mathcal{N})$. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "h"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & A[1] \\
C(t)[-1] \arrow[r, "g"] & B \arrow[r, "t"] & B' \arrow[r] & C(t)
\end{tikzcd}$$

Now let $A' = C(h)[-1]$ and $s = i(h)[-1] : A' \to A$, then $f \circ s = g \circ h \circ s = g \circ 0 = 0$. The reverse implication is similar. □

Example 6.5. Let $\mathcal{D} = \mathcal{K}(\mathcal{A})$ and let $\mathcal{N} = \{A^\bullet \in \mathcal{K}(\mathcal{A}) \mid H^k(A^\bullet) = 0 \ \forall k \in \mathbb{Z}\}$ the triangulated subcategory of acyclic/exact complexes. Then $\mathcal{N}$ is a null system: (N1),(N2) are obvious. (N3) follows from $H^0 : \mathcal{K}(\mathcal{A}) \to \mathcal{A}$ being a cohomological functor (i.e. ther eis a long exact sequence). Recall: $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if $C(f) \in \mathcal{N}$, hence $\mathcal{S}(\mathcal{N})$ is the family of quasi-isomorphisms.

More generally: Let $\mathcal{D}$ be any triangulated category and $H : \mathcal{D} \to \mathcal{A}$ a cohomological functor.

Theorem 6.6. $\mathcal{S} = \{f : A \to B \mid \forall i : H^i(f)$ *is an isomorphism*$\}$ *is multiplicative and* $\mathcal{D} \to \mathcal{S}^{-1}\mathcal{D}$ *is a morphism of triangulated categories. In particular,* $\mathcal{S}^{-1}\mathcal{D}$ *is triangulated.*

Proof. $\mathcal{S}$ being multiplicative is similar to the proof in example 6.5. Define shifts on $\mathcal{S}^{-1}\mathcal{D}$ as the shifts on $\mathcal{D}$ for objects and $(fs^{-1})[1] = f[1]s[1]^{-1}$. Define distinguished triangles as triangles that are isomorphic to the images of distinguished triangles in $\mathcal{D}$. Then check that all the axioms hold, which is painful. □

Corollary 6.7 (Universal property). *Let $F : \mathcal{D} \to \mathcal{L}$ a triangulated functor between triangulated categories such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, where $\mathcal{S}$ is a multiplicative system arising from a cohomological functor. Then F factors uniquely through $\mathcal{S}^{-1}\mathcal{D}$, and $\mathcal{S}^{-1}\mathcal{D} \to \mathcal{L}$ is triangulated.*

Proof. Combine Theorem 6.6 and Proposition 5.6. □

Definition 6.8. Let $\mathcal{D} = \mathcal{K}(\mathcal{A})$, $\mathcal{N}$ as defined in Example 6.5. Then $\mathcal{S}(\mathcal{N})^{-1}\mathcal{K}(\mathcal{A}) = \mathcal{D}(\mathcal{A}) = \mathcal{D}er(\mathcal{A})$ is called the derived category.

Example 6.9. Let $\mathcal{A}$ be semisimple abelian, i.e. all exact sequences split. For example take $\mathcal{A} = \mathtt{Vec}_K$. Consider complexes where all differentials are zero. This defines a full subcategory $\mathcal{C}(\mathcal{A}) \supseteq \mathcal{C}^0(\mathcal{A}) \cong \prod_{i=-\infty}^{\infty} \mathcal{A}[i]$. Taking cohomology defines a functor $\mathcal{C}(\mathcal{A}) \to \mathcal{C}^0(\mathcal{A})$, which takes quasi-isomorphisms to isomorphisms, hence factors through $\kappa : \mathcal{D}(\mathcal{A}) \to \mathcal{C}^0(\mathcal{A})$.
<u>Claim:</u> κ is an equivalence of categories.
Proof: Let $A^\bullet \in \mathcal{C}(\mathcal{A})$, define $B^k = \operatorname{im}(d^{k-1})$, $Z^k = \ker(d^k)$ and $H^k = H^k(A^\bullet) = Z^k/B^k$. Since $\mathcal{A}$ is semisimple, we get $A^k = B^k \oplus H^k \oplus B^{k+1}$ with differentials $d(b^k, h^k, b^{k+1}) \mapsto (b^{k+1}, 0, 0)$. Define $f : A^\bullet \to \bigoplus H^i(A^\bullet), (b^k, h^k, b^{k+1}) \mapsto h^k$ and $g : \bigoplus H^i(A^\bullet) \to A^\bullet, h^k \mapsto (0, h^k, 0)$. Furthermore consider $\varphi : \mathcal{C}^0(\mathcal{A}) \hookrightarrow \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. Then $\kappa \circ \varphi = \mathrm{id}$. Conversely, $\varphi \circ \kappa$ sends $A^\bullet$ to $\bigoplus H^k(A^\bullet)$, and f, g provide isomorphisms $A^\bullet \cong \varphi(\kappa(A^\bullet))$ in $\mathcal{D}(\mathcal{A})$.
For example, $\mathcal{D}(\mathtt{Vec}_K) = \mathtt{grVec}_K$.

Proposition 6.10. *Let $\mathcal{C}$ be a category, $\mathcal{C}'$ a full subcategory, $\mathcal{S}$ a multiplicative system of $\mathcal{C}$, $\mathcal{S}'$ the family of morphisms in $\mathcal{S}$ which belong to $\mathcal{C}'$. Assume that $\mathcal{S}'$ is again multiplicative in $\mathcal{C}'$ and one of the following conditions holds:*

(1) If $(s : X \to Y) \in S$ with $Y \in \mathcal{C}'$, then there exists $g : W \to X$ with $W' \in \mathcal{C}'$ and $s \circ g \in S$.

(2) (1) with all arrows reversed.

Then $(\mathcal{S}')^{-1}\mathcal{C}'$ is a full subcategory of $\mathcal{S}^{-1}\mathcal{C}$.

Proof. Note that there is an obvious inclusion functor. We show that this functor is fully faithful. For fullness, let $(X', s, f) : X \to Y$ in $\mathcal{S}^{-1}\mathcal{C}$ with $X, Y \in \mathcal{C}'$. By condition (1) there exists $g : W \to X'$ such that $s \circ g \in S'$. Consider $(W, s \circ g, f \circ g) = (X', s, f)$, which is a morphism in $(\mathcal{S}')^{-1}\mathcal{C}'$. □

Proposition 6.11. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system in $\mathcal{D}$, $\mathcal{D}'$ a full subcategory of $\mathcal{D}$ and $\mathcal{N}' = \mathcal{N} \cap \mathcal{D}'$. Then*

(1) $\mathcal{N}'$ is a null system in $\mathcal{D}'$

(2) Assume that any morphism $B \to C$ in $\mathcal{D}$ with $B \in \mathcal{D}'$, $C \in \mathcal{N}$ factors through $\mathcal{N}'$. Then $\mathcal{D}'/\mathcal{N}'$ is a full subcategory of $\mathcal{D}/\mathcal{N}$.

Proof. part (1) is clear. For (2), verify (1) of 6.10: Let $(s : A \to B) \in \mathcal{S}(\mathcal{N})$, $B \in \mathcal{D}'$. By assumption $C(s) \in \mathcal{N}$, so $f : B \to C(s)$ factors as $\beta \circ \alpha : C \to C' \to C(s)$ with $C' \in \mathcal{N}'$. From (TR4) one can construct a distinguished triangle $E := C(\alpha)[-1] \xrightarrow{t} A \to C(\beta)[-1] \to C(\alpha)$. (TR4) again yields a distinguished triangle $C(t) \to C(s \circ t) \to C(s) \to C(t)[1]$, and $C(t) = C(\beta)[-1]$ implies that $C(t)$ is in $\mathcal{N}$, hence $C(s \circ t) \in \mathcal{N}$. □

Applying this proposition to $\mathcal{K}^+(\mathcal{A}), \mathcal{K}^-(\mathcal{A}), \mathcal{K}^b(\mathcal{A})$, we obtain corresponding full triangulated subcategories $\mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A}), \mathcal{D}^b(\mathcal{A})$.

Proposition 6.12. *The category $\mathcal{D}^*(\mathcal{A})$, where $* \in \{+,-,b\}$, is equivalent to the full subcategory of $\mathcal{D}(A)$ of objects $A^\bullet$ s.t. $H^k(A^\bullet) = 0$ for $k \ll 0, k \gg 0, |k| \gg 0$, respectively.*

Proof. If $H^j(A^\bullet) = 0$ for $j < k$, then $A^\bullet \to \tau^{\geq k}(A^\bullet)$ is a quasi-isomorphism. □

Definition 6.13. Let $\mathcal{A}$ be an abelian category, $A \in \mathcal{A}$. Then A is projective if $\mathrm{Hom}(A, -)$ is exact and injective if $\mathrm{Hom}(-, A)$ is exact. Equivalently, A is projective, if epis $X \to Y$ lift along maps $A \to Y$, and injective, if monos $X \to Y$ lift along maps $X \to A$. Denote by $\mathcal{I}, \mathcal{P}$ the full subcategories of injective and projective objects.

Definition 6.14. $\mathcal{A}$ has enough projectives if for any object $A \in \mathcal{A}$ there exists an epimorphism $P \to A$ with P projective. Dually, $\mathcal{A}$ has enough injectives if for every $A \in \mathcal{A}$ there exists a monomorphism $A \to I$ with I injective.

In this case each object in $\mathcal{A}$ has a projective/injective resolution.

Lemma 6.15. *Let $s : I^\bullet \to A^\bullet$ be a quasi-isomorphism with $I^\bullet \in \mathcal{K}^+(\mathcal{I})$, $A \in \mathcal{K}^+(\mathcal{A})$. Then there exists a morphism of complexes $t : A^\bullet \to I^\bullet$ such that $t \circ s$ is homotopic to $\mathrm{id}_{I^\bullet}$.*

Proof. Since s is a quasi-isomorphism, $C(s)$ is acyclic. We claim that any morphism from an acyclic complex to a left-bounded complex of injectives is null-homotopic. Indeed, wlog assume we are in the situation

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, "\delta^0"] & C^1 \arrow[r, "d^1"] \arrow[d, "\delta^1"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

By the lifting property of injectives, we get a map $k^0 : C^1 \to I^0$ with $\delta^0 = k^0 \circ d^0$. Now $\delta^1 - d^0_{I^\bullet} \circ k^0$ factors through $\mathrm{coker}(d^0)$, so the mono $\mathrm{coker}(d^0) \to C^2$ lifts along $\mathrm{coker}(d^0) \to I^1$ to $k^1 : C^2 \to I^1$. Proceed inductively and check that the result is a nullhomotopy.

In general, if $A \to B \to C \to A[1]$ is a triangle with $(C \to A[1]) = 0$, then $B = A + C$. Apply this to $I^\bullet \to A^\bullet \to C(s) \to I^\bullet[1]$ to obtain a projection $t : A^\bullet \to I^\bullet$. □

Corollary 6.16. *The natural functor $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$ is fully faithful.*

Proposition 6.17. *If $\mathcal{A}$ has enough injectives, this is an equivalence of categories.*

Definition 6.18. Write $X \oplus_Z Y = \mathrm{coker}(Z \to X \oplus Y)$ for the pushout of X and Y over Z.

Proof. We need to show: For any complex $A^\bullet \in \mathcal{K}^+(\mathcal{A})$ there exists $I^\bullet \in \mathcal{K}^+(\mathcal{A})$ with a quasi-isomorphism $t : A^\bullet \to I^\bullet$. Using that $\mathcal{A}$ has enough injectives, construct a diagram

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r, "d^0"] \arrow[d, "t^0"] & A^1 \arrow[rrr] \arrow[d, "a"] \arrow[dr, dashed, "t^1"] & & & A^2 \arrow[r] \arrow[d] \arrow[drr, dashed, "t^2"] & \cdots & \\
0 \arrow[r] & I^0 \arrow[r, "b"] & I^0 \oplus_{A^0} A^1 \arrow[r, "c"] & I^1 \arrow[r] & \mathrm{coker}(c \circ b) \arrow[r] & \mathrm{coker}(c \circ b) \oplus_{A^1} A^2 \arrow[rr] & & I^2
\end{tikzcd}$$

proceeding inductively as indicated. From technical arguments it follows that t is a quasi-isomorphism. □

It is still not clear whether derived categories actually exist, i.e. whether the collection of morphisms between two objects is a set. Recall $\mathrm{Hom}_{\mathcal{S}^{-1}\mathcal{C}}(X,Y) = \{(X', s, t) \mid s : X' \to X, t : X' \to Y\}$. If $\mathcal{C}$ is small, then everything is clear, because X' ranges over a set, but otherwise this is very unclear. One solution (stacksproject) is to only consider small categories. You can also redefine sets conveniently (Grothendieck) or prove by hand that for certain categories everything works. There are some general theorems on the existence of derived categories, e.g. for model categories. We consider only the following case:

Definition 6.19. A multiplicative system $\mathcal{S}$ is locally small (on the left) if for each X there exists a set $\mathcal{S}_X \subseteq \mathcal{S}$ such that for every $X_1 \to X$ there is a map $X_2 \to X_1$ such that $X_2 \to X_1 \to X$ is in $\mathcal{S}_X$.

Lemma 6.20. *If $\mathcal{S}$ is locally small, then $\mathrm{Hom}_{\mathcal{S}^{-1}\mathcal{C}}(X,Y)$ is a set for all objects X, Y. In particular, the derived category exists.*

Proof. Weibel, 10.3.6 □

Proposition 6.21. *Let R be a ring. Then $\mathcal{D}er(\mathcal{A})$ exists for $\mathcal{A} = R{-}\mathtt{Mod}$, presheaves, sheaves, ...*

Proof. Let $A^\bullet \in \mathcal{C}(R{-}\mathtt{Mod})$. Choose an infinite cardinal number κ larger than the cardinality of all the A^i and R. Call a cochain complex petite if all underlying sets have cardinality less than κ. Then there exists a <u>set</u> of isomorphism classes of petite cochain complexes. Then one can show that there exists a set of isomorphism classes of quasi isomorphisms $q : A' \to A$ with A' petite. Hence, given a quasi isomorphism $B \to A$ it suffices to show that B contains a petite subcomplex quasi-isomorphic to A. The details are left to Weibel, 10.4.4. □

As a final "outlook" in this section, we want to sketch the construction of derived categories in a more general setting.

Definition 6.22. A pair of morphisms $A \xrightarrow{i} B \xrightarrow{p} C$ in an additive category is exact if i is a kernel of p and p is a cokernel of i.

Definition 6.23. A deflation is a morphism which occurs on the right in some exact pair. An inflation is a morphism which occurs on the left in some exact pair. An exact pair is also called a conflation.

Definition 6.24 (Quillen)**.** An exact category is an additive category endowed with a class of exact pairs closed under isomorphisms and satisfying

(Ex0) The identity of the zero object is a deflation.

(Ex1) The composition of two deflation is a deflation.

(Ex1$^{\mathrm{op}}$) The composition of two inflations is an inflation.

(Ex2) Given maps $p : B \to C$ and $c : C' \to C$ such that p is a deflation can be completed to a pullback diagram

$$\begin{array}{ccc} B' & \xrightarrow{p'} & C' \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ B & \xrightarrow{p} & C \end{array}$$

such that p' is a deflation.

(Ex2$^{\mathrm{op}}$) The dual statement of Ex2.

Example 6.25. Let $\mathcal{E} \subseteq \mathcal{A}$ be a full subcategory of an abelian category which is closed under extension. This is an exact category with the class of exact pairs $M' \to M \to M''$ the pairs such that $0 \to M' \to M \to M'' \to 0$ exact in $\mathcal{A}$. Conversely, if $\mathcal{E}$ is any exact category, then $\mathcal{E}$ is a full subcategory of the abelian category of left exact functors from $\mathcal{E}$ to $\mathtt{Ab}$. More concretely, for $\mathcal{A} = R\mathtt{-Mod}$, take $\mathcal{E}$ be the full subcategory of projective of finitely generated modules.

Keller and Nieman showed that the derived category of an exact category can be constructed. Let $\mathcal{E}$ be exact. A complex N over $\mathcal{E}$ is acyclic in degree n if the differential d^{n-1} can be factored as $d^{n-1} = i^{n-1} \circ p^{n-1}$ such that p^{n-1} is a cokernel for d^{n-2} and a deflation and i^{n-1} is a kernel for d^n and an inflation. Let $\mathcal{N} \subseteq \mathcal{K}(\mathcal{E})$ of acyclic complexes. A morphism s in $\mathcal{K}(\mathcal{E})$ is a quasi-isomorphism if there exists a distinguished triangle $N \to X \xrightarrow{s} X' \to N[1]$ with $N \in \mathcal{N}$. Then let $\mathcal{S} = \mathcal{S}(\mathcal{N})$ be the multiplicative system of quasi-isomorphisms, and one can define $\mathcal{D}er(\mathcal{E}) = \mathcal{S}^{-1}\mathcal{K}(\mathcal{E})$. Most statements in this section hold in this setting as well, but the proofs are more technical. Details can be found in Keller's paper on derived categories in the handbook of algebra.

7 Derived functors

Given an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, we ask the question when it can be extended to a triangulated functor $DF : \mathcal{D}er(\mathcal{A}) \to \mathcal{D}er(\mathcal{B})$. Obviously, F extends to a functor $F : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$. In fact, F preserves cochain homotopy equivalences, so we even get a functor $F : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$. In order to get an induced functor on the derived level, F would need to turn quasi-isomorphisms into quasi-isomorphisms. But this is, in general, far from true, of course.

Proposition 7.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be exact. Then the induced functor $F : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ maps quasi-isomorphisms to quasi-isomorphisms, hence descends to a functor $D(F) : \mathcal{D}er(\mathcal{A}) \to \mathcal{D}er(\mathcal{B})$.*

Proof. Let $A^\bullet$ be acyclic. Then $F(A^\bullet)$ is acyclic as well. Hence let $s : A^\bullet \to B^\bullet$ be a quasi-isomorphism. Then $C(s)$ is acyclic, so $F(C(s)) = C(F(s))$ is acyclic as well, so $F(s)$ is a quasi-isomorphism. $D(F)$ now exists by the universal property of the derived category. $\square$

Definition 7.2. Let $F : \mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{B})$ be a functor of triangulated categories. Then a (total) right derived functor of F on $\mathcal{K}^+(\mathcal{A})$ is a triangulated functor $RF : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er(\mathcal{B})$ together with a natural transformation ξ from $qF : \mathcal{K}^+(\mathcal{A}) \xrightarrow{F} \mathcal{K}(\mathcal{B}) \xrightarrow{q} \mathcal{D}er(\mathcal{B})$ to $(RF)q : \mathcal{K}^+(\mathcal{A}) \xrightarrow{q} \mathcal{D}er^+(\mathcal{A}) \xrightarrow{RF} \mathcal{D}er(\mathcal{B})$ which is unversal in the following sense: If $G : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er(\mathcal{B})$ is any other triangulated functor together with a natural transformation $\zeta : qF \Rightarrow Gq$, then there exists a unique natural tranformation $\eta : RF \to G$ such that $\zeta = \eta \circ \xi$

Remark 7.3. Such a functor, if it exists, is unique up to natural isomorphism.

Remark 7.4. If $\mathcal{K}' \subseteq \mathcal{K}^+(\mathcal{A})$ is a so called localizing triangulated subcategory (without definition), then there will be a nautral tranformation from the right derived functor RF' built with $\mathcal{K}'$ instead of $\mathcal{K}^+(\mathcal{A})$ to the restriction of RF. We write $R^bF, R^+F, R^-F, R'F$ etc. to indicate the subcategory we are working in.

Definition 7.5. Similarly, a (total) left derived functor is a triangulated functor $LF : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er(\mathcal{B})$ together with a natural transformation $\xi : (LF)q \to qF$ satisfying the dual universal property.

Remark 7.6. Since $LF = (RF^{\mathrm{op}})^{\mathrm{op}}$, any statement about RF can be dualized to a statement for LF.

Let $\mathcal{A}$ be abelian with enough injective objects. Let $\mathcal{A}'$ be abelian and $F : \mathcal{A} \to \mathcal{A}'$ be left exact. We had an equivalence of triangulated categories $T : \mathcal{K}^+(\mathcal{I}) \cong \mathcal{D}er^+(\mathcal{A})$, i.e. we have a commutative diagram

$$\begin{tikzcd}
\mathcal{K}^+(\mathcal{I}) \arrow[r] \arrow[dr, "T"'] & \mathcal{K}^+(\mathcal{A}) \arrow[r, "\mathcal{K}(F)"] \arrow[d, "q_{\mathcal{A}}"] & \mathcal{K}^+(\mathcal{A}') \arrow[d, "q_{\mathcal{A}'}"] \\
& \mathcal{D}er^+(\mathcal{A}) & \mathcal{D}er^+(\mathcal{A}')
\end{tikzcd}$$

Theorem 7.7. *In this setting, the right derived functor of F exists and is given by*

$$RF = q_{\mathcal{A}'} \circ \mathcal{K}(F) \circ T^{-1},$$

i.e.

(1) there exists a natural transformation $q_{\mathcal{A}'} \circ \mathcal{K}(F) \Rightarrow RF \circ q_{\mathcal{A}}$,

(2) RF is a triangulated functor,

(3) Any triangulated functor $G : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er(\mathcal{A})$ together with a natural transformation $q_{\mathcal{A}'} \circ \mathcal{K}(F) \Rightarrow G \circ q_{\mathcal{A}}$ factors through a unique morphism $RF \to G$.

Furthermore, if $I^\bullet \in \mathcal{K}^+(\mathcal{I})$, then $RF(\mathcal{I}^\bullet) \cong \mathcal{K}(F)(\mathcal{I}^\bullet)$.

Proof. Let $\mathcal{A}^\bullet \in \mathcal{D}er^+(\mathcal{A}), \mathcal{I}^\bullet = T^{-1}(\mathcal{A}^\bullet)$. Then there is a natural transformation $id \Rightarrow T \circ T^{-1}$, hence a functorial isomorphism $\mathbb{A}^\bullet \to I^\bullet$ in $\mathcal{D}er^+(\mathcal{A})$. This means there is a quasi-isomorphism $C^\bullet \to A^\bullet$ and a map $C^\bullet \to I^\bullet$ for some $C^\bullet \in \mathcal{K}^+(\mathcal{A})$. If we have a quasi-isomorphism $s : B^\bullet \to \widetilde{B}^\bullet$, we can apply $\operatorname{Hom}(-, \mathcal{I}^\bullet)$ to the associated distinguished triangle. Since $\operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(C^\bullet, I^\bullet) = 0$ for acyclic complexes $C^\bullet$, this yields an isomorphism $\operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(B^\bullet, I^\bullet) \cong \operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(\widetilde{B}^\bullet, I^\bullet)$. This allows us to conclude that there is a map $A^\bullet \to I^\bullet$ in $\mathcal{K}^+(\mathcal{A})$ which is unique and independent of the choice of $C^\bullet$. Hence it yields a functorial map $\mathcal{K}(F)(\mathcal{A}^\bullet) \to K(\mathcal{F})(I^\bullet) = RF(I^\bullet)$. Claim (2) is clear since all functors in the definition are triangulated. We skip (3) and (4), which are easy.[1] □

Similarly, if $\mathcal{A}$ has enough projective, then we get the dual statement, i.e. left derived functors of right exact functors exist.

Definition 7.8. Lef $RF : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er^+(\mathcal{A}')$ be the right derived functor of the left exact functor $F : \mathcal{A} \to \mathcal{A}'$. Then one defines $R^iF(A) = H^iRF(A)$ for $A \in \mathcal{A}$. These induced functors $R^iF : \mathcal{A} \to \mathcal{A}'$ are called higher derived functors.

Remark 7.9. Note that $R^iF(A) = 0$ for $i < 0$ and $R^0F(A) = F(A)$, since by our construction of RF, we can calculate the higher derived functors in the classical way: $R^iF(A) = H^i(F(I^\bullet))$, where $A \to I^\bullet$ is an injective resolution of A. One says A is F-acyclic if $R^iF(A) = 0$ for $i > 0$.

Remark 7.10. Historic viewpoint: Given some left-exact functor $F : \mathcal{A} \to \mathcal{A}'$ between abelian categories and a short exact sequence $0 \to A \to B \to C \to 0$, is there a canonical way to extend $0 \to F(A) \to F(B) \to F(C)$ to a long exact sequence? For nice enough $\mathcal{A}$, this is given by the hyperderived functors $\mathbb{R}^iF : \mathcal{A} \to \mathcal{A}'$. One can show that this construction agrees with our higher derived functors:

Lemma 7.11. *In the setting above, any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ gives rise to a long exact sequence*

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to \cdots$$

[1] This is confusing. Look up Kan-extensions. Also, this doesn't use left-exactness. But in practice we usually only care in this case, see the following remarks.

Proof. The short exact sequence induses a distinguished triangle in $\mathcal{D}er^+(\mathcal{A})$. Apply RF to get a distinguished triangle in $\mathcal{D}er(\mathcal{A}')$. Taking H^0 and using that $H^0(A^\bullet[i]) = H^i(A^\bullet)$, we get the result. □

We give, without proof, two variants of theorem 7.7 with weaker assumptions: Let $F : \mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{A}')$ be triangulated. Then the right derived functor RF exists[2] whenever there exists a full triangulated subcategory $K_F \subseteq \mathcal{K}^+(\mathcal{A})$ such that F maps acyclics in K_F to acyclics, and any object $A^\bullet \in \mathcal{K}^+(\mathcal{A})$ is quasi-isomorphic to a complex in K_F.

A class of objects $I_F \subseteq \mathcal{A}$ is adapted to F if $F(A^\bullet)$ is acyclic for all acyclic $A^\bullet \in \mathcal{K}^+(\mathcal{A})$ with $A^k \in I_F$ for all k and any $A \in \mathcal{A}$ can be embedded into an element of I_F. Then the localization of $K^+(I_F)$ by quasi-isomorphisms is equivalent to $\mathcal{D}er^+(\mathcal{A})$ and one can define RF as before. (E.g. $\mathcal{I}$ is adapted to any left exact functor.)

Lemma 7.13. *Let $F_1 : \mathcal{A} \to \mathcal{A}'$, $F_2 : \mathcal{A}' \to \mathcal{A}''$ be left-exact functors between abelian categories. Assume there exist adapted subcategories I_{F_1}, I_{F_2} as above such that $F_1(I_{F_1}) \subseteq I_{F_2}$. Then all right-derived functors exist and there is an isomorphism $R(F_2 \circ F_1) \cong RF_2 \circ RF_1$.*

Proof. Existence is clear by the statement above. Let $A^\bullet \in \mathcal{D}er^+(\mathcal{A})$. If $A^\bullet$ is isomorphic to some $I^\bullet \in \mathcal{K}^+(I_{F_1})$, then $R(F_2 \circ F_1)(A) \to (RF_2 \circ RF_1)(A)$ is an iso, since the left side is $\mathcal{K}(F_2) \circ \mathcal{K}(F_1)(I^\bullet)$ and the right side is $RF_2(\mathcal{K}(F_1)(\mathcal{I})) = \mathcal{K}(F_2)(\mathcal{K}(F_1)(\mathcal{I}))$. □

Remark 7.14. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and suppose that the hyper-derived functors exist and $\mathbb{R}^j F = 0$ for all $j \geq i$ for some i. Then the total derived functor RF exists on $\mathcal{D}er(\mathcal{A})$ and its restriction to $\mathcal{D}er^+(\mathcal{A})$ is R^+F. (without proof)

Remark 7.15. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er(\mathcal{B})$ exists. Then $R^iF = 0$ for $i < 0$, which implies that R^0F is left-exact. Further, $F \mapsto R^0F$ is an isomorphism if and only if F is left-exact.

The Ext functor

Definition 7.16. Let $\mathcal{A}$ be abelian with enough injectives. Then $\operatorname{Hom}_{\mathcal{A}}(A, -)$ is left-exact for all $A \in \mathcal{A}$. Hence its right-derived functor exists. It is called $\operatorname{Ext}(A, -) = R\operatorname{Hom}_{\mathcal{A}}(A, -) : \mathcal{D}er^+(\mathcal{A}) \to \mathcal{D}er(\mathtt{Ab})$. Its higher derived functors are denoted $\operatorname{Ext}^i(A, B) = H^i \operatorname{Ext}(A, B)$.

Lemma 7.17. *There is a natural isomorphism of functors*

$$\operatorname{Ext}^i(X, Y) \cong \operatorname{Hom}_{\mathcal{D}er(\mathcal{A})}(X, Y[i])$$

Proof. To compute $R\operatorname{Hom}(A, B)$, replace B by an injective resolution $I^\bullet$. Then $\operatorname{Ext}^i(A, B)$ is the i-th cohomology of the complex $\operatorname{Hom}^i(A, I^\bullet)$.

General construction. Given $A^\bullet, B^\bullet \in \mathcal{C}(\mathcal{A})$, define the "inner Hom complex" $\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ as follows: $\operatorname{Hom}^n(A^\bullet, B^\bullet) = \prod_{i\in\mathbb{Z}} \operatorname{Hom}(A^i, B^{i+n})$ with differentials $df = d_B \circ f - (-1)^n f \circ d_A$. Then $\ker(d^i) = \operatorname{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, B^\bullet[i])$ and $\operatorname{im}(d^i)$ is given by the morphisms homotopic to the zero morphism. Hence $H^i \operatorname{Hom}^\bullet(A^\bullet, B^\bullet) = \operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, B^\bullet[i])$.

Back to the specific case, we have $\operatorname{Ext}^i(A, B) = \operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(A, I^\bullet[i])$, and in $\mathcal{D}er(\mathcal{A})$, we know $B \cong I^\bullet$, and

$$\operatorname{Hom}_{\mathcal{D}er(\mathcal{A})}(A, B[i]) \cong \operatorname{Hom}_{\mathcal{D}er(\mathcal{A})}(A, I^\bullet[i]) = \operatorname{Hom}_{\mathcal{K}(\mathcal{A})}(A, I^\bullet[i]),$$

hence we are done. □

[2]Maybe only on a full subcategory?

Natural occurences of the Ext-functor:

(1) Group cohomology: Let G be a group and M a G-module. Let $M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$. Then $M \mapsto M^G$ is left-exact and equal to $\mathrm{Hom}_G(\mathbb{Z}, -)$, where $\mathbb{Z}$ is considered a trivial G-module. Define $H^i(G, M) = \mathrm{Ext}^i(\mathbb{Z}, M)$.

(2) Lie algebra cohomology: Let g be a Lie algebra over a field k, and M a g-module. Define $M^g = \{m \in M \mid xm = 0 \text{ for all } x \in g\}$. Note $g-\mathtt{Mod} = \mathcal{U}(g)-\mathtt{Mod}$ ($\mathcal{U}(g)$ is the universal enveloping algebra). Under this identification, $M^g = \mathrm{Hom}_{\mathcal{U}(g)}(k, -)$ as above, and we define $H^i(g, M) = \mathrm{Ext}^i_{\mathcal{U}(g)}(k, M)$.

(3) Hochschild cohomology: Let A be a k-algebra, and M an (A, A)-bimodule. Then $M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}$. Again, this can be identified with $Hom_{A^e}(A, -)$ with $A^e = A \otimes_k A^{\mathrm{op}}$. Now define $HH^i(A, M) = \mathrm{Ext}^i_{A^e}(A, M)$.

Definition 7.18. Let $\mathcal{A}$ be abelian, $X, Y \in \mathcal{D}er(\mathcal{A})$. Then $\mathrm{Ext}^i(X, Y) := \mathrm{Hom}_{\mathcal{D}er(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{\mathcal{D}er(\mathcal{A})}(X[-i], Y)$. If $A, B \in \mathcal{A}$, then $\mathrm{Ext}^i(A, B) = \mathrm{Ext}^i(A[0], B[0])$.

From the Ext-functor, we get long exact sequences attached to distinguished triangles. For example, if $Y \to Y' \to Y'' \to Y[1]$ is distinguished, we have

$$\cdots \to \mathrm{Ext}^i(X, Y) \to \mathrm{Ext}^i(X, Y') \to \mathrm{Ext}^i(X, Y'') \to \mathrm{Ext}^{i+1}(X, Y) \to \cdots$$

As before, $\mathrm{Ext}^i(X, Y) = \mathrm{Hom}_{\mathcal{C}(A)}(X, I^\bullet[i])$ where $I^\bullet$ is an injective resolution of Y.

Definition 7.19. Let $\mathcal{A}$ be abelian, $A, B \in \mathcal{A}$. A degree i Yoneda extension of B by A is an exact sequence

$$0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to B \to 0$$

in $\mathcal{A}$. Two Yoneda extionsions E, E' are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Z_{i-1} \arrow[r] & \cdots \arrow[r] & Z_0 \arrow[r] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Z''_{i-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & Z''_0 \arrow[r] \arrow[u] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & Z'_{i-1} \arrow[r] & \cdots \arrow[r] & Z'_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where the middle row is again a Yoneda extension. This defines an equivalence relation. Denote with $\widetilde{\mathrm{Ext}}^i(B, A)$ the set of equivalence classes of degree i Yoneda extensions. If E is a degree i Yoneda extension, let

$$s : (\cdots \to 0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots) \to B[0]$$

This is a quasi-isomorphism. Let

$$f : (\cdots \to 0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots) \to A[i].$$

Define $\delta(E) = fs^{-1} : B[0] \to A[i] \in \mathrm{Ext}^i(B, A)$.

Theorem 7.20. *Them map δ induces an isomorphism* $\mathrm{Ext}^i(B, A) \cong \widetilde{\mathrm{Ext}}^i(B, A)$.

Proof. Let $fs^{-1} : B[0] \to A[i] \in \mathrm{Ext}^i(B, A)$ for some quasi-isomorphism $s : L^\bullet \to B^[0]$ and a morphism $f : L^\bullet \to A[i]$. Replacing $L^\bullet$ by $\tau_{\leq 0} L^\bullet$, we may assume $L^j = 0$ for $j > 0$. Set $Z_{i-1} = (A \oplus L^{-i+1})/L^{-i}$ and $Z_j = L^{-j}$ for $j < i-1$. Then $0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to B \to 0$ is a degree

i Yoneda extension. Check that δ maps this extension back to fs^{-1}. It is easy to see that δ descends to $\widetilde{\operatorname{Ext}}^i(B, A)$, so it remains to show that δ is injective. Suppose $\delta(E) = \delta(E')$. From $fs^{-1} = f's'^{-1}$ we know there is a diagram

$$\begin{tikzcd}
 & L^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
B[0] & \tilde{L}^\bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A[i] \\
 & L'^\bullet \arrow[ul, "s'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

Construct the Yoneda extension for $\tilde{L}^\bullet$ as above. Using the maps $\tilde{L}^\bullet \to L^\bullet, L'^\bullet$, this gives maps between the corresponding extensions. $\square$