

# Algebraic Topology

## Serre spectral sequence, characteristic classes and bordism

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# 1 Informal introduction

One of the big goals of homotopy theory is to compute

$$[X, Y]_{\bullet} = \{\text{base-point preserving cont. maps } X \rightarrow Y\} / \text{homotopy}$$

for  $X$  and  $Y$  pointed CW-complexes. CW-complexes are build out of spheres, hence the building blocks are the sets  $[S^n, S^k]_{\bullet} = \pi_n(S^k, *)$ . For  $n \geq 1$ , there are groups, abelian if  $n > 1$ . What do we know about these groups?

- $\pi_n(S^k, *) = 0$  for  $n < k$  by cellular approximation.
- $\pi_n(S^n, *) \cong \mathbb{Z}$  by the Hurewicz theorem and  $H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$
- $X$  is  $(n-1)$ -connected CW-complex: Then  $\pi_n(X, *) \cong H_n(X, \mathbb{Z})$ .
- $\pi_k(S^1, *) = 0$  for  $k \geq 2$  by covering space theory (universal cover of  $S^1$  is  $\mathbb{R}$ , which is contractible).
- $\pi_3(S^2, *) \neq 0$ , since the attaching map of the 4-cell for  $\mathbb{CP}^2$  is a map  $\eta : S^3 \rightarrow S^2 \cong \mathbb{CP}^1$ . If this was null-homotopic, then we would have  $\mathbb{CP}^2 \sim S^2 \vee S^4$ , which contradicts the ring structure on  $H^*(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}[x]/x^3$ .
- $\pi_k(S^n, *) \rightarrow \pi_{k+1}(S^{n+1}, *) \rightarrow \pi_{k+2}(S^{n+2}, *) \rightarrow \dots$  eventually stabilizes by the Freudenthal suspension theorem.

To go beyond this, we need a new tool, the Serre spectral sequence. To motivate its usefulness, consider the following strategy: There exists a map  $f : S^2 \rightarrow K(\mathbb{Z}, 2)$  which induces an isomorphism  $f_* : \pi_2(S^2, *) \rightarrow \pi_2(K(\mathbb{Z}, 2), *)$ . We can take its homotopy fibre  $H = \text{hofb}_x(f)$  (2-connected cover of  $S^2$ ). Then there is a fiber sequence  $H \rightarrow S^2 \xrightarrow{f} K(\mathbb{Z}, 2)$  and a long exact sequence in homotopy

$$\begin{aligned} \dots \rightarrow \pi_4(K(\mathbb{Z}, 2), *) \rightarrow \pi_3(H, *) \rightarrow \pi_3(S^2, *) \rightarrow \pi_3(K(\mathbb{Z}, 2), *) \rightarrow \pi_2(H, *) \rightarrow \pi_2(S^2, *) \rightarrow \\ \rightarrow \pi_2(K(\mathbb{Z}, 2), *) \rightarrow \pi_1(H, *) \rightarrow \pi_1(S^2, *) \rightarrow \dots \end{aligned}$$

from which we conclude  $\pi_3(H, *) \cong \pi_3(S^2, *)$  and  $\pi_1(H, *) = \pi_2(H, *) = 0$ , i.e.  $H$  is 2-connected and the higher homotopy groups agree with the ones of  $S^2$ . By the Hurewicz theorem,  $\pi_3(S^2, *) = H_3(H, \mathbb{Z})$ . Hence we want to find a way to compute  $H_*(H, *)$  from  $H_*(S^2, \mathbb{Z})$  and  $H_*(K(\mathbb{Z}, 2), \mathbb{Z})$ .

This will also help to compute  $\pi_n(S^k, *)$  in other ways (for example we will show that  $\pi_n(S^k, *)$  is finite unless  $n = k$  or  $n = 2k - 1$  and  $k$  even). Furthermore, the Serre spectral sequence will allow us to compute the (co-)homology of spaces like  $U(n)$ ,  $SU(n)$ ,  $\Omega S^n$ ,  $K(\mathbb{Z}/2, n)$  etc. and (re-)prove structural theorems like Hurewicz, Freudenthal suspension, Thom isomorphisms and more.

So, given a fiber sequence  $F \rightarrow Y \rightarrow X$ , what could the relationship between the homology groups of  $F$ ,  $Y$  and  $X$  be?

**Example 1.1.** Consider the easiest case  $F \rightarrow X \times F \xrightarrow{\text{pr}_X} X$ , the trivial filtration. Then the Alexander-Whitney map induces an isomorphism

$$H_n(X \times F, \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(X, H_q(F)).$$

This is the kind of result we want: It computes the homology of the total space in terms of the homology of  $X$  and  $F$ .

**Example 1.2** (Hopf fibration).  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$ :

$n$	$H_n(S^3, \mathbb{Z})$	$\bigoplus_{p+q=n} H_p(S^2, H_q(S^1, \mathbb{Z}))$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}$
2	0	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0

Hence clearly the Künneth formula from the previous example is "too big" to describe the homology in this case. However, consider the "2-step"-filtration  $S^1 \subseteq S^3$  which satisfies  $\tilde{H}_n(S^3/S^1, \mathbb{Z}) \cong \mathbb{Z}$  for  $n = 2, 3$  and 0 otherwise. Hence  $H_\bullet(S^1, \mathbb{Z}) \oplus H_\bullet(S^3/S^1, \mathbb{Z})$  agrees with the right-hand side of the table above. This does not agree with  $H_*(S^3, \mathbb{Z})$ , because the long exact sequence corresponding to  $S^1 \rightarrow S^3 \rightarrow S^3/S^1$  does not split into nice short exact sequences. Concretely, the boundary map  $\tilde{H}_2(S^3/S^1, \mathbb{Z}) \rightarrow H_1(S^1, \mathbb{Z})$  is an isomorphism, hence these two terms do not contribute to  $H_\bullet(S^3, \mathbb{Z})$ .

It turns out that something similar holds for all fibre sequences  $F \rightarrow Y \rightarrow X$ : There exists a filtration

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_m \subseteq \dots \subseteq C_*(Y, \mathbb{Z})$$

on  $C_*(Y, \mathbb{Z})$  such that  $H_{p+q}(F_p/F_{p-1}) \cong C_p^{\text{cell}}(X, H_q(F, \mathbb{Z}))$ . To then understand  $H_\bullet(Y, \mathbb{Z})$ , one needs to understand the cancellations in the associated long exact sequences. This is best encoded in a spectral sequence.

## 2 The Serre Spectral Sequence

**Definition 2.1.** A (homologically, Serre-graded) *spectral sequence* is a triple  $(E^\bullet, d^\bullet, h^\bullet)$ , where

- $(E^r)_{r \geq 2}$  is a sequence of  $\mathbb{Z}$ -bigraded abelian groups. We write  $E_{p,q}^r$  for the  $(p, q)$ -graded part of  $E^r$ .  $E^r$  is called the  $r$ -th *page* of the spectral sequence.
- $(d^r : E^r \rightarrow E^r)$  is a sequence of morphisms, called *differentials*, of bidegree  $(-r, r-1)$  satisfying  $d^r \circ d^r = 0$ .
- $h^r : H_\bullet(E^r) \rightarrow E^{r+1}$  is a sequence of bigrading-preserving isomorphisms. Here  $H_\bullet(E^r)$  denotes the homology with respect to  $d^r$ , which inherits a bigrading from  $E^r$ .

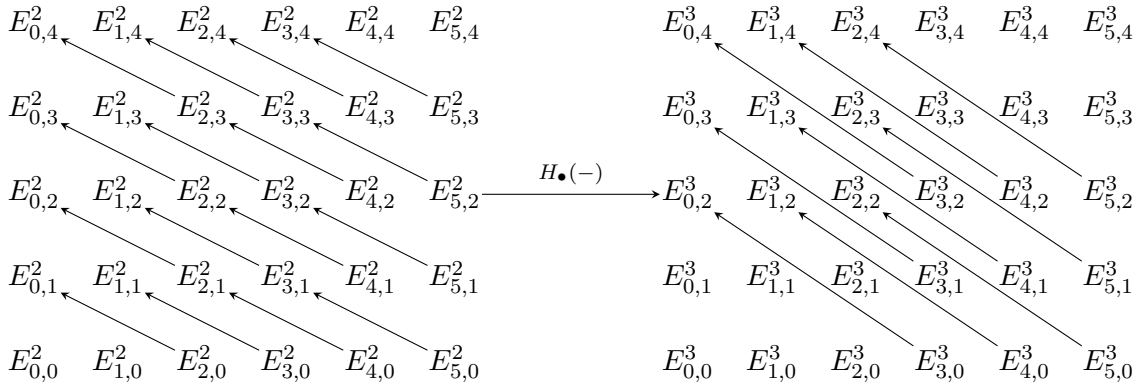


Figure 1: The second and third page of a spectral sequence

**Definition 2.2.** We say that a spectral sequence is *1st quadrant* if all abelian groups  $E_{p,q}^2$  are trivial whenever  $p < 0$  or  $q < 0$ .

**Lemma 2.3.** For a first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$ , we have  $E_{p,q}^r = 0$  if  $p < 0$  or  $q < 0$  for all  $r \geq 2$ . Moreover, for a given  $(p, q) \in \mathbb{Z}^2$ , the map  $h$  induces an isomorphism  $E_{p,q}^r \rightarrow E_{p,q}^{r+1}$  for  $r > r_0 = \max(p, q + 1)$ , i.e. the groups  $E_{p,q}^r$  stabilize as  $r \rightarrow \infty$ .

*Proof.* The first statement follows directly from the existence of the isomorphisms  $h$  by induction on  $r$ . For the second statement, if  $r > r_0$ , then the target of the differential  $d^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$  is trivial, hence every element of  $E_{p,q}^r$  is a cycle. Moreover, the domain of the incoming differential  $d^r : E_{p+r,q-r+1}^r \rightarrow E_{p,q}^r$  is trivial. Hence  $E_{p,q}^r \cong H_\bullet(E_{p,q}^r) \xrightarrow{h} E_{p,q}^{r+1}$   $\square$

**Definition 2.4.** For a first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$ , we define the  $E^\infty$ -page as the bigraded abelian group  $E_{p,q}^\infty = E_{p,q}^{r_0+1}$  with  $r_0 = \max(p, q + 1)$ . By the previous lemma,  $E_{p,q}^\infty \cong E_{p,q}^r$  whenever  $r > r_0$ .

By a filtered object in an abelian category  $\mathcal{A}$  we mean an object  $H \in \mathcal{A}$  with a sequence of inclusions

$$0 = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \dots \subseteq F^n \subseteq \dots \subseteq H.$$

We will apply this to  $\mathcal{A}$  the category of graded abelian groups and  $H = H_*(E, \mathbb{Z})$ .

**Definition 2.5.** A first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$  is said to *converge* to a filtered object in graded abelian groups  $(H, F)$  if there is a chosen isomorphism  $E_{p,q}^\infty \cong F_{p+q}^p / F_{p+q}^{p-1}$  for all  $p, q$  and  $F_n^p = H_n$  if  $n \leq p$ . In this case we write  $E_{p,q}^2 \Rightarrow H$ .

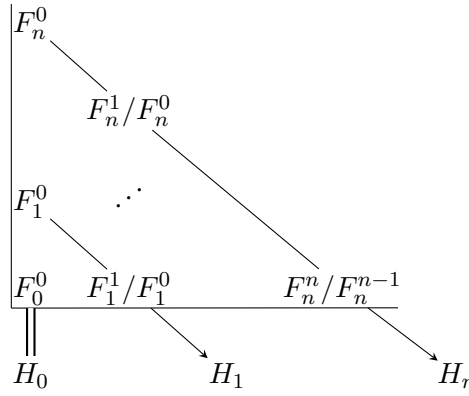


Figure 2: Visualization of  $E^\infty$  as filtrations of the  $H_i$  for a convergent spectral sequence  $E_{p,q}^2 \Rightarrow H$

**Remark.** Convergence is really a *datum* of the necessary isomorphism and not a property. Convergent spectral sequences are often simply encoded as  $E_{p,q}^2 \Rightarrow H$ , but this suppresses not only this data, but also the higher pages, the differentials, and the filtration on  $H$ .

We now want to introduce the Serre spectral sequence for the homology of fibre sequences.

**Definition 2.6.** Let  $f : Y \rightarrow X$  be a continuous map of topological spaces and  $x \in X$  a point. The *homotopy fibre*  $\text{hofb}_x(f)$  of  $f$  at  $x$  is defined to be

$$\text{hofb}_x(f) = P_x X \times_X Y$$

where  $P_x X = \{\gamma : [0, 1] \rightarrow X \mid \gamma(1) = x\}$  is the based path space of  $X$ . It comes with a map  $P_x X \rightarrow X$  given by  $\gamma \mapsto \gamma(0)$ . In words:  $\text{hofb}_x(f)$  is the space of pairs  $(\gamma, y)$  where  $y \in Y$  and  $\gamma$  is a path in  $X$  from  $f(y)$  to  $x$ . We note that  $P_x X$  is contractible by the homotopy

$$H : P_x X \times [0, 1] \rightarrow P_x X, \quad (\gamma, t) \mapsto s \mapsto \gamma((1-t)s + t)$$

**Example 2.7.** If  $f : * \rightarrow X$ , then  $\text{hofb}_x(f) = \Omega_x X$ .

**Definition 2.8.** A *fibre sequence* of topological spaces is a sequence  $F \xrightarrow{i} Y \xrightarrow{f} X$ , a basepoint  $x \in X$ , a homotopy  $h : F \rightarrow X^{[0,1]}$  from the composite  $f \circ i$  to the constant map  $c_x : F \rightarrow X$  such that the induced map  $F \rightarrow \text{hofb}_x(f)$ ,  $z \mapsto (h(z), i(z))$  is a weak homotopy equivalence.

Recall: A weak homotopy equivalence is a map inducing isomorphisms on  $\pi_n(-, x)$  for all  $n \in \mathbb{N}$  and all basepoints  $x$ .

**Example 2.9.** 1. Let  $f : Y \rightarrow X$  be any continuous map,  $x \in X$ . Then the pair  $(\text{hofb}_x f \rightarrow Y \rightarrow X, H)$ , where  $H$  is the homotopy from the definition of the homotopy fibre above, is a fibre sequence. Every fibre sequence is equivalent to this in the following sense: Given  $(F \rightarrow Y \rightarrow X, h)$ , there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\simeq} & \text{hofb}_x(f) \\ \downarrow & & \downarrow \\ Y & \xlongequal{\quad} & Y \\ \downarrow f & & \downarrow f \\ X & \xlongequal{\quad} & X \end{array}$$

In particular,  $\Omega_x X \rightarrow x \rightarrow X$  is a fibre sequence, where  $h : \Omega_x X \times [0, 1] \rightarrow X$  is the evaluation map. If one instead chooses the constant homotopy, one does not obtain a fibre sequence (unless the space is

contractible). This is because the induced map  $\Omega_x X \rightarrow \text{hofb}_x(f) = \Omega_x X$  is constant and hence usually not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces  $F$  and  $X$ ,  $x \in X$ , the pair  $(F \rightarrow F \times X \rightarrow X, \text{const})$  is a fibre sequence, the *trivial fibre sequence*. To see that, note that  $\text{hofb}_x(\text{pr}_X) = F \times P_x X$  with induced map

$$F \rightarrow F \times P_x X, \quad y \mapsto (y, \text{const}),$$

which is a homotopy equivalence as  $P_x X$  is contractible.

3. Let  $p : E \rightarrow B$  be a fibre bundle with fibre  $F = p^{-1}(b)$  for some  $b \in B$ . Then the sequence  $F \rightarrow E \rightarrow B$  with the constant homotopy is a fibre sequence. This is a special case of the next example.

4. Recall that  $p : E \rightarrow B$  is a Serre fibration if in every commutative diagram of the form

$$\begin{array}{ccc} D^n \times 0 & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ D^n \times I & \longrightarrow & B \end{array}$$

there exists a lift  $D^n \times I \rightarrow E$  making both triangles commute. Given a Serre fibration  $p : E \rightarrow B$  and  $b \in B$ , the sequence  $F = p^{-1}(b) \rightarrow E \rightarrow B$  with the constant homotopy is a fibre sequence. (see exercises) Note: Every fibre sequence is also equivalent to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$ . It arises by letting  $S^1 = U(1)$  act on  $S^2 \subseteq \mathbb{C}^2$  via  $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$ , with quotient space  $\mathbb{CP}^1 \cong S^2$ .

6. Example 5 generalizes to fibre bundles  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  with limit case  $S^1 \rightarrow S^\infty \rightarrow \mathbb{CP}^\infty$ , which is equivalent to  $\Omega \mathbb{CP}^\infty \rightarrow * \rightarrow \mathbb{CP}^\infty$ .

We are now ready to state the existence of the Serre spectral sequence.

**Theorem 2.10** (Serre). *For every fibre sequence  $(F \xrightarrow{\iota} Y \xrightarrow{p} X, h)$  with  $X$  simply-connected and abelian group  $A$ , there exists a spectral sequence of the following form*

$$E_{p,q}^2 = H_p(X, H_q(F, A)) \implies H_{p+q}(Y, A)$$

As noted before, this information does not include the differentials and the higher pages, as well as the filtrations on  $H_\bullet(Y, A)$  and the identifications of its subquotients with the  $E^\infty$ -page.

One edge case is easy to state: The map

$$H_n(F, A) = H_0(X, H_n(F, A)) = E_{0,n}^2 \twoheadrightarrow E_{0,n}^\infty \hookrightarrow H_n(Y, A)$$

agrees with the factorization  $H_n(F, A) \twoheadrightarrow \text{im } \iota_* \hookrightarrow H_n(Y, A)$ .

We now assume this theorem and give some sample computations.

**Example 2.11.** We revisit the Hopf fibration  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$ .  $S^2$  is simply connected, so we get a spectral sequence. The  $E^2$ -page is  $H_p(S^2, H_q(S^1, A))$ , which looks like

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ A & 0 & A & 0 \\ & \swarrow & & \\ A & 0 & A & 0 \end{array} \\ p \rightarrow \end{array}$$

There is one potentially non-trivial  $d^2$ -differential, namely  $d^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$ . All higher differentials  $d^r$ ,  $r > 2$ , are trivial for degree reasons. Hence the  $E^\infty$ -page looks as follows:

$$\begin{array}{c}
 \begin{array}{cccc}
 q \uparrow & 0 & 0 & 0 & 0 \\
 \text{coker}(d^2) & 0 & A & 0 & \\
 A & 0 & \ker(d^2) & 0 & \\
 & \xrightarrow{p} & & & 
 \end{array}
 \end{array}$$

We know that  $H_n(S^3, A) = A$  for  $n = 0, 3$  and 0 else. From the  $E^\infty$ -page we thus get  $H_0(S^3, A) = A$ ,  $H_1(S^3, A) = \text{coker}(d^2)$ ,  $H_2(S^3, A) = \ker(d^2)$ ,  $H_3(S^3, A) = A$ . Hence  $d^2$  must be an isomorphism.

**Lemma 2.12.** *There is a fibre bundle*

$$U(n-1) \xrightarrow{i} U(n) \rightarrow S^{2n-1},$$

where  $U(n)$  denotes the topological group of unitary  $n \times n$ -matrices and  $i$  is the standard inclusion which adds a trivial  $\mathbb{C}$ -summand.

*Proof.* The group  $U(n)$  acts on  $\mathbb{C}^n$  by definition. This action restricts to the unit sphere  $S^{2n-1} \subseteq \mathbb{C}^n$ . Furthermore, this action is transitive, because every vector of length 1 can be extended to an orthonormal basis. Hence  $S^{2n-1}$  is in bijection with the orbit space  $U(n)/\text{Stab}(x)$ , for any  $x \in S^{2n-1}$ . For  $x = (0, \dots, 0, 1)$ , the stabilizer equals  $i(U(n-1))$ . We obtain a continuous bijective map  $U(n)/U(n-1) \rightarrow S^{2n-1}$ ,  $[A] \mapsto A(0, \dots, 0, 1)^t$ , which is a homeomorphism since its domain is quasi-compact and its codomain is Hausdorff. Finally, we use the fact that for a smooth, free action of a compact Lie group  $G$  on a manifold  $M$ , the map  $M \rightarrow M/G$  is always a fibre bundle (in fact a  $G$ -principal bundle).  $\square$

**Example 2.13.** We consider the case  $n = 2$ , i.e. the fibre sequence  $S^1 \cong U(1) \hookrightarrow U(2) \rightarrow S^3$ . We want to compute the homology of  $U(2)$  via the Serre spectral sequence  $E_{p,q}^2 = H_p(S^3, H_q(S^1, \mathbb{Z}))$ . All differentials on all pages have to be trivial for degree reasons. (The spectral sequence "collapses".) Hence  $E^\infty = E^2$  and every antidiagonal has at most one non-trivial term, so we can read off  $H_n(U(2), \mathbb{Z}) = \mathbb{Z}$  for  $n = 0, 1, 3, 4$  and 0 else. In fact, one can show that  $U(2) \cong S^3 \times U(1)$ , so the homology could alternatively be computed with the Künneth theorem.

**Example 2.14.** Next we consider the fibre sequence  $U(2) \hookrightarrow U(3) \rightarrow S^5$  with  $E^2$ -page  $E_{p,q}^2 = H_p(S^5, H_q(U(2), \mathbb{Z}))$ , which looks like

$$\begin{array}{c}
 \begin{array}{cccccc}
 q \uparrow & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \swarrow & & & & & \\
 \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \swarrow & & & & & \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \swarrow & & & & & \\
 \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \swarrow & & & & & \\
 \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \xrightarrow{p} & & & & & 
 \end{array}
 \end{array}$$

The first potentially non-trivial differential is  $d^5 : E_{0,5}^2 \rightarrow E_{0,4}^2$ . At this point we cannot decide what this differential is. All higher differentials are again trivial for degree reasons, and all filtrations collapse to

$$H_n(U(3), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, 3, 5, 8, 9, \\ \text{coker}(d^5) & \text{for } n = 4, \\ \text{ker}(d^5) & \text{for } n = 5, \\ 0 & \text{else.} \end{cases}$$

**Example 2.15.** We consider  $U(3) \rightarrow U(4) \rightarrow S^7$ . The  $E_{p,q}^2 = H_p(S^7, H_q(U(3), \mathbb{Z}))$ -page is

$$\begin{array}{ccccccc}
& & & & & & q \\
& & & & & & \uparrow \\
& & & & & & \mathbb{Z} \\
& & & & & & \mathbb{Z} \\
& & & & & & 0 \\
& & & & & & \mathbb{Z} \\
& & & & & & \vdots \\
& & & & & & ? \\
& & & & & & ? \quad \dots \quad 0 \quad \dots \quad ? \\
& & & & & & \mathbb{Z} \\
& & & & & & \vdots \\
& & & & & & \mathbb{Z} \\
& & & & & & 0 \\
& & & & & & \mathbb{Z} \\
& & & & & & \mathbb{Z} \\
& & & & & & \downarrow \\
& & & & & & p \\
& & & & & & 0 \qquad \qquad \qquad 7
\end{array}$$

In the previous examples we used the Serre spectral sequence to compute the homology of the total space of the fibre sequence. We now show that it can also be used to compute the homology of the base space or fibre.

$$\begin{array}{cc} q & \\ \uparrow & \\ 0 & 0 \\ \mathbb{Z} & ? \\ \mathbb{Z} & ? \\ \downarrow & \\ p & \end{array}$$



Since  $H_1(S^{2n+1}, \mathbb{Z}) = 0$ , there must be a surjective  $d^2$ -differential  $d^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$ . But since  $H_2(S^{2n+1}, \mathbb{Z}) = 0$ , this differential must also be injective. Hence

$$\mathbb{Z} \cong E_{2,0}^2 = H_2(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong H_2(\mathbb{CP}^n, \mathbb{Z}).$$

Furthermore, we see that  $E_{1,0}^2 = H_1(\mathbb{CP}^n, \mathbb{Z}) = 0$ . Using  $H_0(S^1, \mathbb{Z}) = H_1(S^1, \mathbb{Z})$ , this implies  $E_{1,1}^2 = 0$  and  $E_{2,1} = \mathbb{Z}$ . Now we see that the 2-page looks like

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} & ? \\ \mathbb{Z} & 0 & \mathbb{Z} & ? \end{array} \\ p \rightarrow \end{array}$$

By the same argument, we can deduce  $d^2 : E_{4,0} \rightarrow E_{2,1}$  is an isomorphism, i.e.  $H_4(\mathbb{CP}^n, H_0(S^1, \mathbb{Z})) \cong \mathbb{Z}$ , and  $E_{3,0} = E_{3,1} = 0$  and so on. Since  $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ , we cannot conclude that the  $\mathbb{Z}$  in bidegree  $(2n, 1)$  must be the image of a differential. There are two possibilities: If  $d^2 : E_{2n+2,0}^2 \rightarrow E_{2n+1}^2$  is the trivial map, then  $E_{2n+2,0}^2 = 0$  and then by induction  $E_{p,q}^2 = 0$  for all  $p > 2n$ . If, on the other hand,  $d^2 : E_{2n+2,0}^2 \rightarrow E_{2n+1}^2$  is non-zero, it has to be surjective: Indeed, since the cokernel is isomorphic to the lowest term of the filtration on  $H_{2n+1}(S^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}$ , and no  $\mathbb{Z}/n\mathbb{Z}$  embeds into  $\mathbb{Z}$ . This then implies  $H_k(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$  for all  $k > 2n$ . This case can be ruled out using that  $\mathbb{CP}^n$  is a  $2n$ -dimensional CW-complex and hence  $H_n(\mathbb{CP}^n, \mathbb{Z}) = 0$  for  $k > 2n$ . In summary, we obtain

$$H_k(\mathbb{CP}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, 2, \dots, 2n, \\ 0 & \text{else.} \end{cases}$$

Next we turn to an example where the Serre spectral sequence can be used to compute the homology of the fibre

**Example 2.17.** We consider the fibre sequence  $\Omega S^3 \rightarrow * \rightarrow S^3$ . On the  $E^2$ -page, we have the entries  $H^p(S^3, H_q(\Omega S^3, \mathbb{Z}))$ , i.e.

$$\begin{array}{c} q \uparrow \\ \begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ ? & 0 & 0 & ? \\ \mathbb{Z} & 0 & 0 & \mathbb{Z} \end{array} \\ p \rightarrow \end{array}$$

The homology of the point is 0 in positive degrees, so we must have  $E_{p,q}^\infty = 0$  unless  $p = q = 0$ . The only non-trivial differentials are  $d^3 : E_{3,q}^3 \rightarrow E_{0,q+2}^3$ , so we conclude that these are isomorphisms. Hence  $H_q(\Omega S^3, \mathbb{Z}) \cong H_{q+2}(S^3, \mathbb{Z})$ . Note  $E_{0,1}^{0,1} = 0$  since this entry cannot be killed by any differential. This implies

$$H_k(\Omega S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

In particular,  $\Omega S^3$  is an infinite-dimensional space.

We now discuss the cohomological version of the Serre spectral sequence and its multiplicative structure. This multiplication also helps in determining differentials, for example for the spectral sequences computing (co-)homology of unitary groups as above.

**Definition 2.18.** A cohomologically graded spectral sequence is a triple  $(E_\bullet, d_\bullet, h_\bullet)$  where  $(E_r)_r$  is a sequence of bigraded abelian groups,  $(d_r : E_r \rightarrow E_r)_r$  is a sequence of differentials  $(d_r \circ d_r = 0)$  of bidegree  $(r, 1 - r)$ , and  $(h_r : H_\bullet(E_r) \rightarrow E_{r+1})_r$  a sequence of bigrading-preserving isomorphisms.

As before, one defines first quadrant ( $E_2^{p,q} = 0$  if  $p < 0$  or  $q < 0$ ) spectral sequences and the  $E_\infty$ -page.

Rather than the filtrations  $0 = F^{-1} \subseteq F^0 \subseteq \cdots \subseteq H$ , one now considers filtrations

$$H = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

**Definition 2.19.** A cohomological first quadrant spectral sequence is said to *converge* to a filtered object  $(H, F)$  in graded abelian groups if there are isomorphisms  $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$  for all  $p, q$ , and  $F_p^n = 0$  for all  $p > n$ . Again we write  $E_2^{p,q} \implies H$ .

**Definition 2.20.** A (commutative) multiplicative structure on a cohomologically graded spectral sequence  $(E_\bullet, d_\bullet, h_\bullet)$  is a bigraded (commutative) ring structure on  $E_r$ , i.e. there are associative maps  $E_r^{p,q} \otimes E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$ , such that  $d_r$  is a graded derivation, i.e.

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{p+q} x \cdot d_r(y)$$

for  $x \in E_r^{p,q}$ . Here (graded) commutative means  $xy = (-1)^{(p+q)(p'+q')}yx$ . As a result,  $H_\bullet(E_r)$  is a bigraded ring and we further require that the  $h_r$  are isomorphisms of bigraded rings. Furthermore, the  $E_\infty$ -page also inherits the structure of a (commutative) bigraded ring.

**Definition 2.21.** A filtration  $\cdots \subseteq F_n \subseteq \cdots \subseteq F_1 \subseteq F_0 = H$  on a graded ring  $H$  is said to be *multiplicative* (or compatible with the multiplicative structure) if  $F_s F_t \subseteq F_{s+t}$ . We say that  $(H, F)$  is a *filtered graded ring*.

It follows that the associated graded object  $\bigoplus F_p/F_{p+1}$  of a filtered graded (commutative) ring is a bigraded (commutative) ring.

**Definition 2.22.** A multiplicative first quadrant spectral sequence  $(E_\bullet, d_\bullet, h_\bullet)$  is said to *converge* to a filtered graded ring  $(H, F)$  if it converges additively and the chosen isomorphism  $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$  is compatible with the graded ring structure.

**Theorem 2.23** (Serre). *For every fibre sequence of spaces  $(F \rightarrow Y \rightarrow X, h)$  with  $X$  simply connected and every abelian group  $A$ , there exists a cohomological first quadrant spectral sequence of the form*

$$E_2^{p,q} = H^p(X, H^q(F, A)) \implies H^{p+q}(Y, A).$$

*If  $A$  is a (commutative) ring, then the spectral sequence is multiplicative and converges multiplicatively, where on the  $E_2$ -page the multiplication is given by  $(-1)^{p'q}$ -times the composite*

$$\begin{aligned} H^p(X, H^q(F, R)) \otimes H^{p'}(X, H^{q'}(F, R)) &\rightarrow H^{p+p'}(X, H^q(F, R) \otimes H^{q'}(F, R)) \\ &\rightarrow H^{p+p'}(X, H^{q+q'}(F, R)). \end{aligned}$$

Note: If  $H^\bullet(F, R)$  or  $H^\bullet(X, R)$  is flat over  $R$  of finite type, then the  $E_2$ -page is isomorphic to the graded tensor product of  $H^\bullet(X, R)$  and  $H^\bullet(F, R)$ .

**Example 2.24.** We reconsider the fibre sequence  $U(1) \rightarrow U(2) \rightarrow S^3$  with  $E_2^{p,q} = H^p(S^3, H^q(U(1), \mathbb{Z}))$ , i.e.

$$\begin{array}{c} \begin{array}{cccc} \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & 0 & \mathbb{Z} \end{array} \\ \begin{array}{c} \uparrow q \\ \xrightarrow{p} \end{array} \end{array}$$

There cannot be non-trivial differentials. As a graded ring, the  $E_2$ -page (and hence also the  $E_\infty$ -page) is isomorphic to  $H^\bullet(S^3, \mathbb{Z}) \otimes H^\bullet(U(1), \mathbb{Z})$ . Let  $x_1 \in H^1(U(1), \mathbb{Z})$  and  $x_3 \in H^3(S^3, \mathbb{Z})$  be generators.

Then  $H^\bullet(U(1), \mathbb{Z}) \cong \bigwedge(x_1)$  and  $H^\bullet(S^3, \mathbb{Z}) \cong \bigwedge(x_3)$ , where  $\bigwedge(M)$  denotes the exterior algebra on a set  $M$ , i.e. the free algebra on  $x \in M$  modulo the relations  $x_i x_j = -x_j x_i$  and  $x_i^2 = 0$  for all  $x_i, x_j \in M$ . Hence, the  $E_2$ -page is isomorphic to  $\bigwedge(x_1, x_3)$ . The  $\mathbb{Z}$  in bidegree  $(3, 1)$  is spanned by  $x_1 x_3$ . The filtration collapses degreewise and hence  $H^*(U(2), \mathbb{Z})$  is exterior on classes  $x_1 \in H^1(U(2), \mathbb{Z})$  and  $x_3 \in H^3(U(2), \mathbb{Z})$  that are uniquely determined by the spectral sequence.

**Example 2.25.** We move on to the fibre sequence  $U(2) \rightarrow U(3) \rightarrow S^5$ . The  $E_2$ -page looks like the homological spectral sequence

$$\begin{array}{cccccc}
 q \uparrow & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
 & & & & & & p \rightarrow
 \end{array}$$

Keeping the notation from the previous example, denote generators of the first column by  $1, x_1, x_3, x_1 x_3$ . Again, the  $E_2$ -page is given by

$$H^\bullet(S^5, \mathbb{Z}) \otimes H^\bullet(U(2), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5),$$

where  $x_5 \in H^5(S^5, \mathbb{Z})$  is a generator. There is one possibly non-trivial differential  $d_5 : E_5^{0,4} \rightarrow E_5^{5,0}$ . However, the product rule implies

$$d_5(x_1 x_3) = d_5(x_1) x_3 + (-1)^{1+0} x_1 d_5(x_3) = 0 + 0 = 0.$$

Again, the filtration collapses and hence  $H^*(U(3), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5)$ .

**Example 2.26.** We revisit  $U(3) \rightarrow U(4) \rightarrow S^7$ . The  $E_2$ -page now is

$$\begin{array}{ccccccc}
 q \uparrow & \mathbb{Z} & & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & 0 & & & & & 0 \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & \mathbb{Z} & & \vdots & & & \mathbb{Z} \\
 & \mathbb{Z} & \cdots & 0 & \cdots & & \mathbb{Z} \\
 & \mathbb{Z} & & \vdots & & & \mathbb{Z} \\
 & 0 & & & & & 0 \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & \mathbb{Z} & & & & & \mathbb{Z} \\
 & & & & & & p \rightarrow \\
 & 0 & & & & & 7
 \end{array}$$

As before, the product rule implies that all  $d_7$  differentials must be trivial, and  $E_2 \cong \bigwedge(x_1, x_3, x_5, x_7)$ . There is a non-trivial filtration on  $H^8(U(4), \mathbb{Z})$  of the form

$$0 \rightarrow \mathbb{Z}(x_1 x_7) \rightarrow H^8(U(4), \mathbb{Z}) \rightarrow \mathbb{Z}(x_3 x_5) \rightarrow 0$$

Additively the sequence splits, but one has to be careful with the multiplicative structure. To resolve this, we need to be precise with the differentiation between the classes  $x_i$  on the  $E_\infty$ -page and the corresponding classes  $\bar{x}_i \in H^*(U(4), \mathbb{Z})$ . Note that the choice of each  $\bar{x}_i$  is unique since the filtration collapses in degrees 0 to 7. Furthermore, we record their filtrations  $\bar{x}_1$  is in  $F_0^1$ ,  $\bar{x}_3$  is in  $F_0^3$ ,  $\bar{x}_5$  is in  $F_0^5$  and  $\bar{x}_7$  is in  $F_7^7$ . It follows that  $\bar{x}_1 \bar{x}_7$  is a generator of  $F_7^8$ , and  $\bar{x}_3 \bar{x}_5$  is a generator of  $F_0^8 / F_1^8$ . Hence,  $H^8(U(4), \mathbb{Z})$  is a free group on  $\bar{x}_1 \bar{x}_7$  and  $\bar{x}_3 \bar{x}_5$ , and it follows that  $H^\bullet(U(4), \mathbb{Z}) \cong \bigwedge(x_1, x_3, x_5, x_7)$ .

**Theorem 2.27.** *For all  $n \in \mathbb{N}$ , there is an isomorphism of graded rings*

$$H^\bullet(U(n), \mathbb{Z}) \cong \bigwedge(x_1, x_3, \dots, x_{2n-1})$$

with  $x_i$  of degree  $i$ .

*Proof.* By induction on  $n$ , the start  $n = 1$  is clear. Let  $n \geq 2$  and assume we know the statement for  $n - 1$ . We consider the Serre spectral sequence for the fibre sequence  $U(n - 1) \rightarrow U(n) \rightarrow S^{2n-1}$ . By induction, its  $E_2$ -page is isomorphic to

$$E_2 \cong H^\bullet(S^{2n-1}, \mathbb{Z}) \otimes H^\bullet(U(n - 1), \mathbb{Z}) = \bigwedge(x_{2n-1}) \otimes \bigwedge(x_1, x_3, \dots, x_{2n-3}).$$

Here,  $x_i$  is a generator of  $E_2^{0,i}$  for  $i \leq 2n - 3$  and  $x_{2n-1}$  is a generator of  $E_2^{2n-1,0}$ . The only possibly non-trivial differentials are  $d_{2n-1}$ . For degree reasons,  $d_{2n-1}$  vanishes on all generators  $x_1, x_3, \dots, x_{2n-1}$ . By the product rule, all differentials are 0. Hence the  $E_\infty$ -page is isomorphic to the  $E_2$ -page, and an exterior algebra  $\bigwedge(x_1, x_3, \dots, x_{2n-1})$ .

The filtrations on  $H^\bullet(U(n), \mathbb{Z})$  collapse in degrees  $0, \dots, 2n - 2$ , therefore we obtain unique lifts  $\bar{x}_1, \dots, \bar{x}_{2n-1} \in H^\bullet(U(n), \mathbb{Z})$ . We only know from the spectral sequence that  $x_i^2$  is of lower filtration hence a multiple of  $\bar{x}_{2n-1}$ , but not necessarily that  $\bar{x}_i^2 = 0$ . However we know from the additive structure (all subquotients are free over  $\mathbb{Z}$ ) that  $H^\bullet(U(n), \mathbb{Z})$  is torsionfree. As the multiplication is graded commutative, we hence have  $\bar{x}_i^2 = 0$ . We obtain a ring map  $f : \bigwedge(x_1, \dots, x_{2n-1}) \rightarrow H^\bullet(U(n), \mathbb{Z})$  by sending  $x_i$  to  $\bar{x}_i$ . To check that  $f$  is an isomorphism, define a grading on  $\bigwedge(x_1, \dots, x_{2n-1})$  by setting the degree of  $x_1, \dots, x_{2n-3}$  to be 0, and the degree of  $x_{2n-1}$  to be  $2n - 1$ . This induces a filtration by setting  $F_i$  to be the direct sum of the graded pieces of degree  $2i$ . Then  $f$  is filtration preserving and induces an isomorphism on associated graded pieces. The proof is finished by the following lemma.  $\square$

**Lemma 2.28.** *Let  $A$  and  $B$  be graded abelian groups equipped with filtrations*

$$\dots F_2 \subseteq F_1 \subseteq F_0 = A \quad \text{and} \quad \dots G_2 \subseteq G_1 \subseteq G_0 = B$$

*which are eventually 0 in every degree. If we have a graded filtration-preserving morphism  $f : A \rightarrow B$  that induces an isomorphism on all associated graded pieces  $F_i/F_{i+1} \cong G_i/G_{i+1}$ , then it is an isomorphism.*

*Proof.* This is an iterated 5-lemma argument.  $\square$

**Example 2.29.** We revisit the fibre sequence  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  and use the Serre spectral sequence to compute the ring  $H^\bullet(\mathbb{CP}^n, \mathbb{Z})$ . Arguing analogously to the homological case, the  $E_2$ -page looks as

$$\begin{array}{ccccccc}
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\
\searrow \scriptstyle \mathbb{R} & & \searrow \scriptstyle \mathbb{R} & & \searrow \scriptstyle \mathbb{R} & & \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots & 0 & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & & & 2n
\end{array}$$

**Example 2.30.** Next we compute the ring structure on  $H^\bullet(\Omega S^2, \mathbb{Z})$ . Dual to the homological case, we obtain the following  $E_2$ -page corresponding to the fibre sequence  $\Omega S^3 \rightarrow * \rightarrow S^3$ :

$$\begin{array}{ccccccc}
& & & & & & q \\
& & & & & & \vdots \\
& & & & & & \mathbb{Z} \\
& & & 0 & 0 & & \mathbb{Z} \\
& & & \searrow & & & \\
& & & 0 & & & 0 \\
& & & \mathbb{Z} & & & \mathbb{Z} \\
& & & \searrow & & & \\
& & & 0 & & & 0 \\
& & & \mathbb{Z} & & & \mathbb{Z} \\
& & & \searrow & & & \\
& & & 0 & & & 0 \\
& & & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
& & & & & & \vdots \\
& & & & & & \mathbb{Z}
\end{array}$$

**Remark:** There is also a ring structure on the homology of  $H_*(\Omega S^3, \mathbb{Z})$  induced by the  $H$ -space (in fact  $A_\infty$ - or  $E_1$ -space) structure by concatenation of loops. One can show that with this ring structure,  $H_*(\Omega S^3, \mathbb{Z})$  is a polynomial ring on a generator in degree 2. More generally, if  $H_*(X, \mathbb{Z})$  is free over  $\mathbb{Z}$ , then  $H_*(\Omega \Sigma X, \mathbb{Z})$  is the tensor algebra on  $H_*(X, \mathbb{Z})$  (Bott-Samelson theorem) and the map  $H_*(X, \mathbb{Z}) \rightarrow H_*(\Omega \Sigma X, \mathbb{Z}) = T(H_*(X, \mathbb{Z}))$  is induced by the natural map  $X \rightarrow \Omega \Sigma X$ .

**Example 2.31.** We consider the map  $S^3 \rightarrow K(\mathbb{Z}, 3)$  classifying a generator of  $H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$ . (equivalently, inducing an isomorphism on  $\pi_3(-, *)$ ). Let  $X$  denote the homotopy fibre, so that  $X \rightarrow S^3 \rightarrow$

[illegible]
$$\begin{array}{cccc}
q & \vdots & & \vdots \\
0 & 0 & 0 & \mathbb{Z}/4\mathbb{Z} \\
0 & & & 0 \\
0 & & & \mathbb{Z}/3\mathbb{Z} \\
0 & & & 0 \\
0 & & & \mathbb{Z}/2\mathbb{Z} \\
0 & & & 0 \\
\mathbb{Z} & 0 & 0 & 0 \\
& p & & 
\end{array}$$
$$\tilde{H}^n(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k\mathbb{Z} & \text{if } n = 2k + 1, \\ 0 & \text{else} \end{cases}$$

**Corollary 2.32.** *We have  $\pi_4(S^3, *) \cong \mathbb{Z}/2\mathbb{Z}$  and  $\pi_4(S^2, *) \cong \mathbb{Z}/2\mathbb{Z}$ .*

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We now know the following homotopy groups of spheres:

	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$
$S^1$	$\mathbb{Z}$	0	0	0	0	0
$S^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
$S^3$	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		
$S^4$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
$S^5$	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$S^6$	0	0	0	0	0	$\mathbb{Z}$

where  $\pi_5(S^4, *) = \pi_6(S^5, *) = \mathbb{Z}/2\mathbb{Z}$  follows from the Freudenthal suspension theorem.

### 3 Construction of the Serre spectral sequence

We focus on the cohomological version. Roughly speaking we proceed via

double complexes  $\Rightarrow$  filtered complexes  $\Rightarrow$  exact couple  $\Rightarrow$  spectral sequence.

**Definition 3.1.** An exact couple is a pair of abelian groups  $(A, E)$  together with a triangle

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \nwarrow k & \nearrow j \\ & E & \end{array}$$

which is exact, i.e. exact at each of the three corners.

Define  $d_1 : E \rightarrow E$  via  $d_1 = j \circ k$ . We have  $d_1^2 = j \circ k \circ j \circ k = j \circ 0 \circ k = 0$ , so  $d_1$  is a differential. We can define  $H(E) = \ker(d_1) / \text{im}(d_1)$ , and claim that there is a new triangle

$$\begin{array}{ccc} A_2 & \xrightarrow{i_2} & A_2 \\ & \nwarrow k_2 & \nearrow j_2 \\ & E_2 & \end{array} \quad (*)$$

defined via  $E_2 = H(E)$ ,  $A_2 = \text{im}(i) \subseteq A$ ,  $i_2 = i|_{A_2}$ . For  $a \in A_2$  write  $a = i(b)$  for some  $b \in A$ . Then  $j_2(a) := [j(b)]$ . This is well-defined:  $d_1(j(b)) = j(k(j(b))) = 0$ , and if  $i(b) = i(b')$ , then  $b - b' \in \ker i = \text{im } k$ , so  $j(b - b') \in \text{im}(j \circ k) = \text{im } d_1$ . Finally,  $k_2([e]) := k(e)$ : Since  $j(k(e)) = 0$ , exactness implies  $k(e) \in \text{im } i$ , and if  $e \in \text{im } d_1$ , then  $e \in \text{im } j$  and  $k(e) = 0$ .

**Lemma 3.2.** The triangle  $(*)$  is again an exact couple.

*Proof.* Diagram chase (omitted). □

Hence we can iterate and obtain a sequence of exact couples  $(A_n, E_n)$  with maps  $i_n, j_n, k_n$ . In particular we obtain a sequence of abelian groups  $E_n$  with differentials  $d_n = j_n \circ k_n$  and isomorphisms  $H(E_n) \cong E_{n+1}$ . This is like a spectral sequence, except we are missing the bigrading.

For the Serre spectral sequence the two gradings play different roles: A filtration degree ( $x$ -axis), and the difference between the cohomological degree and the filtration degree.

**Definition 3.3.** An unrolled exact couple is a collection of pairs  $(A^s, t^s)_{s \in \mathbb{Z}}$  of abelian groups with maps

$$\begin{array}{ccccccc} \cdots & \xrightarrow{i} & A^{s+1} & \xrightarrow{i} & A^s & \xrightarrow{i} & A^{s-1} \rightarrow \cdots \\ & & \swarrow j & \nwarrow k & \swarrow j & \nwarrow k & \swarrow j \\ & & \cdots & & E^s & & E^{s-1} \end{array}$$

Every unrolled exact couple gives an exact couple via  $A = \bigoplus_s A^s$ ,  $E = \bigoplus_s E^s$  combined in a single triangle. We obtain a cochain complex

$$\cdots \xrightarrow{j \circ k} E^{s-1} \xrightarrow{j \circ k} E^s \xrightarrow{j \circ k} E^{s+1} \rightarrow \cdots$$

Hence  $H_*(E)$  inherits a grading, i.e.  $H(E) = \bigoplus_j H^j(E)$ . Generally we can write  $E_r = \bigoplus_s E_r^s \cong \bigoplus_s H^s(E_{r-1})$ . Assume  $e \in E^s$ , we can chase through its "life" in the spectral sequence: If  $d_1(e) \neq 0$ , then  $e$  does not define a class in  $H^\bullet(E)$ . If  $d_1(e) = 0$ , then  $[e] \in H^s(E) = E_2^s$ . In the unrolled picture,  $d_2[e] = j_2 k_2(e)$  is computed as follows: By exactness,  $k(e) \in \text{im}(i)$ , say  $k(e) = i(b)$ , and  $d_2([e]) = [j(b)]$ . If  $d_2([e]) \neq 0$ ,  $e$  does not define a class in  $H(E_2)$ . If  $d_2(e) = 0$ , we can continue in this way. In general, if  $k(e) = i^r(b)$  for some  $r \geq 0$  and  $b \in A^{s+r+1}$ , then  $e$  defines a class in  $E_{r+1}^s = H^s(E_r)$  and  $d_{r+1}([e])$  is  $j(b)$ . If  $e$  survives in every step, it is called a permanent cycle. We note: If  $e \in E_r^s$ , then  $d_r([e])$  is represented by some element of  $E_r^{s+r}$ , i.e.  $d_r$  raises the filtration degree by  $r$ .

A filtered cochain complex is a cochain complex  $C^\bullet$  together with a sequence of subcomplexes

$$\cdots \subseteq F^2 C^\bullet \subseteq F^1 C^\bullet \subseteq F^0 C^\bullet = C^\bullet$$

For convenience, we extend the filtration grading to  $\mathbb{Z}$  via  $F^s C^\bullet = C^\bullet$  for  $s < 0$ . The associated graded complex is the collection of subquotients

$$\text{gr}^s C^\bullet = F^s C^\bullet / F^{s+1} C^\bullet.$$

The corresponding short exact sequences induce a long exact sequence

$$\cdots \rightarrow H^t(F^{s+1} C^\bullet) \rightarrow H^t(F^s C^\bullet) \rightarrow H^t(\text{gr}^s(C^\bullet)) \rightarrow H^{t+1}(F^{s+1} C^\bullet) \rightarrow \cdots$$

Taking the direct sum over all  $t$ , we obtain

$$\begin{array}{ccccccc} \cdots & \xrightarrow{i} & H^\bullet(F^{s+1} C^\bullet) & \xrightarrow{i} & H^\bullet(F^s C^\bullet) & \xrightarrow{i} & H^\bullet(F^{s-1} C^\bullet) \rightarrow \cdots \\ & & \swarrow j & \nwarrow k & \swarrow j & \nwarrow k & \swarrow j \\ & & \cdots & & H^\bullet(\text{gr}^s(C^\bullet)) & & H^\bullet(\text{gr}^s(C^\bullet)) \end{array}$$

in which each triangle is exact. We observe that  $i$  and  $j$  preserve the cohomological degree, but  $k$  raises it by 1. We hence obtain an unrolled exact couple with an additional cohomological degree, and an exact couple with

$$A = \bigoplus_{s,t} H^t(F^s C^\bullet), \quad E = \bigoplus_{s,t} H^t(\text{gr}^s C^\bullet)$$

and an associated spectral sequence. What does it converge to? We define a filtration on  $H^\bullet(C)$  by setting  $F^s H^t(C^\bullet) = \text{im}(H^t(F^s C^\bullet) \rightarrow H^t(C^\bullet))$ .

**Theorem 3.4.** *If for every  $t$ , the cohomology  $H^t(F^s C^\bullet)$  is 0 for  $s$  large enough, the spectral sequence associated to the exact couple constructed above converges to  $(H^\bullet C^\bullet, F^s H^\bullet(C^\bullet))$ , with  $E_1$ -page  $E_1^{s,t} = H^t(\text{gr}^s(C^\bullet))$ .*



The gradings of this spectral sequence are different to the one for the Serre spectral sequence:  $d_r$  raises the filtration degree by  $r$  and the cohomological degree by 1.

If  $C^\bullet$  is concentrated in nonnegative degree, we get a first quadrant spectral sequence with the cohomological degree remaining constant along rows. For the Serre spectral sequence, we use cohomological degree minus filtration degree for the vertical axis. This regrading remains in the first quadrant because all terms with filtration degree larger than cohomological degree are trivial.

*Proof.* (of 3.4) In the exact couple,  $A_r$  is the direct sum over all  $s$  of

$$A_r^s := \text{im}(i_r^{r-1} : H^\bullet(F^{s+r-1}C) \rightarrow H^\bullet(F^sC))$$

For  $t \in \mathbb{Z}$  we set  $n_t \in \mathbb{N}$  to be the minimum over all  $n$  such that  $H^t(F^n C) = 0$ . Then for  $r \geq n_t + 1$ , either  $s > 0$ , i.e.  $s + r - 1 > n_t$ , so  $H^t(F^{s+r-1}C) = 0$ , so  $A_r^s = 0$ , or  $s \leq 0$ , then  $H^t(F^s C) = H^t C$ , and  $A_r^s = F^{s+r-1} H^t C$ . So

$$A_r^t = \bigoplus_{s \leq 0} F^{s+r-1} H^t(C) = \bigoplus_{0 \leq p \leq n_t} F^p H^t(C).$$

This is independent of  $r$  if  $r \geq n_t + 1$ . Let  $A_\infty^t$  be this value. The map  $i_r : A_r^t \rightarrow A_r^t$  is the direct sum over the inclusions  $F^{p+1} H^t(C) \rightarrow F^p H^t(C)$ . In particular, it is injective, so by exactness  $k_r : E_r^{t-1} \rightarrow A_r^t$  must be zero. Thus for  $r$  large, all differentials are zero and the terms  $E_r^t$  stabilize as well. Moreover, by exactness of

$$\begin{array}{ccc} A_\infty & \xrightarrow{i_r} & A_\infty \\ & \nwarrow 0 & \nearrow j_r \\ & E_\infty & \end{array}$$

we have

$$E_\infty^t \cong \text{coker}(i_r) \cong \bigoplus_{p \leq n_t} F^p H^t(C) / F^{p+1} H^t(C).$$

□

**Example 3.5.** Let  $X$  be a CW-complex with skeleta

$$\text{sk}_0 X \subseteq \text{sk}_1 X \subseteq \dots \subseteq \text{sk}_n X \subseteq \dots$$

We can filter  $C^\bullet(X, A)$  by

$$F^s C^\bullet(X, A) = \ker(C^\bullet(X, A) \rightarrow C^\bullet(\text{sk}_s X, A)) =: C^\bullet(X, \text{sk}_s X, A)$$

Then we obtain a spectral sequence with  $E_1$ -page

$$H^\bullet(C^\bullet(X, \text{sk}_s X, A) / C^\bullet(X, \text{sk}_{s+1} X, A)) \cong H^t(\text{sk}_{s+1} X, \text{sk}_s X, A) \cong \tilde{H}^t(\text{sk}_{s+1} X / \text{sk}_s X, A)$$

It converges to  $H^\bullet(C^\bullet(X, A)) = H^\bullet(X, A)$ . Note that on the  $E_1$ -page only the entries on the main diagonal are nontrivial, and these are exactly the cellular cochain complex. For degree reasons, the spectral sequence degenerates at the  $E_2$ -page, where the entries along the main diagonals are the homology of the cellular cochain complex, i.e. the cellular cohomology. Thus, this reproves that the cellular cochain complex computes cohomology.

**Example 3.6.** Let  $p : Y \rightarrow X$  be a Serre fibration with  $X$  a CW-complex. Then we can filter  $Y$  via the preimages  $p^{-1}(\text{sk}_s X)$  and obtain a filtration on  $C^\bullet(Y, A)$ . The resulting spectral sequence "is" the Serre spectral sequence. However, some aspects (in particular, the multiplicative structure) are clearer in the construction with double complexes.

### The Spectral Sequence of a Double Complex

**Definition 3.7.** A double complex is a bigraded abelian group  $C^{\bullet,\bullet}$  equipped with two differentials

$$\delta_h : C^{p,q} \rightarrow C^{p+1,q}, \quad \text{and} \quad \delta_v : C^{p,q} \rightarrow C^{p,q+1}$$

satisfying  $\delta_h^2 = 0 = \delta_v^2$  and  $\delta_h \delta_v = \delta_v \delta_h$ , i.e. there is an infinite commutative diagram

$$\begin{array}{ccccccc} C^{p-1,q+1} & \xrightarrow{\delta_h} & C^{p,q+1} & \longrightarrow & C^{p+1,q+1} & \longrightarrow & \dots \\ \delta_v \uparrow & & \uparrow & & \uparrow & & \\ C^{p-1,q} & \longrightarrow & C^{p,q} & \longrightarrow & C^{p+1,q} & \longrightarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ C^{p-1,q-1} & \longrightarrow & C^{p,q-1} & \longrightarrow & C^{p+1,q-1} & \longrightarrow & \dots \end{array}$$

with all rows and columns complexes. The "vertical cohomology groups"  $H_{\delta_v}^q(C^{p,\bullet})$  inherit a "horizontal" differential  $\delta_h : H_{\delta_v}^q(C^{p,\bullet}) \rightarrow H_{\delta_v}^q(C^{p+1,\bullet})$ . We write  $H_{\delta_h}^p H_{\delta_v}^q(C^{\bullet,\bullet})$  for the resulting cohomology groups.

**Example 3.8.**  $D_1, D_2$  cochain complexes. Then the tensor products  $D_1^p \otimes D_2^q$  form a double complex with  $\delta_h$  from  $D_1$  and  $\delta_v$  from  $D_2$ .

**Definition 3.9.** Let  $(C^{\bullet,\bullet}, \delta_h, \delta_v)$  a double complex. Its *total complex*  $\text{Tot}(C)$  is the cochain complex  $\text{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q}$  and  $\delta = \delta_h + (-1)^p \delta_v$ .

Note: The  $(-1)^p$  is needed to guarantee  $\delta^2 = 0$ .

A double complex  $C^{\bullet,\bullet}$  can be filtered by

$$F^s(C)^{p,q} = \begin{cases} C^{p,q} & \text{if } p \geq s, \\ 0 & \text{else.} \end{cases}$$

This induces a filtration on  $\text{Tot}(C)$  via  $F^s \text{Tot}(C) = \text{Tot}(F^s(C))$ . Then  $\text{gr}_s \text{Tot}(C)^t = C^{s,t-s}$  with differential  $(-1)^s \delta_v$ . The sign doesn't affect cohomology, hence  $H^t(\text{gr}_s \text{Tot}(C)) = H_{\delta_v}^{t-s}(C^{s,\bullet})$ . We hence get a spectral sequence with  $E_1$ -page  $E_1^{s,t} = H^1(\text{gr}_s \text{Tot}(C)) = H_{\delta_v}^{t-s}(C^{s,\bullet})$ . Moreover,  $d_1$  agrees with the horizontal differentials  $\delta_h$  (exercise), so  $E_2^{s,t} \cong H_{\delta_h}^s H_{\delta_v}^{t-s}(C^{\bullet,\bullet})$ . If  $C^{p,q} = 0$  if  $p < 0$  or  $q < 0$ , then the  $E_1$ -page is concentrated in degrees  $t \geq s \geq 0$  and the spectral sequence converges to the  $H^\bullet(\text{Tot}(C))$ . It is hence customary to reindex to  $(s, t-s)$  and obtain a first quadrant spectral sequence with Serre grading  $H_{\delta_h}^s H_{\delta_v}^t(C) \Rightarrow H^{s+t}(\text{Tot}(C))$ .

Remark: Swapping the horizontal and vertical direction yields a different spectral sequence also converging to  $H^\bullet(\text{Tot}(C))$ .

### Diress's construction of the Serre spectral sequence

Let  $f : E \rightarrow B$  be a Serre fibration. A singular  $(p, q)$ -simplex of  $f$  is a commutative diagram

$$\begin{array}{ccc} \Delta^p \times \Delta^q & \longrightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow f \\ \Delta^p & \longrightarrow & B \end{array}$$

Let  $C_{p,q}(f)$  be the free abelian group on all singular  $(p, q)$ -simplices. There is a differential  $\delta_h : C_{p,q}(f) \rightarrow C_{p-1,q}(f)$  by taking the alternating sum of the faces of the  $p$ -simplex, i.e. if  $d_i : \Delta^{p-1} \rightarrow \Delta^p$

is the  $i$ -th face of  $\Delta^p$ , we consider

$$\begin{array}{ccccc} \Delta^{p-1} \times \Delta^q & \xrightarrow{d_i \times \text{id}} & \Delta^p \times \Delta^q & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f \\ \Delta^{p-1} & \xrightarrow{d_i} & \Delta^p & \longrightarrow & B \end{array}$$

Similarly, one obtains a vertical differential  $\delta_v : C_{p,q}(f) \rightarrow C_{p,q-1}(f)$  as the alternating sum of the faces of the  $q$ -simplex, obtained as

$$\begin{array}{ccccc} \Delta^p \times \Delta^{q-1} & \xrightarrow{\text{id} \times d_i} & \Delta^p \times \Delta^q & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f \\ \Delta^p & \longrightarrow & \Delta^p & \longrightarrow & B \end{array}$$

By dualizing we obtain a double complex  $C^{\bullet,\bullet}(f, A) = \text{Hom}(C_{\bullet,\bullet}(f), A)$  for every coefficient group  $A$ . We have  $C^{p,q}(f, A) = 0$  for  $p < 0$  or  $q < 0$ , hence we obtain a Serre-graded first quadrant spectral sequence with  $E_2$ -page  $E_2^{s,t} = H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet,\bullet}(f, A))$  converging to  $H^\bullet(\text{Tot}(C^{\bullet,\bullet}(f, A)))$ . This is the Serre spectral sequence. We have to show that the  $E_2$ -page and the limit agree with  $H^\bullet(B, H^\bullet(F, A))$  and  $H^\bullet(E, A)$ , respectively.

We start with  $H^\bullet(\text{Tot}(C^\bullet(f, A))) \cong H^\bullet(E, A)$ . For this we swap the horizontal and vertical directions and obtain another spectral sequence  $H_{\delta_v}^s H_{\delta_h}^t(C(f, A)) \Rightarrow H^\bullet(\text{Tot}(C(f, A)))$ . Claim: The  $E_2$ -page is 0 if  $s \neq 0$  and  $H_{\delta_v}^0 H_{\delta_h}^t(C(f, A)) \cong H^t(E, A)$ .

We fix  $s \geq 0$  and consider diagrams

$$\begin{array}{ccc} \Delta^t \times \Delta^s & \longrightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow f \\ \Delta^t & \longrightarrow & B \end{array}$$

We can rewrite this to

$$\begin{array}{ccc} \Delta^t & \longrightarrow & \text{map}(\Delta^s, E) \\ \downarrow & & \downarrow \text{map}(\Delta^s, f) \\ B & \xrightarrow{\text{const}} & \text{map}(\Delta^s, B) \end{array}$$

This in turn is equivalent to a single map

$$\Delta^t \rightarrow B \times_{\text{map}(\Delta^s, B)} \text{map}(\Delta^s, E) =: P,$$

i.e. a singular  $t$ -simplex of the space  $P$ . One checks that in fact  $C^{s,\bullet}(f, A)$  is isomorphic to  $C^\bullet(P, A)$ . Now,  $\Delta^s$  is contractible, hence  $B \rightarrow \text{map}(\Delta^s, B)$  is a homotopy equivalence. Since  $f$  is a Serre fibration, it follows that  $P \rightarrow \text{map}(\Delta^s, E)$  is a weak homotopy equivalence. In particular,  $H_{\delta_h}^t(C^{\bullet,\bullet}(f, A)) \cong H^t(E, A)$  for all  $t \geq 0$ . Under these identifications, every face map of  $\Delta^s$  induces the identity on these groups. Hence the complex computing  $H_{\delta_v}^s H_{\delta_h}^t(E, A)$  equals

$$H^t(E, A) \xrightarrow{0} H^t(E, A) \xrightarrow{\text{id}} H^t(E, A) \xrightarrow{0} H^t(E, A) \xrightarrow{\text{id}} \dots$$

thus

$$H_{\delta_v}^s H_{\delta_h}^t(C(f, A)) = \begin{cases} H^t(E, A) & s = 0 \\ 0 & s > 0. \end{cases}$$

It follows that  $E_\infty = E_2$  and  $H^t(\text{Tot}(C(f, A))) \cong H^t(E, A)$ .

It remains to compute the  $E_2$ -term.

**Definition 3.10.** The *fundamental groupoid*  $\pi_1 X$  of a space  $X$  is the category with objects the points of  $X$  and morphisms  $\text{Hom}_{\pi_1 X}(x, y) = \{\text{homotopy classes of paths } \gamma \text{ from } x \text{ to } y\}$ . Composition is the concatenation of paths.

$\pi_1 X$  is a groupoid, i.e. every morphism is invertible. Furthermore,  $\text{Hom}_{\pi_1 X}(x, x) = \pi_1(X, x)$ .

**Definition 3.11.** A *local system*  $M$  on  $X$  is a functor  $M : \pi_1 X \rightarrow \text{Ab}$ . We write  $M_x$  for  $M(x)$

Note: If  $X$  is path-connected,  $\pi_1 X$  is equivalent to the groupoid with one object  $x \in X$  and automorphisms  $\pi_1(X, x)$ . Hence a local system is equivalent datum to an abelian group  $A$  with action by  $\pi_1(X, x)$ .

If  $X$  is even simply-connected, every local system is isomorphic to the constant local system for an abelian group  $A$ .

**Example 3.12.** Let  $f : E \rightarrow B$  be a Serre fibration,  $A$  an abelian group and  $q \in \mathbb{N}$ . We write  $F_x := f^{-1}(x)$  for the fibre of  $x \in X$ . We obtain a local system  $x \mapsto H_q(F_x, A)$ . On homotopy classes of paths  $[\gamma : x \rightarrow y]$ , this is defined as follows: Consider the pullback

$$\begin{array}{ccc} F_\gamma & \longrightarrow & E \\ \downarrow & & \downarrow f \\ I & \xrightarrow{\gamma} & B \end{array}$$

which comes with weak homotopy equivalences  $F_x \rightarrow F_\gamma, F_y \rightarrow F_\gamma$ . Hence on homology we obtain an induced map

$$H_p(\gamma, A) : H_q(F_x, A) \xrightarrow{\cong} H_q(F_\gamma, A) \xleftarrow{\cong} H_q(F_y, A).$$

To show compatability with composition and homotopy invariance, consider pullbacks of  $f$  along maps  $\Delta^2 \rightarrow B$ .

**Example 3.13.** Similarly, one obtains a local system  $x \mapsto H^q(F_x, A)$ .

**Example 3.14.** Let  $M$  be a topological manifold of dimension  $n$ , then the assignment  $x \mapsto M_x = H_n(M, M \setminus \{x\}, \mathbb{Z})$  extends to a local system. For a path  $\gamma : x \rightarrow y$ , cover the path by contractible open sets  $U_i$  and use that  $H_n(M, M \setminus \{x\}, \mathbb{Z}) \cong H_n(M, M \setminus U_i, \mathbb{Z})$ . iteratively.  $M$  is orientable iff this local system is isomorphic to the constant one on  $\mathbb{Z}$ .

Next we define (co-)homology with coefficients in a local system  $M$  on  $X$ . We set  $C_n(X, M) = \bigoplus_{\sigma : \Delta^n \rightarrow X} M_{\sigma_0}$  where  $\sigma_0 \in X$  is the image of the 0th vertex of  $\Delta^n$ . There is a differential

$$d : C_n(X, M) \rightarrow C_{n-1}(X, M), \quad d(\sigma, m) = (\sigma \circ d_0, (\sigma_{0,1})_*(m)) + \sum_{i=1}^n (-1)^i (\sigma \circ d_i, m)$$

where  $\sigma_{0,1}$  is the image under  $\sigma$  of any path from the 0th vertex to the 1st vertex. Note that  $\sigma \circ d_0$  has 0th vertex equal to the first vertex of  $\sigma$ , while all the other  $\sigma \circ d_i$  have  $(\sigma \circ d_i)_0 = \sigma_0$ .

We omit the proof that this defines a chain complex.

**Definition 3.15.** For a local system  $M$ , we define  $H_\bullet(X, M)$  as the homology of this chain complex.

If  $M$  is constant, this recovers the ordinary homology. A map of local systems  $M \rightarrow N$  induces maps  $H_\bullet(X, M) \rightarrow H_\bullet(X, N)$ . A map of spaces  $f : X' \rightarrow X$  and a local system  $M$  on  $X$ , we obtain a pullback local system  $f^*M$  and a map  $H_\bullet(X', f^*M) \rightarrow H_\bullet(X, M)$ , where  $(f^*M)_x = M_{f(x)}$ .