Hodge Theory

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Contents

1 Overview and basic definitions

The aim of Hodge theory is to try to understand non-linear objects (e.g. projective varieties or Kähler manifolds) using linear objects (vector spaces, subspaces, lattices, etc.).

We will move freely between Algebraic Geometry (polynomial functions on $\mathbb{C}^n$, $\mathbb{C}[x_1, \ldots, x_n]$) and Complex Geometry (holomorphic functions on $\mathbb{C}^n$ or open subsets $U \subseteq \mathbb{C}^n$).

Definition 1.1. An *affine algebraic variety* is a vanishing locus

$$V(f_1, \ldots, f_m) = \{x \in \mathbb{C}^n \mid f_i(x) = 0 \text{ for all } i\}.$$

of some polynomials $f_i \in \mathbb{C}[x_1, \ldots, x_n]$.

Example 1.2. $y^2 = x(x-1)(x-2)$ in $\mathbb{C}^2$.

In general, an algebraic variety is covered by affine algebraic varieties, whose transition functions are polynomial maps.

Definition 1.3. $\mathbb{CP}^n = \{\text{lines through the origin in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1} \setminus \{0\}/x \sim \lambda \times x$.

Consider $f_i \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous. Then $f_i(\lambda x) = \lambda^{\deg f_i} f_i(x)$, so it makes sense to talk about zeroes of homogeneous polynomials in $\mathbb{CP}^n$.

Definition 1.4. A *projective variety* is $V(f_1, \ldots, f_m) \subseteq \mathbb{CP}^n$, $f_i \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous.

Example 1.5. $V(xy) \subseteq \mathbb{C}^2$ is the union of the two coordinate axes.

Definition 1.6. A *complex manifold* is a topological space X with local homeomorphisms onto open sets in $\mathbb{C}^n$, such that transition functions are holomorphic. In the case of $n = 1$, X is called a *Riemann surface*.

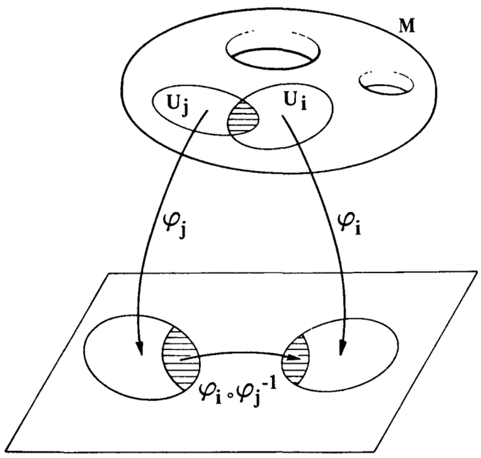


Figure 1: Two charts φ_i, φ_j of a manifold M

Example 1.7. $\mathbb{CP}^1 = \{[1 : y] \mid y \in C\} \cup \{[x : 1] \mid x \in \mathbb{C}\} =: U_1 \cup U_2$, where both factors are clearly isomorphic to $\mathbb{C}$. Now $[1 : y] = [x : 1]$ iff $xy = 1$. Now under the isomorphisms $U_1 \cap U_2$ gets identified with $\mathbb{C}^\times$, and $t \mapsto t^{-1}$ is holomorphic on $\mathbb{C}^\times$. This also shows that $\mathbb{CP}^1$ is homeomorphic to S^2.

2 Riemann surfaces of algebraic curves

2.1 The genus one case: Complex Tori

Example 2.1 (Complex Tori)**.** Consider $\mathbb{C}/\Lambda$ where Λ is a subgroup of $\mathbb{Z}$ isomorphic to $\mathbb{Z}^2$ and discrete, e.g. take $\Lambda = \mathbb{Z}[i]$. Focusing on the fundamental region $[0,1]+[0,1]i$, one sees that $\mathbb{C}/\Lambda$ topologically is a torus. For charts, for a point $z \in \mathbb{C}/\Lambda$ pick a representative in $\mathbb{C}$ with a neighbourhood. The transition maps then work out to be simple translations.

From a different point of view, homogenize the equation $y^2 = x(x-1)(x-\lambda)$, $\lambda \neq 0, 1$ from example 1.2 to $y^2 z = x(x-z)(x-\lambda z)$ to get a projective variety in $\mathbb{CP}^2$, which adds a unique additional point $[0:1:0]$.

Consider the "multiform function" $f(x) = \sqrt{x(x-1)(x-\lambda)}$. This clearly has zeroes at 0, 1 and λ, but its other values are not uniquely specified[1]. Picking one value, say $f(\frac{1}{2})$, also determines the value of f in a neighbourhood of that point, if we want f to be continuous. In fact, if one analytically continues f along the circle $x = \frac{1}{2}e^{i\theta}$, $\theta \in [0, 2\pi]$, we get $f(x) = \frac{1}{\sqrt{2}}e^{i\theta/2}\sqrt{(x-1)(x-\lambda)}$, where the latter square root can be chosen to be well-defined on, say, $|z| < \frac{2}{3}$. Hence $f(e^{2\pi i}x) = -f(x)$, which is a problem. To fix this, Riemann's idea was to enlargen the region of definition to two linked complex planes so one can circle around the origin twice without running into problems. This introduces cuts in the planes where they are connected, but on this object f is a well-defined function. Topologically, a plane with two cuts (one from 0 to 1 and one from λ to ∞) is a open cylinder, and glueing two of these together yields, again, a torus.

In conclusion, we came up with different ways to construct a compact Riemann surface of genus 1: The quotient $\mathbb{C}/\Lambda$ versus the projective variety $y^2 z = x(x-z)(x-\lambda z)$ or the "domain" of the function $\sqrt{x(x-1)(x-\lambda)}$ in the above sense. When are $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $zy^2 = x(x-z)(x-\lambda z)$ the same Riemann surface?

Definition 2.2. An *isomorphism of Riemann surfaces* $f : X \to Y$ is a homeomorphism which is biholomorphic in local charts.

Question: Given a one-dimensional complex torus $\mathbb{C}/\Lambda$, can we find polynomial equations describing the same Riemann surface?

Weierstrass answered this question by building functions x and y on $\mathbb{C}/\Lambda$.

Proposition 2.3. *There does not exist a holomorphic nonconstant function $f : \mathbb{C}/\Lambda \to \mathbb{C}$.*

Proof. Any such f gives $\widetilde{f} : \mathbb{C} \to \mathbb{C}/\Lambda \to \mathbb{C}$ with $\widetilde{f}$ bounded and entire, hence constant. □

Building a meromorphic function on $\mathbb{C}/\Lambda$ is equivalent to finding $f : \mathbb{C} \to \mathbb{CP}^1$ such that $f(x+\lambda) = f(x)$ for $\lambda \in \Lambda$. Define

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda\setminus\{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

This function converges and is invariant under the action of the lattice. One computes its derivative as $\wp'(z) = -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}$. Note that $\wp$ is even and $\wp'$ is odd. For the series expansion around 0 one gets

$$\wp(z) = \frac{1}{z^2} + c_1 z^2 + c_2 z^4 + \ldots \quad \text{and} \quad \wp'(z) = -2(\frac{1}{z^3} - c_1 z - \ldots)$$

and one can verify $\wp'(z)^2 = 4\wp(z)^3 + g_2\wp(z) + g_3$ for $g_2 = -20c_1$ and some constant $g_3 \in \mathbb{C}$ (verify using the series expansion that $\wp'^2 - 4\wp^3 - g_2\wp$ is biperiodic and holomorphic).

[1] Assume λ is in a general position

Proposition 2.4. *There exists a polynomial relation $\wp'(z)^2 = 4\wp(z)^3 + g_2\wp(z) + g_3$ for some constants $g_2, g_3 \in \mathbb{C}$.* □

Consider the map $\varphi : \mathbb{C}/\Lambda \to \mathbb{CP}^2$, $z \mapsto [\wp(z) : \wp'(z) : 1]$. (For $z = 0$, we get $0 \mapsto [0 : 1 : 0]$.) Now $\operatorname{im}\varphi \subseteq V(x_1^2x_2 - 4x_0^3 - g_2x_0x_2^2 - g_3x_2^3) =: V(f)$. We claim that φ is injective and surjective on $V(f)$.

Proof. $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is 2 to 1 because $\wp^{-1}(\infty) = 2[0]$ and the multiplicity is the number of inverse images of $\wp$ near ∞. So $\mathbb{C}/\Lambda \to \mathbb{CP}^1$ is the quotient map by the $\mathbb{Z}^2$-action $z \mapsto -z$. Assume $\wp(z) = \wp(w)$ and $\wp'(z) = \wp'(w)$ for some $z \neq w$. By the above, $z = -w$ and $\wp'(z) = 0$. If $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$, since $\wp'$ is odd we have $\wp'(\frac{1}{2}v_1) = \wp'(\frac{1}{2}v_2) = \wp'(\frac{1}{2}(v_1 + v_2)) = 0$. Since $\wp'^{-1}(\infty) = 3[0]$, by the same argument as before 0 has at most 3 preimages, hence $z \in \{\frac{1}{2}v_1, \frac{1}{2}v_2, \frac{1}{2}(v_1 + v_2)\}$, and hence $z = -z = w$. This proves that φ is injective.

For surjectivity, we use the open mapping theorem: If $f : C \to D$ is a holomorphic map of Riemann surfaces, then $\operatorname{im} f$ is open. Hence $\operatorname{im}\varphi$ is open. Since $\mathbb{C}/\Lambda$ is compact, we also have that $\operatorname{im}\varphi$ is closed. Thus φ is surjective. □

This answers the question how to go from a lattice to a cubic. Now let us think about the reverse direction.

Definition 2.5. A holomorphic 1-form ω on a Riemann surface Σ is a compatible collection of expressions $\{f(z)dz\}$ f holomorphic, ranging over the charts of Σ.

Spelt out, this means whenever we have charts $\varphi_1 : U_1 \to \mathbb{C}$ and $\varphi_2 : U_2 \to \mathbb{C}$ with expressions $f_1(z)dz$ and $f_2(z)dz$ on U_1 and U_2, respectively, with transition map $w = \varphi_2 \circ \varphi_1$, we want $f_2(w(z))d(w(z)) = f_1(z)dz$, i.e. $f_1(z) = f_2(w(z))w'(z)$.

Now define a holomorphic 1-form on $V(y^2 - x(x-1)(x-\lambda))$ by $\omega = \frac{dx}{y}$. When $x \neq 0, 1, \lambda, \infty$, then x is a local coordinate. Then $y \neq 0$ and everything is fine. If $x = 0$, then $w = \sqrt{x}$ is a local holomorphic coordinate. Then $x = w^2$ and $y = w\sqrt{(w^2-1)(w^2-\lambda)}$ as well as $dx = 2wdw$. Together,

$$\frac{dx}{y} = \frac{2}{\sqrt{(w^2-1)(w^2-\lambda)}}dw,$$

where the fraction is a holomorphic function of w near 0. The same arguments work for $x = 1$ and $x = \lambda$. At ∞, we had $w = x^{-\frac{1}{2}}$ as a holomorphic function and similar calculations show that everything works out. ω is nowhere vanishing: In a local chart $z, \omega = f(z)dz$, then $f(z) \neq 0$.

Proposition 2.6. *Any holomorphic 1-form on a Riemann surface Σ is closed as a $\mathbb{C}$-valued differentiable 1-form.*

There is a map $d : \{\text{diff. } p\text{-forms}\} \to \{\text{diff. } p+1\text{-forms}\}$ given by

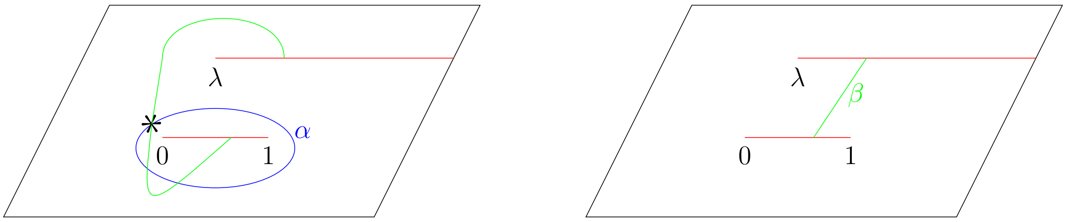
$$f dx_1 \wedge \cdots \wedge dx_p \mapsto \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p.$$

Write $\omega = f(z)dz = f(x+iy)(dx+idy)$. Then $d\omega$ computes as

$$d\omega = i\frac{\partial f}{\partial x}dx \wedge dy + \frac{\partial f}{\partial y}dy \wedge dx = \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = 0$$

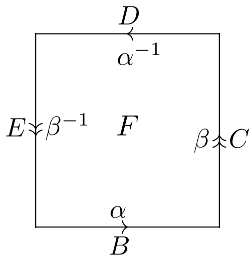
Consider $A(p) = \int_*^p \omega$ as a "function" on $\Sigma = V(y^2 - x(x-1)(x-\lambda)$. $A(p)$ depends on the chosen path. If γ_1, γ_2 are two homotopic paths from $*$ to p, then $\int_{\gamma_1}\omega = \int_{\gamma_2}\omega$ by Stokes theorem. Hence A depends only on the homotopy class of the chosen path. If γ_1, γ_2 are two homotopy classes of paths from

$*$ to p, then $\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\gamma_2^{-1} \circ \gamma_1} \omega$ and $\gamma_2^{-1} \circ \gamma_1 \in \pi_1(\Sigma, *) \cong \mathbb{Z}^2$, since Σ is a torus. Set $v_1 = \int_\alpha \omega$, $v_2 = \int_\beta \omega$, where α, β are generators of $\pi_1(\Sigma, *)$, as indicated in the picture:



Then A is a single valued function with target $\mathbb{C}/\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$. v_1 and v_2 are called the "Abelian" integrals. We can explicitly write $v_1 = 2\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$ and $v_2 = 2\int_0^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$. Claim: $v_1, v_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

Proof. Cut along α, β. You get a square F, denote its sides as in the figure.



Then $-i\int_\Sigma \omega \wedge \bar{\omega} > 0$, since locally, if $\omega = f(z)dz$, then

$$-i\omega \wedge \bar{\omega} = -if\bar{f}dz \wedge d\bar{z} = 2f\bar{f}dx \wedge dy.$$

On the other hand, $\int_\Sigma \omega \wedge \bar{\omega} = \int_F \omega \wedge \bar{\omega} = \int_F d(A) \wedge \bar{\omega} = \int_F d(A \cdot \bar{\omega}) = \int_{\partial F} A\bar{\omega}$ by Stokes. Note $\bar{\omega}|_B = \bar{\omega}|_{-D}$ and the same for C, E. Similarly $A|_B - A|_{-D}$ is equal to the constant function $\int_\beta \omega$ and $A|_C - A|_{-E} = \int_{-\alpha} \omega$. Hence

$$\begin{aligned} \int_\Sigma \omega \wedge \bar{\omega} &= \int_B A\bar{\omega} - \int_{-D} A\bar{\omega} + \int_C A\bar{\omega} - \int_{-E} A\bar{\omega} = \int_B \left(\int_\beta \omega\right)\bar{\omega} + \int_C \left(\int_{-\alpha} \omega\right)\bar{\omega} \\ &= \int_\alpha \bar{\omega} \int_\beta \omega - \int_\alpha \omega \int_\beta \bar{\omega} = \bar{v}_1 v_2 - v_1 \bar{v}_2. \end{aligned}$$

Putting everything together, we have $-i(\bar{v}_1 v_2 - v_1\bar{v}_2) > 0$, i.e. $\mathrm{Im}(v_1\bar{v}_2) > 0$. $\square$

So $\Lambda := \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ is a lattice and $A : \Sigma \to \mathbb{C}/\Lambda$ is a locally invertible map into a torus. Hence A is a covering map and Σ compact implies the fibres of A are finite, i.e. Σ is a finite covering of $\mathbb{C}/\Lambda$. With some covering theory, this implies $\Sigma = \mathbb{C}/\Lambda'$, where $\Lambda' \subseteq \Lambda$ is a finite index sublattice.

Next we ask: Given lattices $\Lambda, \Lambda' \subseteq \mathbb{C}$, when are $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ isomorphic as Riemann surfaces?

Proposition 2.7. *$\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ if and only if there exists $c \in \mathbb{C}^\times$ s.t. $c\Lambda = \Lambda'$.*

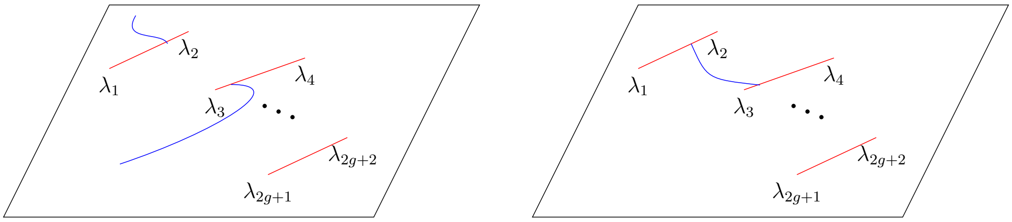
Proof. Let $i : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be an iso. and assume $i(0) = 0$. Lift i to the universal cover $\tilde{\imath} : \mathbb{C} \to \mathbb{C}$ isomorphism with $\tilde{\imath}(0) = 0$. This implies that $\tilde{i}$ is linear, i.e. $\tilde{i}(z) = cz$. Since $i(\Lambda) = \Lambda'$, it follows that $c\Lambda = \Lambda'$. The converse follows similarly. $\square$

Given any lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \subseteq \mathbb{C}$, multiply it by $\frac{1}{v_2}$ to get $\Lambda' \cong \mathbb{Z}\tau \oplus \mathbb{Z}$ where $\tau = \frac{v_1}{v_2}$. Assume $\operatorname{Im}(\tau) > 0$ (otherwise replace τ by $-\tau$). Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ be the upper half-plane. Then $\tau, \tau' \in \mathbb{H}$ define the same complex torus if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Hence the space of 1-dimensional complex tori is in bijection to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

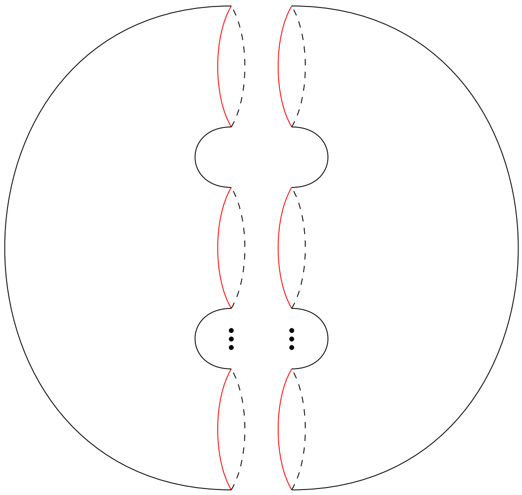
2.2 Curves of higher genus

Consider a Riemann surface Σ of genus g.

Example 2.8 (Hyperelliptic curves)**.** $y^2 = (x-\lambda_1)\cdots(x-\lambda_{2g+2})$. This corresponds to the Riemann surface Σ of the function $f(x) = \sqrt{(x-\lambda_1)\cdots(x-\lambda_{2g+2})}$, which has a unique analytic continuation to $\mathbb{C}$ without the $g+1$ cuts between λ_{2i-1} and λ_{2i}, $i = 1, \ldots, g+1$.



The result is a genus g Riemann surface and the local chart near $x = \lambda_i$ is $\sqrt{x-\lambda_i}$.



Again consider $\omega = \frac{dx}{y}$. For the same reasons as before, ω is holomorphic at $y = 0$. Near ∞, $w = \frac{1}{x}$ is a local corrdinate, and

$$\omega = \frac{d(1/w)}{\sqrt{(1/w-\lambda_1)\cdots(1/w-\lambda_{2g+2})}} = \frac{-1/w^2 dw}{1/w^{g+1}\sqrt{h(w)}} = -w^{g-1}h(w)^{-1/2}dw$$

for some holomorphic function h. Hence for $0 \le r \le g-1$, even $\omega_r = \frac{x^r dx}{y}$ is a holomorphic 1-form on Σ, so there is a g-dimensional vector space $\bigoplus_{r=0}^{g-1} \mathbb{C}\omega_r$ of holomorphic 1-forms.
Fact: Let $\Omega^1(\Sigma)$ be the $\mathbb{C}$-vector space of holomorphic 1-forms on a genus g compact Riemann surface Σ. Then $\dim_{\mathbb{C}} \Omega^1(\Sigma) = g$. We will prove this later.

Exercise 2.9. What is the genus of $y^3 = x^6 - 1$?

3 De-Rham Cohomology

Let M be a real manifold of dimension d. Let $\bigwedge^p(M) := \{\text{smooth } p\text{-forms on } M\}$, that is smooth p-forms on an open cover that agree on intersections, where $\bigwedge^p(U) = \{\sum_{|I|=p} f_I dx_i \wedge \cdots \wedge dx_p\}$ with the f_I smooth. Now consider the de-Rham complex

$$0 \to \bigwedge^0(M) \xrightarrow{d_0} \bigwedge^1(M) \xrightarrow{d_1} \bigwedge^2(M) \to \cdots \to \bigwedge^d(M) \to 0$$

where

$$d(f dx_I) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

Now define the de-Rham cohomology as the homology of this complex, i.e.

$$H^p_{\mathrm{dR}}(M, \mathbb{C}) = \ker(d_p) / \operatorname{im}(d_{p-1}).$$

Theorem 3.1 (De Rham)**.** $H^p_{dR}(M, \mathbb{C}) \cong H^p_{sing}(M, \mathbb{C})$.

Here, the map is defined as follows: Let $[\omega] \in H^p_{\mathrm{dR}}(M, \mathbb{C})$ be represented by $\omega \in \bigwedge^p(M)$ which is exact: $d_p\omega = 0$. Then define $[\omega] \mapsto (\sigma \mapsto \int_\sigma \omega)$.

Now consider M a complex manifold of $\mathbb{C}$-dimension d.

Definition 3.2. The *smooth* (p, q)*-forms* on $U \subseteq \mathbb{C}^n$ are defined as

$$\bigwedge^{p,q}(U) = \left\{ \sum_{\substack{|I|=p,\\ |J|=q}} f_{I\bar{J}} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge d\bar{x}_{j_1} \cdots \wedge d\bar{x}_{j_q} \right\}$$

where the $f_{I\bar{J}}$ are smooth functions and the x_i are the coordinates of $\mathbb{C}^n$. The smooth $(p+q)$-forms of a manifold M are forms locally of the type $\bigwedge^{p,q}(U)$.

This is well-defined because the pullback of a (p, q)-form under a holomorphic map is a (p, q)-form.

Example 3.3. $z\bar{z}dz$ is a smooth $(1, 0)$-form on $\mathbb{C}$. It corresponds to $(x^2 + y^2)(dx + idy) \in \bigwedge^1(\mathbb{R}^2)$. Similarly, $\bar{z}d\bar{z} \in \bigwedge^{0,1}(\mathbb{C})$. If (z, w) are the coordinates of $\mathbb{C}^2$, then $dz \wedge d\bar{w} + d\bar{z} \wedge dw \in \bigwedge^{1,1}(\mathbb{C})$.

Lemma 3.4. $\bigwedge^k(M_\mathbb{R}) = \bigoplus_{p+q=k} \bigwedge^{p,q}(M)$*, where* $M_\mathbb{R}$ *is* M *considered as a real manifold.*

Proof. If $z_1, \dots, z_n$ are local complex coordinates and $x_1, y_1, \dots, x_n, z_n$ the corresponding local real coordinates, then $z_i = x_i + iy_i$ and $x_i = \frac{1}{2}(z_i + \bar{z}_i)$, $y_i = \frac{1}{2}(z_i - \bar{z}_i)$, so one can directly translate expressions from each set into an expression from the other set. □

On a one-dimensional complex manifold Σ, the only spaces of (p, q)-forms to consider are $\bigwedge^{0,0}(\Sigma) = \{\text{loc. smooth functions}\}$, $\bigwedge^{1,0}(\Sigma)$, $\bigwedge^{0,1}(\Sigma)$ and $\bigwedge^{1,1}(\Sigma)$. Hence the de-Rham complex is

$$0 \to \bigwedge^{0,0}(\Sigma) \xrightarrow{d} \bigwedge^{0,1}(\Sigma) \oplus \bigwedge^{1,0}(\Sigma) \xrightarrow{d} \bigwedge^{1,1}(\Sigma)$$

and the first exterior derivative is $d = \partial_z \oplus \partial_{\bar{z}}$, i.e. given by $f \mapsto \frac{\partial f}{\partial z} d\bar{z} + \frac{\partial f}{\partial \bar{z}} dz$. A holomorphic 1-form on Σ is locally expressible as $\omega = f(z)dz$ with f holomorphic, $\omega \in \bigwedge^{1,0}(\Sigma)$. As before, we see that ω is closed:

$$d(f(z, \bar{z})dz) = \frac{\partial f}{\partial z} \underbrace{dz \wedge dz}_{=0} + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{=0} d\bar{z} \wedge dz = 0.$$

For a smooth (p,q)-form $\omega \in \bigwedge^{p,q}(M)$, M a d-dimensional complex manifold, write

$$df = \underbrace{\frac{\partial f}{\partial z_1}dz_1 + \ldots + \frac{\partial f}{\partial z_d}dz_d}_{=:\partial f} + \underbrace{\frac{\partial f}{\partial \bar{z}_1}d\bar{z}_1 + \ldots + \frac{\partial f}{\partial \bar{z}_d}dz_d}_{=:\overline{\partial} f}$$

Then $d : \bigwedge^k(M) \to \bigwedge^{k+1}(M)$ decomposes into a sum $d = \partial + \overline{\partial}$ where

$$\partial(f_{I,\bar{J}}dz_I \wedge d\bar{z}_J) = \partial f_{I,\bar{J}} \wedge dz_I \wedge d\bar{z}_J$$

and similarly for $\overline{\partial}$. We get a double complex

$$\begin{array}{ccccc}
\bigwedge^{0,2}(M) & & & & \\
\bar{\partial}\Big\uparrow & & & & \\
\bigwedge^{0,1}(M) & \xrightarrow{\partial} & \bigwedge^{1,1}(M) & & \\
\bar{\partial}\Big\uparrow & & \bar{\partial}\Big\uparrow & & \\
\bigwedge^{0,0}(M) & \xrightarrow{\partial} & \bigwedge^{1,0}(M) & \xrightarrow{\partial} & \bigwedge^{2,0}(M)
\end{array}$$

with the squares commuting up to sign. Also it is easy to check that $\bar{\partial} \circ \bar{\partial} = 0 = \partial \circ \partial$. The total complex of a double complex $(E^{p,q}, d_1, d_2)$ is $(\bigoplus_{p+q=k} E^{p,q}, d_1 + d_2)$. In this case, we get

$$\left(\bigoplus_{p+q=k} \bigwedge^{p,q}(M), \partial + \bar{\partial}\right) = (\bigwedge^k(M), d)$$

the original de Rham complex on M.

Example 3.5.

$$\bar{\partial}(z\bar{z}d\bar{w} + wd\bar{z}) = zd\bar{z} \wedge d\bar{w} + 0 \in \bigwedge^{0,2}(\mathbb{C}^2).$$

Definition 3.6. The *Dolbeault cohomology*

$$H^{p,q}(M) := \frac{\ker(\bar{\partial} : \bigwedge^{p,q}(M) \to \bigwedge^{p,q+1}(M))}{\mathrm{im}(\bar{\partial} : \bigwedge^{p,q-1}(M) \to \bigwedge^{p,q}(M))}$$

This is an analogue of de Rham cohomology when M is a complex manifold.

Theorem 3.7 (Poincaré $\bar{\partial}$-lemma)**.** *Let* $0 \in U \subseteq \mathbb{C}^n$ *be an open set,* $\omega \in \bigwedge^{p,q}(U)$*. Then, if* $\bar{\partial}\omega = 0$*, then there exists an open subset* $0 \in V \subseteq U$ *and an* $\alpha \in \bigwedge^{p,q-1}(V)$ *such that* $\bar{\partial}\alpha = \omega|_V$*.*

Proof. (for $n = 1$). Let $\Delta \subseteq \mathbb{C}$ be the unit disk. Let $gd\bar{z} \in \bigwedge^{0,1}(\overline{\Delta})$. Then $\bar{\partial}gd\bar{z}$ is automatically 0. Then

$$f(z,\bar{z}) := \frac{1}{2\pi i}\int_\Delta \frac{g(w,\bar{w})}{w-z}dw \wedge d\bar{w}$$

satisfies $\bar{\partial}f = gd\bar{z}$: Write $g = g_1 + g_2$ such that $\operatorname{supp} g_1 \subseteq B_{2\varepsilon}(z)$ and $\operatorname{supp}(g_2) \subseteq B_\varepsilon(z)^c$. Now

$$f = \frac{1}{2\pi i}\left(\int_\Delta \frac{g_1}{w-z}dw \wedge d\bar{w} + \underbrace{\int_\Delta \frac{g_2}{w-z}dw \wedge d\bar{w}}_{\bar{\partial}(-)=0}\right)$$

and
$$\int_\Delta \frac{g_1}{w-z} dw\wedge d\bar w = \int_{B_{2\varepsilon}(0)} \frac{g_1(z+u)}{u} du\wedge d\bar u = \int_0^{2\pi}\int_0^{2\varepsilon} g_1(z+u)e^{-i\theta} dr d\theta$$
is clearly the integral of a smooth function, hence smooth. One calculates
$$\begin{aligned}
2\pi i\bar\partial f &= \bar\partial \int_\Delta \frac{g_1}{w-z} dw\wedge d\bar w = \lim_{\mu\to 0} \bar\partial \int_{B_{2\varepsilon}(z)-B_\mu(z)} \frac{g_1}{w-z} dw\wedge d\bar w \\
&= \lim_{\mu\to 0} \int_{B_{2\varepsilon}(z)-B_\mu(z)} \underbrace{\frac{\partial g_1}{\partial \bar w}(z)\frac{1}{w-z} dw\wedge d\bar w}_{d\eta \text{ where } \eta = -\frac{g(w)dw}{w-z}} \\
&\overset{\text{Stokes}}{=} \lim_{\mu\to 0}\left(\int_{C_{2\varepsilon}(z)} \frac{-g_1(w)dw}{w-z} + \int_{C_{\mu(z)}} \frac{g_1(w)}{w-z}dw\right) \\
&= 2\pi i g_1(z) = 2\pi i g(z)
\end{aligned}$$

□

Let Σ be a Riemann surface of genus g. Let α_i, β_i, $i = 1,\dots,g$ be the loops as indicated in the picture.

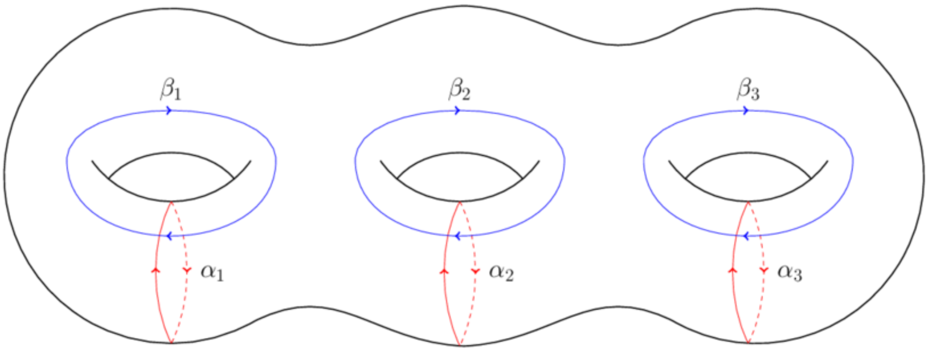


Figure 2: A genus 3 surface with a basis for its first homology

They form a basis of $H_1(\Sigma,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. By Poincaré duality, $H_1(\Sigma,\mathbb{Z}) \overset{PD}{\cong} H^1(\Sigma,\mathbb{Z})$. From the cohomological product structure one gets a pairing on $H_1(\Sigma,\mathbb{Z})$ by $\alpha\cdot\beta = \int_\Sigma PD(\alpha)\smile PD(\beta)$, which is the intersection form: Represent α,β by transversely intersecting cycles. Then $\alpha\cdot\beta = \sum_{p\in\alpha\cap\beta} \underbrace{\mathrm{or}_p(\alpha,\beta)}_{\in\pm 1}$.

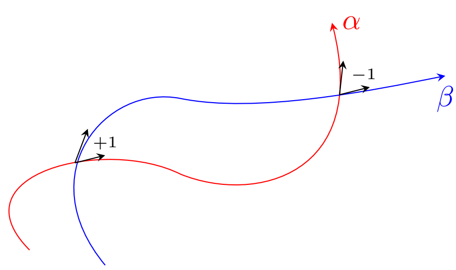


Figure 3: Two cycles α,β with intersection form $\alpha\cdot\beta = 1-1 = 0$.

The Intersection matrix of the chosen basis $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$ is given by $\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} \in \mathbb{Z}^{2g\times 2g}$. Thus $H_1(\Sigma,\mathbb{Z})$ has the structure of a symplectic lattice, with an alternating map $(-,-) : \Lambda\otimes\Lambda \to \mathbb{Z}$.

Choose generators α_i, β_i of $\pi_1(\Sigma, *)$ that are homologous to the α_i, β_i from before. Then $\pi_1(\Sigma, *) = \langle \alpha_i, \beta_i \mid \prod_i [\alpha_i, \beta_i] \rangle$ Recall $\dim \Omega^1(\Sigma) = g$. Let $\omega, \omega' \in \Omega^1(\Sigma)$.

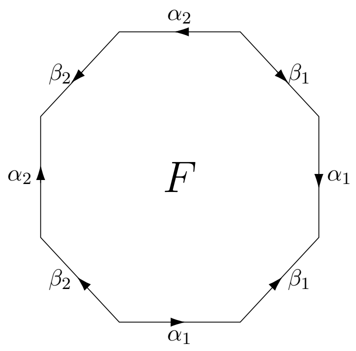


Figure 4: Cut up torus of genus 2

Theorem 3.8 (Riemann Bilinear Relations)**.** *There exists a unique basis of* $\Omega^1(\Sigma) = \bigoplus_{i=1}^g \mathbb{C}\omega_i$ *such that* $\int_{\alpha_j} \omega_i = \delta_{ij}$ *and* $(\int_{\beta_j} \omega_i)_{i,j}$ *is a symmetric* $g \times g$*-matrix with positive definite imaginary part.*

Lemma 3.9.

$$\text{Symmetry:} \qquad \sum_{i=1}^{g} \int_{\alpha_i} \omega \int_{\beta_i} \omega' - \int_{\alpha_i} \omega' \int_{\beta_i} \omega = 0.$$

$$\text{Positivity:} \qquad \sum_{i=1}^{g} \int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\beta_i} \omega \int_{\alpha_i} \bar{\omega}_i > 0.$$

Proof. Let $A(p) = \int_*^p \omega$ where $*$ is one of the vertices of F. $A(p)$ is holomorphic on F and welldefined since $d\omega = 0$. Further $dA = \omega$ on F.

$$0 = \sum_{\Sigma} \omega \wedge \omega' = \int_F \omega \wedge \omega' = \int_F d(A\omega') = \int_{\partial F} A\omega'.$$

Now $\int_\alpha A\omega' + \int_{\alpha_1^{-1}} A\omega' = (A|_{\alpha_1} - A|_{\alpha_1^{-1}}) \int_{\alpha_1} \omega' = \int_{\beta_1} \omega \int_{\alpha_1} \omega'$. Doing this for all i gives

$$\int_{\partial F} A\omega' = \sum_{i=1}^{g} \int_{\beta_i} \omega \int_{\alpha_i} \omega' - \int_{\alpha_i} \omega \int_{\beta_i} \omega'.$$

$\square$