Hodge Theory

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1 Overview and basic definitions

The aim of Hodge theory is to try to understand non-linear objects (e.g. projective varieties or Kähler manifolds) using linear objects (vector spaces, subspaces, lattices, etc.).

We will move freely between Algebraic Geometry (polynomial functions on $\mathbb{C}^n$, $\mathbb{C}[x_1, \dots, x_n]$) and Complex Geometry (holomorphic functions on $\mathbb{C}^n$ or open subsets $U \subseteq \mathbb{C}^n$).

Definition 1.1. An *affine algebraic variety* is a vanishing locus

$$V(f_1, \dots, f_m) = \{x \in \mathbb{C}^n \mid f_i(x) = 0 \text{ for all } i\}.$$

of some polynomials $f_i \in \mathbb{C}[x_1, \dots, x_n]$.

Example 1.2. $y^2 = x(x-1)(x-2)$ in $\mathbb{C}^2$.

In general, an algebraic variety is covered by affine algebraic varieties, whose transition functions are polynomial maps.

Definition 1.3. $\mathbb{CP}^n = \{\text{lines through the origin in } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1} \setminus \{0\}/x \sim \lambda \cdot x$.

Consider $f_i \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous. Then $f_i(\lambda x) = \lambda^{\deg f_i} f_i(x)$, so it makes sense to talk about zeroes of homogeneous polynomials in $\mathbb{CP}^n$.

Definition 1.4. A *projective variety* is $V(f_1, \dots, f_m) \subseteq \mathbb{CP}^n$, $f_i \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous.

Example 1.5. $V(xy) \subseteq \mathbb{C}^2$ is the union of the two coordinate axes.

Definition 1.6. A *complex manifold* is a topological space X with local homeomorphisms onto open sets in $\mathbb{C}^n$, such that transition functions are holomorphic. In the case of $n = 1$, X is called a *Riemann surface*.

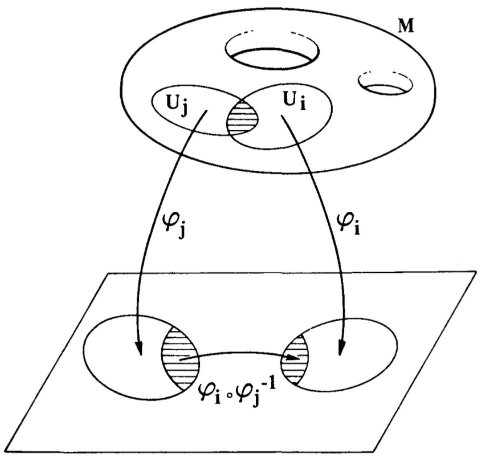


Figure 1: Two charts φ_i, φ_j of a manifold M

Example 1.7. $\mathbb{CP}^1 = \{[1 : y] \mid y \in \mathbb{C}\} \cup \{[x : 1] \mid x \in \mathbb{C}\} =: U_1 \cup U_2$, where both factors are clearly isomorphic to $\mathbb{C}$. Now $[1 : y] = [x : 1]$ iff $xy = 1$. Under the isomorphisms to $\mathbb{C}$, $U_1 \cap U_2$ gets identified with $\mathbb{C}^\times$, and $t \mapsto t^{-1}$ is holomorphic on $\mathbb{C}^\times$. This also shows that $\mathbb{CP}^1$ is homeomorphic to S^2.

2 Riemann surfaces of algebraic curves

2.1 The genus one case: Complex Tori

Example 2.1 (Complex Tori)**.** Consider $\mathbb{C}/\Lambda$ where Λ is a subgroup of $\mathbb{Z}$ isomorphic to $\mathbb{Z}^2$ and discrete, e.g. take $\Lambda = \mathbb{Z}[i]$. Focusing on the fundamental region $[0,1]+[0,1]i$, one sees that $\mathbb{C}/\Lambda$ topologically is a torus. For charts, for a point $z \in \mathbb{C}/\Lambda$ pick a representative in $\mathbb{C}$ with a neighbourhood. The transition maps then work out to be simple translations.

From a different point of view, homogenize the equation $y^2 = x(x-1)(x-\lambda)$, $\lambda \neq 0, 1$ from example 1.2 to $y^2 z = x(x-z)(x-\lambda z)$ to get a projective variety in $\mathbb{CP}^2$, which adds a unique additional point $[0:1:0]$.

Consider the "multiform function" $f(x) = \sqrt{x(x-1)(x-\lambda)}$. This clearly has zeroes at 0, 1 and λ, but its other values are not uniquely specified[1]. Picking one value, say $f(\frac{1}{2})$, also determines the value of f in a neighbourhood of that point, if we want f to be continuous. In fact, if one analytically continues f along the circle $x = \frac{1}{2}e^{i\theta}$, $\theta \in [0, 2\pi]$, we get $f(x) = \frac{1}{\sqrt{2}}e^{i\theta/2}\sqrt{(x-1)(x-\lambda)}$, where the latter square root can be chosen to be well-defined on, say, $|z| < \frac{2}{3}$. Hence $f(e^{2\pi i}x) = -f(x)$, which is a problem. To fix this, Riemann's idea was to enlargen the region of definition to two linked complex planes so one can circle around the origin twice without running into problems. This introduces cuts in the planes where they are connected, but on this object f is a well-defined function. Topologically, a plane with two cuts (one from 0 to 1 and one from λ to ∞) is a open cylinder, and glueing two of these together yields, again, a torus.

In conclusion, we came up with different ways to construct a compact Riemann surface of genus 1: The quotient $\mathbb{C}/\Lambda$ versus the projective variety $y^2 z = x(x-z)(x-\lambda z)$ or the "domain" of the function $\sqrt{x(x-1)(x-\lambda)}$ in the above sense. When are $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $zy^2 = x(x-z)(x-\lambda z)$ the same Riemann surface?

Definition 2.2. An *isomorphism of Riemann surfaces* $f : X \to Y$ is a homeomorphism which is biholomorphic in local charts.

Question: Given a one-dimensional complex torus $\mathbb{C}/\Lambda$, can we find polynomial equations describing the same Riemann surface?

Weierstrass answered this question by building functions x and y on $\mathbb{C}/\Lambda$.

Proposition 2.3. *There does not exist a holomorphic nonconstant function $f : \mathbb{C}/\Lambda \to \mathbb{C}$.*

Proof. Any such f gives $\widetilde{f} : \mathbb{C} \to \mathbb{C}/\Lambda \to \mathbb{C}$ with $\widetilde{f}$ bounded and entire, hence constant. □

Building a meromorphic function on $\mathbb{C}/\Lambda$ is equivalent to finding $f : \mathbb{C} \to \mathbb{CP}^1$ such that $f(x+\lambda) = f(x)$ for $\lambda \in \Lambda$. Define

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

This function converges and is invariant under the action of the lattice. One computes its derivative as $\wp'(z) = -2\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$. Note that $\wp$ is even and $\wp'$ is odd. For the series expansion around 0 one gets

$$\wp(z) = \frac{1}{z^2} + c_1 z^2 + c_2 z^4 + \dots \quad \text{and} \quad \wp'(z) = -2(\frac{1}{z^3} - c_1 z - \dots)$$

and one can verify $\wp'(z)^2 = 4\wp(z)^3 + g_2\wp(z) + g_3$ for $g_2 = -20c_1$ and some constant $g_3 \in \mathbb{C}$ (verify using the series expansion that $\wp'^2 - 4\wp^3 - g_2\wp$ is biperiodic and holomorphic).

[1] Assume λ is in a general position

Proposition 2.4. *There exists a polynomial relation $\wp'(z)^2 = 4\wp(z)^3 + g_2\wp(z) + g_3$ for some constants $g_2, g_3 \in \mathbb{C}$.* $\square$

Consider the map $\varphi : \mathbb{C}/\Lambda \to \mathbb{CP}^2$, $z \mapsto [\wp(z) : \wp'(z) : 1]$. (For $z = 0$, we get $0 \mapsto [0 : 1 : 0]$.) Now $\operatorname{im}\varphi \subseteq V(x_1^2x_2 - 4x_0^3 - g_2x_0x_2^2 - g_3x_2^3) =: V(f)$. We claim that φ is injective and surjective on $V(f)$.

Proof. $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is 2 to 1 because $\wp^{-1}(\infty) = 2[0]$ and the multiplicity is the number of inverse images of $\wp$ near ∞. So $\mathbb{C}/\Lambda \to \mathbb{CP}^1$ is the quotient map by the $\mathbb{Z}^2$-action $z \mapsto -z$. Assume $\wp(z) = \wp(w)$ and $\wp'(z) = \wp'(w)$ for some $z \neq w$. By the above, $z = -w$ and $\wp'(z) = 0$. If $\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2$, since $\wp'$ is odd we have $\wp'(\frac{1}{2}v_1) = \wp'(\frac{1}{2}v_2) = \wp'(\frac{1}{2}(v_1 + v_2)) = 0$. Since $\wp'^{-1}(\infty) = 3[0]$, by the same argument as before 0 has at most 3 preimages, hence $z \in \{\frac{1}{2}v_1, \frac{1}{2}v_2, \frac{1}{2}(v_1 + v_2)\}$, and hence $z = -z = w$. This proves that φ is injective.

For surjectivity, we use the open mapping theorem: If $f : C \to D$ is a holomorphic map of Riemann surfaces, then $\operatorname{im} f$ is open. Hence $\operatorname{im}\varphi$ is open. Since $\mathbb{C}/\Lambda$ is compact, we also have that $\operatorname{im}\varphi$ is closed. Thus φ is surjective. $\square$

This answers the question how to go from a lattice to a cubic. Now let us think about the reverse direction.

Definition 2.5. A holomorphic 1-form ω on a Riemann surface Σ is a compatible collection of expressions $\{f(z)dz\}$ f holomorphic, ranging over the charts of Σ.

Spelt out, this means whenever we have charts $\varphi_1 : U_1 \to \mathbb{C}$ and $\varphi_2 : U_2 \to \mathbb{C}$ with expressions $f_1(z)dz$ and $f_2(z)dz$ on U_1 and U_2, respectively, with transition map $w = \varphi_2 \circ \varphi_1$, we want $f_2(w(z))d(w(z)) = f_1(z)dz$, i.e. $f_1(z) = f_2(w(z))w'(z)$.

Now define a holomorphic 1-form on $V(y^2 - x(x-1)(x-\lambda))$ by $\omega = \frac{dx}{y}$. When $x \neq 0, 1, \lambda, \infty$, then x is a local coordinate. Then $y \neq 0$ and everything is fine. If $x = 0$, then $w = \sqrt{x}$ is a local holomorphic coordinate. Then $x = w^2$ and $y = w\sqrt{(w^2-1)(w^2-\lambda)}$ as well as $dx = 2wdw$. Together,

$$\frac{dx}{y} = \frac{2}{\sqrt{(w^2-1)(w^2-\lambda)}}dw,$$

where the fraction is a holomorphic function of w near 0. The same arguments work for $x = 1$ and $x = \lambda$. At ∞, we had $w = x^{-\frac{1}{2}}$ as a holomorphic function and similar calculations show that everything works out. ω is nowhere vanishing: In a local chart $z, \omega = f(z)dz$, then $f(z) \neq 0$.

Proposition 2.6. *Any holomorphic 1-form on a Riemann surface Σ is closed as a $\mathbb{C}$-valued differentiable 1-form.*

There is a map $d : \{\text{diff. } p\text{-forms}\} \to \{\text{diff. } p+1\text{-forms}\}$ given by

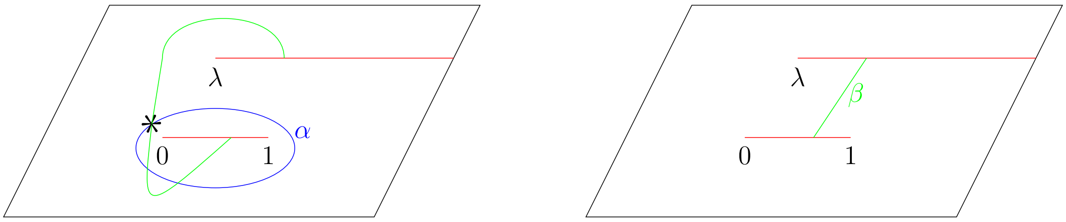
$$f dx_1 \wedge \cdots \wedge dx_p \mapsto \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_p.$$

Write $\omega = f(z)dz = f(x+iy)(dx + idy)$. Then $d\omega$ computes as

$$d\omega = i\frac{\partial f}{\partial x}dx \wedge dy + \frac{\partial f}{\partial y}dy \wedge dx = \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = 0$$

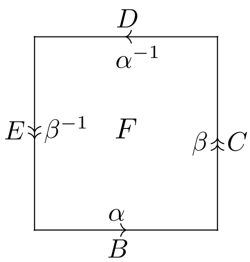
Consider $A(p) = \int_*^p \omega$ as a "function" on $\Sigma = V(y^2 - x(x-1)(x-\lambda)$. $A(p)$ depends on the chosen path. If γ_1, γ_2 are two homotopic paths from $*$ to p, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ by Stokes theorem. Hence A depends only on the homotopy class of the chosen path. If γ_1, γ_2 are two homotopy classes of paths from

$*$ to p, then $\int_{\gamma_1}\omega - \int_{\gamma_2}\omega = \int_{\gamma_2^{-1}\circ\gamma_1}\omega$ and $\gamma_2^{-1}\circ\gamma_1 \in \pi_1(\Sigma,*) \cong \mathbb{Z}^2$, since Σ is a torus. Set $v_1 = \int_\alpha \omega$, $v_2 = \int_\beta \omega$, where α, β are generators of $\pi_1(\Sigma,*)$, as indicated in the picture:



Then A is a single valued function with target $\mathbb{C}/\mathbb{Z}v_1 \oplus \mathbb{Z}v_2$. v_1 and v_2 are called the "Abelian" integrals. We can explicitly write $v_1 = 2\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$ and $v_2 = 2\int_0^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$. Claim: $v_1, v_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

Proof. Cut along α, β. You get a square F, denote its sides as in the figure.



Then $-i\int_\Sigma \omega\wedge\bar\omega > 0$, since locally, if $\omega = f(z)dz$, then

$$-i\omega\wedge\bar\omega = -if\bar f dz\wedge d\bar z = 2f\bar f dx\wedge dy.$$

On the other hand, $\int_\Sigma \omega\wedge\bar\omega = \int_F \omega\wedge\bar\omega = \int_F d(A)\wedge\bar\omega = \int_F d(A\cdot\bar\omega) = \int_{\partial F} A\bar\omega$ by Stokes. Note $\bar\omega|_B = \bar\omega|_{-D}$ and the same for C, E. Similarly $A|_B - A|_{-D}$ is equal to the constant function $\int_\beta \omega$ and $A|_C - A|_{-E} = \int_{-\alpha}\omega$. Hence

$$\begin{aligned}\int_\Sigma \omega\wedge\bar\omega &= \int_B A\bar\omega - \int_{-D} A\bar\omega + \int_C A\bar\omega - \int_{-E} A\bar\omega = \int_B\left(\int_\beta \omega\right)\bar\omega + \int_C\left(\int_{-\alpha}\omega\right)\bar\omega \\ &= \int_\alpha \bar\omega\int_\beta\omega - \int_\alpha\omega\int_\beta\bar\omega = \bar v_1 v_2 - v_1\bar v_2.\end{aligned}$$

Putting everything together, we have $-i(\bar v_1 v_2 - v_1\bar v_2) > 0$, i.e. $\mathrm{Im}(v_1\bar v_2) > 0$. $\square$

So $\Lambda := \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ is a lattice and $A : \Sigma \to \mathbb{C}/\Lambda$ is a locally invertible map into a torus. Hence A is a covering map and Σ compact implies the fibres of A are finite, i.e. Σ is a finite covering of $\mathbb{C}/\Lambda$. With some covering theory, this implies $\Sigma = \mathbb{C}/\Lambda'$, where $\Lambda' \subseteq \Lambda$ is a finite index sublattice.

Next we ask: Given lattices $\Lambda, \Lambda' \subseteq \mathbb{C}$, when are $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ isomorphic as Riemann surfaces?

Proposition 2.7. $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ *if and only if there exists* $c \in \mathbb{C}^\times$ *s.t.* $c\Lambda = \Lambda'$.

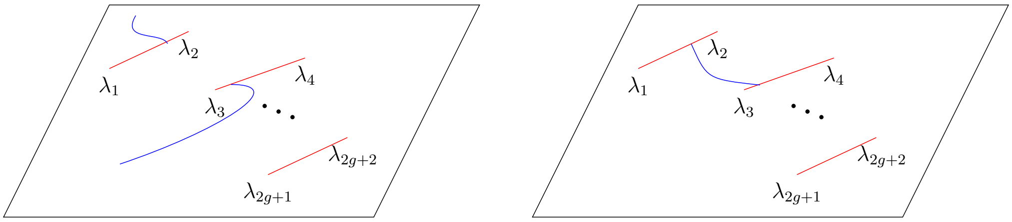
Proof. Let $i : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be an iso. and assume $i(0) = 0$. Lift i to the universal cover $\tilde\imath : \mathbb{C} \to \mathbb{C}$ isomorphism with $\tilde\imath(0) = 0$. This implies that $\tilde i$ is linear, i.e. $\tilde i(z) = cz$. Since $i(\Lambda) = \Lambda'$, it follows that $c\Lambda = \Lambda'$. The converse follows similarly. $\square$

Given any lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \subseteq \mathbb{C}$, multiply it by $\frac{1}{v_2}$ to get $\Lambda' \cong \mathbb{Z}\tau \oplus \mathbb{Z}$ where $\tau = \frac{v_1}{v_2}$. Assume $\mathrm{Im}(\tau) > 0$ (otherwise replace τ by $-\tau$). Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$ be the upper half-plane. Then $\tau, \tau' \in \mathbb{H}$ define the same complex torus if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Hence the space of 1-dimensional complex tori is in bijection to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

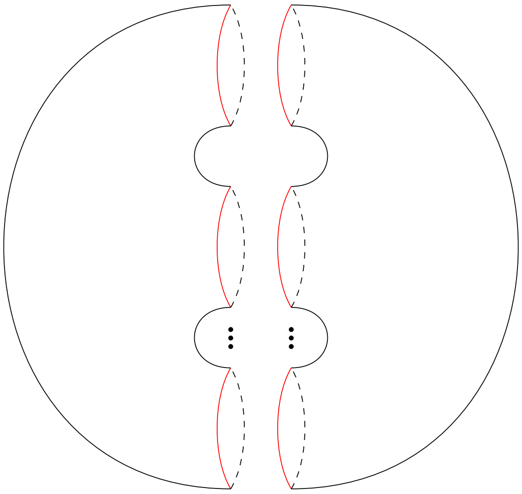
2.2 Curves of higher genus

Consider a Riemann surface Σ of genus g.

Example 2.8 (Hyperelliptic curves)**.** $y^2 = (x-\lambda_1)\cdots(x-\lambda_{2g+2})$. This corresponds to the Riemann surface Σ of the function $f(x) = \sqrt{(x-\lambda_1)\cdots(x-\lambda_{2g+2})}$, which has a unique analytic continuation to $\mathbb{C}$ without the $g+1$ cuts between λ_{2i-1} and λ_{2i}, $i = 1, \ldots, g+1$.



The result is a genus g Riemann surface and the local chart near $x = \lambda_i$ is $\sqrt{x-\lambda_i}$.



Again consider $\omega = \frac{dx}{y}$. For the same reasons as before, ω is holomorphic at $y = 0$. Near ∞, $w = \frac{1}{x}$ is a local corrdinate, and

$$\omega = \frac{d(1/w)}{\sqrt{(1/w-\lambda_1)\cdots(1/w-\lambda_{2g+2})}} = \frac{-1/w^2 dw}{1/w^{g+1}\sqrt{h(w)}} = -w^{g-1}h(w)^{-1/2}dw$$

for some holomorphic function h. Hence for $0 \le r \le g-1$, even $\omega_r = \frac{x^r dx}{y}$ is a holomorphic 1-form on Σ, so there is a g-dimensional vector space $\bigoplus_{r=0}^{g-1} \mathbb{C}\omega_r$ of holomorphic 1-forms.
Fact: Let $\Omega^1(\Sigma)$ be the $\mathbb{C}$-vector space of holomorphic 1-forms on a genus g compact Riemann surface Σ. Then $\dim_{\mathbb{C}} \Omega^1(\Sigma) = g$. We will prove this later.

Exercise 2.9. What is the genus of $y^3 = x^6 - 1$?

2.3 De-Rham Cohomology

Let M be a real manifold of dimension d. Let $\bigwedge^p(M) := \{\text{smooth } p\text{-forms on } M\}$, that is smooth p-forms on an open cover that agree on intersections, where $\bigwedge^p(U) = \{\sum_{|I|=p} f_I dx_i \wedge \cdots \wedge dx_p\}$ with the f_I smooth. Now consider the de-Rham complex

$$0 \to \bigwedge^0(M) \xrightarrow{d_0} \bigwedge^1(M) \xrightarrow{d_1} \bigwedge^2(M) \to \cdots \to \bigwedge^d(M) \to 0$$

where

$$d(f dx_I) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

Now define the de-Rham cohomology as the homology of this complex, i.e.

$$H^p_{\mathrm{dR}}(M, \mathbb{C}) = \ker(d_p)/\operatorname{im}(d_{p-1}).$$

Theorem 2.10 (De Rham). $H^p_{dR}(M, \mathbb{C}) \cong H^p_{sing}(M, \mathbb{C})$.

Here, the map is defined as follows: Let $[\omega] \in H^p_{\mathrm{dR}}(M, \mathbb{C})$ be represented by $\omega \in \bigwedge^p(M)$ which is exact: $d_p\omega = 0$. Then define $[\omega] \mapsto (\sigma \mapsto \int_\sigma \omega)$.

Now consider M a complex manifold of $\mathbb{C}$-dimension d.

Definition 2.11. The *smooth* (p,q)*-forms* on $U \subseteq \mathbb{C}^n$ are defined as

$$\bigwedge^{p,q}(U) = \left\{ \sum_{\substack{|I|=p,\\ |J|=q}} f_{I\bar{J}} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge d\bar{x}_{j_1} \cdots \wedge d\bar{x}_{j_q} \right\}$$

where the $f_{I\bar{J}}$ are smooth functions and the x_i are the coordinates of $\mathbb{C}^n$. The smooth (p,q)-forms of a manifold M are forms locally of the type $\bigwedge^{p,q}(U)$.

This is well-defined because the pullback of a (p,q)-form under a holomorphic map is a (p,q)-form.

Example 2.12. $z\bar{z}dz$ is a smooth $(1,0)$-form on $\mathbb{C}$. It corresponds to $(x^2+y^2)(dx+idy) \in \bigwedge^1(\mathbb{R}^2)$. Similarly, $\bar{z}d\bar{z} \in \bigwedge^{0,1}(\mathbb{C})$. If (z,w) are the coordinates of $\mathbb{C}^2$, then $dz \wedge d\bar{w} + d\bar{z} \wedge dw \in \bigwedge^{1,1}(\mathbb{C})$.

Lemma 2.13. $\bigwedge^k(M_\mathbb{R}) = \bigoplus_{p+q=k} \bigwedge^{p,q}(M)$*, where* $M_\mathbb{R}$ *is* M *considered as a real manifold.*

Proof. If $z_1, \dots, z_n$ are local complex coordinates and $x_1, y_1, \dots, x_n, z_n$ the corresponding local real coordinates, then $z_i = x_i + iy_i$ and $x_i = \frac{1}{2}(z_i + \bar{z}_i)$, $y_i = \frac{1}{2}(z_i - \bar{z}_i)$, so one can directly translate expressions from each set into an expression from the other set. ☐

On a one-dimensional complex manifold Σ, the only spaces of (p,q)-forms to consider are $\bigwedge^{0,0}(\Sigma) = \{\text{loc. smooth functions}\}$, $\bigwedge^{1,0}(\Sigma)$, $\bigwedge^{0,1}(\Sigma)$ and $\bigwedge^{1,1}(\Sigma)$. Hence the de-Rham complex is

$$0 \to \bigwedge^{0,0}(\Sigma) \xrightarrow{d} \bigwedge^{0,1}(\Sigma) \oplus \bigwedge^{1,0}(\Sigma) \xrightarrow{d} \bigwedge^{1,1}(\Sigma)$$

and the first exterior derivative is $d = \partial_z \oplus \partial_{\bar{z}}$, i.e. given by $f \mapsto \frac{\partial f}{\partial z} d\bar{z} + \frac{\partial f}{\partial \bar{z}} dz$. A holomorphic 1-form on Σ is locally expressible as $\omega = f(z)dz$ with f holomorphic, $\omega \in \bigwedge^{1,0}(\Sigma)$. As before, we see that ω is closed:

$$d(f(z,\bar{z})dz) = \frac{\partial f}{\partial z} \underbrace{dz \wedge dz}_{=0} + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{=0} d\bar{z} \wedge dz = 0.$$

For a smooth complex-valued function $f \in \bigwedge^{0,0}(M)$ write

$$df = \underbrace{\frac{\partial f}{\partial z_1}dz_1 + \ldots + \frac{\partial f}{\partial z_d}dz_d}_{=:\partial f} + \underbrace{\frac{\partial f}{\partial \bar{z}_1}d\bar{z}_1 + \ldots + \frac{\partial f}{\partial \bar{z}_d}dz_d}_{=:\overline{\partial} f}$$

Then $d : \bigwedge^k(M) \to \bigwedge^{k+1}(M)$ decomposes into a sum $d = \partial + \overline{\partial}$ where

$$\partial(f_{I,\bar{J}}dz_I \wedge d\bar{z}_J) = \partial f_{I,\bar{J}} \wedge dz_I \wedge d\bar{z}_J$$

and similarly for $\overline{\partial}$. We get a double complex

$$\begin{array}{ccccc}
\bigwedge^{0,2}(M) & & & & \\
\bar{\partial}\Big\uparrow & & & & \\
\bigwedge^{0,1}(M) & \xrightarrow{\partial} & \bigwedge^{1,1}(M) & & \\
\bar{\partial}\Big\uparrow & & \bar{\partial}\Big\uparrow & & \\
\bigwedge^{0,0}(M) & \xrightarrow{\partial} & \bigwedge^{1,0}(M) & \xrightarrow{\partial} & \bigwedge^{2,0}(M)
\end{array}$$

with the squares commuting up to sign. Also it is easy to check that $\bar{\partial} \circ \bar{\partial} = 0 = \partial \circ \partial$. The total complex of a double complex $(E^{p,q}, d_1, d_2)$ is $(\bigoplus_{p+q=k} E^{p,q}, d_1 + d_2)$. In this case, we get

$$\left(\bigoplus_{p+q=k} \bigwedge^{p,q}(M), \partial + \bar{\partial}\right) = (\bigwedge^k(M), d)$$

the original de Rham complex on M.

Example 2.14.

$$\bar{\partial}(z\bar{z}d\bar{w} + wd\bar{z}) = zd\bar{z} \wedge d\bar{w} + 0 \in \bigwedge^{0,2}(\mathbb{C}^2).$$

Definition 2.15. The *Dolbeault cohomology*

$$H^{p,q}(M) := \frac{\ker(\bar{\partial} : \bigwedge^{p,q}(M) \to \bigwedge^{p,q+1}(M))}{\operatorname{im}(\bar{\partial} : \bigwedge^{p,q-1}(M) \to \bigwedge^{p,q}(M))}$$

This is an analogue of de Rham cohomology when M is a complex manifold.

Theorem 2.16 (Poincaré $\bar{\partial}$-lemma)**.** *Let* $0 \in U \subseteq \mathbb{C}^n$ *be an open set,* $\omega \in \bigwedge^{p,q}(U)$*. Then, if* $\bar{\partial}\omega = 0$*, then there exists an open subset* $0 \in V \subseteq U$ *and an* $\alpha \in \bigwedge^{p,q-1}(V)$ *such that* $\bar{\partial}\alpha = \omega|_V$*.*

Proof. (for $n = 1$). Let $\Delta \subseteq \mathbb{C}$ be the unit disk. Let $gd\bar{z} \in \bigwedge^{0,1}(\overline{\Delta})$. Then $\bar{\partial}gd\bar{z}$ is automatically 0. Then

$$f(z, \bar{z}) := \frac{1}{2\pi i}\int_\Delta \frac{g(w, \bar{w})}{w - z}dw \wedge d\bar{w}$$

satisfies $\bar{\partial}f = gd\bar{z}$: Write $g = g_1 + g_2$ such that $\operatorname{supp} g_1 \subseteq B_{2\varepsilon}(z)$ and $\operatorname{supp}(g_2) \subseteq B_\varepsilon(z)^c$. Now

$$f = \frac{1}{2\pi i}\left(\int_\Delta \frac{g_1}{w - z}dw \wedge d\bar{w} + \underbrace{\int_\Delta \frac{g_2}{w - z}dw \wedge d\bar{w}}_{\bar{\partial}(-)=0}\right),$$

and

$$\int_\Delta \frac{g_1}{w - z}dw \wedge d\bar{w} = \int_{B_{2\varepsilon}(0)} \frac{g_1(z + u)}{u}du \wedge d\bar{u} = \frac{i}{2}\int_0^{2\pi}\int_0^{2\varepsilon} g_1(z + u)e^{-i\theta}drd\theta$$

is clearly the integral of a smooth function, hence smooth. One calculates

$$\begin{aligned}2\pi i\bar{\partial} f &= \bar{\partial}\int_{\Delta}\frac{g_1}{w-z}dw\wedge d\bar{w} = \lim_{\mu\to 0}\bar{\partial}\int_{B_{2\varepsilon}(z)-B_\mu(z)}\frac{g_1}{w-z}dw\wedge d\bar{w}\\
&= \lim_{\mu\to 0}\left(\int_{B_{2\varepsilon}(z)-B_\mu(z)}\underbrace{\frac{\partial g_1}{\partial\bar{w}}(w)\frac{1}{w-z}dw\wedge d\bar{w}}_{d\eta \text{ where } \eta=-\frac{g(w)dw}{w-z}}\right)d\bar{z}\\
&\overset{\text{Stokes}}{=}\lim_{\mu\to 0}\left(\int_{C_{2\varepsilon}(z)}\frac{-g_1(w)dw}{w-z}+\int_{C_{\mu(z)}}\frac{g_1(w)}{w-z}dw\right)d\bar{z}\\
&= 2\pi i g_1(z)d\bar{z} = 2\pi i g(z)d\bar{z}\end{aligned}$$

□

Let Σ be a Riemann surface of genus g. Let α_i, β_i, $i = 1, \dots, g$ be the loops as indicated in the picture.

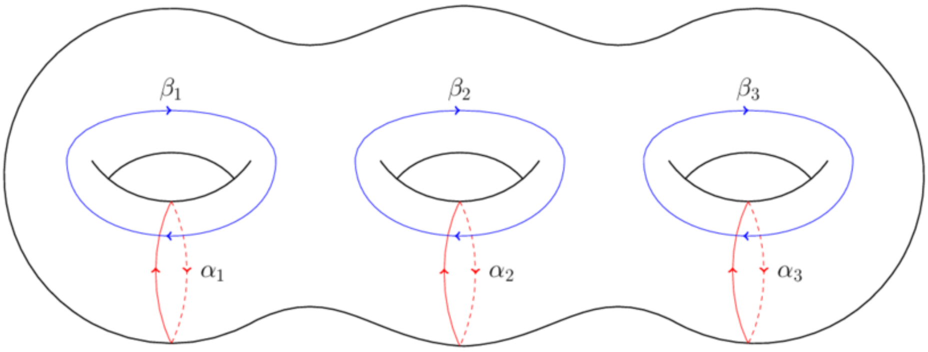


Figure 2: A genus 3 surface with a basis for its first homology

They form a basis of $H_1(\Sigma,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. By Poincaré duality, $H_1(\Sigma,\mathbb{Z}) \overset{PD}{\cong} H^1(\Sigma,\mathbb{Z})$. From the cohomological product structure one gets a pairing on $H_1(\Sigma,\mathbb{Z})$ by $\alpha\cdot\beta = \int_\Sigma PD(\alpha)\smile PD(\beta)$, which is the intersection form: Represent α,β by transversely intersecting cycles. Then $\alpha\cdot\beta = \sum_{p\in\alpha\cap\beta}\underbrace{\mathrm{or}_p(\alpha,\beta)}_{\in\pm 1}$.

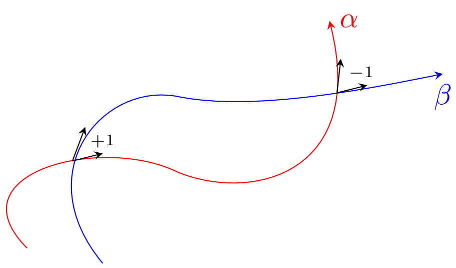


Figure 3: Two cycles α,β with intersection form $\alpha\cdot\beta = 1-1 = 0$.

The Intersection matrix of the chosen basis $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$ is given by $\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\in\mathbb{Z}^{2g\times 2g}$. Thus $H_1(\Sigma,\mathbb{Z})$ has the structure of a symplectic lattice, with an alternating map $(-,-):\Lambda\otimes\Lambda\to\mathbb{Z}$.

Choose generators α_i,β_i of $\pi_1(\Sigma,*)$ that are homologous to the α_i,β_i from before. Then $\pi_1(\Sigma,*) = \langle\alpha_i,\beta_i \mid \prod_i[\alpha_i,\beta_i]\rangle$ Recall $\dim\Omega^1(\Sigma) = g$. Let $\omega,\omega'\in\Omega^1(\Sigma)$.

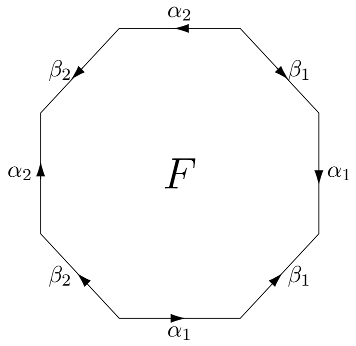


Figure 4: Cut up torus of genus 2

Theorem 2.17 (Riemann Bilinear Relations)**.** *There exists a unique basis $\Omega^1(\Sigma) = \bigoplus_{i=1}^g \mathbb{C}\omega_i$ such that $\int_{\alpha_j} \omega_i = \delta_{ij}$ and $(\int_{\beta_j} \omega_i)_{i,j}$ is a symmetric $g \times g$-matrix with positive definite imaginary part.*

Lemma 2.18.

$$\textit{Symmetry:} \qquad \sum_{i=1}^g \int_{\alpha_i} \omega \int_{\beta_i} \omega' - \int_{\alpha_i} \omega' \int_{\beta_i} \omega = 0.$$

$$\textit{Positivity:} \qquad i\sum_{i=1}^g \int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\beta_i} \omega \int_{\alpha_i} \bar{\omega} > 0.$$

Proof. Let $A(p) = \int_*^p \omega$ where $*$ is one of the vertices of F. $A(p)$ is holomorphic on F and welldefined since $d\omega = 0$. Further $dA = \omega$ on F.

$$0 = \int_\Sigma \omega \wedge \omega' = \int_F \omega \wedge \omega' = \int_F d(A\omega') = \int_{\partial F} A\omega'.$$

Now, as in the case of elliptic curves,

$$\int_{\alpha_1} A\omega' + \int_{\alpha_1^{-1}} A\omega' = \int_{\alpha_1} (A|_{\alpha_1} - A|_{\alpha^{-1}})\omega' = (A|_{\alpha_1} - A|_{\alpha_1^{-1}}) \int_{\alpha_1} \omega' = -\int_{\beta_1} \omega \int_{\alpha_1} \omega'.$$

Doing this for all i gives

$$\int_{\partial F} A\omega' = -\sum_{i=1}^g \int_{\beta_i} \omega \int_{\alpha_i} \omega' - \int_{\alpha_i} \omega \int_{\beta_i} \omega'.$$

This shows symmetry. For positivity, we have

$$0 < i\int_\Sigma \omega \wedge \bar{\omega} = i\int_F \omega \wedge \bar{\omega} = i\int_F d(A\bar{\omega}) = i\int_{\partial F} A\bar{\omega}$$

As before,

$$\int_{\alpha_i} A\bar{\omega} + \int_{\alpha^{-1}} A\bar{\omega} = -\int_{\beta_i} \omega \int_{\alpha_i} \bar{\omega}.$$

Doing this for all sides of F gives the result. □

Corollary 2.19. *There is no $\omega \in \Omega^1(\Sigma) \setminus 0$ such that $\int_{\alpha_i} \omega = 0$ for all i.* □

Corollary 2.20. $\dim_{\mathbb{C}} \Omega^1(\Sigma) \leq g$.

Proof. $\Omega^1(\Sigma) \to \mathbb{C}^n$, $\omega \mapsto (\int_{\alpha_i} \omega)_i$ is an injective linear map. □

Corollary 2.21. *If Σ is the Riemann surface of $\sqrt{(x-\lambda_1)\cdots(x-\lambda_{2g+2}} = y$, then*

$$\Omega^1(\Sigma) = \bigoplus_{r=0}^{g} \mathbb{C}\frac{x^r\,dr}{y}.$$

□

Proof. (of 3.8) Assume again $\Omega^1(\Sigma) = g$, which we will prove later. The map in the proof of 3.10 is then an iso, hence we can choose a basis that satisfies $\int_{\alpha_j} \omega_i = \delta_{ij}$.

Consider the "period matrix of Σ" $P = (P_{ij}) = (\int_{\beta_j} \omega_i)$. P is symmetric: Let ω_k, ω_l be elements of the normalized basis. Then

$$0 = \sum_{j=1}^{g} \int_{\alpha_j} \omega_k \int_{\beta_j} \omega_l - \int_{\alpha_j} \omega_l \int_{\beta_j} \omega_l = \sum_{j=1}^{g} \delta_{jk} \int_{\beta_j} \omega_l - \delta_{jl} \int_{\beta_l} \omega_k = \int_{\beta_k} \omega_l - \int_{\beta_l} \omega_k = P_{lk} - P_{kl}.$$

Let $\omega = c_1\omega_1 + \ldots + c_g\omega_g$ with $c_i \in \mathbb{R}$ not all 0. Then

$$\begin{aligned} 0 < i \sum_{j=1}^{g} \int_{\alpha_j} \omega \int_{\beta_j} \bar{\omega} - \int_{\beta_j} \omega \int_{\alpha_j} \bar{\omega} &= i \sum_{j,k=1}^{g} c_j \int_{\beta_j} \bar{c}_k \bar{\omega}_k - c_k \int_{\beta_j} \bar{\omega}_k \bar{c}_j \\ &= 2\,\mathrm{Im}\left(c_j \bar{c}_k \int_{\beta_j} \omega_k \right) = 2(\vec{c})^\dagger\,\mathrm{Im}\,P\vec{c} \end{aligned}$$

So $\mathrm{Im}\,P$ is positive definite as a bilinear form. □

Lemma 2.22. *Assuming* $\dim_{\mathbb{C}} \Omega^1(\Sigma) = g$, *then* $H^1_{dR}(\Sigma, \mathbb{C}) \cong \Omega^1(\Sigma) \oplus \overline{\Omega^1(\Sigma)}$

Proof. Every $\omega \in \Omega^1(\Sigma)$ is d-closed, hence so is every $\bar{\omega} \in \overline{\Omega^1(\Sigma)}$. So we have a map

$$\Omega^1(\Sigma) \oplus \overline{\Omega^1(\Sigma)} \xrightarrow{\varphi} H^1_{dR}(\Sigma, \mathbb{C}) \cong H_1(\Sigma, \mathbb{C})^* = \mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), \mathbb{C})$$

given by $\omega \mapsto \gamma \mapsto \int_\gamma \omega$ It is represented by the $2g \times 2g$-matrix (rows $\omega_i, \bar{\omega}_i$, columns α_j, β_j)

$$\begin{pmatrix} I & P \\ I & \bar{P} \end{pmatrix} \rightsquigarrow \begin{pmatrix} I & P \\ 0 & \bar{P} - P \end{pmatrix}$$

where $\bar{P} - P = -2i\,\mathrm{Im}\,P$ is positive definite, hence has full rank. Thus φ is injective and since the dimensions agree, it is an isomorphism. □

Let $* \in \Sigma$ be a base point. As in the genus 1 case, we can define a multivalued holomorphic map $p \mapsto (\int_*^p \omega_i)_i \in \mathbb{C}^g$. If γ, γ' are paths from $*$ to p, then $\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\gamma - \gamma'} \omega$ with $\gamma - \gamma' \in H_1(\Sigma, \mathbb{Z})$. Thus the value of our function is unique up to the free abelian group generated by $(\int_{\alpha_i} \omega_j)_j$ and $(\int_{\beta_i} \omega_j)_j$, $i, = 1, \ldots, g$. This is exactly $\mathbb{Z}^g \oplus P\mathbb{Z}^g \subseteq \mathbb{C}^g$. Since $\mathrm{Im}\,P > 0$, this is a discrete subgroup of $\mathbb{C}^g$ (Exercise).

Definition 2.23. The *Jacobian* of Σ is $\mathrm{Jac}(\Sigma) = \mathbb{C}^g / \mathbb{Z}^g \oplus P\mathbb{Z}^g$. This is a compact complex manifold of dimension g. Further, the *Abel-Jacobi map*

$$AJ : \Sigma \to \mathrm{Jac}(\Sigma), \quad p \mapsto \left(\int_*^p \omega_1, \ldots, \int_*^p \omega_g \right)$$

is single-valued, well-defined and holomorphic.

Note that $\mathrm{Jac}(\Sigma)$ is diffeomorphic to $(S^1)^{2g}$.

3 Holomorphic Vector Bundles

Definition 3.1. Let M be a complex manifold. A holomorphic vector bundle $\pi : \mathcal{E} \to M$ is a complex manifold $\mathcal{E}$ of rank r that has local trivializations

$$\begin{tikzcd} \mathcal{E}|_U \arrow[rr, "\pi"] \arrow[dr, "h_U"'] & & U \\ & U \times \mathbb{C}^r \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

with biholomorphic maps h_U such that $h_V \circ h_U^{-1} : (U \cap V) \times \mathbb{C}^r \to (U \cap V) \times \mathbb{C}^r$ is linear on every fibre and the induced map $t_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ is holomorphic.

Example 3.2. $\mathbb{C}^r \times M \to M$ is the trivial vector bundle.

Definition 3.3. A *section* of $\mathcal{E}$ over $U \subseteq M$ is a holomorphic map $s : U \to \mathcal{E}|_U$ such that $\pi \circ s = \mathrm{id}$. Denote the space of sections over U by $\mathcal{E}(U)$.

There is a holomorphic vector bundle Ω^p on M such that $\Omega^p(U)$ consists of the holomorphic p-forms on U: On a coordinate chart $U \hookrightarrow \mathbb{C}^n$, $\Omega^p(U) = \{\sum_{|I|=p} f_I dz_I \mid f_I \text{ holomorphic}\} = \bigoplus_{|I|=p} \mathrm{Hol}(U) \cdot dz_I$ with the transition functions the usual coorinate change. This is a holomorphic vector bundle of rank $r = \binom{\dim_{\mathbb{C}} M}{k}$ with $\Omega^k|_U \cong \mathbb{C}^r \times U$ trivialized by $\{dz_{i_1} \wedge \cdots \wedge dz_{i_k}\}$ in a local coordinate chart $(z_1, \ldots, z_d)$ with $d = \dim M$. If $\varphi_U : U \to \mathbb{C}^d, \varphi_V : V \to \mathbb{C}^d$ are two charts with local coordinates z, w, respectively, let us compute the coorinate change t_{UV} for Ω^1. This is the change of coordinates on 1-forms $\sum f_i dz_i \rightsquigarrow \sum g_i dw_i$. One obtains

$$\begin{pmatrix} \frac{dw_1}{dz_1} & \frac{dw_2}{dz_1} & \cdots & \\ \vdots & & & \vdots \\ & & \cdots & \frac{dw_d}{dz_d} \end{pmatrix} = \mathrm{Jac}(\varphi_V \circ \varphi_U^{-1}).$$

Note that the usual constructions on vector spaces, like $\mathrm{Hom}, \otimes, (-)^\vee$, exist for vector bundles. With this in mind, we can write $\Omega^k = \bigwedge^k \Omega^1$.

Note: Similarly to holomorphic vector bundles, one can define e.g. smooth vector bundles by requiring that the manifolds, trivializations and transition maps involved are smooth.

Given a complex manifold M, then $\bigwedge^{p,q}(M) \to M$ is a smooth vector bundle with trivialization given by $\{dz_I \wedge d\bar{z}_J \mid |I| = p, |J| = q\}$. In particular, $\bigwedge^{p,0}(M)$ is not the same as ω^p, since the first is considered as a smooth manifold: Looking at sections,

$$\textstyle\bigwedge^{p,0} = \{\sum f dz_I \mid f \text{ smooth}\} \quad \text{but} \quad \omega^p(U) = \{\sum f_I dz_I \mid f_I \text{ holomorphic}\}.$$

Definition 3.4. The holomorphic line bundle $\Omega^{\dim M}$ is called the canonical bundle K_M.

Crash Course on Sheaves

Let X be a topological space. A sheaf of abelian groups (or with values in a category $\mathcal{C}$) F on X is an assignment $\{\text{open sets of } X\} \to \mathtt{Ab}$, $U \mapsto F(U)$, together with restriction maps $\rho_{UV} : F(U) \to F(V)$ for all open $V \subseteq U$ such that $\rho_{UU} = \mathrm{id}_{F(U)}$, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$, $\rho(\emptyset) = 0$, and such that given $\{s_i \in F(U_i)\}$ with $\rho_{ij}(s_i) = \rho_{ji}(s_j)$ there exists a unique $s \in F(\bigcup U_i)$ with $s|_{U_i} = s_i$. Sheaves of abelian groups on X form an abelian category. A homomorphism $\varphi : F \to G$ is a collection of homomorphisms $\varphi(U)$ in $\mathcal{C}$ such that $\rho_{UV}\varphi(U) = \varphi(V)\rho(UV)$.

Any complex holomorphic vector bundle gives a sheaf E via $E(U)$ the holomorphic sections over U. This is even a sheaf of $\mathbb{C}$-vector spaces. For $\Omega^0 = \mathbb{C} \times M$, the trivial bundle, the corresponding sheaf $\mathcal{O} = \Omega^0$ is the sheaf of holomorphic functions $U \mapsto \mathcal{O}(U)$. Let $\mathcal{O}^*$ be the sheaf of nonvanishing holomorphic functions

$$\mathcal{O}^*(U) = \{f : U \to \mathbb{C}^* \mid f \text{ holomorphic}\}.$$

This is a sheaf of abelian groups.

Example 3.5. Let X be any topological space, A an abelian group. Then the constant sheaf on X with value A is the sheaf

$$\underline{A} : U \mapsto A(U) = \{\text{loc. const. functions } U \to A\}$$

For instance, if $X = \mathbb{R}; A = \mathbb{Z}$, then $\underline{\mathbb{Z}}((0,1) \cup (2,3)) = \mathbb{Z} \oplus \mathbb{Z}$. For M a complex manifold, define a morphism $\exp : \mathcal{O} \to \mathcal{O}^\times$ via $\mathcal{O}(U) \to \mathcal{O}^*(U)$, $f \mapsto e^f$. We want to compute the kernel K and cokernel Q of this map. For the cokernel, first set $Q^{\text{pre}}(U) = \mathcal{O}^*(U)/\exp(\mathcal{O}(U))$. For example, if $M = \mathbb{C}$ and $U = \mathbb{C}^*$, then $\mathcal{O}^*(\mathbb{C}^*)/\exp(\mathcal{O}(\mathbb{C}^*)) \cong \mathbb{Z}$, since given $f \in \mathcal{O}^*(\mathbb{C}^*)$ one can take the logarithm locally and analytically continue. Walking around the origin once, the difference is a logarithm of 1, i.e. an element of $2\pi i\mathbb{Z}$. This shows that there exists a unique $n \in \mathbb{Z}$ such that fz^n has a well-defined $\log$ on $\mathbb{C}^*$. On the other hand, for all $V \subseteq \mathbb{C}^*$ sufficiently small, $Q^{\text{pre}}(V) = 1$. Hence Q^{pre} is not a sheaf. Hence sheafify Q^{pre} to get

$$Q(U) = \{\text{compatible sections of } Q^{\text{pre}}(V_i) \text{ for a suff. small cover } U = \bigcup V_i\}.$$

Hence $Q = 1$. Similiarly, $K = K^{\text{pre}} = \underline{2\pi i\mathbb{Z}}$, since the kernel presheaf is already a sheaf. This gives a short exact sequence of abelian groups

$$0 \to 2\pi i\underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^\times \to 0,$$

the exponential exact sequence.

Čech Cohomology

Let $\mathcal{U} = \{U_i\}$ be an open cover of X and F a sheaf on X. Set

$$C^p_{\mathcal{U}}(X, F) = \bigoplus_{i_0 < \ldots < i_p} F(U_{i_0} \cap \cdots \cap U_{i_p}).$$

For example, if $X = S^1$, $F = \underline{\mathbb{Z}}$ and $\mathcal{U} = \{U_0, U_1\}$ with $U_0 = S^1 \setminus \{-1\}$, $U_1 = S^1 \setminus \{1\}$, then $C^0_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_0) \oplus \underline{\mathbb{Z}}(U_1) = \mathbb{Z} \oplus \mathbb{Z}$, and $C^1_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_0 \cap U_1) = \mathbb{Z}^2$. There is a coboundary $\partial^p : C^p_{\mathcal{U}}(X, F) \to C^{p+1}_{\mathcal{U}}(X, F)$ given by

$$(s_{i_0,\ldots,i_p}) \mapsto \left(\sum_{j=0}^{p+1} (-1)^j s_{i_0,\ldots,\hat{i}_j,\ldots,i_{p+1}} \right) |_{U_{i_0} \cap \ldots \cap U_{i_{p+1}}}$$

In the example, the map $C^0_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) \to C^1_{\mathcal{U}}(S^1, \underline{\mathbb{Z}})$ is given by $(a, b) \mapsto a|_{U_{01}} - b|_{U_{01}} = (a - b, a - b)$. Define the Čech cohomology with respect to $\mathcal{U}$ as the cohomology of this complex, i.e. $H^p_{\mathcal{U}}(X, F) = H^p(C^\bullet_{\mathcal{U}}(X, F), \partial^\bullet)$. In our case, $H^p_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \mathbb{Z}$ if $p = 0, 1$ and 0 else. This cohomology is dependent ono $\mathcal{U}$. For example, if $\mathcal{U} = \{S^1\}$, then $H^p_{\mathcal{U}}(S^1, \underline{\mathbb{Z}}) = \mathbb{Z}$ for $p = 0$ and 0 otherwise. If $\mathcal{V}$ refines $\mathcal{U}$, then one gets a natural map $H^p_{\mathcal{U}}(X, F) \to H^p_{\mathcal{V}}(X, F)$ by using the restriction maps.

Definition 3.6. The Čech cohomology is

$$H^p(X, F) = \operatorname{colim}_{\mathcal{U}} H^p_{\mathcal{U}}(X, F)$$

If $\mathcal{U}$ is a good cover of M, i.e. all intersections of the U_i are contractible, then $H^p(X,F) = H^p_{\mathcal{U}}(X,F)$. For any short exact sequence $0 \to F \to G \to H \to 0$ of sheaves on X, we get a long exact sequence

$$0 \to H^0(X,F) \to H^0(X,G) \to H^0(X,H) \to H^1(X,F) \to \cdots$$

Let $E \to M$ be a holomorphic vector bundle. Consider the smooth vector bundlees $E \otimes \bigwedge^{0,p}$ for $p \geq 0$. There is an exact sequence

$$0 \to E \to E \otimes_{\mathcal{O}} \bigwedge^{0,0} \xrightarrow{1\otimes\bar\partial} E \otimes_{\mathcal{O}} \bigwedge^{0,1}$$

where the middle term is just the sheaf of smooth sections of E. Remember that the Poincare $\bar\partial$-lemma says that if $\alpha \in \bigwedge^{0,p}(U)$ s.t. $\bar\partial\alpha = 0$, then there exists $V \subseteq U$ open and $\beta \in \bigwedge^{0,p-1}(V)$ such that $\bar\partial\beta = \alpha|_V$. Hence on $U \subseteq M$ a trivializing chart for E, we have $E|_U = \mathcal{O}_U^{\oplus r}$, hence restricted to U, the Poincare lemma says that the sequence

$$0 \to \mathcal{O}_U^{\oplus r} \to \bigwedge^{0,0}(U)^{\oplus r} \to \bigwedge^{0,1}(U)^{\oplus r} \to \cdots$$

is exact. Hence this works globally and we obtain

Proposition 3.7. *There exists an exact sequence of sheaves*

$$0 \to E \to E \otimes_{\mathcal{O}} \bigwedge^{0,0} \xrightarrow{1\otimes\bar\partial} E \otimes_{\mathcal{O}} \bigwedge^{0,1} \xrightarrow{1\otimes\bar\partial} E \otimes_{\mathcal{O}} \bigwedge^{0,2} \xrightarrow{1\otimes\bar\partial} \cdots$$

This complex is called the Dolbeault complex *for E.*

Before that, we should check

Proposition 3.8. *$\bar\partial$ is well-defined, independent of the coordinate chart.*

Proof. Let $s_U \otimes \omega_U \in E \otimes \bigwedge^{0,q}(U)$ and $s_V \otimes \omega_V \in E \otimes \bigwedge^{0,q}(V)$. Let $\varphi_U : U \to \mathbb{C}^d$ and $\varphi_V : V \to \mathbb{C}^d$ be the coordinate charts and $t_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$ the transition function of E, which is holomorphic. We have $s_V \otimes \omega_W = t_{UV} s_U \otimes (\varphi_U \circ \varphi_V^{-1})^* \omega_U$. Now $\bar\partial\omega_v = \bar\partial(\varphi_U \circ \varphi_V^{-1})^*\omega_U$ because $\bar\partial(\varphi_U \circ \varphi_V^{-1}) = 0$. □

We want to prove the following

Theorem 3.9. *Let V be any C^∞-vector bundle on a smooth manifold X. Then $H^i(X,V) = 0$ for $i > 0$.*

Definition 3.10. A sheaf F on X is *flasque* if for any $V \subseteq U$ opens, ρ_{UV} is surjective. F is *soft* if for any $Z \subseteq X$ closed, $F(X) \to F(Z)$ is surjective, where $F(Z) := \operatorname{colim}_{U \supseteq Z} F(U)$.

For example, take $V = C^\infty$ the trivial bundle over $\mathbb{R}$, then $\frac{1}{x}$ does not lie in the image of $C^\infty(\mathbb{R}) \to C^\infty((0,1))$. So C^∞-vector bundles are usually not flasque. However, C^∞ is soft on $\mathbb{R}$ (exercise). This also holds for any C^∞-vector bundle on a manifold.

Lemma 3.11. *A flasque sheaf is soft.* □

Lemma 3.12. *Let $0 \to F \to G \to H \to 0$ be an exact sequence with F soft. Then $0 \to F(X) \to G(X) \to H(X) \to 0$ is exact.* □

Corollary 3.13. *If F and G are soft, so is H.*

Proof. Let $Z \subseteq X$ be closed and $s \in H(Z)$. By lemma 3.12, there exists $t \in G(Z)$ that maps to s. By assumption, there is a $\widetilde{t} \in G(X)$ restricting to t. Then the image of $\widetilde{t}$ in $H(X)$ restricts to s. □

Proposition 3.14. *If $0 \to F_0 \to F_1 \to \dots$ is an exact sequence of soft sheaves then $0 \to F_0(X) \to F_1(X) \to \dots$ is also exact.* □

Recall: An injective sheaf I is one such that for any $\varphi : A \to I$ and any inclusion $A \to B$, there is an extionsion $\widetilde{\varphi} : B \to I$ such that $\widetilde{\varphi}|_A = \varphi$.

Theorem 3.15. *Sheaves of abelian groups on a paracompact space admit injective resolutions, and sheaf cohomology can be computed as the homology of any such resolution.*

Proof. Omitted. □

Injective sheaves are flasque (exercise) and hence soft. Thus by proposition 3.14, $H^i(X, F) = 0$ if F is soft. This proves theorem 3.9. In particular, $H^i(X, E \otimes \bigwedge^{0,q}) = 0$ for all $i > 0$. So the Dolbeault resolution is an acyclic resolution of E.

Proposition 3.16. *An acyclic resolution of E computes $H^i(E)$: Given $0 \to E \to F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots$ acyclic, then*

$$H^i(E) = \frac{\ker(F_i(X) \to F_{i+1}(X))}{\operatorname{im}(F_{i-1}(X) \to F_i(X))}$$

Proof. Split up the resolution into short exact sequences $0 \to \ker d_i \to F_i \to \ker d_{i+1} \to 0$ and use the associated long exact sequences in cohomology. □

This implies *Dolbeault's theorem*

$$H^{p,q}(X) = \frac{\ker(\bar{\partial} : \bigwedge^{p,q}(X) \to \bigwedge^{p,q+1}(X)}{\operatorname{im}(\bar{\partial} : \bigwedge^{p,q-1}(X) \to \bigwedge^{p,q}(X))} \cong H^q(X, \Omega^p)$$

from the Dolbeault complex for $E = \Omega^p$.

Line Bundles on $\mathbb{CP}^n$

Definition 3.17. The *tautological line bundle* $\mathcal{O}(-1)$ of $\mathbb{CP}^n$ is the total space of lines through 0 in $\mathbb{C}^{n+1}$

As complex manifolds, we have $\mathcal{O}(-1) \cong \operatorname{Bl}_0 \mathbb{C}^{n+1} \xrightarrow{\pi} \mathbb{CP}^n$, where π is the blow-up map followed by the natural projection.

Consider the dual $\mathcal{O}(1) \cong \mathcal{O}(-1)^*$. Let $U \subseteq \mathbb{CP}^n$ be open. Then $\mathcal{O}(1)(U)$ consists of holomorphically varying families of linear functions on the tautological lines through U. That is,

$$\mathcal{O}(1)(U) = \{f : \pi^{-1}(U) \to \mathbb{C} \text{ holomorphic} \mid f(\lambda x) = \lambda f(x)\}.$$

In particular, $\mathcal{O}(1)(\mathbb{CP}^n)$ is the set of linear forms on $\mathbb{C}^{n+1}$.

Next we can define $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ and $\mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k}$ for $k > 0$. This yields a collection of line bundles such that

$$\mathcal{O}(m)(U) = \{f : \pi^{-1}(U) \to \mathbb{C} \mid f(\lambda x) = \lambda^m f(x)\}.$$

For $U = \mathbb{CP}^n$ one has $\mathcal{O}(m)(\mathbb{CP}^n) \cong \mathbb{C}[x_0, \ldots, x_n]^{(m)}$.

More generally, if $X \subseteq \mathbb{CP}^n$ is a projective variety, we can restrict $\mathcal{O}(m)$ to X, which we denote $\mathcal{O}_X(m)$.

3.1 Line Bundles associated to divisors

Definition 3.18. Let X be a smooth projective variety. A divisor on X is a $\mathbb{Z}$-linear combination of irreducible, codimension 1 subvarieties. More generally, if X is a complex manifold, take the $\mathbb{Z}$-linear combination of all closed subsets of X that are locally cut out by a simgle holomorphic function.

If X is a Riemann surface, then codimension 1 subvarieties are points, hence $\mathrm{Div}(X) = \bigoplus_{p \in X} \mathbb{Z}[p]$.

Associate to any divisor D on X the holomorphic line bundle $\mathcal{O}_X(D)$ constructed as follows: For curves, declare $\mathcal{O}_X(D)|_U = \mathcal{O}_U$ where $U = X \setminus \operatorname{supp} D$. Let V_i be an open neighbourhood of $P_i \in \operatorname{supp} D$ that doesn't contain any other element of $\operatorname{supp} D$. Now $\mathcal{O}_X(D)|_{V_i} \cong \mathcal{O}_V$. Take $t_{UV_i} : U \cap V \to \mathbb{C}^\times$ to be $t_{UV_i} = z_i^{n_i}$, where $z_i : V_i \to \mathbb{C}$ is the local coordinate $P_i \to 0$ and n_i is the coefficient of $[P_i]$ in D.

In general, if f is a meromorphic function, define $\operatorname{div}(f) = \sum_P \operatorname{ord}_p(f)[P] \in \mathrm{Div}(X)$. Then if V is a small open, write $D \cap V = \operatorname{div}(f)$ for a suitable meromorphic function f and use this f to define the transition function. (Fact from Hartshorne: If X is smooth, every Weil divisor is Cartier.)

Definition 3.19. $D = \sum n_i[X_i] \in \mathrm{Div}(X)$ is effective if $n_i \geq 0$ for all i.

If D is effective, there is a section $s_D \in \mathcal{O}_X(D)(X)$ given by $h_U(s_D|_U) = 1$ where h_U is the trivialization to $U \times \mathbb{C}$. On $U \cap V_i$, we need $h_V(s_D|_V) = t_{UV}(1) = z_i^{n_i}$, which by assumption extends holomorphically over $P_i \in V_i$. If D is not effective, s_D defines a meromorphic section.

Proposition 3.20. *Let $\mathcal{L} \to X$ be a holomorphic line bundle over a Riemann surface X. Then if $s \in \mathcal{L}(X)$ (or if s is a meromorphic section),* $\operatorname{ord}_p(s)$ *is well-defined, where* $\operatorname{ord}_p(s) := \operatorname{ord}_p(h_U(s))$ *for any trivializing chart $U \ni p$.*

Proof. If V is any other such chart, then

$$\operatorname{ord}_p(h_V(s)) = \operatorname{ord}_p(t_{UV}(h_U(s))) = \operatorname{ord}_p t_{UV} + \operatorname{ord}_p h_U(s) = \operatorname{ord}_p h_U(s)$$

since t_{UV} is invertible. □

Proposition 3.21. *$\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X(D')$ iff there is a meromorphic function f on X such that* $\operatorname{div}(f) = D - D'$.

Proof. Let $s, s' \in \mathcal{L}(X)$ for $\mathcal{L}$ a holomorphic line bundle, $s, s' \neq 0$. Then s/s' is a well-defined meromorphic function, defined on an open chart U as $h_U(s)/h_U(s')$, since on a chart V,

$$\frac{h_V(s)}{h_V(s')} = \frac{t_{UV} h_U(s)}{t_{UV} h_U(s')} = \frac{h_U(s)}{h_U(s')}.$$

If $\mathcal{L} = \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, then $\operatorname{div}(s_D/s_{D'}) = D - D'$.

Conversely, suppose there is a meromorphic $f : X \to \mathbb{C}$ such that $\operatorname{div}(f) = D - D'$. On $(\operatorname{supp} D \cup \operatorname{supp} D')^c$, multiplication by f induces an isomorphism. One checks that this extends to the whole line bundles. □

Definition 3.22. Let $\mathrm{PDiv}(X) = \{\operatorname{div} f \mid f \in \mathbb{C}(X)^*\} \subseteq \mathrm{Div}(X)$ be the subgroup of *principal divisors*. The *divisor class group* is $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$.

Aside: Let $X = \operatorname{Spec} \mathcal{O}_K$ for $K/\mathbb{Q}$ a number field. As in number theory, $\mathrm{Div}(X) = \bigoplus_{0 \neq \mathfrak{p} \subseteq \mathcal{O}_k} \mathbb{Z}[\mathfrak{p}]$ and $\mathrm{PDiv}(X) = \{\sum n_\mathfrak{p}[\mathfrak{p}] \mid \prod \mathfrak{p}^{n_\mathfrak{p}} = (a) \text{ for some } a \in K\}$. Then $\mathrm{Cl}(X)$ is the class group of the number field, e.g. $\mathrm{Cl}(\operatorname{Spec} \mathbb{Z}[i]) = 1$ or $\mathrm{Cl}(\operatorname{Spec} \mathbb{Z}[\sqrt{-5}]) = \{1, (1 + \sqrt{-5}, 2)\}$.

Is every holomorphic line bundle $\mathcal{L} \to X$ of the form $\mathcal{L} \cong \mathcal{O}_X(D)$ for some divisor D? Equivalently, does every $\mathcal{L} \to X$ admit a meromorphic section $s \in \mathrm{Mero}(X, \mathcal{L})$? (If so, $\mathcal{L} \cong \mathcal{O}(\operatorname{div} s)$.)

Definition 3.23. Let $\operatorname{Pic} X$ denote the set of isomorphism classes of holomorphic line bundles $\mathcal{L} \to X$, which is a group under the tensor product.

Indeed, if $\mathcal{L}_1 \to X$ and $\mathcal{L}_2 \to X$ are line bundles, their tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2 \to X$ naturally admits the structure of a line bundle: For U, V trivializing charts of both $\mathcal{L}_i$, given the transition functions $t^i_{UV} : U \cap V \to \mathcal{C}^\times$ for $\mathcal{L}^i$, the transition function for $\mathcal{L}_1 \otimes \mathcal{L}_2$ is $t_{UV} = t^1_{UV} t^2_{UV}$.

Note that by proposition 3.21, the map $\mathrm{Cl}(X) \to \mathrm{Pic}(X), [D] \mapsto \mathcal{O}_X(D)$ is a well-defined group homomorphism.

Example 3.24. $\mathrm{Cl}(\mathbb{CP}^n) = \mathbb{Z}$. Let $D \subseteq \mathbb{CP}^n$ be a divisor. It has the form $D = V(f_d)$ where $f_d \in \mathbb{C}[x_0, \ldots, x_n]^{(d)} \setminus 0$. (see below) If $D' = V(g_d)$ is another divisor of the same degree, then $D - D' = \mathrm{div}(f_d/g_d)$ with $f_d/g_d \in \mathbb{C}(\mathbb{CP}^n)^\times$, so $[D] = [D']$.

Chow's Lemma: If X is a projective variety over $\mathbb{C}$, every closed analytic subspace of X is algebraic.

Example 3.25. $y - e^x = 0$ in $\mathbb{C}^2$ is not algebraic, but $\mathbb{C}^2$ is not projective. What about its closure in $\mathbb{CP}^1 \times \mathbb{CP}^1$? Since e^x has a transcentental pole at ∞, it obtains almost all values in any neighbourhood of ∞. Therefore, the closure contains $\infty \times \mathbb{CP}^1$. So it is not an analytic subspace of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

3.2 Hermitian Metrics

Let $E \to X$ be a holomorphic vector bundle.

Definition 3.26. A *hermitian metric* is a smoothly varying hermitian metric $h_x.\overline{E}_x \otimes E_x \to \mathbb{C}$. In other words, $h \in C^\infty(X, \overline{E}^* \otimes E^*)$, where E^* is the dual line bundle, with transition functions t^{-1}_{UV}, if t_{UV} is the transition function on E, and $h(\overline{e}, e) > 0$ for $e \in E(U)$ nonvanishing.

Example 3.27. Let $D = \sum n_p[P] \in \mathrm{Div}(C)$, C a Riemann surface. For $P_i \in \operatorname{supp} D$, let $p \in W_i \subset V_i$ be opens and $U = (\bigcup W_i)^c$. Then $\mathcal{O}(D)|_U = \mathcal{O}_U$.

Let $s_D \in \mathrm{Mero}(C, \mathcal{O}(D))$ such that $\mathrm{div}(s_D) = D$. Define $h_x(\overline{s}_D, s_D) = 1$ on $x \in (\bigcup V_i)^c$ and $h_x(\overline{s}_D, s_D) = |x|^{2n_i}$ in the chart $W_i \ni P_i$ with local coordinate x. In the trivialization $\mathcal{O}(D)|_{V_i} \cong \mathcal{O}_{V_i}$, $h_x(1,1) = 1$. Then smoothly interpolate $h(\overline{s}_D, s_D)$ on the annuli $V_i \setminus W_i$

If h is a hermitian metric $E \to X$, then $h(\bar{s}, t) \in C^\infty(X)$, for $s, t \in C^\infty(X, E)$.

Proposition 3.28. *Let $\mathcal{L} \to X$ be a holomorphic line bundle with a hermitian metric h. Let $s \in \mathcal{L}(U)$ be a local holomorphic nonvanishing section (s generates $\mathcal{L}|_U$). Then $\frac{i}{2\pi}\partial\bar\partial \log h(s, \bar{s})$ is independent of s.*

Proof. If $s' \in \mathcal{L}(U)$ is some other generator, then $s' = fs$ for some $f \in \mathcal{O}^\times(U)$. Then

$$\frac{i}{2\pi}\partial\bar\partial \log h(\overline{fs}, fs) = \frac{i}{2\pi}\partial\bar\partial\big(\log h(\bar{s}, s) + \log \bar{f} + \log f\big) = \frac{i}{2\pi}\partial\bar\partial \log h(\bar{s}, s),$$

since $\log f$ is killed by $\bar\partial$ and $\log \bar{f}$ by ∂. □

Set $c_1(\mathcal{L}) = [\frac{i}{2\pi}\partial\bar\partial \log h(s, \bar{s})] \in H^2_{dR}(X, \mathbb{C})$, the *first Chern class* of $\mathcal{L}$. This is independent of h: If h' is another hermitian metric on $\mathcal{L}$, then $h' = ch$ for $c \in C^\infty(X)$, $c > 0$. Hence $\frac{i}{2\pi}\partial\bar\partial \log h'(s, \bar{s}) = \frac{i}{2\pi}\partial\bar\partial(\log c + \log h(s, \bar{s})$ and $\partial\bar\partial \log c = d\bar\partial \log c$ is exact.

Example 3.29. We want to compute $c_1(\mathcal{O}(D))$ for a divisor D with the hermitian form defined above. Note that $\partial\bar\partial \log h(\bar{s}, s)$ is supported in the annuli $V_i \setminus W_i$, say these opens are chosen such that in the trivializations, these annuli have the form $\frac{\varepsilon}{2} < |z| < \varepsilon$: In the local frame $s_D = z^{n_k}$, we have $h(1,1) = 1$, so $h(\bar{z^{n_k}}, z^{n_k}) = |z|^{2n_k}$. Note $H^2(C, \mathbb{C}) = \mathbb{C}[p]$, $[p]$ the fundamental class of a point, and the coefficient of $[p]$ is

$$\begin{aligned}
\frac{i}{2\pi}\int_C \partial\bar\partial \log h &= \sum_k \frac{i}{2\pi}\int_{\frac{\varepsilon}{2}<|z_k|<\varepsilon} d\bar\partial \log h \\
&= \sum_k \frac{i}{2\pi}\int_{|z_k|=\varepsilon} \bar\partial \log \underbrace{h(\bar s_D, s_D)}_{=1} - \int_{|z_k|=\varepsilon/2} \bar\partial \log \underbrace{h(\bar s_D, s_D)}_{=z^{n_k}\overline{z^{n_k}}} \\
&= \sum_i \frac{i}{2\pi}(-n_k)\int_{|z_k|=\varepsilon/2}\frac{d\bar z}{\bar z} = \sum_k \frac{i}{2\pi}(-n_k)\overline{2\pi i} = \sum_k n_k = \deg D.
\end{aligned}$$

Thus $c_1(\mathcal{O}(D)) = (\deg D)[p]$.

Recall the exponential exact sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 1$. A holomorphic line bundle defines a class $[t] \in H^1(X, \mathcal{O}^\times)$: Let t_{UV} be the transition functions. These satisfy $t_{U_0U_1}t^{-1}_{U_0U_2}t_{U_1U_2} = 1$, so $t = \{t_{U_iU_j}\} \in Z^1(X, \mathcal{O}^*)$. The long exact sequence in cohomology associated to the exponential exact sequence maps $[t]$ to $c_1(\mathcal{L})$, in particular, $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$.

On a higher dimensional complex manifold X, we have $c_1(\mathcal{O}(D)) \in H^2(X,\mathbb{C})$. Let $i : \Sigma \hookrightarrow X$ be a closed smooth topological oriented surface. Then

$$[\Sigma] \in H_2(\Sigma,\mathbb{Z}) \xrightarrow{i_*} H_2(X,\mathbb{Z}) \xrightarrow{\text{P.D.}} H^{2d-2}(X,\mathbb{Z})$$

Represent $[\Sigma]$ by a surface transversly intersecting D. We may further assume that the arcs of Σ at intersection points are holomorphic. Define a hermitian form as before. Then

$$c_1(\mathcal{O}(D))([\Sigma]) = \int_\Sigma i^* c_1(\mathcal{O}(D)) = \sum n_k[\Sigma]\cdot[P_k],$$

so $c_1(\mathcal{O}(D)) = \sum n_k PD[P_k]$. Suppressing the Poincaré dual, we have proven

Proposition 3.30. *For a divisor $D = \sum n_k[P_k]$ on a complex manifold, we have*

$$c_1(\mathcal{O}(D)) = \sum_k n_k[P_k] \in H^2(X,\mathbb{Z}).$$

Recall the Dolbeault complex for a holomorphic line bundle $\mathcal{L} \to C$, C a Riemann surface:

$$0 \to C^\infty(C,\mathcal{L}) \xrightarrow{\bar\partial} C^\infty(C, \mathcal{L}\otimes \textstyle\bigwedge^{0,1}) \to 0$$

with $\ker(\bar\partial) = H^0(C,\mathcal{L})$ and $\operatorname{coker}(\bar\partial) = H^1(C,\mathcal{L})$. Choose a hermitian metric h on $\mathcal{L}$, and let $d\nu$ be a volume form on C.

Definition 3.31. Let $s, t \in C^\infty(C,\mathcal{L})$. We define a pairing $\langle s,t\rangle := \int_C h(\bar s, t)d\nu$

Observe $\langle s,s\rangle = \int_C h(\bar s, s)d\nu$, so if s is not identically 0, then $\langle s,s\rangle > 0$. Hence $\langle\cdot,\cdot\rangle$ is a positive-definite hermitian form on $C^\infty(C,\mathcal{L})$. However, $(C^\infty(C,\mathcal{L}), \langle\cdot,\cdot\rangle)$ is not a complete inner product space. Its completion is defined to be $L^2(C,\mathcal{L})$.

Definition 3.32. Similarly, let $s\otimes\alpha, t\otimes\beta \in C^\infty(C,\mathcal{L}\otimes\bigwedge^{0,1})$ and define

$$\langle s\otimes\alpha, t\otimes\beta\rangle = i\int_C h(\bar s, t)\bar\alpha\wedge\beta.$$

Again, $\langle s\otimes\alpha, s\otimes\alpha\rangle = \int_C h(\bar s, s)(i\bar\alpha\wedge\alpha)$ is positive when $s\otimes\alpha \neq 0$, so $\langle\cdot,\cdot\rangle$ defines a hermitian form.

Let $\varphi : V \to W$ be a map of inner product spaces. Its adjoint $\varphi^\dagger$ is defined by the property $\langle w, \varphi(v)\rangle = \langle \varphi^\dagger(w), v\rangle$. We compute the adjoint of $\bar\delta$. Let $s \otimes \alpha \in C^\infty(C, \mathcal{L} \otimes \bigwedge^{0,1})$ and $t \in C^\infty(C, \mathcal{L})$. Then

$$\int_C h \cdot \overline{s \cdot \alpha} \cdot \bar\partial t = \langle s \otimes \alpha, \bar\partial t\rangle = \langle \bar\partial^\dagger(s \otimes \alpha), t\rangle = \int_C h(\overline{\bar\partial^\dagger(s \otimes \alpha)}, t) d\nu$$

Notice $h \cdot \bar s \cdot \bar\alpha \in C^\infty(\mathcal{L}^{-1} \otimes \bigwedge^{1,0}) \cong C^\infty(\mathcal{L}^{-1} \otimes K_C)$, which has a natural $\bar\delta$-operator. Integration by parts yields

$$\langle s \otimes \alpha, \bar\partial t\rangle = -\int_C \bar\delta(h \cdot \bar s \cdot \bar\alpha) \cdot t = -\int_C \frac{\bar\partial(h \cdot \bar s \cdot \bar\alpha)}{d\nu} \cdot t d\nu = \left\langle \overline{-h^{-1}\frac{\bar\partial(h \cdot \bar s \cdot \bar\alpha)}{d\nu}}, t \right\rangle.$$

Therefore,

$$\bar\partial^\dagger(s \otimes \alpha) = \overline{\frac{-h^{-1}\bar\partial(h \cdot \bar s \cdot \bar\alpha)}{d\nu}}.$$

Theorem 3.33. *It is a fact from Analysis that $C^\infty(\mathcal{L} \otimes \bigwedge^{0,1}) \cong (\operatorname{im} \bar\partial) \oplus (\operatorname{im} \bar\partial)^\perp$.*

Assume $\langle s \otimes \alpha, \bar\partial t\rangle = 0$ for all t. This is equivalent to $\langle \bar\partial^\dagger(s \otimes \alpha), t\rangle = 0$, hence to $\bar\partial^\dagger(s \otimes \alpha) = 0$. Looking at the formula for $\bar\partial^\dagger$, this is true precisely if $\bar\partial(h \cdot \bar s \cdot \bar\alpha) = 0$, so $h \cdot \bar s \cdot \bar\alpha \in H^0(C, \mathcal{L}^{-1} \otimes K_C)$. So

$$H^1(C, \mathcal{L}) \cong \operatorname{coker}(\bar\partial) \cong (\operatorname{im} \bar\partial)^\perp \cong H^0(C, \mathcal{L}^{-1} \otimes K_C).$$

This is Serre duality. Note the missing dual, which stems from having chosen inner products, thus identifying spaces with their duals. More canonically:

Given $s \otimes \alpha$ representing an element of $H^1(C, \mathcal{L})$ by $C^\infty(\mathcal{L} \otimes \bigwedge^{0,1})$, there is a pairing with $C^\infty(\mathcal{L}^{-1} \otimes \bigwedge^{1,0}) \ni \varphi$ given by $\int_C s \cdot \alpha \cdot \varphi$ inducing a perfect pairing $H^0(C, \mathcal{L}^{-1} \otimes K_C) \times H^1(C, \mathcal{L}) \to \mathbb{C}$, i.e.

Proposition 3.34 (Serre duality). *Let C be a Riemann surface and $\mathcal{L}$ a holomorphic line bundle. Then*

$$H^1(C, \mathcal{L}) \cong H^0(C, \mathcal{L}^{-1} \otimes K_C)^\vee.$$

Proposition 3.35. *Let $\mathcal{L} \to C$ be a holomorphic line bundle on a compact Riemann surface C. Then $H^0(C, \mathcal{L})$ is finite-dimensional.*

Proof. If $h^0(C, \mathcal{L}) = \infty$, then choose local trivialization of $\mathcal{L}|_{U_i} \cong \mathcal{O}_U$. There exists some nonzero $s \in H^0(C, \mathcal{L})$ such that $\operatorname{ord}_p(s) > d$ for any d: Indeed, choose linearly independent sections $s_1, s_2, \ldots \in \mathcal{O}_U(U)$, which can be written as elements of $\mathbb{C}[\![z]\!]$. Then $\varphi : \mathbb{C}[\![z]\!] \to \mathbb{C}[\![z]\!]/(z^d)$, then $\ker \varphi \cap \langle s_i \rangle$ is nontrivial. But now,

$$\int_C \mathcal{L} = \deg \mathcal{L} = \sum_q n_q \operatorname{ord}_q(s) > d,$$

which is impossible. ☐

Corollary 3.36. $H^1(C, \mathcal{L}) \cong H^0(C, \mathcal{L}^{-1} \otimes K_C)^\vee$ *is finite-dimensional.* ☐

Note that $H^i(C, \mathcal{L}) = 0$ for $i > 1$.

Definition 3.37. Let $\chi(C, \mathcal{L}) = \chi(\mathcal{L}) = h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) \in \mathbb{Z}$.

Let $p \in C$ be a point, then we have a short exact sequence of sheaves

$$0 \to \mathcal{L} \xrightarrow{\otimes s_p} \mathcal{L} \otimes \mathcal{O}(p) \to \mathbb{C}_p \to 0$$

since $\operatorname{coker}(\otimes s_p)(U) = 0$ if $p \notin U$ and on a small neighbourhood V of p, the map is multiplication by the local coordinate z. Here, $\mathbb{C}_p$ is the skyscraper sheaf with value $\mathbb{C}$ at p. Hence we get a long exact exact sequence in cohomology

$$0 \to H^0(\mathcal{L}) \to H^0(\mathcal{L}(p)) \to \underbrace{H^0(\mathbb{C}_p)}_{\cong \mathbb{C}} \to H^1(\mathcal{L}) \to H^1(\mathcal{L}(p)) \to H^1(\mathbb{C}_p) = 0.$$

Since $\dim(-)$ is additive on exact sequences, it follows that $\chi(\mathcal{L}(p)) = \chi(\mathcal{L}) + 1$.

Proposition 3.38. *Any holomorphic line bundle $\mathcal{L} \to \mathbb{C}$ has a meromorphic section.*

Proof. By the above, there is $d \in \mathbb{Z}$ such that $\chi(\mathcal{L}(dp)) > 0$, so there exists a section $s \in H^0(\mathcal{L}(dp))$ and $s \otimes s_{d[p]}^{-1} \in \operatorname{Mero}(C, \mathcal{L})$. □

This finally proves that any holomorphic line bundle $\mathcal{L} \to C$ is isomorphic to $\mathcal{O}(D)$ for some divisor D. (We showed previously that if there is a meromorphic section t, then $\mathcal{L} \cong \mathcal{O}(\operatorname{div} t)$.)

Corollary 3.39. $\operatorname{Cl}(C) \cong \operatorname{Pic}(C)$

Theorem 3.40 (Riemann-Roch)**.** *Let $\mathcal{L} \to C$ be a holomorphic line bundle. Then $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(\mathcal{O})$.*

Proof. $\mathcal{L} = \mathcal{O}(D)$ for some $D = \sum n_P[P]$. Proceed by induction on $\sum |n_P|$. If $n_{P_1} > 0$, then from

$$0 \to \mathcal{O}(D - [P_1]) \to \mathcal{O}(D) \to \mathbb{C}_{P_1} \to 0$$

we get $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D - [P_1]) + 1 = (\deg \mathcal{L} - 1) + \chi(\mathcal{O}) + 1$ and similarly for $n_{P_1} < 0$. □

Furthermore, $\chi(\mathcal{O}) = h^0(\mathcal{O}) - h^1(\mathcal{O}) = 1 - h^0(K_C)$ by Serre duality, and

$$\chi(K_C) = h^0(K_C) - h^0(K_C^{-1} \otimes K_C) = h^0(K_C) - 1 = \deg K_C + \chi(\mathcal{O}) = \deg K_C + 1 - h^0(K_C),$$

so $h^0(K_C) - 1 = \frac{1}{2} \deg K_C$. Recall $K_C \cong \Omega_C^1 \cong T_C^*$, where T_C is the holomorphic tangle bundle. Thus $\deg K_C = -\deg T_C$.

We use the Poincare-Hopf formula: Let M be a manifold over $\mathbb{R}$ and V a vector field which is non-degenerate at all critical points (p where $V = 0$). Then $\chi_{\text{top.}}(M) = \sum_p \operatorname{ind}_p(V)$ with the index $\operatorname{ind}_p(V) \in \{\pm 1\}$, and $\deg T_C = \chi_{\text{top.}}(C)$:

Let $s \in \operatorname{Mero}(C, T_C)$. Locally near a pole of s, we have $s = z^{-k} \frac{\partial}{\partial z}$. On the unit circle, we have $z^{-k} = \bar{z}^k$, so define $\widetilde{s}$ as s away of poles of s and $\bar{z}^k \frac{\partial}{\partial z}$ in neighbourhoods of poles, and then smooth out. Finally, perturb $\widetilde{s}$ slightly (say $\bar{z}^k \rightsquigarrow (\bar{z}^k - \varepsilon)$ etc.) to $\widetilde{\widetilde{s}}$ to make sure it is non-degenerate and by construction $\operatorname{ind}(\widetilde{\widetilde{s}}) = \deg(\operatorname{div} s) = \deg T_C$.

Putting everything together, we have $\deg K_C = -\deg T_C = \chi_{\text{top}}(C) = 2g - 2$, so $h^0(K_C) = g$. This simplifies Riemann-Roch to $\chi(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$.

Definition 3.41. A hermitian manifold is a complex manifold together with a hermitian metric on the holomorphic tangent bundle TX.

If $p \in X$ and $\varphi : U \to \mathbb{C}^d, p \to 0$ is a chart, then $T_X(U) = \{\sum_{i=1}^d f_i(z_1, \ldots, z_d) \frac{\partial}{\partial z_i} \mid f_i$ holomorphic on $U\}$.

Let $T_{\mathbb{R},X}$ denote the tangent bundle of the real manifold underlying X: $T_{\mathbb{R},X}(U) = \{f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i} \mid f_i, g_i$ smooth$\}$.

Definition 3.42. An *almost compex structure* on a real manifold M is given by an endomorphism $J : T_{\mathbb{R},M} \to T_{\mathbb{R},M}$ such that $J^2 = -\operatorname{id}$.

Example 3.43. If X is a complex manifold, $T_{\mathbb{R},X}$ has an almost complex structure: On a complex chart φ identifying $T_{\mathbb{R},X,P} = \mathbb{C}_0^n \cong \mathbb{R}^{2n}$. Take J to be multiplication by i. In particular, $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$.

Proposition 3.44. *Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map such that $J^2 = -\operatorname{id}$. Then J has an n-dimensional eigenspace with eigenvalue i and an n-dimensional eigenspace with eigenvalue $-i$ on $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$.*

Then the holomorphic tangent bundle T_X is isomorphic to the $+i$-eigenspace, and $T_{\mathbb{R},X} \otimes \mathbb{C} \cong T^{1,0} \oplus T^{0,1}$ with $T^{1,0}$ spanned by $\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}$ and $T^{0,1}$ generated by $\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}$. Furthermore, $J : T_{\mathbb{R}} \to T_{\mathbb{R}}$ induces a map $J : \bigwedge_{\mathbb{R}}^1 \to \bigwedge_{\mathbb{R}}^1$, which is spanned by dx_j, dy_j, such that $\bigwedge^{1,0}$ and $\bigwedge^{0,1}$ are the $\pm i$-eigenspaces, respectively.

Observe that a Riemannian metric on $T_{\mathbb{R}}$ induces hermitian metric on $T^{1,0}$ and $T^{0,1}$. If $(z_1, \dots, z_n)$ are local complex coordinates on X such that $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ are an orthonormal basis for the Riemannian metric, we obtain an induced orthonormal basis on $T^{1,0}$ and $T^{0,1}$, given respectively as $\{\frac{1}{\sqrt{2}}\frac{\partial}{\partial z_j}\}$ and $\{\frac{1}{\sqrt{2}}\frac{\partial}{\partial \bar{z}_j}\}$. In turn, this induces a hermitiain metric on $\bigwedge^{p,q} = \bigwedge^p(\bigwedge^{1,0}) \otimes \bigwedge^q(\bigwedge^{0,1})$. In total, we have an orthonormal basis of $\bigwedge^{p,q}$ by $\frac{1}{\sqrt{2}}^{p+q} dz_I \wedge d\bar{z}_J$ where $\frac{\partial}{\partial z_j}$ is an orthonormal basis for hermitian metric on TX.

4 Kähler Geometry

Consider $V \cong \mathbb{R}^{2n}$ a vector space over $\mathbb{R}$ of dimension $2n$.

Definition 4.1. An *almost complex structure* (short: AC-structure) is a $J \in \mathrm{End}(V)$ such that $J^2 = -\mathrm{id}$.

This gives a decomposition $V = \otimes_{\mathbb{R}} \mathbb{C} \cong V^{1,0} \oplus V^{0,1}$ into the $(+i)$-eigenspace and the $(-i)$-eigenspace. Recall that a metric on V is a positive definite symmetric bilinear form $g \in \mathrm{Sym}^2(V^*)$.

Definition 4.2. A metric g is *compatible* with an AC-structure J if $g(Jv, Jw) = g(v, w)$ for all $v, w \in V$.

For example, take $V = \langle x_j, y_j \rangle_{j=1,\dots,n}$, $Jx_j = y_j$ and $g(x_j, x_k) = \delta_{jk} = g(y_j, y_k)$, $g(x_j, y_k) = 0$ for all j, k.

Definition 4.3. A pair (J, g) on V such that g compatible with J is called an *almost hermitian structure*.

Observe that (J, g) induce a hermitian form on $V^{1,0}$ and $V^{0,1}$: $h(u, v) = g_{\mathbb{C}}(\bar{u}, v)$, where $g_{\mathbb{C}} \in \mathrm{Sym}^2(V^* \otimes_{\mathbb{R}} \mathbb{C})$. If g is compatible with J, then $V^{1,0} \perp_h V^{0,1}$.

More generally, let $M = M^{2n}$ be a $2n$-dimensional manifold over $\mathbb{R}$. An almost complex structure is a morphism $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$, $J \in C^\infty(\mathrm{End}(T_{\mathbb{R}}M))$ satisfying $J^2 = -\mathrm{id}$. An almost hermitian structure is a pair (J, g) with J an almost complex structure and j a metric on $T_{\mathbb{R}}M$ compatible with J. Finally, a hermitian structure is a structure of a complex manifold of $\mathbb{C}$-dimension n on M such that J equals multiplication by i on $T_{\mathbb{R}}M$. Equivalently, M is a complex manifold with a compatible Riemannian metric g.

Proposition 4.4. *Let (J, g) be an almost hermitian structure on a vector space $V \cong \mathbb{R}^{2n}$. Then $\omega(v, w) = g(Jv, w)$ is an alternating non-degenerate form.*

Proof. $\omega(v, w) = g(Jv, w) = g(w, Jv) = g(Jw, J^2 v) = g(Jw, -v) = -g(Jw, v) = -\omega(w, v)$ and $\omega(v, Jv) = g(Jv, Jv) = 0$ if and only if $Jv = 0$ if and only if $v = 0$. □

For $\mathbb{R}^{2n} \cong \langle x_j, y_j \rangle$ an orthonormal basis and $Jx_j = y_j$ as before, we have $\omega = x_1^* \wedge y_1^* + \ldots + x_n^* \wedge y_n^*$.

Now consider M a $2n$-dimensional $\mathbb{R}$-manifold with an almost-hermitian structure (J, g). Let $v, w \in T_{\mathbb{R}}(U)$. Define $\omega \in \bigwedge^2 T_{\mathbb{R}}^*(U)$ as $\omega(v, w) := g(Jv, w)$. Then ω defines a C^∞-section of $\bigwedge^2 T_{\mathbb{R}}^*$, i.e. a real 2-form.

Definition 4.5. A hermitian manifold (M, J, g) is called *Kähler* if the corresponding $\omega \in \bigwedge^2 T_{\mathbb{R}}^*(M)$ is closed (under d).

Recall that given J, we have a decomposition $R_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1}$ into eigenspaces. If $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}_j$ is an orthonormal basis of $T_{\mathbb{R}}M$ where $x_j + iy_j = z_j$, then the hermitian metric on $T^{1,0}$ has orthonormal basis $\{\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})\}_j = \{\frac{1}{\sqrt{2}}\frac{\partial}{\partial z_j}\}_j$. Similarly, an orthonormal basis on $T^{0,1}$ is $\{\frac{1}{\sqrt{2}}\frac{\partial}{\partial \bar{z}_j}\}_j$. Set $dz_j = dx_j \wedge idy_j$, which is in the $(+i)$-eigenspace of $J : T_{\mathbb{R}}^* \otimes \mathbb{C} \to T_{\mathbb{R}}^* \otimes \mathbb{C}$, then annoyingly $(dz_j)\frac{\partial}{\partial z_j} = 2$. Hence $\{\frac{1}{\sqrt{2}}dz_j\}_j$ is an orthonormal basis of $\bigwedge^{1,0}$ and $\{\frac{1}{\sqrt{2}}d\bar{z}_j\}$ is an orthonormal basis of $\bigwedge^{0,1}$. Since $\bigwedge^{p,q} = \bigwedge^p \bigwedge^{1,0} \otimes \bigwedge^q \bigwedge^{0,1}$, we have that $\{\sqrt{2}^{-(p+q)} dz_I \wedge d\bar{z}_J\}_{|I|=p,|J|=q}$ is an orthonormal basis of $\bigwedge^{p,q}$. From g we also get a volume form on M:

$$\mathrm{vol}(g) = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dy_n = 2^{-n}(idz_1 \wedge d\bar{z}_1) \wedge \ldots \wedge (idz_n \wedge d\bar{z}_n).$$

Definition 4.6. The *hodge star operator* $* : \bigwedge^{p,q}(M) \to \bigwedge^{n-p,n-q}(M)$ is defined as the unique operator satisfying the following equivalent definitions.

(1) For all $\alpha, \beta \in \bigwedge^{p,q}(M)$, one has $\alpha \wedge *\beta = h(\beta, \alpha)\,\mathrm{vol}(g)$,

(2) If $\beta = \sum_{I,J} \beta_{I\bar{J}} dz_I \wedge d\bar{z}_J$, then

$$*\beta = 2^{p+q-n} \sum_{I,J} \varepsilon_{I,\bar{J}} \overline{\beta}_{I,\bar{J}} dz_{I^c} \wedge d\bar{z}_{J^c}$$

where $\varepsilon_{I,\bar{J}} \in \{\pm 1\}$ is defined via the equation

$$2^{-n}(dz_I \wedge d\bar{z}_J) \wedge (dz_{I^c} \wedge d\bar{z}_{J^c}) = \varepsilon_{I,\bar{J}}\,\mathrm{vol}(g).$$

For example, if (V, J, g) is an almost hermitian vector space $V \cong \mathbb{R}^{2n}$, then

$$*(dz_1) = +dz_2 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$$

Proposition 4.7. *$*$ is complex anti-linear and $**\beta = (-1)^{p+q}\beta$.* □

Assume that M is compact. Define an inner product on $\bigwedge^{p,q}(M)$ (depending on g) by $\langle s, t\rangle := \int_M h(s,t)\,\mathrm{vol}(g)$.

Proposition 4.8. *$\langle s, s\rangle \geq 0$ with equality iff $s = 0$.* □

Proposition 4.9. *The adjoint of $\bar{\partial} : \bigwedge^{p,q-1}(M) \to \bigwedge^{p,q}(M)$ is given by*

$$\bar{\partial}^\dagger = - * \bar{\partial} * .$$

Correction to Conventions To get rid of the annoying factors in the bases above, it is better to set

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

which we will do from now on. Then $dz = dx + i dy$ pairs with $\frac{\partial}{\partial z}$ to be 1 and $\frac{\partial}{\partial z}(z) = 1$ instead of 2.

Proof. (of proposition 3.53).

$$\langle \eta, \bar{\partial}\psi\rangle = \int_M \bar{\partial}\psi \wedge *\eta = -(-1)^{p+q} \int_M \psi \wedge \bar{\partial}(*\eta) + \int_M \bar{\partial}(\psi \wedge *\eta).$$

Note that $\psi \wedge *\eta$ is a $(n, n-1)$-form, hence $\bar{\partial}(\psi \wedge *\eta) = d(\psi \wedge *\eta)$. Thus the second integral vanishes and

$$\langle \bar{\partial}\psi, \eta\rangle = -(-1)^{2(p+q)} \int_M \psi \wedge ** \bar{\partial}(*\eta) = \langle - * \bar{\partial} * \eta, \psi\rangle.$$

□

Definition 4.10. The *$\bar{\partial}$-Laplacian* on (M, J, g) is the operator

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} : \textstyle\bigwedge^{p,q} \to \bigwedge^{p,q}$$

Proposition 4.11. *$\psi \in \ker(\Delta_{\bar{\partial}})$ if and only if $\bar{\partial}\psi = 0$ and $\bar{\partial}^\dagger = 0$.*

Proof. The "if"-part is trivial. So suppose $\Delta_{\bar{\partial}}\psi = 0$. Then $\langle \Delta_{\bar{\partial}}\psi, \psi\rangle = 0$, i.e.

$$0 = \langle \bar{\partial}\bar{\partial}^\dagger\psi, \psi\rangle + \langle \bar{\partial}^\dagger\bar{\partial}\psi, \psi\rangle = \langle \bar{\partial}^\dagger\psi, \bar{\partial}^\dagger\psi\rangle + \langle \bar{\partial}\psi, \bar{\partial}\psi\rangle \geq 0$$

with equality iff $\bar{\partial}\psi = 0 = \bar{\partial}^\dagger\psi$. □

We assume that $\bigwedge^{p,q}(M) = \operatorname{im}\Delta_{\bar\partial} \oplus (\operatorname{im}\Delta_{\bar\partial})^\perp$, which is true, but its justification requires a lot of analysis. Note that $\operatorname{im}(\Delta_{\bar\partial})^\perp = \ker(\Delta_{\bar\partial})$: If $s \in (\operatorname{im}\Delta_{\bar\partial})^\perp$, then $\langle s, \Delta_{\bar\partial} s\rangle = 0$, so $s \in \ker(\bar\partial) \cap \ker(\bar\partial^\dagger) = \ker\Delta_{\bar\partial}$, and conversely, if $s \in \ker\Delta_{\bar\partial}$, then

$$\langle s, \bar\partial\bar\partial^\dagger t + \bar\partial^\dagger\bar\partial t\rangle = \langle \bar\partial^t s, \bar\partial^d t\rangle + (\bar\partial s, \bar\partial t\rangle = 0 + 0 = 0.$$

Definition 4.12. $\mathcal{H}^{p,q}(M) = \ker\Delta_{\bar\partial}$ are the *harmonic* (p,q)-forms.

From the definitions, we have $\bigwedge^{p,q}(M) = \operatorname{im}(\Delta_{\bar\partial}) \oplus \mathcal{H}^{p,q}(M)$, and $\operatorname{im}(\Delta_{\bar\partial}) = \operatorname{im}\bar\partial \oplus \operatorname{im}\bar\partial^\dagger$: If $\alpha = \bar\partial s + \bar\partial^\dagger t \in \operatorname{im}\bar\partial \oplus \operatorname{im}\bar\partial^\dagger$ and $\beta \in \ker\Delta_{\bar\partial}$, then

$$\langle \bar\partial s + \bar\partial^\dagger t, \beta\rangle = \langle s, \bar\partial^\dagger\beta\rangle + \langle t, \bar\partial\beta\rangle = 0 + 0 = 0$$

by the proposition, so $\alpha \in \ker(\Delta_{\bar\partial})^\perp = (\operatorname{im}\Delta_{\bar\partial})^{\perp\perp} = \operatorname{im}\Delta_{\bar\partial}$ and $\langle\bar\partial s, \bar\partial^\dagger t\rangle = \langle \bar\partial\bar\partial s, t\rangle = 0$. In total, we have

Theorem 4.13 (Hodge decomposition)**.** $\bigwedge^{p,q}(M) = \ker(\Delta_{\bar\partial}) \oplus \operatorname{im}\bar\partial \oplus \operatorname{im}\bar\partial^\dagger$

We also have $\ker\bar\partial = \ker\Delta_{\bar\partial} \oplus \operatorname{im}\bar\partial$: Let $\alpha \in \bigwedge^{p,q}(M)$. Write $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ as in the theorem, so $\bar\partial\alpha = \bar\partial\alpha_2 = \bar\partial\bar\partial^\dagger\beta$ for some β. But $\bar\partial\bar\partial^\dagger\beta = 0$ if and only if $\langle\bar\partial\bar\partial^\dagger\beta, \beta\rangle = 0$, i.e. iff $\bar\partial^\dagger\beta = \alpha_2 = 0$. Similarly, $\ker\bar\partial^\dagger = \ker\Delta_{\bar\partial} \oplus \operatorname{im}\bar\partial^\dagger$. (Note $\bar\partial^\dagger\bar\partial^\dagger = *\bar\partial * *\bar\partial* = \pm * \bar\partial\bar\partial * = 0$.)

Recall the complex $0 \to \Omega^p \to \bigwedge^{p,0} \xrightarrow{\bar\partial} \bigwedge^{p,1} \to \cdots$, which is a soft resolution, hence we saw that $H^q(M, \Omega^p) \cong H^{p,q}(M) = \frac{\ker\bar\partial}{\operatorname{im}\bar\partial}$. Using the above decomposition $\ker\bar\partial = \mathcal{H}^{p,q}(M) + \operatorname{im}\bar\partial$, we get $H^{p,q}(M) \cong \mathcal{H}^{p,q}(M)$.

Remark 4.14. Exactly the same arguments work in the real setting as well: Let M be a compact oriented real manifold of dimension n and g a Riemannian metric on M. g defines a metric on $T_{\mathbb{R}}M = TM$ and on $T*M = \bigwedge^1$ and on $\bigwedge^k(M) = \bigwedge^k(\bigwedge^1(M))$. One defines the Hodge star $* : \bigwedge^k(M) \to \bigwedge^{n-k}(M)$ as $f dx_I \mapsto f dx_{I^c}$. The same proof as above shows that $d^\dagger = *d*$. Defining $\Delta_d = dd^\dagger + d^\dagger d$ as before, and again making the reasonable assumption that $\bigwedge^k(M) = \operatorname{im}(d) + \operatorname{im}(d)^\perp$, we get the relations $\mathcal{H}^k(M) := \ker(\Delta_d) = H^k_{dR}(M, \mathbb{R})$ and $\bigwedge^k(M) = \mathcal{H}^k(M) \oplus \operatorname{im} d \oplus \operatorname{im} d^\dagger$.

We now aim to justify the "reasonable assumptions" made above, or at least give an outline of a proof of them. Let M be a compact complex manifold, g a compatible metric, and $\alpha \in \bigwedge^{p,q}(M)$. Then

$$\|\alpha\|^2 = \langle\alpha, \alpha\rangle = \int_M \alpha \wedge *\alpha = \int_M h(\alpha, \alpha)\operatorname{vol}(g).$$

Given any normed vector space, we can take the completion (defined so that any Cauchy sequence converges), and we denote the completion of $\bigwedge^{p,q}(M)$ with respect to $\|\cdot\|$ by $L_2^{p,q}(M) = W^{p,q}_{2,0}(M)$. We can make the analogous definitions for real manifolds.

Example 4.15. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the torus with the standard metric. For $f \in C^\infty(\mathbb{T}^n)$,

$$\|f\|^2 = \int_{\mathbb{T}^n} f\bar f dx_1 \wedge \ldots \wedge dx_n.$$

Write $f(x) = \sum_{m\in\mathbb{Z}^n} a_m e^{2\pi i x\cdot m}$ the Fourier expansion of f. Then

$$L_2(C^\infty(\mathbb{T}^n)) = \{(a_m)_{m\in\mathbb{Z}^n} \mid \sum_m |a_m|^2 < \infty\}.$$

We define Sobolev norms on $C^\infty(\mathbb{T}^n)$ as

$$\|f\|^2_{2,k} := \sum_{|I|\le k} \|\frac{\partial^{|I|} f}{\partial x_I}\|^2$$

and define $W_{2,k}(C^\infty(\mathbb{T}^n))$ as the completion with respect to $\|\cdot\|_{2,k}$. As before, one can identify these with $\{(a_m)_{m\in\mathbb{Z}^n} \mid \sum_{m\in\mathbb{Z}^n}(1+|m|^2+\ldots+|m|^{2k})|a_m|^2 < \infty\}$.

We want to generalize this example. Let $E \to M$ be a vector bundle with a (hermitian or riemannian) metric. Trivialize E on open sets U such that $h_U : E|_U \xrightarrow{\cong} (C^\infty)^{\oplus r}$, $r = \operatorname{rank}(E)$. Choose charts $\varphi_U : U \to \mathbb{T}^n$ and a partition of unity $\{f_U\}$ subordinate to this open cover. For $\alpha \in E(U)$ define $\|\alpha\|^2_{h_U,f_U,\varphi_U} := \sum_U \|h_U(f_U\alpha)\|^2$, where the norm on the right side is the standard L^2-norm on $\mathbb{T}^n$ via φ_U.

Exercise 4.16. This norm is equivalent to $\|\alpha\|$, so it defines the same completion.

Further, for $\alpha \in E(M)$ define $\|\alpha\|^2_{2,k} = \sum_U \|h_U(f_U\alpha)\|^2_{2,k}$. Again, the completion $W_{2,k}(E(M))$ is independent of the choice of extra data.

Now consider $\Delta_d : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$, sending $\sum a_m e^{2\pi i x\cdot m}$ to $\sum |m|^2 e^{2\pi i x\cdot m}$, so Λ_d naturally extends to a map $W_{2,k}(\mathbb{T}^m) \to W_{2,k-2}(\mathbb{T}^m)$. More generally, Δ_d defines a map $W_{2,k}(\bigwedge^p(M)) \to W_{2,k-2}(\bigwedge^p(M))$. We want to "invert" Λ_d. Let $P_{\mathbb{T}^n} : W_{2,k-2}(\mathbb{T}^n)$ be defined as

$$(a_m)_m \mapsto (\frac{a_m}{|m|^2} \text{ if } m \neq 0,\ 0 \text{ if } m = 0).$$

Then $\Delta_d \circ P_{\mathbb{T}^n} = \mathrm{id} - a_0$ and $P_{\mathbb{T}^n} \circ \Delta_d = \mathrm{id} - a_0$. This is called a "pseudo-inverse" in elliptic operators.

More generally, consider any Riemannian metric g on $\mathbb{T}^n$. Then

$$\Delta_d(f) = \det(g)^{-1/2} \sum_{i,j} \frac{\partial}{\partial x_j}\left(g^{ij} \deg(g)^{1/2} \frac{\partial f}{\partial x_i}\right),$$

where $g^{ij} = g(d\bar{z}_j, z_i)$. Appropriately modify P for this metric: The leading order term of the differential operator Δ_d is $\sum_{i,j} g^{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f$. The symbol of Δ_d is thus $\sum_{i,j} g^{ij} y_i y_j$, which is nonvanishing since g is positive definite.[2] The key property of $P_{\mathbb{T}^n}$ for a general metric $g = (g_{ij})$ on $\mathbb{T}^n$ is that $P_{\mathbb{T}^n} \circ \Delta_d - \mathrm{id}$ and $\Delta_d \circ P_{\mathbb{T}^n} - \mathrm{id}$ are regularity increasing operators, i.e. map $W_{2,k}(\mathbb{T}^n) \to W_{2,k+1}(\mathbb{T}^n)$.

On a general compact manifold M, we extend $\Lambda_d : \bigwedge^p(M) \to \bigwedge^p(M)$ to $W^p_{2,k}(M) \to W^p_{2,k-2}(M)$ and take $P : W^p_{2,k-2}(M) \to W^p_{2,k}(M)$ as $P(\alpha) = \sum P^g_{\mathbb{T}^n}(f_U\alpha)$. Let $\alpha \in W^p_{2,k}(M)$ in $\ker(\Delta_d)$. Then $(P \circ \Delta_d - \mathrm{id})(\alpha) = -\alpha$, so $\alpha \in W^p_{2,k+1}(M)$ since $P \circ \Delta_d - \mathrm{id}$ is regularity increasing. Iterating, we get $\alpha \in \bigcap_k W^p_{2,k}(M) = \bigwedge^p(M)$. Some more calculations and facts on elliptic operators yield that $\operatorname{im}(\Delta_d) \subseteq \bigwedge^p(M)$ is closed. This justifies the "plausible assumptions" we did earlier.

We now return to Kähler manifolds. Let M be a compact complex manifold, g a Kähler metric, and $\omega(v,w) = g(Jv,w)$ the associated closed 2-form. We first give some examples:

Example 4.17. 1. $M \cong \mathbb{C}^n$, J the standard complex structure and g the standard Riemannian metric on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Then $\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$. Let $\Lambda \subseteq \mathbb{C}^n$ be a lattice of rank $2n$ over $\mathbb{Z}$. Then $\mathbb{C}^{2n}/\Lambda$ is a manifold, and g, ω descend, giving a compact Kähler manifold. An example where this descent goes wrong is $(\mathbb{C}^2 \setminus \{0\})/(z,w) \sim (2z, 2w)$. We will see soon that this is in fact not a Kähler manifold.
2. Any Riemann surface C with any choice of compatible metric, since $\omega \in \bigwedge^2 T^*_{\mathbb{R}}C$ is automatically closed.
3. Given M, N Kähler, $M \times N$ is Kähler: If g_M, g_N are the metrics and ω_M, ω_N the 2-forms on M, N, then $g = g_M + g_N \in \operatorname{Sym}^2 T^*_{\mathbb{R}}(M \times N)$, and $\omega = \pi^*_M\omega_M + \pi^*_N\omega_N$ where π_M, π_N are the projections to M and N. 4. Let (M, g, J) Kähler and $N \subseteq M$ a complex submanifold. Then $J|_{T_pN}$ gives the

[2] For a differential operator $D = \sum_{|I|\leq k} f_I \frac{\partial^I}{\partial x_I}$, its *symbol* is $\operatorname{symbol}(D) = \sum_{|I|=k} f_I y^I$. We say the symbol is nonvanishing if $\operatorname{symbol}(D) \neq 0$ on $\sum y_i^2 = 1$. In this case, we call D an *elliptic* operator.

AC-structure on N for $p \in N \subseteq M$, $g|_N$ is a metric, and $\omega_N = \iota^*\omega$ where ι is the inclusion. So N is Kähler.
5. $M = \mathbb{CP}^n$ is a complex manifold. The Fubini-Study-form on $\mathbb{CP}^n$ is defined as

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(|z|^2)$$

where $z = [1 : z_1 : \ldots : z_n]$. Recall that $\mathcal{O}(-1) \to \mathbb{CP}^n$ is the tautological line bundle. This has a hermitian metric induced by the standard hermitian metric on each tautological line. It induces a hermitian metric on its dual $\mathcal{O}(1)$. From the definitions, $c_1(\mathcal{O}(1)) = [\omega_{FS}] = [D]$ where $\mathcal{O}(1) = \mathcal{O}(D)$. We have $x_0 \in H^0(\mathbb{CP}^n, \mathcal{O}(1))$, so $D = V(x_0) \in \mathbb{CP}^n$ is a hyperplane. Define g such that $g(Jv, w) = \omega_{FS}(v, w)$. Clearly $g \in \operatorname{Sym}^2 T^*_{\mathbb{R}}$. One checks on affine charts that this is indeed a metric. So $\mathbb{CP}^n$ is Kähler.

Example 4.18. Consider $\mathbb{C}^n$ with the standard metric. Then one calculates

$$\Delta_{\bar{\partial}} f dz_I \wedge d\bar{z}_J = \sum_k \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} dz_I \wedge d\bar{z}_J$$

and $\frac{\partial^2 f}{\partial z_k \partial \bar{z}_k} = \frac{1}{4}(\frac{\partial^2}{\partial^2 x_k} + \frac{\partial^2}{\partial^2 y_k})$, so that $2\Delta_{\bar{\partial}} = \Delta_d$ on $\mathbb{C}^n$ with the standard metric. We will prove soon that this holds in general for Kähler manifolds.

Warning: Given a point $p \in (M, J, g)$ on a hermitian (or even Kähler) manifold, it is in general not possible to choose complex coordinate charts $p \in U \to \mathbb{C}^d$ such that $g|_U$ is the pullback of the standard metric on $\mathbb{C}^d$.

Definition 4.19. Define $L : \bigwedge^{p,q}(M) \to \bigwedge^{p+1,q+1}(M),\ \eta \mapsto \eta \wedge \omega$ with adjoint $L^\dagger$

Theorem 4.20. *The key identity $[L^\dagger, d] = i(\partial - \bar{\partial})^\dagger$ holds.*

Proof. Omitted. □

Observe $[L, d] = 0$, since $(Ld - dL)\eta = d\eta \wedge \omega - d(\eta \wedge \omega) = 0$ as $d\omega = 0$. Hence also $[L^\dagger, d^\dagger] = 0$.

Proposition 4.21. *The following identities hold for Kähler manifolds:*

(a) $[L^\dagger, \Delta_d] = 0$ *and* $[L, \Delta_d] = 0$.

(b) $\partial\bar{\partial}^\dagger + \bar{\partial}^\dagger\partial = 0$.

(c) $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$.

Proof. (a) The second equality follows from the first, since Δ_d is self-adjoint. The first is a direct calculation:

$$\begin{aligned} L^\dagger(dd^\dagger + d^\dagger d) &= (dL^\dagger + i(\partial - \bar{\partial})^\dagger)d^\dagger + d^\dagger L^\dagger d \\ &= dd^\dagger L^\dagger + (i(\partial - \bar{\partial}))d^\dagger + d^\dagger dL^\dagger + d^\dagger(i(\partial - \bar{\partial})^\dagger) \\ &= (dd^\dagger + d^\dagger d)L^\dagger \end{aligned}$$

(b) From the key identity, $[L^\dagger, \partial] = i\bar{\partial}^\dagger$, so

$$-i(\partial\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}) = \partial(L^\dagger\partial - \partial L^\dagger) + (L^\dagger\partial - \partial L^\dagger)\partial) \overset{\partial^2=0}{=} 0.$$

(c) Again, a direct computation yields

$$\begin{aligned}\Delta_d = dd^\dagger + d^\dagger d &= (\partial + \bar\partial)(\partial^\dagger + \bar\partial^\dagger) + (\partial^\dagger + \bar\partial^\dagger)(\partial + \bar\partial) \\ &= \partial\partial^\dagger + \bar\partial\bar\partial^\dagger + \partial^\dagger\partial + \bar\partial^\dagger\bar\partial = \Delta_\partial + \Delta_{\bar\partial}\end{aligned}$$

since by the lemma, the cross-terms cancel to 0. It remains to show that $\Delta_\partial = \Delta_{\bar\partial}$:

$$\begin{aligned}\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial &= -i(\partial[L^\dagger, \bar\partial] + [L^\dagger, \bar\partial]\partial) \\ &\overset{\text{check}}{=} -i(\bar\partial[L^\dagger, \partial] + [L^\dagger, \partial]\bar\partial) = \Delta_{\bar\partial}\end{aligned}$$

□

Summary of Important Results so far

Theorem (Hodge Theorem, ver. 1)**.** *Let (M, g) be a compact hermitian manifold.*

($\mathbb{C}$1) $\bigwedge^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \operatorname{im}\bar\partial \oplus \operatorname{im}\bar\partial^\dagger$ *where* $\mathcal{H}^{p,q}(M) = \ker(\Delta_{\bar\partial})$.

($\mathbb{C}$2) $\mathcal{H}^{p,q}(M) \cong H^{p,q}(M) = \ker\bar\partial / \operatorname{im}\bar\partial \cong \check{H}^q(M, \Omega^p)$.

Theorem (version for $\mathbb{R}$)**.** *Let (M, g) be a compact oriented Riemannian manifold.*

($\mathbb{R}$1) $\bigwedge^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \operatorname{im}\bar d \oplus \operatorname{im} d^\dagger$ *where* $\mathcal{H}^{p,q}(M) = \ker(\Delta_d)$.

($\mathbb{R}$2) $\mathcal{H}^k(M) \cong H^k_{dR}(M)$.

Theorem (Kähler identities)**.** *Let (M, g) be a compact complex Kähler manifold.*

(K1) $\Delta_d = 2\Delta_{\bar\partial} = 2\Delta_\partial$.

(K2) If $L = \cdot \wedge \omega : \bigwedge^{p,q}(M) \to \bigwedge^{p+1,q+1}(M)$ and $L^\dagger$ its adjoint, then $[L, \Delta_d] = [L^\dagger, \Delta_d] = 0$, as well as $[\Pi^{p,q}, \Delta_d] = 0$, where $\Pi^{p,q} : \bigoplus_{p,q} \bigwedge^{p,q}(M) \to \bigwedge^{p,q}(M)$ is the projection.

The last statement follows from Δ_d preserving the bidegree (p, q), which follows from (K1). We can now combine our previous results to prove the important

Theorem 4.22 (Hodge Decomposition Theorem)**.** *Let (M, g) be a compact complex Kähler manifold. Then*

(1) $H^k_{dR}(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M)$ *("Hodge decomposition")*

(2) $H^{q,p}(M) \cong \overline{H^{p,q}(M)}$ *("Hodge symmetry")*

Proof. By ($\mathbb{R}$2), $H^k_{dR}(M, \mathbb{R}) \cong \mathcal{H}^k(M, \mathbb{R})$, and tensoring with $\mathbb{C}$ yields $H^k_{dR}(M, \mathbb{C}) \cong \mathcal{H}^k(M, \mathbb{C})$. Let $\alpha \in \mathcal{H}^k(M, \mathbb{C})$. Write $\alpha = \sum_{p+q=k} \alpha_{pq}$ with $\alpha_{pq} \in \bigwedge^{p,q}(M)$. Then $\Delta_d\alpha = 0$ iff $\Delta_d\alpha_{p,q} = 0$ for all p, q, since Δ_d preserves bidegree. Hence by $(K1)$, $\Delta_{\bar\partial}\alpha_{pq} = 0$ for all p, q, so $\alpha_{p,q} \in \mathcal{H}^{p,q}(M)$. This proves (1).

For (2), we have

$$\alpha \in H^{p,q}(M) \quad\Leftrightarrow\quad \Delta_d\alpha = 0 \quad\Leftrightarrow\quad \Delta_d\bar\alpha = 0 \quad\Leftrightarrow\quad \bar\alpha \in \mathcal{H}^{q,b}(M),$$

hence $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$, and the result follows by ($\mathbb{C}$2). □

Remark 4.23. The space $\mathcal{H}^{p,q}(M) \subseteq \bigwedge^k(M)$ depends on the metric g. On the other hand, we have a canonical isomorphism $\mathcal{H}^{p,q}(M) \cong H^{p,q}(M)$, which is independent of the metric. So the space of harmonic forms for different metrics are canonically isomorphic.

Corollary 4.24. *Let $b_k(M) = \dim_{\mathbb{C}} H^k(M,\mathbb{C})$ be the Betti-numbers. Then the odd Betti-numbers $b_{2k+1}(M)$ are even.*

Proof. $H^{2k+1}(M,\mathbb{C}) = H^{2k+1,0}(M) \oplus \ldots \oplus H^{k+1,k}(M) \oplus \overline{H^{k+1,k}(M)} \oplus \ldots \oplus H^{0,2k+1}(M)$ with $\dim H^{i,j}(M) \oplus \overline{H^{i,j}(M)} = 2\dim H^{i,j}(M)$ even. $\square$

Example 4.25. Let C be a compact Riemann surface and g any hermitian metric. Then $b_1(C)$ is even, which agrees with $b_1(C) = h^1(C,\mathbb{C}) \cong \mathbb{C}^{2g}$.

Example 4.26. Let $M = \mathbb{C}^2 \setminus \{0\}/(x,y) \mapsto (2x,2y)$. Then M is diffeomorphic to $S^3 \times S^1 = \{|x|^2+|y|^2 = 1\} \times \mathbb{R}_{>0}/2^{\mathbb{Z}}$. Hence by Künneth, $H^1(M,\mathbb{C}) \cong H^1(S^3,\mathbb{C}) \otimes H^0(S^1,\mathbb{C}) \oplus H^0(S^3,\mathbb{C}) \otimes H^1(S^1,\mathbb{C}) = \mathbb{C}$, so $b_1(M) = 1$. Hence there exists no Kähler metric on M.

One can visualize the Hodge decomposition with the *Hodge diamond*, the grid of numbers $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}(M)$, in the following form:

$$\begin{array}{ccccccc}
 & & & h^{d,d} & & & \\
 & & h^{d,d-1} & & h^{d-1,d} & & \\
 & h^{d,d-2} & & h^{d-1,d-1} & & h^{d-2,d} & \\
\ddots & & & & & & \ddots \\
h^{d,0} & h^{d-1,1} & & \cdots & & h^{1,d-1} & h^{0,d} \\
\ddots & & & & & & \ddots \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

Hodge symmetry says that this diamond is symmetric along its vertical axis. By Poincare duality, there is a perfect pairing $H^i(M,\mathbb{C}) \otimes H^{2n-i}(M,\mathbb{C}) \to \mathbb{C}$, sending $[\alpha] \otimes [\beta]$ to $\int_M \alpha \wedge \beta$. Let $\alpha = \alpha_{pq} \in \mathcal{H}^{p,q}(M) \subseteq \bigwedge^i(M)$ and $\beta = \sum_{p'+q'=i} \beta_{n-p',n-q'} \in \mathcal{H}^{2n-i}(M) \subseteq \bigwedge^{2n-i}(M)$. Then $\alpha \wedge \beta = \alpha \wedge \beta_{n-p,n-q}$, i.e. the Poincare Duality respects the bidegree. Hence $H^{n-p,n-q}(M) \cong H^{p,q}(M)^*$, so that $h^{p,q} = h^{n-p,n-q}$. That means that the Hodge diamond is symmetric under rotation of 180°, and combining the two symmetries results in symmetry along the horizontal axis.

Example 4.27. 1. Let C be a compact genus g curve. Its Hodge diamond is completely determined by the symmetries as

$$\begin{array}{ccc}
 & 1 & \\
g & & g \\
 & 1 &
\end{array}$$

and we can read off the Betti numbers $b_0 = 1,\ b_1 = 2g,\ b_2 = 1$, as well as $h^{1,0}(C) = H^0(C,K_C) \cong \mathbb{C}^g$.

2. Consider $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \ldots \sqcup \mathbb{C}^0$, which is a cell decomposition. Using cellular homology and the fact that there are only cells in even dimensions, $H_k(\mathbb{CP}^n,\mathbb{Z}) = \mathbb{Z}$ for k even, $0 < k < n$ and 0 otherwise. By Poincare duality, $b_{2k+1} = 0$ and $b_{2k} = 1$ for $0 < k < n$. Hence the Hodge diamond has 1s along its vertical axis and 0 everywhere else.

M is Calabi-Yau if $K_m = \Omega^n = \mathcal{O}$. Then $h^i(M, \mathcal{O}) \cong h^{n-i}(M, \mathcal{O}^* \otimes K_M) = h^{n-i}(M, \mathcal{O})$, so $h^{0,i} = h^{0,n-i}$, i.e. the edges of the diamond are each symmetric.

Definition 4.28. A *Hodge structure* of weight k consists of a finitely generated abelian group $V_{\mathbb{Z}}$ and a $\mathbb{C}$-vector space decomposition $V_{\mathbb{Z}} \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}$ such that $H^{q,p} = \overline{H^{p,q}}$.

The prime example of a Hodge structure is $V_{\mathbb{Z}} = H^k_{sing}(M, \mathbb{Z})$ for M a compact Kähler manifold. Then $V_{\mathbb{Z}} \otimes \mathbb{C} \cong H^k_{sing}(M, \mathbb{C}) \cong H^k_{dR}(M, \mathbb{C})$, and we can set $H^{p,q} = H^{p,q}(M)$. This is a Hodge structure of weight k.

Example 4.29. (Weight 1 Hodge structure). Let $V_{\mathbb{Z}} \cong \mathbb{Z}^{2g}$, then $V_{\mathbb{Z}} \otimes \mathbb{C} \cong \mathbb{C}^{2g}$. Let $H^{1,0} \subseteq V_{\mathbb{Z}} \otimes \mathbb{C}$ be a g-dimensional complex sub-vectorspace such that $H^{1,0} \cap (V_{\mathbb{Z}} \times \mathbb{R}) = \{0\}$. Then $V_{\mathbb{Z}} \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ is a Hodge structure, where $H^{0,1} := \overline{H^{1,0}}$.

Theorem 4.30 (Lefschetz Hyperplane Theorem)**.** *Let $X \subseteq \mathbb{CP}^d$ be an n-dimensional smooth projectie variety over $\mathbb{C}$. Let $Y = X \cap H$ be a smooth intersection with a hyperplane. Then the map $H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ induced by the inclusion is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.*

Corollary 4.31. *The Lefschetz Hyperplane Theorem is also valid for hypersurfaces H of degree k in $\mathbb{CP}^d$.*

Proof. Consider the Veronese embedding $\mathbb{CP}^d \hookrightarrow \mathbb{CP}^D$, $[x_0 : \ldots : x_d] \mapsto [x_0^k : x_0^{k-1}x_1 : \ldots : x_d^k]$, with the entries ranging over all monomials of degree k. Take $X \hookrightarrow \mathbb{CP}^d \hookrightarrow \mathbb{CP}^D$ and let $H \subseteq \mathbb{CP}^D$ be defined by $a_0z_0 + \ldots + a_Dz_D = 0$ for $a_i \in \mathbb{C}$. Then $X \cap H \cong X \cap V(a_0x_0^k + \ldots + a_dx_d^k)$. $\square$

T_xX and T_xH are transverse subspaces of $T_x\mathbb{CP}^d$ for all $x \in X \cap H$, so $X \cap H$ is a smooth complex submanifold of X of codimension 1.

Theorem 4.32 (Bertini)**.** *A generic hyperplane section of $X \subseteq \mathbb{CP}^d$ is itself smooth.*

Example 4.33. Let $X = \mathbb{CP}^d$ and $Y = V(f)$ where $f \in \mathbb{C}[x_0, \ldots, x_d]^{(k)}$ is homogeneous of degree d such that Y is smooth and $V(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_d}) = \emptyset$. Let $i \leq n-2 = \dim_{\mathbb{C}} Y - 1$. Note that $\dim_{\mathbb{R}} Y = 2\dim_C Y$, so $H^{\dim_{\mathbb{C}} Y}(Y, \mathbb{Z})$ is the "middle cohomology and $H^i(Y, \mathbb{Z}) \cong H^i(X, \mathbb{Z})$ for $i < \dim_{\mathbb{C}} Y$. Recall that the Hodge numbers of projective space are $h^{i,i}(\mathbb{CP}^d) = 1$ and $h^{i,j}(\mathbb{CP}^d) = 0$ for $i \neq j$. By Lefschetz, we have $h^i(Y) = h^i(\mathbb{CP}^d)$ for $i \leq d-2$. Oberseve that if $\alpha \in \bigwedge^{p,q}(X)$, then $\alpha|_Y \in \bigwedge^{p,q}(Y)$. Also $\alpha \in \mathcal{H}^{p,q}(X)$ implies $\alpha|_Y \in \mathcal{H}^{p,q}(Y)$ (with the induced Kähler metric), and by symmetry we also know that $h^i(Y) = h^{2d-2-i}(Y)$ for $d-1 < i < 2d-2$. So the interesting cohomology is h^{d-1}.

Key to the proof of the Lefschetz Hyperplane Theorem is

Theorem 4.34 (Andreotti-Frankel)**.** *Let $Y \subseteq \mathbb{C}^r$ be a closed complex submanifold of dimension n. Then Y has the homotopy type of a CW complex of real dimension n.*

Corollary 4.35. *$H^i(Y, \mathbb{Z}) = 0$ for $i > n$.*

Proof. A Morse function on a manifold M is $f : M \to \mathbb{R}$ such that all critical points are non-degenerate (i.e. if $(df)_m : T_mM \to T_{f(m)}\mathbb{R}$ is zero, then $Q_m = (\frac{\partial^2 f}{\partial x_i \partial x_j})$ is a non-degenerate matrix). If Q_m is non-degenerate, then it is diagonalizable with eigenvalues ± 1. The index of a critical point m is the number of -1's.

Using a Morse function, we can build a CW complex isomorphic to M, with cells of dimension $\leq n$. $\square$

Proposition 4.36. *Let $Y \subseteq \mathbb{C}^r$ be a closed complex submanifold. Then $f_c : Y \to \mathbb{R}$, $y \mapsto |y - c|^2_{\mathbb{C}}$, is a*

Morse function for almost all $c \in \mathbb{C}^r$.

Lemma 4.37. *Any critical point of f_c has index $\leq n = \dim_{\mathbb{C}} Y$.*

Proof. Using the implicit function theorem, write Y locally near a critical point y of f_c as

$$Y = \{(z_1, \ldots, z_n, f_1(z), \ldots, f_{r-n}(z)) \mid z_i \in \mathbb{C}\}$$

with f_i holomorphic and $\operatorname{ord}(f_i) \geq 2$ and $f_i \in \mathcal{O}(z^2)$ at $c = (1, 0, \ldots, 0)$.

Let $g_c(z) = (1 - 2\operatorname{Re}(f_1(z))) + \sum_{i \geq 2} |z_i|^2 + |f_i(z)|^2$, where the last summands are of order 4. Write $f_1(z) = \sum_{ij} Q_{ij} z_i z_j + \ldots$ Then

$$\operatorname{Hess}(g_c) = \frac{\partial^2 g_c}{\partial a_i \partial a_j}|_0 = 2\mathbb{1} - 2\operatorname{Hess}(\operatorname{Re}(\sum Q_{ij} z_i z_j)) = 2\mathbb{1} - M,$$

where M diagonalizes to a matrix with diagonal entries $1, -1, 1, -1, \ldots$, since if we diagonalize $\sum Q_{ij} z_i z_j$ by complex linear change of variables such that $\sum Q_{ij} z_i z_j = \sum z_i^2$, then $\operatorname{Re}(\sum z_i^2) = \sum (x_i^2 - y_i^2)$. So the index of any critical point of g_c is at most $n = \dim_{\mathbb{C}} Y$. □

Corollary 4.38. *If Y is an affine variety over $\mathbb{C}$ of $\dim_{\mathbb{C}} Y = n$, then $H^i(Y, \mathbb{Z}) = 0$ and $H_i(Y, \mathbb{Z}) = 0$ for $i > n$.*

Proof. (of the Lefschetz Hyperplane theorem). Let $X \subseteq \mathbb{CP}^d$ be a smooth projective variety and $Y = X \cap H$ a smooth hyperplane section. Consider the long exact sequence of the pair (X, Y) (with $\mathbb{Z}$-coefficients)

$$\cdots \to H^i(X, Y) \to H^i(X) \to H^i(Y) \to H^{i+1}(X, Y) \to \cdots$$

By Poincare duality for manifolds with boundary (aka Poincare-Lefschetz duality), we have $H^i(X, Y) = H_{2n+2-i}(X \backslash Y)$ and $H^{i+1}(X, Y) \cong H_{2n+2-(i+1)}(X \backslash Y)$. Now $X \backslash Y) = X \backslash (X \cap H) \in \mathbb{CP}^d \backslash H \cong \mathbb{C}^d$ is a smooth closed submanifold. By the previous corollary, we get $H_j(X, Y) = 0$ in the necessary degrees. □

Proposition 4.39. *Let $V \subseteq \mathbb{CP}^{n+1}$ be a smooth hypersurface of degree d. Then*

$$\chi_{top}(V) = (n+2) - \frac{1}{d}(1 + (-1)^{n+1}(d-1)^{n+2}).$$

This is useful since we know the Hodge numbers above and below middle dimension, so this gives the middle Betti number.

Proof. Oberser that the answer only depends on d and n, since any two V's with the same d and n are diffeomorphic. So let $V_{d,n} = V(x_0^d + \ldots + x_{n+1}^d)$. Consider the map $\varphi : V_{d,n} \to \mathbb{CP}^n$, $[x_0 : \ldots : x_{n+1}] \mapsto [x_0 : \ldots : x_n]$. If $x_0^d + \ldots + x_n^d = 0$, then φ has one inverse image, and if $x_0^d + \ldots + x_n^d \neq 0$, then φ has d inverse images. We have

$$\chi_{\text{top}} = \chi_{\text{top}}(V_{d,n} \setminus S) + \chi_{\text{top}}(S) = d\chi_{\text{top}}(\mathbb{CP}^n \setminus V_{d,n-1}) = d(n-1) + (1-d)\chi_{\text{top}}(V_{d,n-1}),$$

and the claim follows by induction. □

Theorem 4.40 (Adjunction formula)**.** *If $Y \subseteq X$ is a smooth divisor, then $K_Y \cong K_X \otimes \mathcal{O}_X(Y)|_Y$*

Proof. Locally, $Y = V(z_1)$ for $z_1, \ldots, z_n$ local coordinates on X. In this neighbourhood, $dz_1 \wedge \ldots \wedge dz_n$ is a generator of the line bundle K_X. $\mathcal{O}_X(Y)$ is locally generated by z_1^{-1}, so $K_X \otimes \mathcal{O}_X(Y)$ is locally generated by $\frac{dz_1}{z_1} \wedge dz_2 \wedge \ldots \wedge dz_n$. Consider the map

$$K_X \otimes \mathcal{O}_X(Y) \to K_Y, \quad f\frac{dz_1}{z_1} \wedge dz_2 \wedge \ldots dz_n \overset{Res}{\mapsto} 2\pi i f dz_2 \wedge \ldots \wedge dz_n.$$

The kernel of this map is exactly I_Y, the sheaf of functions vanishing at Y. Hence

$$K_Y \cong K_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_X/I_Y \cong K_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_Y \cong K_X \otimes \mathcal{O}_X(Y)|_Y.$$

$\square$

Example 4.41. 1. Let $C \subseteq \mathbb{CP}^2$ be a smooth curve of degree d. Then $(K_C \cong K_{\mathbb{CP}^2} \otimes \mathcal{O}_{\mathbb{CP}^2}(C))|_C$. By abuse of notation, we use $K_{\mathbb{CP}^2}$ to denote both the canonical bundle and the canonical divisor. Hence $K_C = (-3H + dH)|_C$ and $\deg K_C = d(d-3)$ by Bezout's theorem. On the other hand, $\deg K_C = 2g - 2$, so $g = \binom{d-1}{2}$, which completely determines the Hodge diamond in terms of d.

2. Next, consider the smooth hypersurface of degree 2 in $\mathbb{CP}^3$, $X = V(ad - bc) \subseteq \mathbb{CP}^3_{[a:b:c:d]}$ with isomorphism

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \to X, \quad ([x:y],[z:w]) \mapsto [xz : xw : yz : yw].$$

Note that the Künneth formula is compatible with the Hodge decomposition, i.e.

$$H^{p,q}(X \times Y) \cong \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{p_1,q_1}(X) \otimes H^{p_2,q_2}(Y)$$

for Kähler manifolds X, Y. So the Hodge diamond of our hypersurface X is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

3. Degree 3 hypersurfaces in CP^3: $X \cong \mathrm{Bl}_{p_1,\ldots,p_6}(\mathbb{CP}^2)$. Then $K_X = \mathcal{O}_X(-4+3) \cong \mathcal{O}_X(-1)$. Thus $h^0(X, K_X) = 0$:

Lemma 4.42. *If X is a compact complex manifold, $L \to X$ a holomorphic line bundle with $H^0(X, L) \neq 0$. Then $L = \mathcal{O}_X$ or $H^0(X, L^{-1}) = 0$.*

Proof. Let $0 \neq s \in H^0(X, L)$. If $t \in H^0(X, L^{-1})$, $t \neq 0$. Then $st \in H^0(X; \mathcal{O}_X)$. If $V(st) \neq \emptyset$, then $st = 0$, which is a contradiction. If $V(st) = \emptyset$, then $V(s) = \emptyset$, so $L \cong \mathcal{O}_X$. $\square$

So X has no holomorphic 2-forms, and in the Hodge diamond, $h^{2,0}(X) = 0$. By Lefschetz and symmetry, we can fill in everything but $h^{1,1}$ as

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & ? & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Finally, $\chi_{\text{top}}(X_{3,2}) = (2+2) - \frac{1}{3}(1 + (-1)(3-1)^{2+2}) = 9$. Using $\chi_{\text{top}}(X) = h^0 - h^1 + h^2 - h^3 + h^4$, we get $h^{2,2}(X) = 7$. This agrees with the description as blow-up, since

Fact 4.43. *If S is a smooth surface, then $H^2(\mathrm{Bl}_p S, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}[E]$, where E is the exceptional divisor.*

4. Let $X = V(x_0^4 + x_1^4 + x_2^4 + x_3^4) \subseteq \mathbb{CP}^3$. Then $K_X = \mathcal{O}_X(-4+4) = \mathcal{O}_X$. This is called a K3-surface. Hence $h^0(X, K_X) = h^0(X, \mathcal{O}_X) = 1$. Using the additional symmetry along the edges and $\chi_{\text{top}}(X) = 24$ as before, we get the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

5. Let $X = V(f) \subseteq \mathbb{CP}^4$ be a cubic 3-fold, f cubic. Then $K_X = \mathcal{O}_X(-2)$, so $h^{3,0}(X) = 0$. Again by Lefschetz and symmetry, we can fill almost the entire Hodge diamond, and from $\chi_{\text{top}}(X) = -6$ we get the final entries and obtain

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

6. Let $X \subseteq \mathbb{CP}^4$ be a quintic 3-fold. Then $K_X = \mathcal{O}_X$. So X is Kalabi-Yau, and we again compute the Hodge diamond using this additional symmetry as

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

7. Let $X \subseteq \mathbb{CP}^5$ be a cubic 4-fold. Then $K_X \cong \mathcal{O}_X(-3)$. Hence $h^0(K_X) = h^{4,0}(X) = 0$. This leaves three numbers missing in the Hodge diamond, $h^{3,1}(X) = h^{1,3}(X)$ and $h^{2,2}(X)$. Once we have one of these, the Euler characteristic allows us to finish the diamond. For this, consider the exact sequence $0 \to T_x \to T_{\mathbb{CP}^5}|_X \to N_X \to 0$, where N_X is the normal (line) bundle of X. We have $N_X = \mathcal{O}_X(X) = \mathcal{O}_{\mathbb{CP}^5}(X)|_X$. In this case, $N_X = \mathcal{O}_X(3)$. Dualize this exact sequence to get

$$0 \to N_X^* = \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{CP}^5}|_X \to \Omega^1_X \to 0.$$

The long exact sequence in sheaf cohomology yields

$$\cdots \to H^3(\Omega^1_{\mathbb{CP}^5}|_X) \to H^3(\Omega^1_X) \to H^4(N_X^*) \to H^4(\Omega^1_{\mathbb{CP}^5}|_X) \to \cdots$$

and we compute the terms as

$$H^4(X, N_X^*) \cong H^4(X, \mathcal{O}_X(-3)) \overset{\text{Serre}}{\cong} H^0(X, \mathcal{O}_X)^*.$$

Next we show $h^3(\Omega^1_{\mathbb{CP}^5}|_X) = 0 = h^4(\Omega^1_{\mathbb{CP}^5}|_X)$, which will show $h^{3,1}(X) = 1$. For this, consider the short exact sequence $0 \to \Omega^1_{\mathbb{CP}^5}(-3) \to \Omega^1_{\mathbb{CP}^5} \to \Omega^1_{\mathbb{CP}^5}|_X \to 0$ and the long exact sequence gives exact sequences $H^i(\Omega^1_{\mathbb{CP}^5}) \to H^i(\Omega^1_{\mathbb{CP}^5}|X) \to H^{i+1}(\Omega^1_{\mathbb{CP}^5}(-3))$ for $i = 3, 4$, where the left term vanishes by our computation of the Hodge diamond for $\mathbb{CP}^n$. So it remains to show that $h^i(\Omega^1_{\mathbb{CP}^5}(-3)) = 0$ for $i = 4, 5$. The final exact sequence we need is the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^n} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{CP}^n} \to 0,$$

where the maps are $1 \mapsto (x_0, \ldots, x_n)$ and $(l_0, \ldots, l_n) \mapsto \sum l_i \frac{\partial}{\partial x_i}$ Dualizing and twisting gives us the exact sequence $0 \to \Omega^1_{\mathbb{CP}^5}(-3) \to \mathcal{O}(-4)^{\oplus 6} \to \mathcal{O}(-3) \to 0$ and the claim follows from the long exact sequence in cohomology, the fact that $h^i(\mathbb{CP}^n, \mathcal{O}(k)) = 0$ for $0 < i < n$ and Serre duality. Thus we finally get the Hodge diamond

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

5 Lie Algebras

Definition 5.1. A *Lie algebra* is a vector space $\mathfrak{g}$ over a field k together with a bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y]$$

called *bracket* satisfying

(1) skew-symmetry, i.e. $[x, y] + [y, x] = 0$ for all $x, y \in \mathfrak{g}$,

(2) the Jacobi-identity for all $x, y, z \in \mathfrak{g}$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Example 5.2. Let G be a Lie group over $\mathbb{R}$ or $\mathbb{C}$ (smooth complex manifold with smooth holomorphic addition and negation. Let $e \in G$ be the identity element. Then $\mathfrak{g} = T_eG$ is a Lie algebra: There is an identification of T_eG with left G-invariant vector fields, and the commutator of vector fields defines a bracket on T_eG.

Example 5.3. For V a vector space, $\mathrm{gl}(V) = \mathrm{End}(V)$ is a Lie algebra with commutator as bracket. If $\dim V = n < \infty$, then $\mathrm{gl}(V) \cong M_{n\times n}(k)$.

Example 5.4. For $G = \mathrm{SL}_2(\mathbb{C}) \subseteq \mathrm{GL}_2(\mathbb{C})$, the construction in example 5.2 yields

$$\mathrm{sl}_2 := T_eG = \{A \in M_{2\times 2}(\mathbb{C}) \mid \mathrm{tr}\, A = 0\} \subseteq M_{2\times 2}(\mathbb{C}).$$

It is generated as a $\mathbb{C}$-vector space by the standard generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and one calculates

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

Definition 5.5. A homomorphism of Lie algebras is a linear map $\mathfrak{g} \to \mathfrak{h}$ compatible with the bracket. A representation of a Lie algebra g on a vector space V is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathrm{gl}(V)$. A representation V of $\mathfrak{g}$ is called irreducible if there are no proper non-trivial subspace $W \subseteq V$ preserved by $\mathfrak{g}$. An ideal of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{i} \subseteq \mathfrak{g}$ such that $[\mathfrak{g}] \subseteq \mathfrak{i}$. A Lie algebra is simple if it is non-abelian (i.e. $[,] \neq 0$) and contains no non-zero proper ideals. A Lie algebra is called semi-simple if it is a direct sum of simple Lie algebras.

Example 5.6. sl_2 is simple. gl_n is not semi-simple.

Theorem 5.7. *Every finite dimensional representation of a semi-simple Lie-algebra decomposes uniquely into a direct sum of irreducible representations.*

Example 5.8. gl_n admits a representation on $V = \mathbb{C}^n$. This induces a representation on all tensor constructions of V, for example on $V \otimes V$, one gets $A(v \otimes w) = Av \otimes w + v \otimes Aw$.

sl_2 has a standard representation on $\mathbb{C}^2$, hence an induced representation on $\mathrm{Sym}^n\, \mathbb{C}^2 \cong \mathbb{C}^{k+1}$. Explicitly, choose a basis $e, f \in V$, so that

$$\mathrm{Sym}^k\, \mathbb{C}^2 \cong \mathbb{C}e^k \oplus \mathbb{C}e^{k-1}f \oplus \ldots \oplus \mathbb{C}f^k.$$

Define a map $\mathrm{sl}_2 \to \mathrm{gl}_{k+1}$ by

$$X \mapsto f\frac{\partial}{\partial e}, \quad Y \mapsto e\frac{\partial}{\partial f}, \quad H \mapsto f\frac{\partial}{\partial e} \circ e\frac{\partial}{\partial f} - e\frac{\partial}{\partial f} \circ f\frac{\partial}{\partial e}$$

Check that this is a representation. One computes

$$H(e^i f^j) = ((i+1)j - i(j+1))e^i f^j = (j-i)e^i f^j.$$

The $\langle e^i f^j \rangle$ are eigenspaces of H of eigenvalue $j-i$. For general sl_2-representations, the eigenspaces of H are called *weight spaces*, the eigenvalues are called *weights*.

Exercise 5.9. Those are all the irreducible representations of sl_2!

Remark 5.10. The lowest weight space is $\langle e^k \rangle$ with weight $-k$. All other weight spaces are obtained from this one by applying the "raising operator" X. Each application of X raises the weight by 2.

5.1 sl_2-representations and Hodge theory

Let (M,ω) be a compact Kähler manifold of complex dimension n with $\omega \in \bigwedge^{1,1}(M)$ the (closed) Kähler form. Recall the Lefschetz operator

$$L : \bigwedge^{p,q}(M) \to \bigwedge^{p+1,q+1}(M), \quad \alpha \mapsto \alpha \bigwedge \omega$$

with adjoint $L^\dagger$.

Proposition 5.11. *The action of $[L, L^\dagger]$ on $\bigwedge^{p,q}(M)$ is given by multiplication by $p+q-n$.*

Proof. This can be checked fibre-wise, since L and $L^\dagger$ come from maps of bundles. One can check: For every $m \in M$, $(L^\dagger)_m$ is adjoint to L_m with respect to $\langle \alpha_m, \beta_m \rangle = (\beta_m \wedge *\alpha_m)/(\omega^n)_m$. Now without loss of generality, $\bigwedge^{p,q}(M)_m$ are spanned by $dz_I \wedge d\bar{z}_J$, $|I| = p, |J| = q$, and $\omega_m = dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n$. In this setting, we can calculate everything:

$$L_m(dz_I \wedge d\bar{z}_J) = \sum_{k \notin I \cup J} dz_{I\cup\{k\}} \wedge d\bar{z}_{J\cup\{k\}},$$

$$L_m^\dagger(dz_I \wedge d\bar{z}_J) = \sum_{k \in I \cap J} dz_{I\setminus\{k\}} \wedge d\bar{z}_{J\setminus\{k\}}.$$

(Check that the formula for $L_m^\dagger$ is correct.) Next

$$L_m^\dagger \circ L_m(dz_I \wedge d\bar{z}_J) = \sum_{k \in I\cup J} \sum_{l \in (I\cap J)\cup\{k\}} dz_{I\cup\{k\}\setminus\{l\}} \wedge d\bar{z}_{J\cup\{k\}\setminus\{l\}}, \tag{$*$}$$

$$L_m \circ L_m^\dagger(dz_I \wedge d\bar{z}_J) = \sum_{l \in I\cap J} \sum_{k \in (I\cup J)\setminus\{l\}} dz_{I\cup\{k\}\setminus\{l\}} \wedge d\bar{z}_{J\cup\{k\}\setminus\{l\}} \tag{$**$}$$

Oberve that all summands where $k \neq l$ appear in both formulas, so vanish in the commutator:

$$(L_m \circ L_m^\dagger - L_m^\dagger L_m)(dz_I \wedge d\bar{z}_J) = (\#\text{summands with } k = l \text{ in } * - \text{ those in } **)dz_I \wedge d\bar{z}_J$$

and this coefficient is $|I \cap J| - (n - |I \cup J|) = p + q - n$. □

Definition 5.12. Let $h : \bigwedge^*(M) \to \bigwedge^*(M)$ be the operator given by multiplication by $(k-n)$ in $\bigwedge^k(M)$. Define an sl_2-representation $\mathrm{sl}_2 \to \mathrm{gl}(\bigwedge^*(M))$ by

$$X \mapsto L^\dagger, \quad Y \mapsto L, \quad H \mapsto h.$$

Note that by what we just proved, this is an (infinite-dimensional) representation. We did not use that ω is closed. Recall that, for Kähler manifolds, $[L, \Delta_d] = 0 = [L^\dagger, \Delta_d]$ and clearly $[h, \Delta_d] = 0$. So we get an induced representation on harmonic forms $\mathcal{H}^*(M)$.

Theorem 5.13 (Hard Lefschetz)**.** *The map $L^k : H^{n-k}(M) \to H^{n+k}(M)$ is an isomorphism. If we define the k-th* primitive cohomology *of M to be $P^k(M) = \ker(L^{n-k+1} : H^k(M) \to H^{2n-k+2})$, then $H^*(M) = \bigoplus_{i,k\geq 0} L^i P^k(M)$*

Proof. $H^*(M)$ is a finite-dimensional sl_2-representation by the above. So by theorem 5.7 and exercise 5.9, we can write it as a direct sum of irreducible representations, which have weight-spaces which differ by weight 2 and are centered around 0. By the above proposition, weight k corresponds to cohomological degree $n-k$. The result follows from that.

More formally, for the first statement, it suffices to show that L^k is an isomorphism on each irreducible sl_2-subrepresentation. Choose one such subrepresentation, say with weight spaces $\mathbb{C}e^r \oplus \ldots \oplus \mathbb{C}f^r$. We know that $H(e^i f^j) = (j-i)e^i f^j$. Comparing with h, this means that $e^r \in H^{n-r}(M)$ and $f^r \in H^{n+r}(M)$. We know that $Y^r : \mathbb{C}e^r \to \mathbb{C}f^r$ is an isomorphism, and similarly $Y^k : \mathbb{C}e^k f^{r-k} \to \mathbb{C}e^{r-k} f^k$. Since L^k corresponds to Y^k, L^k is an isomorphism on this irreducible subrepresentation.

For the second statement, let $v \in H^*(M)$ and choose the irreducible sl_2-subrepresentation V containing v. Then $v = L^i v'$ for some primitive $v' \in H^{k-2i}(M)$. It suffices to show $v' \in P^{k-2i}(M)$, i.e. $L^{n+(k-2i)+1} v' = 0$. This follows from the fact that V has top weight $n+k-2i$. ☐

Example 5.14. Let M be a K3 surface. We have computed its Hodge diamond before, so we can see that the decomposition into irreducible subrepresentations contains one of the form

$$H^0(M) \oplus \mathbb{C}\omega \oplus H^4(M),$$

and everything else is concentrated in degree 2, so there are 21 more irreducible subrepresentations on $H^2(M)$.

Example 5.15. Abelian surfaces. The Hodge diamond is

$$\begin{matrix} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{matrix}$$

and we can read off that there is one 3-dimensional, four 2-dimensional and five one-dimensional irreducible subrepresentations of the sl_2-representation.

Definition 5.16. The primitive part of the cohomology of M is

$$H^{p,q}_{\text{prim}}(M) = P^{p,q}(M) := P^{p+q}(M) \cap H^{p,q}(M).$$

Corollary 5.17. *If $p+q \leq n$, then $h^{p,q} \geq h^{p-1,q-1}$.*

Proof. Every irreducible sl_2-subrepresentation contributing to $h^{p-1,q-1}$ also contributes to $h^{p,q}$. ☐

Corollary 5.18. *$L : H^k(M) \to H^{k+2}(M)$ is injective for $k < n$ and surjective for $k \geq n$.*

5.2 Hodge-Riemann Relations

Let M be a Kähler manifold with Kähler form ω. Define a bilinear form Q on $H^k(M)$ via

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \omega^{n-k} = \langle \alpha, L^{n-k}\beta \rangle.$$

Remark 5.19. The Hodge and Lefschetz decompositions are "orthogonal" with respect to Q: We have $Q(H^{p,q}(M), H^{p',q'}(M)) = 0$ unless $p = q'$ and $p' = q$, because otherwise $\alpha \wedge \beta \wedge \omega^{n-k}$ is not of type (n, n), so is 0. Further $Q(L^i P^{k-2i}(M), L^j P^{k-2j}(M)) = 0$ unless $i = j$. Let $\alpha = L^i\alpha', \beta = L^j\beta'$ with $\alpha' \in P^{k-2i}(M), \beta' \in P^{k-2j}(M)$. Assume $i < j$. Then

$$\alpha \wedge \beta \wedge \omega^{n-k} = L^i\alpha' \wedge L^{n-k+j}\beta = \pm L^{n-k+i-j}\alpha' \wedge \beta'.$$

Now α' is primitive of degree $k - 2i$, and $i + j > 2i$. Thus $L^{n-k+i-j}\alpha' = 0$ and $Q(\alpha, \beta) = 0$.

Theorem 5.20 (Hodge-Riemann relations)**.** *Let* $0 \neq \alpha \in P^{p,q}(M)$*. Let* $k = p + q$*. Then*

$$(-1)^{k(k-1)/2} i^{p-q} Q(\alpha, \bar{\alpha}) > 0.$$

Proof. Recall $\|\alpha\| = \langle \alpha, \alpha \rangle = \int_M \alpha \wedge *\alpha > 0$.

Lemma 5.21 (Weil's identity)**.** *For* α *primitive, one has*

$$*\alpha = \frac{(-1)^{k(k-1)/2} i^{p-q}}{(n-k)!} L^{n-k}\bar{\alpha}$$

Plucking this formula in, this gives

$$0 < \int_m \alpha \wedge *\alpha = C\alpha \wedge \bar{\alpha} \wedge \omega^{n-k} = CQ(\alpha, \bar{\alpha})$$

where C is the constant in Weil's identity. □

Proof. of the lemma. Let $x \in M$, choose a basis $dz_1, \dots, dz_n$ for $\Omega_{M,x}$ such that $\omega = \sum_i dz_i \wedge d\bar{z}_i$. Since α is primitive, $\alpha \wedge \omega^{n-k+1} = 0$. In local coordinates write $\alpha = \sum f_{IJ} dz_I \wedge d\bar{z}_J$. One needs to check that

$$*\alpha = 2^{k-n} \sum \varepsilon_{IJ} \overline{f_{IJ}} dz_{I^c} d\bar{z}_{J^c} = \frac{(-1)^{k(k-1)/2} i^{p-q}}{(n-k)!} \sum \overline{f_{IJ}} dz_I \wedge d\bar{z}_J \wedge \left(\sum_i dz_i \wedge d\bar{z}_i \right)^{n-k},$$

which is left as an exercise. □

Example 5.22. Let M be a Riemann surface with genus g. If $\omega \in H^{0,1}(M)$, then $-i\int_M \omega \wedge \bar{\omega} > 0$, and if $\omega \in H^{1,0}(M)$, then $i\int_M \omega \wedge \bar{\omega} > 0$, which we have seen previously.

Example 5.23. Let (S, ω) be a Kähler surface. Then $H^{1,1}(S) = LH^0(S) \oplus P^{1,1}(S)$. Since S is Kähler, $\int_M \omega \wedge \omega > 0$. For the other factor, if $\alpha \in P^{1,1}(S)$, then $Q(\alpha, \bar{\alpha}) < 0$ by the Hodge-Riemann relation. In summary, Q is positive definite on $\mathbb{C}\omega$ and negative definite on $P^{1,1}(S)$.

Theorem 5.24 (Hirzebruch signature / Hodge index theorem)**.** *Let* (S, ω) *be a compact Kähler surface. Then the intersection product on* $H^{1,1}(S, \mathbb{R}) := H^{1,1}(S) \cap H^2(S, \mathbb{R})$ *has signature* $(1, h^{1,1} - 1)$

Proof. $Q(\alpha, \bar{\beta})$ on $H^{1,1}(S, \mathbb{R})$ is $\int_M \alpha \wedge \beta$, which is just the cup-product/intersection product in de-Rham cohomology. Since $H^{1,1}(S) = \overline{H^{1,1}(S)}$, $\dim_{\mathbb{R}} H^{1,1}(S, \mathbb{R}) = \dim_{\mathbb{C}} H^{1,1}(S) = h^{1,1}$. By the Hodge-Riemann relations, the signature of $Q(\alpha, \beta)$ is $(1, h^{1,1} - 1)$ as claimed. □

Example 5.25. Let M be a K3 surface. Combining the above with the Hodge-Riemann relations for $H^{2,0}(M)$ and $H^{0,2}(M)$ (where everything is primitive), one finds that the signature of the intersection product is $(3, 19)$.

6 Polarized Hodge Structures

Let (M, ω) be a compact Kähler manifold. Assume that $\omega \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$. For example, for $M = \mathbb{CP}^n$, $[\omega] = c_1(\mathcal{O}(1))$ is integral. The same holds for closed submanifolds. If one sets $P^k(M, \mathbb{Z})$ as $P^k(M) \cap H^k(M, \mathbb{Z})/(\text{torsion})$, then $P^k(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = P^k(M)$. Furthermore, Q is $\mathbb{Z}$-valued on $P^k(M, \mathbb{Z})$.

Definition 6.1. An integral polarized Hodge structure of weight k is a finitely generated $\mathbb{Z}$-module V such that $V \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$ with $V^{p,q} = \overline{V^{q,p}}$ and a bilinear form Q satisfying

(1) $Q(V^{p,q}, V^{p',q'}) = 0$ unless $p = q'$ and $p' = q$,

(2) $(-1)^{k(k-1)/2} i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ for all $0 \neq \alpha \in V^{p,q}$,

(3) Q is symmetric if k is even and antisymmetric if k is odd.

If (M, ω) is compact Kähler and $\omega \in H^2(M, \mathbb{Z})$, then $(P^k(M, \mathbb{Z}), Q)$ is an integral polarized Hodge structure.

We want to understand polarized weight 1 Hodge structures of type $(h^{1,0}, h^{0,1}) = (1, 1)$ on $(V = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta, \cdot)$ with $\alpha \cdot \beta = 1$. Then $[H^{1,0}] \in \mathbb{P}(V_{\mathbb{C}}) \cong \mathbb{CP}^1$ defines a Hodge structure if $H^{1,0} \cap \overline{H^{1,0}} = \{0\}$, i.e. iff $[H^{1,0}] \in \mathbb{P}(V_{\mathbb{C}}) \setminus \mathbb{P}(V_{\mathbb{R}}) = \mathbb{H} \cup -\mathbb{H}$.

For this to be polarized, we need $i^{1-0}(-1)^{1 \cdot 0/2} Q(v, \bar{v}) > 0$ for $v \in H^{1,0} \setminus \{0\}$. The positivity on $H^{0,1}$ is equivalent to this, so it is sufficient to check one of them. Let $v = a\alpha + b\beta$. Then we need

$$0 < i(a\alpha + b\beta)(\bar{a}\alpha + \bar{b}\beta) = i(a\bar{b} - \bar{a}b) \iff \operatorname{Im}(\bar{a}b) > 0 \iff \operatorname{Im}(\frac{b}{a}) > 0.$$

But $\frac{b}{a}$ is a projective coordinate on $\mathbb{CP}^1$, so this means $[H^{1,0}] \in \mathbb{H}$.

Conversely, if $\tau \in \mathbb{H}$, setting $H^{1,0} = \mathbb{C}(\alpha + \tau\beta)$ gives a polarized Hodge structure.

Example 6.2. If C is an elliptic curve, then $H^1(C, \mathbb{Z}) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, Q is the intersection form.

More generally, given a genus g curve C, we need a standard symplectic basis $\{\alpha_i, \beta_i\}$ of $H^1(C, \mathbb{Z})$ to identify $H^1(C, \mathbb{Z})$ with a fixed reference symplectic lattice $(V_{\mathbb{C}}, \mathbb{Q})$.

So let us look at the space of all polarized weight 1 Hodge structures of type (g, g) on (V, Q), $V \cong \mathbb{Z}^{2g}$. Similar to before, $[H^{1,0}] \in \operatorname{Gr}(g, V_{\mathbb{C}})$ and we need $H^{1,0} \cap \overline{H}^{1,0} = 0$, i.e. $H^{1,0} \cap V_{\mathbb{R}} = 0$. Note that this is an open condition, so $\{\text{Hodge strucutres on } V \text{ of type } (g, g)\} \subseteq \operatorname{Gr}(g, V_{\mathbb{C}})$ is open. For the polarization, we need in particular $H^{1,0} \perp H^{1,0}$, i.e. $Q|_{H^{1,0}} = 0$.

Definition 6.3. If $(\mathbb{C}^{2g}, \cdot)$ is a symplectic vector space, a g-dimensional subspace $L \subseteq \mathbb{C}^{2g}$ such that $\cdot|_L = 0$ is called a *Lagrengian*. More generally, a subspace on which a quadratic form restricts to 0 is called *isotropic*.

The easiest example of a Lagrengian in V is $\mathbb{C}\alpha_1 \oplus \ldots \oplus \mathbb{C}\alpha_g$. We need $H^{1,0}$ to be Lagrengian, so $[H^{1,0}] \in \operatorname{LGr}(V_{\mathbb{C}}, Q)$, the Lagrengian Grassmanian. It is easy to see that $H^{1,0}$ is Lagrengian iff $H^{0,1} = \overline{H^{1,0}}$ is, so we don't need to check the latter separately. The positivity condition says that $iQ(v, \bar{v}) > 0$ for all $0 \neq v \in H^{1,0}$. Observe that this automatically implies that $H^{1,0}$ and $\overline{H^{1,0}}$ are transverse, since otherwise $iQ(v, \bar{v}) = iQ(v, v) = 0$ for $0 \neq v \in H^{1,0} \cap V_{\mathbb{R}}$. Now let $e_1, \ldots, e_g$ be a basis of $H^{1,0}$. Then positivity is equivalent to $iQ(e_j, \bar{e}_k)$ being positive definite. Since the eigenvalues of a matrix vary continuously, there is an open subset $N \subseteq \operatorname{LGr}(V_{\mathbb{C}}, Q)$ with $[H^{1,0}] \in N$ such that positivity holds for all $L' \in N$.

Definition 6.4. The *Siegel upper half-space* $\mathcal{H}_g \subseteq \mathrm{LGr}(V_\mathbb{C}, Q)$ is the open subset of $[L] \in \mathrm{LGr}(V_\mathbb{C}, Q)$ such that $iQ(l, \bar{l}) > 0$ for all $l \in L \setminus 0$. By the arguments above, this is exactly the space of Hodge structures on $(V, \mathbb{Q})$.

Definition 6.5. Fix a type $\underline{h} = (h^{k,0}, \ldots, h^{0,k})$ and (V, Q), V a lattice and Q an alternating/symmetric nondegenerate bilinear form depending on the partiy of k. Then the *period domain* is the space of all Hodge structures of the given type polarized by Q.

Definition 6.6. The *Hodge filtration* $F^\bullet$ is the descending filtration $F^i = \bigoplus_{p\geq i} H^{p,q}$, i.e.

$$F^0 = V_\mathbb{C} \supseteq F^1 = \bigoplus_{i=0}^{k-1} H^{k-i,i} \supseteq \ldots \supseteq F^{k-1} = H^{k,0} \oplus H^{k-1,1} \supseteq F^k = H^{k,0}$$

Note that $\overline{F^i} = \bigoplus_{p\geq i} H^{q,p}$, so that $H^{p,q} = F^p \cap \overline{F^q}$ and $F^{p+1} \cap \overline{F^q} = 0$. Hence $F^\bullet$ recovers the Hodge decomposition.

Proposition 6.7. *Let $F^\bullet$ define a polarized Hodge structure of type $\underline{h}$ on (V, Q). Then*

$$(F^p)^\perp = F^{k-p+1}$$

Proof. Since $H^{p,q}$ pairs non-trivially only with $H^{q,p}$, every summand in F^p is orthogonal to every summand in F^{k-p+1}. This shows $F^{k-p+1} \subseteq (F^p)^\perp$. Since Q is not degenerate, $\dim(F^p)^\perp = \operatorname{codim} F^p$. On the other hand $\dim F^{k-p+1} = \operatorname{codim} F^p$ by Hodge symmetry. $\square$

Proposition 6.8. *The condition $(F^p)^\perp = F^{k-p+1}$ implies orthogonality.*

Proof. Exercise. $\square$

Define the *flag variety*

$$\mathrm{Fl}(\underline{h}, V_\mathbb{C}) = \{\text{desc. filtrations } F^\bullet \text{ on } V_\mathbb{C} \text{ s.t. } h^{p,q} = \dim F^p/F^{p+1}\}.$$

It has a natural structure as a projective algebraic variety as a subvariety of products of Grassmanians. The orthogonal flag variety $\mathrm{OFl}(\underline{h}, V_\mathbb{C}) \subseteq \mathrm{Fl}(\underline{h}, V_\mathbb{C})$ is the closed algebraic subvariety where $(F^p)^\perp = F^{k-p+1}$ for all p. Finally, we have the open condition $i^{p-q}(-1)^{k(k-1)/2} Q(v, \bar{v}) > 0$ for all $0 \neq v \in F^p \cap \overline{F^q}$. In conclusion, the period domain $\mathbb{D}$ is an open subvariety of $\mathrm{OFl}(\underline{h}, V_\mathbb{C})$, in particular $\mathbb{D}$ is an complex manifold.

Proposition 6.9. *If we have a flag $F^\bullet \in \mathbb{D} \subseteq \mathrm{OFl}(\underline{h}, V_\mathbb{C})$, then $V_\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$.*

Proof. By induction, it suffices to show $F^p = H^{p,q} \oplus F^{p+1}$. Since $H^{p,q} = F^p \cap \overline{F^p}$ has the right dimension, it suffices to show that $H^{p,q} \cap F^{p+1} = \{0\}$. Let $0 \neq \alpha \in H^{p,q} \cap F^{p+1} = F^{p+1} \cap \overline{F^q} \subseteq F^{p+1} \cap \overline{F^{q-1}}$ From the positivity condition we get both $i^{p-q}(-1)^{k(k-1)/2} \alpha\bar{\alpha} > 0$ and $i^{(p+1)-(q-1)}(-1)^{k(k-1)/2}\alpha\bar{\alpha} > 0$, but these differ by $i^2 = -1$, contradiction. $\square$

We return to Siegel spaces. Recall That

$$\mathcal{H}_g = \{L \mid il \cdot \bar{l} > 0 \text{ for all } l \in L\} \subseteq \mathrm{LGr}(V_\mathbb{C})$$

is the open subspace of polarized Hodge structures on $(V, \cdot)$ equal to $\mathbb{Z}^{2g}$ with the standard symplectic form of type (g, g). Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be a standard symplectic basis.

Lemma 6.10. *There exists a basis $(\omega_1, \dots, \omega_g)$ of $L \in \mathcal{H}_g$ such that $\omega_i \cdot \alpha_j = \delta_{ij}$. In this basis, $(\omega_i \cdot \beta_j)$ is a symmetric $g \times g$ matrix with complec entries whose imaginary part is positive definite. So*

$$\mathcal{H}_g = \{P \in \mathrm{Sym}_{g\times g}(\mathbb{C}) \mid \operatorname{Im} P > 0\}$$

This is called a Type III hermitian symmetric domain.

Denote by $\mathrm{Sp}(2g, \mathbb{R})$ the isometries on $(V_{\mathbb{R}}, \cdot)$. This group acts transitively on $\mathcal{H}_g$. The stabilizer of L is $U(g)$: $i\alpha \cdot \bar{\beta}$ is a hermitian positive definite form on L and any element $g \in U(L, i\alpha \cdot \beta)$ extends to an element of $\mathrm{Sp}(2g, \mathbb{R})$. Hence

$$\mathcal{H}_g \cong \mathrm{Sp}(2g, \mathbb{R})/U(g).$$

Taking a left invariant metric on $\mathrm{Sp}(2g, \mathbb{R})$ which is right-invariant under $U(g)$, we get a Riemannian metric on $\mathcal{H}_g$ for which $\mathrm{Sp}(2g, \mathbb{R})$ acts by isometries.

The subgroup $\mathrm{Sp}(2g, \mathbb{Z}) \subseteq \mathrm{Sp}(2g, \mathbb{R})$ of isometries of $(V, \cdot)$ acts freely transitively on the standard symplectic bases of V. Define $\mathcal{A}_g := \mathrm{Sp}(2g, \mathbb{Z})\backslash \mathrm{Sp}(2g, \mathbb{R})/U(g)$. Since $\mathrm{Sp}(2g, \mathbb{Z}) \subseteq \mathrm{Sp}(2g, \mathbb{R})$ is discrete and $U(g)$ is compact, the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$ is properly discontinuous. So $\mathcal{A}_g$ inherits a Hausdorff topology from $\mathcal{H}_g$. More precisely, for all $p \in \mathcal{A}_g$ there is a neighbourhood $N \ni p$ and $\widetilde{N} \subseteq \mathcal{H}_g$ open such that N is the quotient of $\widetilde{N}$ by some finite quotient of $\mathrm{Sp}(2g, \mathbb{Z})$.

Definition 6.11. The *Torelli map*

$$\mathcal{M}_g = \{\text{isomorphism classes of genus } g \text{ Riemann surfaces}\} \to \mathcal{A}_g$$

is defined by sending C to the polarized Hodge structure on $H^1(C, \mathbb{Z})$, i.e. we choose a symplectic basis α_i, β_i of $H^1(C, \mathbb{Z})$, giving an isometry $\varphi : (H^1(C, \mathbb{Z}), \cup) \to (V, \cdot)$. This gives an element of $\mathcal{H}_g$ by using φ to give V a polarized Hodge structure. Since φ is unique up to post-compostion by elements of $\mathrm{Sp}(V, \cdot)$. So the corresponding point $\varphi(H^{1,0}) \in \mathcal{H}_g$ is well-defined up to $\mathrm{Sp}(V, \cdot)$, i.e. its class in $\mathcal{A}_g$ is well-defined.

It's a fact from complex geometry that deformations of C as a Riemann surface are identified with $H^1(C, T_C)$. Serre duality gives $h^1(C, T_C) = h^0(C, K_C^{\otimes 2})$ and Riemann-Roch gives

$$h^0(K_C^{\otimes 2}) - h^1(K_C^{\otimes 2}) = \deg(K_C^{\otimes 2}) + 1 - g = 3g - 3.$$

And $h^1(K_C^{\otimes 2}) = h^0(K_C^{-1}) = 0$ if $g \geq 2$, so $h^0(K_C^{\otimes 2}) = 3g - 3$ for $g \geq 2$. Thus $\dim \mathcal{M}_g = 3g - 3$, whereas $\dim \mathcal{A}_g = \dim \mathcal{H}_g = \dim \mathrm{Sym}_{g\times g}(\mathbb{C}) = \binom{g+1}{2}$. Keeping in mind that we studied the case $g = 1$ in chapter 2, we can list the values for small values of g:

g	0	1	2	3	4	5
$\dim \mathcal{M}_g$	0	1	3	6	9	12
$\dim \mathcal{A}_g$	0	1	3	6	10	15

For $g = 2, 3$, one can show that the Torelli map is birational, while the dimensions diverge for $g \geq 4$. In general, one has:

Theorem 6.12 (Torelli 1905)**.** *The Torelli map is injective.*

Even weight Hodge structures

Let $(V, \cdot)$ be a lattice with a non-degenerate symmetric bilinear form. Then $(V_{\mathbb{R}}, \cdot) \cong (\mathbb{R}^{a,b}, Q)$ where Q is a diagonal matrix with 1 in the first a and -1 in the remaining b entries. Let $V_{\mathbb{C}} = \bigoplus H^{p,q}$ be a polarized Hodge structure. Then by positivity $\sum_{2|p} h^{p,q} = a$ and $\sum_{2\nmid p} h^{p,q} = b$. Since $H^{p,q} \oplus H^{q,p}$ is invariant under complex conjugation, there is a real supspace $(H^{p,q} \oplus H^{q,p})_{\mathbb{R}} \subseteq V_{\mathbb{R}}$. The signature of $\alpha \cdot \bar{\alpha}$ on $H^{p,q} \oplus H^{q,p}$ is the same as the signature of $(H^{p,q} \oplus H^{q,p})_{\mathbb{R}}$.

For example, consider polarized Hodge structures of type $(1, n, 1)$ on a lattice $(V, \cdot)$ of signature $(2, n)$. Then $H^{2,0} \subseteq V_{\mathbb{C}}$ is a line, say $H^{2,0} = \mathbb{C}\omega$. Then $F^2 = H^{2,0}$ and $F^1 = (F^2)^{\perp} = H^{2,0} \oplus H^{1,1}$, so $H^{2,0}$ determines the full $F^{\bullet}$. Also notice $F^2 \subseteq F^1$, so $\omega \in \omega^{\perp}$, i.e. $\omega \cdot \omega = 0$. Such an ω is called isotropic. The other spaces are given as $H^{0,2} = \overline{H^{2,0}}$ and $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$. From positivity, $\omega \cdot \bar{\omega} > 0$, which already implies that $\cdot$ is negative definite on $H^{1,1}$. So again everything is uniquely determined by ω. Thus we can identify the period domain $\mathbb{D}$ with

$$\mathbb{P}\{\omega \in V_{\mathbb{C}} \mid \omega \cdot \omega = 0,\ \omega \cdot \bar{\omega} > 0\} \subset V(\omega \cdot \omega) \subset \mathbb{P}V_{\mathbb{C}} \cong \mathbb{CP}^{n+1}$$

$\omega \cdot \omega$ is a nonsingular quadric of dimension n and $\mathbb{D}$ is an open subset. This is called a Type IV Hermitian symmetric domain.

Given a $\mathbb{Z}$-lattice $V = V_{\mathbb{Z}}$ with a symmetric or skew-symmetric $\mathbb{Z}$-valued bilinear form $\cdot$, and a Hodge type $\underline{h} = (h^{k,0}, h^{k-1,1}, \dots, h^{0,k})$, $h^{p,q} = h^{q,p}$, we have a period domain

$$\mathbb{D} = \{F^{\bullet} \in \mathrm{OFl}(\underline{h}, V_{\mathbb{C}}) \mid i^{p-q}(-1)^{k(k-1)/2} \alpha \cdot \bar{\alpha} > 0 \text{ for all } 0 \neq \alpha \in F^p \cap \overline{F^q}\}.$$

Further set $\Gamma = \mathrm{Aut}(V_{\mathbb{Z}}, \cdot) = \{\gamma \in \mathrm{GL}(V_{\mathbb{Z}}) \mid \gamma x \cdot \gamma y = x \cdot y\}$. Γ is an integral form of $\mathrm{Sp}(\sum h^{p,q})$ or $O(\sum_{2|p} h^{p,q}, \sum_{2\nmid p} h^{p,q})$. Write $G_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, \cdot)$, which is either $\mathrm{Sp}(2g, \mathbb{R})$ or $O(a, b, \mathbb{R})$ and $\Gamma \subseteq G_{\mathbb{R}}$ is a discrete subgroup.

Proposition 6.13. *$G_{\mathbb{R}}$ acts transitively on $\mathbb{D}$, and the stabilizer of $F^{\bullet}$ is $\prod_{p<q} U(h^{p,q}) \times O(h^{k/2,k/2})$, where the second factor only exists if k is even.*

Definition 6.14. The *Hodge norm/form* is $i^{p-q}(-1)^{k(k-1)/2}\alpha \cdot \bar{\beta}$ on $\alpha, \beta \in H^{p,q}$. This defines a hermitian positive definite inner product on $V_{\mathbb{C}}$. A *Hodge basis* is an orthonormal basis of $\bigoplus_{p<q} H^{p,q} \oplus (H^{k/2,k/2})_{\mathbb{R}}$ with respect to the Hodge form which is compatible with the direct sum decomposition.

Proof. Let $V_{\mathbb{C}} = \bigoplus \widetilde{H}^{p,q}$ be some other polarized Hodge structure on $V_{\mathbb{Z}}$, let $e_i, \widetilde{e}_i$ be Hodge bases of $\bigoplus H^{p,q}$ and $\bigoplus \widetilde{H}^{p,q}$, respectively. Then $\varphi : e_i \mapsto \widetilde{e}_i$ defines a linear isomorphism $\bigoplus_{p<q} H^{p,q} \to \bigoplus_{p<q} \widetilde{H}^{p,q}$ and $(H^{k/2,k/2})_{\mathbb{R}} \to (\widetilde{H}^{k/2,k/2})_{\mathbb{R}}$. φ extends uniquely to $\psi : V_{\mathbb{C}} \to V_{\mathbb{C}}$ which is real, by declaring $\psi(\bar{v}) = \overline{\varphi(v)}$ for $v \in H^{p,q}$, $p < q$. This $\psi \in G_{\mathbb{R}}$ is then the unique element sending $(H^{p,q}, e_i)$ to $(\widetilde{H}^{p,q}, \widetilde{e}_i)$. In particular, $G_{\mathbb{R}}$ acts transitively on $\mathbb{D}$ and the stabilizer $\mathrm{Stab}_{G_{\mathbb{R}}}(F^{\bullet})$ acts freely transitively on the Hodge bases of $F^{\bullet}$. Thus

$$\mathrm{Stab}_{G_{\mathbb{R}}}(F^{\bullet}) = \prod_{p<q} U(H^{p,q}, \text{hodge}) \times O((H^{k/2,k/2})_{\mathbb{R}}, \text{hodge}).$$

Thus $\mathbb{D} = G_{\mathbb{R}}/K := G_{\mathbb{R}}/\prod_{p<q} U(h^{p,q}) \times O(h^{k/2,k/2})$ by orbit-stabilizer. □

Definition 6.15. $\Gamma \backslash \mathbb{D}$ is called a *Hodge manifold.*

Definition 6.16. A *K3 surface* is a compact complex surface X such that $K_X = \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$. (equivalently $h^1(X, \mathbb{C}) = 0$ or $\pi_1(X) = 0$).

Remark 6.17. Every K3 surface is Kähler, which is a result of Siu (1983). Also, all K3 surfaces are deformation-equivalent, i.e. for X_0, X_1 K3 surfaces, there exists a flat family/holomorphic fibre bundle $\mathcal{X} \to B$ of K3s over a connected base B and points $0, 1 \in B$ such that $\mathcal{X}_0 = X_0$ and $\mathcal{X}_1 = X_1$.

An example of a K3 surface is $X = V(x_0^4 + \ldots + x_3^4) \subseteq \mathbb{CP}^3$. By the π_1-version of the Lefschetz hyperplane theorem, $\pi_1(X) = \pi_1(\mathbb{CP}^3) = 0$.

Corollary 6.18. *Any smooth complete intersection of dimension at least* 2 *in* $\mathbb{CP}^n$ *has trivial* π_1.

Theorem 6.19 (Noether's Formula)**.** *Let* S *be a compact complex surface. Then*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(\chi_{top}(S) + K_S \cdot K_S)$$

Hence in this case, $\chi_{\text{top}}(S) = 12(1 - 0 + 1) = 24$. This gives another way to compute the Hodge diamond of a K3 surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proposition 6.20. *If* $\mathcal{X} \to B$ *is a smooth proper family of compact Kähler manifolds over* B *connected, then* $h^{p,q}(\mathcal{X}_b)$ *is constant for all* $b \in B$.

Proof. The function $b \mapsto h^{p,q}(\mathcal{X}_b) = h^q(\mathcal{X}_b, \Omega^p_{\mathcal{X}_b})$ is upper semicontinuous. On the other hand, $\mathcal{X} \to B$ is a fibre bundle, so $h^k(\mathcal{X}_b) = \sum_{p+q=k} h^{p,q}(\mathcal{X}_b)$ is constant. □

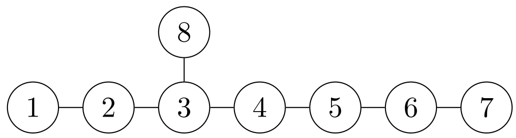
$H^2(X, \mathbb{C})$ has a symmetric bilinear form given by the intersection product. By Poincare duality, this is unimodular, i.e. for all $v \in V = H^2(X, \mathbb{Z})$ primitive (=not a multiple of another lattice point), there is a w such that $v \cdot w = 1$. Equivalently, if (e_i) is a $\mathbb{Z}$-basis, then $(e_i \cdot e_j)$ has determinant ± 1, or $V = V^* := \{w \in V_{\mathbb{R}} \mid w \cdot v \in \mathbb{Z} \text{ for all } v \in V\}$. The Hirzebruch signature theorem says that $\cdot$ has signature $(3, 19)$. Furthermore,

Theorem 6.21 (Wu's formula)**.** *For* S *a compact complex surface,* $v \in V$, $v \cdot K_S = v \cdot v \bmod 2$.

Since $K_X = 0$, $\cdot$ is even. A theorem in Serre's Course in Arithmetic says that an indefinite unimodular lattice is uniquely determined by its signature and parity. Write $\mathrm{I}_{m,n}$ for the unique unimodular lattice of signature (m, n) and $\mathrm{II}_{m,n}$ for the unique unimodular lattice of signature (m, n). We can easily write down $\mathrm{I}_{m,n} = \mathbb{Z}^{m+n}$ with form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix},$$

with 1 in the first m entries and -1 in the remaining n ones. Type $\mathrm{II}_{m,n}$ exists only if $m \equiv n \bmod 8$. Let $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with signature $(1, 1)$. If $n = m$, take copies of H. Otherwise, consider E_8 given by $e_i \cdot e_i = -2$ and $e_i \cdot e_j = 1$ if i and j are connected in the following graph and $e_i \cdot e_j = 0$ otherwise:



Then one checks that E_8 is even unimodular of signature $(0, 8)$. Then $\mathrm{II}_{n,m} = H^{\oplus \min(n,m)} \oplus E_8^{\oplus (n-m)/8}$.

Returning to $V = H^2(X, \mathbb{Z})$ of signature $(3, 19)$, we now see that V is isometric to $H^{\oplus 3} \oplus E_8^{\oplus 2}$. Fix a copy of $\mathrm{II}_{3,19}$.

Definition 6.22. A *marking* of a K3 surface X is an isometry $\varphi : (H^2(X,\mathbb{Z}),\cup) \to (\mathrm{II}_{3,19},\cdot)$. Given a K3 surface and a marking (X,φ), we can extract a Hodge decomposition on $\mathrm{II}_{3,19}$ of type $(1,20,1)$.

Definition 6.23. A *polarized* compact complex manifold is a pair (X,L) where X is a compact complex manifold and $L \to X$ is an *ample* line bundle. That is, for $n \gg 0$, the map $\varphi_{|nL|} : X \to \mathbb{CP}^N$, $x \mapsto [s_0(x) : \ldots : s_N(x)]$, where $\{s_i\}$ is a basis of $H^0(X,nL)$, is an embedding.

If (X,L) is polarized, then $\varphi^*_{|nL|}\omega_{FS}$ is Kähler on X, where ω_{FS} is the Kähler form on $\mathbb{CP}^N$. Write $[\omega_{FS}] = [H] \in H^2(\mathbb{CP}^N,\mathbb{Z})$ for some surface H, then

$$[\varphi^*_{|nL|}\omega_{FS}] = \varphi^*_{|nL|}[\omega_{FS}] = \varphi^*_{|nL|}[H] = [H \cap X] = c_1(nL) = nc_1(L).$$

Dividing by n, $\frac{1}{n}\varphi^*_{|nL|}\omega_{FS}$ is a Kähler form representing $c_1(L) \in H^2(X,\mathbb{Z})$.

Theorem 6.24 (Kodaira Embedding Theorem)**.** *If (X,ω) is a compact Kähler manifold with $[\omega] \in H^2(X,\mathbb{Z})$, then X is a projective variety with $[\omega] = c_1(L)$, $L \to X$ ample.*

Define the Kähler cone $\mathcal{K}_X \subseteq (H^{1,1}(X))_\mathbb{R}$ as the set of cohomology classes representing a Kähler form. $\mathcal{K}_X$ is closed under scaling by $\mathbb{R}^+$ and addition. It is further open: If ω is Kähler and $\alpha \in \bigwedge^{1,1}(X)_\mathbb{R}$ closed, then $\omega + \varepsilon\alpha$ is still Kähler for $\varepsilon \ll 1$. Kondaira's Theorem shows that $\mathcal{K}_X \cap H^2(X,\mathbb{Z}) \neq \{0\}$ if and only if X is projective.

Theorem 6.25 (Lefschetz $(1,1)$-theorem)**.** *The map $c_1 : \mathrm{Pic}(X) = H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})$ surjects onto $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$.*

Proof. Consider the long exact sequence for exponential exact sequence

$$\ldots \to H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{\alpha} H^2(X,\mathcal{O}) \to \cdots$$

Hence it is sufficient to compute $\ker(\alpha)$. The map α is the composition

$$H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \xrightarrow{\pi_{0,2}} H^{0,2}(X) = H^2(X,\mathcal{O}).$$

Here we write $\pi_{p,q}$ for the projection to $H^{p,q}(X)$. Now if $v \in \ker\alpha = \ker\pi_{0,2}$, then since $\bar{v} = v$ also $\pi_{2,0}(v) = 0$, so that $v \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. Conversely, if $v \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$, then in particular $\pi_{0,2}(v) = 0$, so $v \in \ker\alpha$. $\square$

Let (X,L) be a polarized K3 surface. Observe that $L \cdot L = \int_X c_1(L) \wedge c_1(L) \in 2\mathbb{Z}_{>0}$. Call $2d = L \cdot L$ the degree. If $L = \pi^*\mathcal{O}(1)$, this notion agrees with the usual degree.

Definition 6.26. A *marking* on the degree k cohomology of a polarized projective variety (X,L) is an isometry $P^k(X,\mathbb{Z}) \to (V_\mathbb{Z},\cdot)$ to a fixed reference lattice.

Then $\varphi(F^\bullet)$ defines a point in the period domain associated to $(V_\mathbb{Z},\cdot)$ for $\underline{h}$ the Hodge vector of $P^k(X,\mathbb{Z})$.

Proposition 6.27. *Fix $v \in \mathrm{II}_{3,19}$ primitive, satisfying $v \cdot v = 2d$ for some fixed d. For (X,L) a primitive polarized K3 surface, there exists an isometry $\varphi : H^2(X,\mathbb{Z}) \to \mathrm{II}_{3,19}$ sending $c_1(L) \mapsto v$.*

Proof. We know that there exists an isometry $\varphi' : H^2(X,\mathbb{Z}) \to \mathrm{II}_{3,19}$. Let $v' = \varphi'(c_1(L))$, which is a primitive vector of norm $2d$. It thus suffices to show that there is $\gamma \in O(\mathrm{II}_{3,19})$ such that $\gamma(v') = v$. Since $\mathrm{II}_{3,19}$ is unimodular, there is a w' such that $v' \cdot w' = 1$. One can find $u \in \{v',w'\}^\perp \subset \mathrm{II}_{3,19}$ such that $u \cdot u = -w' \cdot w'$. Replacing w' by $w' + u$, we may assume that $w' \cdot w' = 0$. Next, change basis to $(v' - dw', w')$, since $(v' - dw') \cdot (v' - dw') = 0$. We can write $v = e + df$, where $H \cong \mathbb{Z}e \oplus \mathbb{Z}f$ is the first summand in $\mathrm{II}_{3,19} = H \oplus H \oplus H \oplus E_8 \oplus E_8$. Then define an isometry $\gamma : \mathrm{II}_{3,19} \to \mathrm{II}_{3,19}$ sending $v' - dw'$ to e and w' to f, so that $\gamma(v') = v$.

Fact: If L is an unimodular lattice and $M \subseteq L$ is a primitive sublattice, then $M^*/M \cong (M^\perp)^*/M^\perp$. In particular, if M is unimodular, so is $M^\perp$.

From this fact it follows that $(\mathbb{Z}v' \oplus \mathbb{Z}w')^\perp$ is unimodular. It is also even of sign $(2, 18)$. Therefore by uniqueness there exists an isometry $\gamma_0 : \{v', w'\}^\perp \to H \oplus H \oplus E_8 \oplus E_8$. So we can complete the definition of γ as $\gamma = \gamma_0$ on $\{v', w'\}^\perp$. Finally, $\gamma \circ \varphi'$ is the desired isometry. □

From the proof we can conclude that $P^2(X, \mathbb{Z})$ is isometric to the fixed lattice $v^\perp \in \mathrm{I}_{3,19}$. Thus a marked primitively polarized K3 surface (X, L, φ) gives a polarized Hodge strucutre of type $(1, 19, 1)$ on $v^\perp$. The period of (X, L, φ) is

$$\varphi(H^{2,0}) \in \mathbb{D} = \mathbb{P}\{x \in v^\perp \otimes \mathbb{C} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}.$$

Observe that the set of markings of (X, L) fis a torsor over $\Gamma_{2d} = \{\gamma \in O(\mathrm{II}_{3,19}) \mid \gamma(v) = v\}$ acting by $\varphi \mapsto \gamma \circ \varphi$. Therefore (X, L) uniquely determines a point in $\Gamma_{2d}\backslash\mathbb{D}$. We generalize this as follows:

Let $\pi : (\mathcal{X}, \mathcal{L}) \to B$ be a smooth projective polarized family over connected base B, of relative dimension n. Fix $0 \in B$ and fix a lattice $(V_\mathbb{Z}, \cdot)$ isometric to $P^k(\mathcal{X}_0, \mathbb{Z})$ with intersection product $\alpha \cdot \beta = \int_{\mathcal{X}_0} \alpha \wedge \beta \wedge c_1(\mathcal{L}|_{\mathcal{X}_0})^{n-k}$. From this we get the period map $\varphi : B \to \Gamma\backslash\mathbb{D}$, where $\mathbb{D} = \{$polarized Hodge structures on $(V_\mathbb{Z}, \cdot)$ of the same type as $P^k(\mathcal{X}_0\}$, given by $b \mapsto [F_b^\bullet]$, where $F_b^\bullet$ is the hodge filtration of $P^k(\mathcal{X}_b, \mathbb{C})$. To make this well-defined, this requires a path cennecting b to 0 to identify the lattices. But if we have two such paths γ_1, γ_2, then $\gamma_1^{-1} \circ \gamma_2 \in \pi_0(B, 0)$. Such an element $\gamma \in \pi_0(B, 0)$ induces an isometry $\rho(\gamma) \in \Gamma = \mathrm{Aut}(V_\mathbb{Z}, \cdot)$. This defines a group homomorphism $\pi_1(B, 0) \to \Gamma$ called "monodromy representation". In particular, the class of $F_b^\bullet$ doesn't depend on the choice of path.

Theorem 6.28 (Griffiths). *(1) $\varphi : B \to \Gamma\backslash\mathbb{D}$ is holomorphic.*

(2) The image of $d\varphi : TB \to T(\Gamma\backslash\mathbb{D})$ lands in the subbundle $\Xi \in T(\Gamma\backslash\mathbb{D})$ called the horizontal subbundle, to be defined below.

Let $F^p \subseteq V_\mathbb{C}$ be a subspace. What is $T_{[F^p]} \operatorname{Gr}(d, V_\mathbb{C})$? It is canonically isomorphic to $\operatorname{Hom}(F^p, V_\mathbb{C}/F^p)$: $V_\mathbb{C}/F^p$ can be represented by a tranverse subspace. Small perturbations of $[F^p]$ can be considered the graphs of a linear map $F^p \to V_\mathbb{C}/F^p$.

If $[F^\bullet] \in \mathrm{Fl}(\underline{h}, V_\mathbb{C})$, then $T_{[F^\bullet]} \mathrm{Fl}(\underline{h}, V_\mathbb{C}) \hookrightarrow \bigoplus_p \operatorname{Hom}(F^p, V_\mathbb{C}/F^p)$. Now we can define $\Xi \subseteq T\mathbb{D}$ as

$$\Xi_{[F^\bullet]}\mathbb{D} = T_{[F^\bullet]}\mathbb{D} \cap \bigoplus_p \operatorname{Hom}(F^p, F^{p-1}/F^p).$$

This then descends to the horizontal subbundle on $\Gamma\backslash\mathbb{D}$. One has $\Xi = T\mathbb{D}$ if and only if $\underline{h} = (g, g)$ or $\underline{h} = (1, n, 1)$: We elaborate on the second case: Say $H^{2,0} = \mathbb{C}v$ and $H^{0,2} = \mathbb{C}\bar{v}$. We have to rule out perturbations of the form $v \mapsto v + \varepsilon\bar{v}$. But in that case we would have $(v + \varepsilon\bar{v}) \cdot (v + \varepsilon\bar{v}) = 2\varepsilon v \cdot \bar{v}$, contradicting the orthogonality $v \cdot v = 0$.

(2) is called Griffiths transversality.

Let F_{2d} be the moduli space of primitivly polarized K3 surfaces (X, L) of degree $L \cdot L = 2d$. Let $(\mathcal{X}, \mathcal{L}) \to F_{2d}$ be the universal family, such that the fibre over a point is the surface described by that point. We will assume that such an object exists. Then we get a holomorphic map $F_{2d} \to \Gamma\backslash\mathbb{D}$, where

$$\mathbb{D} = \mathbb{P}\{x \in v^\perp \otimes \mathbb{C} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}$$

and $\Gamma = \mathrm{Stab}_{O(\mathrm{II}_{3,19})}(v)$ where $v \in \mathrm{II}_{3,19}$ is a fixed primitive vector with $v \cdot v = 2d$. Recall that we constructed a marking $\sigma : H^2(X, \mathbb{Z}) \to \mathrm{II}_{3,19},\ c_1(\mathcal{L}) \to v$. Then $H^{2,0} \subseteq c_1(\mathcal{L})^\perp$, so $\sigma(H^{2,0}) \subseteq \mathbb{P}(v^\perp \otimes \mathbb{C})$. Hence the above makes sense.

Theorem 6.29 (Piatetski-Shapiro-Shafarevich, 1971)**.** $F_{2d} \to \Gamma\backslash\mathbb{D}$ *is an isomorphism!*

This motivates the notion of a "$\mathbb{Z}$-polarized variation of Hodge structure" ($\mathbb{Z}$-PVHS). Informally, a $\mathbb{Z}$-PVHS over a complex manifold B is a family of polarized ($\mathbb{Z}$-)Hodge structures over B satisfying the Griffiths transversality condition. More, concretely,

Definition 6.30. A $\mathbb{Z}$-PVHS over B is

(1) $\mathbb{Z}$-local system $\mathbb{V}_\mathbb{Z} \to B$, with

(2) a symmetric or skew-symmetric bilinear form $\psi : \mathbb{V}_\mathbb{Z} \otimes \mathbb{V}_\mathbb{Z} \to \mathbb{Z}_B$ and

(3) a filtration $F^\bullet$ of $\mathbb{V}_\mathbb{Z} \otimes \mathcal{O}_B$ by holomorphic subbundles,

such that

(A) $\nabla_v F^p \subseteq F^{p-1}/F^p$ for all p and $v \in T_b B, b \in B$, where ∇_v is the directional derivative of $[F_b^p] \in \mathbb{V}_{\mathbb{Z},b} \otimes \mathbb{C}$, as an element of $\mathrm{Gr}(\mathbb{V}_{\mathbb{Z},b} \otimes \mathbb{C})$.

(B) $(\mathbb{V}_{\mathbb{Z},b}, F_b^\bullet, \psi_b)$ defines a $\mathbb{Z}$-polarized Hodge structure for all $b \in B$.

Note we can uniquely define $\nabla : \mathbb{V}_\mathbb{Z} \otimes \mathcal{O}_B \to \mathbb{V}_\mathbb{Z} \otimes \Omega_B^1$ as satisfying the Leibniz rule and $\nabla = 0$ on $\mathbb{V}_\mathbb{Z} \otimes \mathbb{C}$. One can replace (A) with the condition $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega_B^1$.

Theorem 6.31 (Deligne 1987)**.** *Let B be a smooth quasi-projective variety over $\mathbb{C}$. Then (up to isomorphism) there are only finitely many $\mathbb{Z}$-local systems $\mathbb{V}_\mathbb{Z} \to B$ underlying a $\mathbb{Z}$-PVHS of rank r.*

7 The Hodge Conjecture

Let X be a smooth projective variety of dimension n. If $Y \subseteq X$ is a projective subvariety of codimension k, consider

$$[Y] \in H_{2n-2k}(Y, \mathbb{Z}) \xrightarrow{i^*} H_{2n-2k}(X, \mathbb{Z}) \xrightarrow{PD} H^{2k}(X, \mathbb{Z})$$

Proposition 7.1. $[Y] \in H^{k,k}(X)$.

Proof. Let $\eta \in H^{p,q}(X)$ with $p + q = 2n - 2k$. Then $i^*\eta = 0$ unless $p = q = n - k$. So $[Y] \wedge \eta = \int_Y i^*\eta = 0$ unless $\eta \in H^{n-k,n-k}(X)$. This implies the claim. □

Thus $[Y] \in H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X)$.

Conjecture 7.2 (Hodge)**.** *If $\alpha \in H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X)$, then $\alpha = \sum n_i[Y_i]$, $n_i \in \mathbb{Z}$, $Y_i \subseteq X$ subvarieties of codimension k.*

This is false, and was then corrected to $n_i \in \mathbb{Q}$. This conjecture is open.

Example 7.3. For $k = 1$, we have the map $\mathrm{Pic}(X) \twoheadrightarrow H^2(X, \mathbb{Z}) \cap H^{1,1}(X),\ \mathcal{L} \mapsto c_1(\mathcal{L})$. $\mathcal{L}$ has a meromorphic section s satisfying $[\div(s)] = c_1(\mathcal{L})$. This verifies the Hodge conjecture in this case.

Example 7.4. $k = n-1$. By Hard Lefschetz, we have an isomorphism $H^2(X, \mathbb{Q}) \xrightarrow{\wedge c_1(L)^{n-2}} H^{2n-2}(X, \mathbb{Q})$. Now $c_1(L) \in H^{1,1}(X)$ implies that this map sends $H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$ to $H^{2n-2}(X, \mathbb{Q}) \cap H^{n-1,n-1}(X)$.

If $\alpha \in H^{2n-2}(X,\mathbb{Q}) \cap H^{n-1,n-1}(X)$, then we can thus write $\alpha = \beta \wedge c_1(L)^{n-2} = \sum n_i[Y_i] \wedge c_1(L)^{n-2}$ since we know the Hodge conjecture for H^2. Further let $c_1(L) = [H]$ for some hyperplane H, then $\alpha = \sum n_i[Y_i] \cap H^n = \sum n_i i[Y_i \cap H_1 \cap \ldots \cap H_{n-2}]$ for very general hyperplanes

The "easiest" case for which the conjecture is still open is X a projective 4-fold.