Complex Analysis

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1 Fundamental theorem of algebra:-

Theorem 1 *Every non-constant polynomial with complex co-efficients has a complex root.*

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by i , then of course $-i$ is also a root. In other words, you are imagining an i , which has this property that $i^2 = 1$. And then we can write $x^2 + 1 = (x - i)(x + i)$

Corollary 1.1 *A complex polynomial of degree n has exactly n roots*

2 Some basic notions of topology:-

Definition 1 *Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an* ***open*** *subset of $\mathbb{C}$ if given any point z_0, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C}$ such that $|z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0. Thus, z is in the δ-neighbourhood of z_0,denoted by $B_\delta(z_0)$, which means that z is inside the circle with center z_0 and radius δ .*

Definition 2 *A subset $z \in \mathbb{C}$ is said to be* ***closed*** *if its complement is open. It is a basic fact that there exists no subset of $\mathbb{C}$ that is both open and closed other than $\emptyset$ and $\mathbb{C}$.*

Definition 3 *We define the closure of a subset $S \subseteq \mathbb{C}$ to be the smallest closed set containing S. It is denoted $\bar{S}$. Equivalently, the* ***closure*** *of S is the union of S together with its limit points. (In other words a point belongs to the closure of S if it is arbitrarily close to points in S)*

Definition 4 *A subset $S \subseteq \mathbb{C}$ is said to be* ***path-connected*** *if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them, i.e. a continuous function $f : [0, 1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a* ***domain****.*

Definition 5 *A subset $S \subseteq \mathbb{C}$ is said to be* ***compact*** *if it is closed and bounded.*

Theorem 2 *Any continuous complex valued function $f : S \subseteq \mathbb{C} \to \mathbb{C}$ is bounded, i.e. $\exists M \in \mathbb{R}$ such that $|f(z)| < M \ \forall \ z \in S$, iff S is compact.*

Remark 1 *The rules and definitions of limits, differentiability and continuity are similar to that in real analysis just replacing real numbers with complex numbers.*

Definition 6 *We say that f is* ***holomorphic*** *on Ω if f is differentiable at each point of Ω. f is holomorphic (also called complex analytic) at z_0 if it is holomorphic in some neighbourhood of z_0*

Theorem 3 (Goursat's Theorem) *If f(z) is holomorphic, $f'(z)$ is continuous.*

Remark 2 *If $f : \Omega \to \mathbb{C}$ is a function, then f can be thought of as a function from $\mathbb{R}^2 \to \mathbb{R}^2$, i.e two functions of two real variables $f(xy) = (u(xy), v(xy))$, where u and v are the real and imaginary parts of f . It can be shown that if f is a holomorphic function, then thought of as a real function from $\mathbb{R}^2$ to itself in the above sense, it is infinitely differentiable; equivalently, both u and v are infinitely differentiable functions of x and y (all partial derivatives exist up all orders). We will soon see that the converse is not true; most infinitely differentiable functions from the real plane $\mathbb{R}^2$ to itself are NOT holomorphic. Thus the property of holomorphicity is much stronger than infinite differentiability of the associated real functions (in this case denoted u and v). Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain, Once differentiable implies infinitely differentiable. In fact something much stronger is true, namely the function is defined by Taylor series. We will see this later.*

3 Cauchy-Riemann Equations and Differentiability:-

The differential condition satisfied by a differential function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \tag{1}$$

for $z_0 \in \Omega$. The limit exists as z approaches z_0 along any path. For deriving the CR equations we look at the limit along x and y directions and equate them. So if, $z_0 = a + ib$, $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, we have,

$$f'(z_0) = u_x(a, b) + iv_x(a, b) = v_y(a, b) - iu_y(a, b)$$

$$\therefore u_x = v_y \ \& \ u_y = -v_x \tag{2}$$

These are the C-R equations.
These can be clubbed together to a single equation :

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x} \tag{3}$$

Theorem 4 *For a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, the CR equations being satisfied at a point $z \in \Omega$ is necessary but not sufficient for f to be differentiable at that point.*

If $z = x + iy$, then,

$$x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2i}$$

Using these we can obtain,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{4}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{5}$$

The following operator relations can therefore be stated,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{6}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{7}$$

The CR equations can now be written as:

$$\frac{\partial f}{\partial \bar{z}} = 0 \tag{8}$$

Therefore the CR equations are equivalent to the condition that f be a pure function of z without any terms containing $\bar{z}$. So, for f to be differentiable, it has to be a pure function of z.
We will later see that if f is once differentiable, then it's infinitely differentiable. Then the partial derivatives would exist infinitely and the following equation holds:

$$\frac{\partial^m f}{\partial y^m} = i^m\frac{\partial^m f}{\partial x^m} \tag{9}$$

3.1 A note on differentiability:-

We can of course view $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ as a function of two real variables;

$$f(x, y) = (u(x, y), v(x, y))$$

Now we state the following definition of differentiability from Real Analysis.

Definition 7 *$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (a, b) if $\exists$ a 2×2 matrix $Df(a, b)$ such that:*

$$\lim_{(h,k)\to(0,0)} \frac{||f(a+h, b+k) - f(a, b) - Df(a, b) \begin{bmatrix} h \\ k \end{bmatrix} ||}{||(h, k)||} = 0 \tag{10}$$

We then call $Df(a, b)$ the total derivative of f at (a, b). If f is differentiable, i.e. the total derivative exists, then all the partial derivatives exist, and

$$Df(a, b) = Jf(a, b) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

where Jf is the Jacobian matrix.

Theorem 5 *If $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (u(x, y), v(x, y))$ has continuous partial derivatives, then f is differentiable, and the total derivative Df exists and is equal to the Jacobian matrix, Jf.*

An equivalent definition for complex differentiability:

Definition 8 *A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is differentiable at z_o if:*

$$\lim_{\Delta z \to 0} \frac{f(z_o + \Delta z) - f(z_o) - f'(z_o)\Delta z}{\Delta z} = 0 \tag{11}$$

Now viewing f as a function of two real variables,

$$f'(z_o)\Delta z = Df(a, b) \begin{bmatrix} h \\ k \end{bmatrix} \tag{12}$$

where, $z_o = a + ib$ and $\Delta z = h + ik$. But now, the partial derivatives in $Df(a, b)$ have an additional constraint, they must follow the C-R Equations. And from this we can conclude,

Theorem 6 *A real differentiable function may not be complex differentiable but the converse will always be true†.*

$\dagger$ because we can always find $Df(a, b)$ from $f'(z_o)$ Thus the following holds for any complex function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$:

Complex Differentiability
$\iff$ Real Differentiability + Real and Imaginary parts satisfy C-R equations.
$\iff$ Real and Imaginary parts have continuous partial derivatives and satisfy C-R equations.

In fact a more general theorem holds true:

Theorem 7 *Let f be continuous on Ω. Suppose the partial derivatives exist and satisfy the C-R equations <u>at every point</u> in Ω. Then f is holomorphic in Ω.*

4 Power Series

A power series is defined by the following:

$$f(z) = \sum_{i=0}^{\infty} a_i (z - z_o)^i \tag{13}$$

A radius of convergence exists for any power series;there exists a real number R such that $f(z)$ converges when $|z - z_0| < R$, and diverges when $|z - z_0| > R$.

Definition 9 *The series $\sum_{i=0}^{\infty} a_i$ is said to be absolutely convergent if $\sum_{i=0}^{\infty} |a_i|$ is convergent.*

Theorem 8 *An absolutely convergent series is convergent.*

Theorem 9 *Comparison Test: If $\sum_{i=0}^{\infty} b_i$ is absolutely convergent and if $|a_i| \leq |b_i|$ for all large enough i, then $\sum_{i=0}^{\infty} a_i$ is absolutely convergent.*

Definition 10 *Limit Supremum: For a sequence of real numbers $\{x_n\}$, let y_n be the supremum of the set $\{x_n, x_{n+1}, \dots\}$. Then the sequence $\{y_1, y_2, \dots\}$ is a monotonically decreasing sequence which diverges to ∞ or has a finite limit. This is called the upper limit (also called limit superior, denoted $\limsup\limits_{n\to\infty} x_n$ of the sequence x_n. It can be ∞.*

If $\{x_n\}$ is a convergent sequence converging to l , then l is the lim sup.

Theorem 10 *Cauchy's Root Test: For a series $\sum_{i=0}^{\infty} a_i$ of complex numbers, let $C = \limsup\limits_{i\to\infty} \sqrt[i]{|a_i|}$. Then the series converges absolutely if $C < 1$ and it diverges if $C > 1$. The test is indecisive if $C = 1$.*

Corollary 10.1 *Cauchy-Hadamard Theorem: For the power series $\sum_{i=0}^{\infty} a_i (z - z_o)^i$, let $R = \frac{1}{\limsup\limits_{i\to\infty} \sqrt[i]{|a_i|}}$. Then the power series converges absolutely if $|z - z_o| < R$ and diverges if $|z - z_o| > R$.*

Theorem 11 *Ratio Test: For a series $\sum_{i=0}^{\infty} a_i$, let $L = \limsup\limits_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$. Then if $L < 1$, the series converges absolutely. The series diverges if there exists N such that $\left|\frac{a_{i+1}}{a_i}\right| > 1$ for $i \geq N$*

Remark 3 *$L > 1$ in the above test doesn't imply that the series diverges.*

Remark 4 *If a series converges by the ratio test, then it converges by the root test as well. But not conversely. In fact,*

$$\limsup_{i\to\infty} \sqrt[i]{|a_i|} \leq \limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| \tag{14}$$

Remark 5 *A power series, it's differentiated and integrated series all have the same radius of convergence.*

Theorem 12 (Cauchy-Maclaurin integral test) *For an eventually decreasing non-negative, continuous function f defined on the unbounded interval* $[1, \infty)$, *on which it is eventually monotone decreasing. Then the infinite series* $\sum_{n=1}^{\infty} f(n)$ *converges if and only if the improper integral* $\int_1^{\infty} f(x)dx$ *exists (i.e, is finite)*

5 Analytic Functions

Definition 11 *A function* $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ *is said to be* ***analytic*** *if it is locally given by a convergent power series; i.e., every* $z_o \in \Omega$ *has a neighbourhood contained in* Ω *such that there exists a power series centered at* z_o *which converges to* $f(z)$ *for all* z *in that neighbourhood.*

Analytic functions are infinitely differentiable. Also if $f(z) = \sum_{i=0}^{\infty} a_i(z - z_o)^i$, then $a_i = \frac{f^{(i)}(z_o)}{i!}$. If the lowest power in the expansion is m. Then it is said to be an Analytic function with **mth order zero**.

6 Integration

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{C}$ be a piecewise continuous function. Let $f(t) = u(t) + iv(t)$. We define

$$\int_a^b f(t)dt$$

to be

$$\int_a^b u(t)dt + i\int_a^b v(t)dt$$

where both these integrals are defined to be the usual Riemann integrals.

6.1 Some definitions:-

We say a curve $\gamma(t) = x(t) + iy(t)$ is **smooth** if $\gamma(t) \neq 0$ for all t. Such a curve is also called regular parametrized curve. A **contour** is a curve consisting of a finite number of smooth curves joined end to end. It is said to be **simple** if the parametrization map is one to one except possibly at the end-points. (Intuitively it means that the curve does not cross itself). It is said to be **closed** if the initial and end-point are the same, i.e. $\gamma(a) = \gamma(b)$.

Theorem 13 (Jordan Curve Theorem) *Any simple closed curve in $\mathbb{R}^2$ separates the plane into two connected components. The curve is the common boundary of both of them. Exactly one of the components is bounded.*

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a complex function and C be a contour. We define the integral of f along C to be:

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{15}$$

The usual properties of real line integrals get carried over to the complex analogues.

Theorem 14 *A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ has a primitive iff $\int f(z)dz$ is path independent.*

We get the above theorem from real analysis.

Theorem 15 (Cauchy's Theorem) *Let C be a simple closed contour and let f be a holomorphic function defined on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.*

Remark 6 *We use the Goursat's theorem, Green's theorem and the C-R equations to arrive at this theorem.*

Remark 7 *Note that by Jordan Curve theorem, interior of C makes sense.*

The definition for simply-connectedness is similar to that in real analysis.

Theorem 16 (More general form of Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function on Ω. Let $\mathbb{C}$ be a simple closed contour in Ω. Then $\int_C f(z)dz = 0$.*

In case the path of integration is not differentiable, i.e. $x(t)$ and $y(t)$ are not differentiable functions of the parameter t, one can still make sense of the integral in much the same way as we define Riemann Integral.

Theorem 17 (Even more general form of the Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function on Ω. Let $\mathbb{C}$ be a simple closed rectifiable curve in Ω. Then $\int_C f(z)dz = 0$.*

Definition 12 *Let $\gamma_1 : [0,1] \to \Omega$ and $\gamma_2 : [0,1] \to \Omega$ be two closed contours. We say γ_1 is **continuously deformed** into γ_2 if $\exists\ \gamma : [0,1] \times [0,1] \to \Omega$ such that $\gamma|_{[0,1]\times 0} = \gamma_1$; $\gamma|_{[0,1]\times 1} = \gamma_2$ and $\gamma|_{0\times t} = \gamma|_{1\times t}\ \forall\ t$.*

Theorem 18 (Even stronger form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function on Ω. If γ and γ' are two simple closed contours in Ω which can be continuously deformed into each other, then $\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz$.*

Theorem 19 (Cauchy Integral Formula) *Let f be holomorphic on a set containing a simple closed contour γ and it's interior (oriented positively). If z_o is interior to γ, then,*

$$f(z_o) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_o}dz \tag{16}$$

The Cauchy Integral Formula (CIF) and the Cauchy's theorem can be shown to be equivalent. One implies the other.

7 Some consequences of the CIF

Let f be a holomorphic function in a neighbourhood of a point $z_o \in C$. Let $R > 0$ be such that f is holomorphic in $|z - z_o| < R$. Let γ be a circle of radius r with $r < R$ centered at z_o. From the CIF we can write the following:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}dw$$

for any z such that $|z - z_o| < r$. We can now expand $\frac{1}{w-z} = \frac{1}{w-z_o}\cdot\frac{w-z_o}{w-z} = \frac{1}{w-z_o}\cdot\frac{1}{1-\left[\frac{z-z_o}{w-z_o}\right]}$ and get the following theorem:

Theorem 20 (Holomorphic $\implies$ Analytic) *If f is holomorphic in the disc $|z - z_o| < R$, then, $f(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^n$, where*

$$a_n = \frac{1}{2\pi i}\int_{|w-z_o|=r} \frac{f(w)}{(w - z_o)^{n+1}}dw = \frac{f^{(n)}(z_o)}{n!} \tag{17}$$

for any $r < R$. Any $r < R$ gives same a_n. Thus, the radius of convergence is atleast R.

Theorem 21 (Morera's theorem (Converse to Cauchy's th.)) *It states that if $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is continuous and $\int_\gamma f(z)dz = 0$ for every closed contour γ in $\mathbb{C}$, then f is holomorphic on Ω*

Cauchy'estimate:

From Eq. 17 we can derive the following estimate:
If f is holomorphic in $|z - z_o| < R$ and it's bounded above by $M > 0$ there. Then,

$$|f^{(n)}(z_o)| \leq \frac{n!M}{R^n} \tag{18}$$

Definition 13 *A function defined all over $\mathbb{C}$ is called* ***entire*** *if it is holomorphic everywhere in $\mathbb{C}$. Examples: Polynomials, e^z, $sinz$, $cosz$, etc. Clearly, sums and products of entire functions are entire. The Taylor series expansion convergent over $R \to \infty$.*

From (18) we derive the following theorem:

Theorem 22 (Liouville's theorem) *A bounded above entire function is a constant.*

It also means that a non-constant entire function has to be unbounded. **And now using Liouville's theorem, we can derive the Fundamental theorem of Algebra!!!!**

Definition 14 (Proper maps) *A continuous function $f : \mathbb{C} \to \mathbb{C}$ is said to be proper if $|f(z)| \to \infty$ as $|z| \to \infty$.*

Theorem 23 *Non-constant polynomial functions on $\mathbb{C}$ are proper. Conversely, any proper, holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ is necessarily a non-constant polynomial.*

Corollary 23.1 *Polynomial functions over complex numbers are surjective.*

Similarly sin, cos and various other elementary functions are surjective.

Theorem 24 *Any function whose image doesn't contain the origin is of the form $e^{g(z)}$ for some entire function g(z).*

Definition 15 *If $z_o \in \mathbb{C}$ is any point and γ is any closed contour not passing through z_o, then $\int_\gamma \frac{1}{z-z_o}\,dz$ is an integral multiple of $2\pi i$. This integer is called the **winding number** of γ around z_o and counts the number of times the curve winds around z_o.*

If γ is a finite union of simple closed curves and z_o lies outside γ, then this integer is zero. If γ is a simple closed curve, and z_o lies in the interior of γ then this integer is zero.

8 Logarithms

9 Singularities and Poles

Definition 16 *The set of points in Ω where f is not defined or not holomorphic are called the **singularities** of f.*

Definition 17 *A singular point is said to be **isolated** if the function is holomorphic in a punctured disc around that point. A singularity is **non-isolated** if it is not isolated. That is, in no punctured neighborhood of the singularity is the function holomorphic.*

Definition 18 *If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable. If not we say it is non-removable.*

Theorem 25 *If an isolated singularity at z_o is removable, then $\lim_{z\to z_o} f(z)$ exists.*

Theorem 26 (Riemann's Removable Singularity Theorem) *Converse of the above: If f has an isolated singularity at z_o and $\lim_{z\to z_o} f(z)$ exists, then it is a removable singularity.*

Definition 19 *An isolated singularity z_o is said to be a **pole** if $\lim_{z\to z_o} f(z) = \infty$*

In this case the function $g(z) = \frac{1}{f(z)}$ is holomorphic at z_o with $g(0) = 0$. It follows that $g(z) = (z - z_o)^m h(z)$ for some holomorphic function $h(z)$ defined in a neighbourhood of z_o with $h(z_o) \neq 0$. Such an m and therefore such a $h(z)$ is uniquely defined. Thus $f(z) = (z - z_o)^{-m} f_1(z)$ for some holomorphic function $f_1(z)$ in a punctured neighbourhood of z_o with $f_1(z_o) \neq 0$, and m is called the order of the pole and is a measure of how fast the function blows up at z_o. $f(z)$ is then said to have an **mth order pole** at z_o. m is also therefore the highest exponent in the principal part of the Laurent series expansion of $f(z)$ around z_o. If m is one, we say the pole is a **simple pole**.

Definition 20 *A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. They have isolated singularities.*

Definition 21 *An isolated singularity that is neither a pole nor a removable singularity is called an **essential singularity**. It can also be said to be an mth order pole with $m \to \infty$.*

Definition 22 *A function $f(z)$ is said to have a **branch singularity** at z_o if it has multiple values at that point with no Laurent or Taylor series expansion around that point.*

Theorem 27 (Casorati-Weierstrass theorem) *If z_o is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_o.*

10 Laurent Series

Suppose z_o is an isolated singularity for f. Consider an annulus with radii $R > r$ centered at z_o such that f is holomorphic there. CIF takes the form:

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_o|=R} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{|w-z_o|=r} \frac{f(w)}{w-z}dw \tag{19}$$

The first integral gives rise to $\sum_{n=0}^{\infty} a_n(z - z_o)^n$, where

$$a_n = \frac{1}{2\pi i}\int_{|w-z_o|=R} \frac{f(w)}{(w-z_o)^{n+1}}dw$$

and the second integral after some manipulations gives $\sum_{n=1}^{\infty} b_n(z-z_o)^{-n}$ where

$$b_n = \frac{1}{2\pi i}\int_{|w-z_o|=r} \frac{f(w)}{(w-z_o)^{-n+1}}dw$$

They can together be written as:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_o)^n \tag{20}$$

This is the Laurent series around the isolated singularity z_o. The negative part is called the principal part of the Laurent series.
If we integrate the Laurent series, only a_{-1} remains, other terms vanish. What remains is usually called the residue.

$$a_{-1} = \frac{1}{2\pi i}\int_{\gamma} f(z)dz \tag{21}$$

For a function with mth order pole, the following also holds true:

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_o)^m f(z)]\bigg|_{z=z_o} \tag{22}$$

In general,

$$a_n = \frac{1}{(m+n)!}\frac{d^{m+n}}{dz^{m+n}}[(z-z_o)^m f(z)]\bigg|_{z=z_o} \tag{23}$$

where, $-m \leq n < \infty$
If $n=-m$,

$$a_{-n} = \lim_{z\to z_0}[f(z)(z-z_o)^m] \tag{24}$$

Theorem 28 (Cauchy-Residue Theorem) *Suppose f and γ is given. Suppose there are finitely many isolated singularities of f inside γ; say z_i.*

$$\int_{\gamma} f(z)dz = 2\pi i \cdot \sum_{i=1}^{n} Res(f, z_i) \tag{25}$$

Definition 23 (Isolated singularity at Infinity) *f(z) is said to have isolated singularity at ∞ if f is holomorphic outside a disc of radius R for some R. Equivalently $f(\frac{1}{z})$ has an isolated singularity at 0.*

Definition 24 *f is said to have a zero (resp. removable singularity, pole, essential singularity) at ∞ if $f(\frac{1}{z})$ has a zero (resp. removable singularity, pole, essential singularity) at 0.*

Theorem 29 *An entire functions from $\mathbb{C}$ to $\mathbb{C}$ has a pole at ∞ if and only if it is a non-constant polynomial.*

11 Some other important theorems

Using CIF and the Identity theorem(Riemann's Last th.), we can prove the following:

Theorem 30 (Maximum Modulus Theorem) *A non-constant holomorphic function on an open connected domain never attains its maximum modulus at any point in the domain.*

A nice consequence of the Maximum modulus principle is the following lemma of Schwartz.

Theorem 31 (Schwarz Lemma) *Let* $\mathbb{D} = \{z : |z| < 1\}$ *be the open unit disk and let* $f : \mathbb{D} \to \mathbb{C}$ *be a holomorphic map such that* $f(0) = 0$ *and* $|f(z)| \leq 1$ *on* $\mathbb{D}$. *Then,* $|f(z)| \leq |z| \ \forall \ z \in \ \mathbb{D}$ *and* $|f'(0)| \leq 1$. *Moreover, if* $|f(z)| = |z|$ *for some non-zero* z *or* $|f'(0)| = 1$, *then* $f(z) = az$ *for some* $a \in \mathbb{C}$ *with* $|a| = 1$

The maximum modulus theorem is a special case of a even more powerful theorem called the Open Mapping Theorem.

Theorem 32 (Open Mapping Theorem) *Any non-constant holomorphic function defined on an open set* $\Omega \subseteq \mathbb{C}$ *is open; i.e. maps open subsets of* $\mathbb{C}$ *contained in* Ω *to open subsets of* $\mathbb{C}$.

Theorem 33 (Mittag-Leffler's Theorem) *Given any discrete sequence of points going to infinity, there exists a meromorphic functions with poles exactly along this sequence and having prescribed principal parts at those poles.*

12 Harmonic Functions:-

Definition 25 *A real valued function* $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ *is called harmonic if it is twice continuously differentiable and satisfies* $u_{xx} + u_{yy} = 0$ *on* U. *If* $f = u + iv$ *is holomorphic on* Ω, *then both* u *and* v *are harmonic on* Ω, *by the C-R equations.*

Definition 26 *Suppose* u *and* v *are harmonic functions on* Ω. *We say that* v *is a harmonic conjugate of* u *if* $f = u + iv$ *is holomorphic in* Ω.

v is a harmonic conjugate of u does not mean that u is a harmonic conjugate of v. In fact, if u and v are harmonic conjugates of each other, then they are constant functions.

Theorem 34 *Let* U *be a simply-connected domain in* $\mathbb{C}$ *and let* u *be a harmonic fuction on* U. *Then* u *admits exactly one harmonic conjugate up to a constant*

Corollary 34.1 *Harmonic functions are infinitely differentiable*

Theorem 35 (Mean-Value Property) *Let u be a harmonic function on a disc of radius R. Then for any $r < R$, we have,*

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta \tag{26}$$

In particular, u does not attain it's maximum at any interior point unless it is constant.

Theorem 36 (Identity Principle) *Let u be a harmonic function on a domain $\Omega \subset \mathbb{C}$. If $u = 0$ on a non-empty open subset $U \subseteq \Omega$, then $u = 0$ throughout Ω.*