Lecture 5 – Image Restoration （图像复原）

This lecture will cover:

– Model of Image Degradation Process（图像退化过程模型）

– Noise Reduction（噪声消除）

- Noise Models（噪声模型）
- Spatial Filtering （空间域滤波方法）
- Frequency Domain Filtering（频率域滤波方法）

– Image Restoration（图像复原）

- **Degradation Function（退化函数）**
- **Inverse Filtering (逆滤波)**
- **Wiener Filtering（维纳滤波）**
- **Constrained Least Squares Filtering（约束最小二乘方滤波）**
- **Geometric Mean Filtering（几何均值滤波）**

Image Degradation（图像退化）

For an operator H, let

$$g(x,y) = H[f(x,y)]$$

then, H is linear if

$$H[af_1(x,y) + bf_2(x,y)] = aH[f_1(x,y)] + bH[f_2(x,y)]$$

H is position invariant if

$$H[f(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta)$$

$\boldsymbol{H}$ is a linear, position-invariant process

Image Degradation（图像退化）

The impulse response:

$$h\,(x, \alpha, y, \beta) = H[\delta(x-\alpha, y-\beta)]$$

where $h\,(x, \alpha, y, \beta)$ is called point spread function (PSF,点扩散函数), and

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h\,(x,\alpha,y,\beta)\mathrm{d}\alpha\mathrm{d}\beta$$

In presence of additive noise

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h\,(x,\alpha,y,\beta)\mathrm{d}\alpha\mathrm{d}\beta + \eta(x,y)$$

Model of Image Degradation （图像退化模型）

In Spatial domain:

$$g(x,y) = h\,(x,y) \star f(x,y) + \eta(x,y)$$

Where

$g(x,y)$: a degraded image $f(x,y)$: input image

$h(x,y)$: degradation function $\eta(x,y)$: additive noise term

In Frequency domain:

$$G(u,v) = H\,(u,v)F(u,v) + N(u,v)$$

Image deconvolution(图像去卷积)---linear image restoration

Deconvolution filters(去卷积滤波器)--- filters used in the restoration process

Degradation Function（退化函数）

To estimate degradation function

- Observation
- Experimentation
- Mathematical modeling

Blind Deconvolution(盲去卷积): restore an image by using a degradation function

Observation

- Gather information from the image itself

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

Where

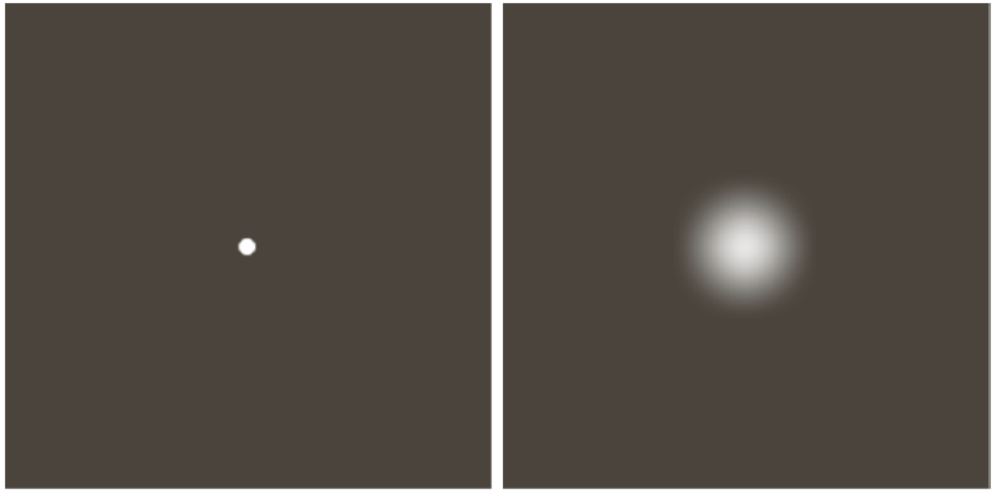
$\hat{F}_s(u,v)$: the processed subimage

$G_s(u,v)$: observed subimage

Experimentation

$$H(u,v) = \frac{G(u,v)}{A}$$

$G(u,v)$:the observed image A: a constant strength of the impulse



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Mathematical modeling

- Based on the physical characteristics: $H(u,v) = e^{-k(u^2+v^2)^{5/6}}$

Mathematical modeling

- By deriving a mathematical model from principles:

$$H(u,v) = \frac{T}{\pi(ua+vb)} \sin \pi(ua+vb) e^{-j\pi(ua+vb)}$$

Discrete Degradation Function

1D discrete degradation model:

$$g(x) = \sum f(m)h(x-m)$$

where A and B are length of $f(x)$ and $h(x)$

consider extend $f(x)$ and $h(x)$ to a period $M \geq A + B - 1$,

$$f_e(x) = \begin{cases} f(x), & 0 \leq x \leq A-1 \\ 0, & A \leq x \leq M-1 \end{cases} \quad \text{and} \quad h_e(x) = \begin{cases} h(x), & 0 \leq x \leq B-1 \\ 0, & B \leq x \leq M-1 \end{cases}$$

then

$$g_e(x) = \sum_{m=0}^{M-1} f_e(m)h_e(x-m)$$

Matrix Form

1D discrete degradation model in matrix: $g = Hf$

where

$$g = [g_e(0) \;\; g_e(1) \;\; g_e(2) \cdots g_e(M-1)]^T$$

$$f = [f_e(0) \;\; f_e(1) \;\; f_e(2) \cdots f_e(M-1)]^T$$

$$H = \begin{bmatrix} h_e(0) & h_e(M-1) & h_e(M-2) & \cdots & h_e(1) \\ h_e(1) & h_e(0) & h_e(M-1) & \cdots & h_e(2) \\ h_e(2) & h_e(1) & h_e(0) & \cdots & h_e(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_e(M-1) & h_e(M-2) & h_e(M-3) & \cdots & h_e(0) \end{bmatrix}$$

2D Degradation Model

consider extend $f(x,y)$ and $h(x,y)$ to a period $M \geq A+C-1$, $N \geq B+D-1$

$$f_e(x,y) = \begin{cases} f(x,y), & 0 \leq x \leq A-1 \\ & 0 \leq y \leq B-1 \\ 0, & A \leq x \leq M-1 \\ & B \leq y \leq N-1 \end{cases}$$ and $$h_e(x,y) = \begin{cases} h(x,y), & 0 \leq x \leq C-1 \\ & 0 \leq y \leq D-1 \\ 0, & C \leq x \leq M-1 \\ & D \leq y \leq N-1 \end{cases}$$

Then 2D discrete degradation model:

$$g_e(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) h_e(x-m, y-n)$$

2D Degradation Model

2D discrete degradation model in matrix: $g = Hf$

Where $g = [g_e^j(x,y)]^T$ and $f = [f_e^j(x,y)]^T$

$$H = \begin{bmatrix} H^0 & H^{M-1} & H^{M-2} & \cdots & H^1 \\ H^1 & H^0 & H^{M-1} & \cdots & H^2 \\ H^2 & H^1 & H^0 & \cdots & H^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H^{M-1} & H^{M-2} & H^{M-3} & \cdots & H^0 \end{bmatrix}$$

and

$$H^j = \begin{bmatrix} h_e(j,0) & h_e(j,N-1) & h_e(j,N-2) & \cdots & h_e(j,1) \\ h_e(j,1) & h_e(j,0) & h_e(j,N-1) & \cdots & h_e(j,2) \\ h_e(j,2) & h_e(j,1) & h_e(j,0) & \cdots & h_e(j,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_e(j,N-1) & h_e(j,N-2) & h_e(j,N-3) & \cdots & h_e(j,0) \end{bmatrix}$$

2D Degradation Model

2D discrete degradation model with noise:

$$g_e(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m, n) h_e(x-m, y-n) + n_e(x, y)$$

2D discrete degradation model in matrix form:

$$g = Hf + n$$

Algebraic Restoration Method（代数复原）

- Unconstrained restoration method

$$J(\hat{f}) = \left\| g - H\hat{f} \right\|^2 \quad \Longrightarrow \quad \frac{\partial J(\hat{f})}{\partial \hat{f}} = -2H^*\left(g - H\hat{f}\right) = 0$$

$$\hat{f} = H^{-1}(H^*)^{-1}H^*g$$

- Constrained restoration method

$$J(\hat{f}) = \left\| Q\hat{f} \right\|^2 + \lambda\left(\left\| g - H\hat{f} \right\|^2\right)$$

$$\Longrightarrow \quad \frac{\partial J(\hat{f})}{\partial \hat{f}} = 2Q^*Q\hat{f} - 2\lambda H^*\left(g - H\hat{f}\right) = 0$$

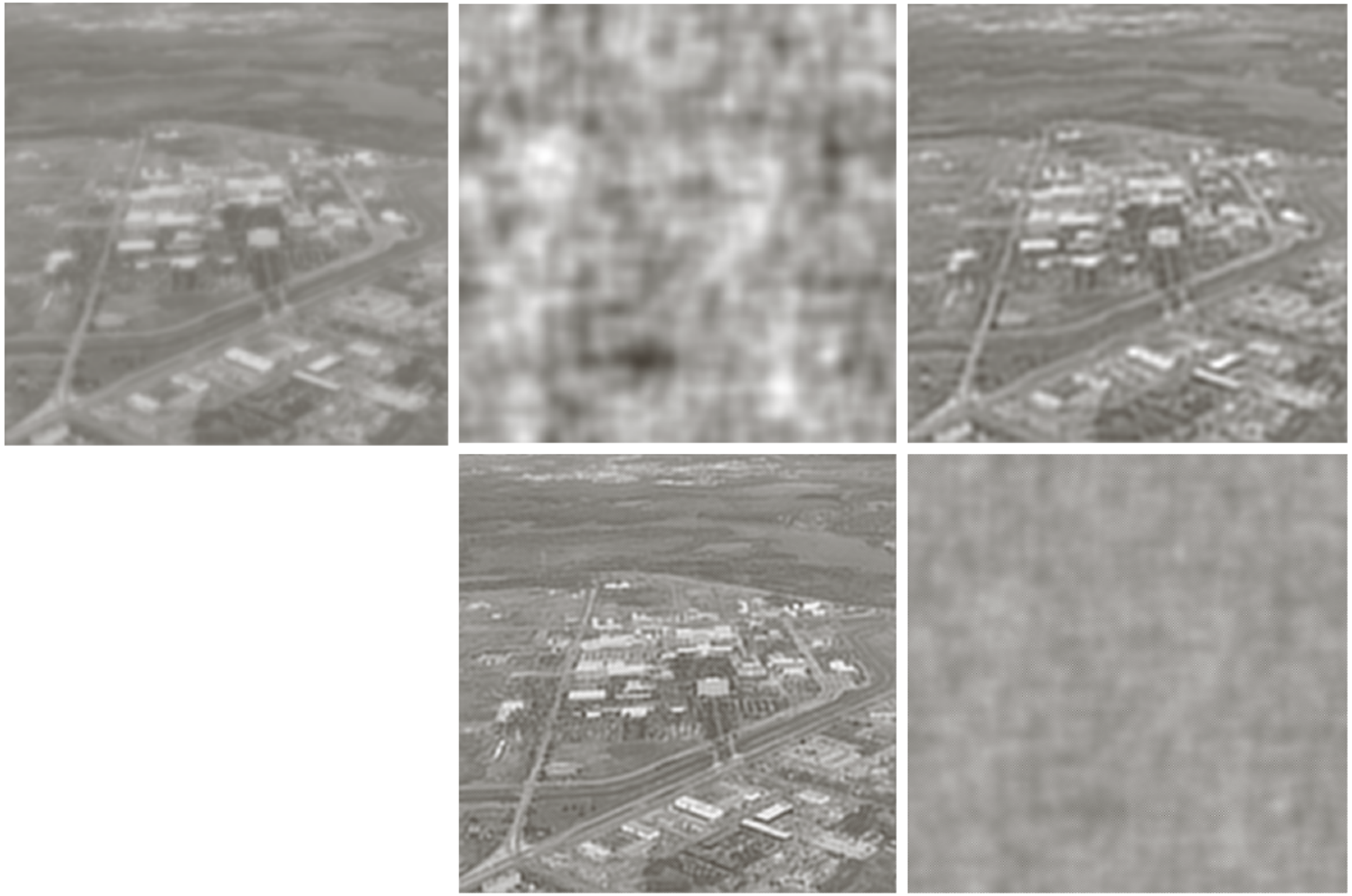
$$\hat{f} = (H^*H + \frac{1}{\lambda}Q^*Q)^{-1}H^*g$$

Inverse Filtering (逆滤波)

$$F(u,v) = \frac{G(u,v)}{H(u,v)}$$

$$= \frac{H(u,v)F(u,v) + N(u,v)}{H(u,v)}$$

$$= F(u,v) + \frac{N(u,v)}{H(u,v)}$$



Wiener Filtering（维纳滤波）

- Expected value of mean square error

$$e^2 = E\left\{\left(f - \hat{f}\right)^2\right\}$$

The estimate of f in frequency domain

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \dfrac{S_n(u,v)}{S_f(u,v)}}\right] G(u,v)$$

$$= \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + \dfrac{S_n(u,v)}{S_f(u,v)}}\right] G(u,v)$$

SNR (信噪比)

- SNR (Signal-to-noise ratio) in Frequency domain:

$$SNR = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \frac{|F(u,v|^2}{|N(u,v|^2}$$

- SNR (Signal-to-noise ratio) in spatial domain:

$$SNR = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \frac{\hat{f}(x,y)^2}{\left[f(x,y) - \hat{f}(x,y)\right]^2}$$

Wiener Filtering（维纳滤波）

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K} \right] G(u,v)$$

Wiener Filtering（维纳滤波）

Constrained Least Squares Filtering（约束最小二乘方滤波）

$$\hat{f} = (H^*H + \frac{1}{\lambda}Q^*Q)^{-1}H^*g \quad \Longrightarrow \quad F(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma|P(u,v)|^2}\right]G(u,v)$$

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Iterative Method

Let a residual (残差) $\boldsymbol{r} = \boldsymbol{g} - \boldsymbol{H}\hat{\boldsymbol{f}} \quad \Longrightarrow \quad \|\boldsymbol{r}\|^2 = \boldsymbol{r}^*\boldsymbol{r}$

Adjust $\boldsymbol{\gamma}$ for $\|\boldsymbol{r}\|^2 = \|\boldsymbol{n}\|^2 \pm \boldsymbol{a}$, where a is an accuracy factor(精确度因子), then an iterative method follows:

1. **Specify an initial value of γ**
2. **Calculate $\|\boldsymbol{r}\|^2$**
3. **There will be three options**
   - Stop if $\|\boldsymbol{r}\|^2 \leq \|\boldsymbol{n}\|^2 \pm \boldsymbol{a}$
   - Increase $\boldsymbol{\gamma}$ if $\|\boldsymbol{r}\|^2 < \|\boldsymbol{n}\|^2 - \boldsymbol{a}$ and return to Step 2
   - Decrease $\boldsymbol{\gamma}$ if $\|\boldsymbol{r}\|^2 > \|\boldsymbol{n}\|^2 + \boldsymbol{a}$ and return to Step 2

Estimation of Noise

The mean of noise:

$$\bar{m} = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} n(x,y)$$

The variance of noise:

$$\sigma_n^2 = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} [n(x,y) - \bar{m}]^2$$

The Power density of noise:

$$\|n\|^2 = MN[\sigma_n^2 + \bar{m}^2]$$

Iterative Method

Geometric Mean Filtering（几何均值滤波）

$$F(u,v)=\left[\frac{H^*(u,v)}{|H(u,v)|^2}\right]^{\alpha}\left[\frac{H^*(u,v)}{|H(u,v)|^2+\beta\left[\frac{S_n(u,v)}{S_f(u,v)}\right]}\right]^{1-\alpha}G(u,v)$$

Where α,β: real positive

$\alpha=1$: inverse filtering

$\alpha=0$: Parametric Wiener filtering (参数维纳滤波器)

$\alpha=0,\beta=1$: Wiener filtering

$\alpha=1/2,\beta=1$: Spectrum equalization filter (谱均衡滤波器)