Lecture 4 – Frequency Domain Transform （频率域变换）

This lecture will cover:

- – 2D Discrete Fourier Transform （傅里叶变换）
- – Frequency Domain Filtering（频率域滤波）
  - Lowpass Filtering（低通滤波器）
  - Highpass Filtering （高通滤波器）
  - Selective Filtering（选择性滤波）
- **– Other Transform**
  - **Discrete Cosine Transform（余弦变换）**
  - **Walsh Transform (沃尔什变换)**
  - **Discrete Wavelet Transform （小波变换）**

Image Transform

The general form:

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v)$$

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} T(u,v) s(x,y,u,v)$$

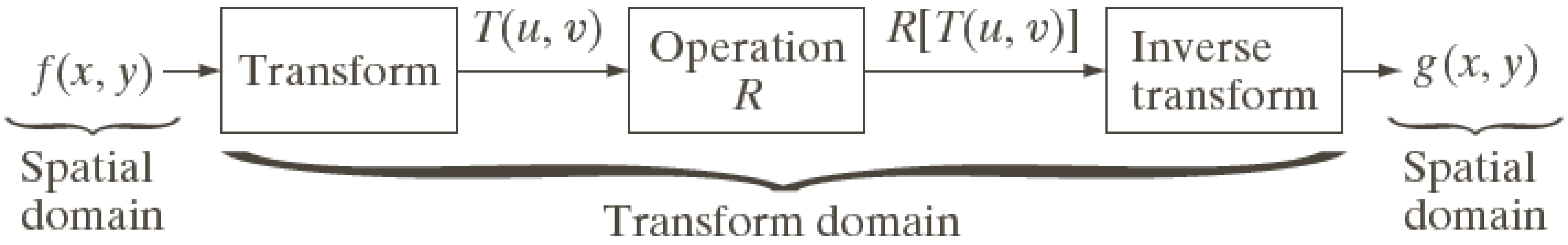
Where

$r(x,y,u,v)$: forward transformation kernel

$s(x,y,u,v)$: inverse transformation kernel

Image Transform

- ➢ **The general approach for operating in the linear transform domain**



Properties

- **Separable**

$$r(x,y,u,v) = r_1(x,u)r_2(y,v)$$

- **Symmetry**

$$r(x,y,u,v) = r_1(x,u)r_1(y,v)$$

Matrix Form

- Forward Transform

$$T = AFA$$

- Inverse Transform

$$F = BTB \qquad \text{if } B = A^{-1}$$

$$\hat{F} = BAFAB \qquad \text{Otherwise}$$

Cosine Transform (余弦变换)

Discrete Cosine Transform (DCT):

$$F(0) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \qquad F(u) = \sqrt{\frac{2}{N}} \sum_{x=0}^{N-1} f(x) \cos \frac{(2x+1)u\pi}{2N}$$

Inverse Discrete Cosine Transform (IDCT):

$$f(x) = \frac{1}{\sqrt{N}} F(0) + \sqrt{\frac{2}{N}} \sum_{u=1}^{N-1} F(u) \cos \frac{(2x+1)u\pi}{2N}$$

2D DCT

Forward Transform:

$$F(0,0) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

$$F(u,0) = \frac{\sqrt{2}}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\frac{(2x+1)u\pi}{2N}$$

$$F(0,v) = \frac{\sqrt{2}}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\frac{(2y+1)v\pi}{2N}$$

$$F(u,v) = \frac{2}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\frac{(2x+1)u\pi}{2N}\cos\frac{(2y+1)v\pi}{2N}$$

2D IDCT

Inverse Transform:

$$f(x,y) = \frac{1}{N}F(0,0)$$

$$+\frac{\sqrt{2}}{N}\sum_{u=1}^{N-1}F(u,0)\cos\frac{(2x+1)u\pi}{2N}$$

$$+\frac{\sqrt{2}}{N}\sum_{v=1}^{N-1}F(0,v)\cos\frac{(2y+1)v\pi}{2N}$$

$$+\frac{2}{N}\sum_{x=1}^{N-1}\sum_{y=1}^{N-1}F(u,v)\cos\frac{(2x+1)u\pi}{2N}\cos\frac{(2y+1)v\pi}{2N}$$

Matrix Form

Analytic form:

$$\begin{cases} F(0) = 0.500f(0) + 0.500f(1) + 0.500f(2) + 0.500f(3) \\ F(1) = 0.653f(0) + 0.271f(1) - 0.271f(2) - 0.653f(3) \\ F(0) = 0.500f(0) - 0.500f(1) - 0.500f(2) + 0.500f(3) \\ F(0) = 0.271f(0) - 0.653f(1) + 0.653f(2) - 0.271f(3) \end{cases}$$

Matrix Form:

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 0.500 & 0.500 & 0.500 & 0.500 \\ 0.653 & 0.271 & -0.271 & -0.653 \\ 0.500 & -0.500 & -0.500 & 0.500 \\ 0.271 & -0.653 & 0.653 & -0.271 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

Forward Matrix form: $[\boldsymbol{F}(\boldsymbol{u})] = [\boldsymbol{A}][\boldsymbol{f}(\boldsymbol{x})]$

Inverse Matrix form: $[\boldsymbol{f}(\boldsymbol{x})] = [\boldsymbol{A}]^{\boldsymbol{T}}[\boldsymbol{F}(\boldsymbol{u})]$

2D DCT Matrix Form

Forward Transform:

$$F(0,0) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \qquad F(u,0) = \frac{\sqrt{2}}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\frac{(2x+1)u\pi}{2N}$$

$$F(0,v) = \frac{\sqrt{2}}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\frac{(2y+1)v\pi}{2N} \qquad F(u,v) = \frac{2}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\frac{(2x+1)u\pi}{2N} \cos\frac{(2y+1)v\pi}{2N}$$

Forward Matrix form: $\boldsymbol{[F(u,v)] = [A][f(x,y)][A]^T}$

Inverse Matrix form: $\boldsymbol{[f(x,y)] = [A]^T[F(u,\mathrm{v})][A]}$

Calculate DCT by DFT

$$F(u)=\sqrt{\frac{2}{N}}\sum_{x=0}^{N-1}f(x)\cos\frac{(2x+1)u\pi}{2N}=\sqrt{\frac{2}{N}}\sum_{x=0}^{N-1}f(x)\mathrm{Re}\left\{e^{-j\frac{(2x+1)u\pi}{2N}}\right\}=\sqrt{\frac{2}{N}}\mathrm{Re}\left\{\sum_{x=0}^{N-1}f(x)e^{-j\frac{(2x+1)u\pi}{2N}}\right\}$$

$$f_e(x)=\begin{cases}f(x), & x=0,1,2,\cdots,N-1\\ 0, & x=N,N+1,N+2,\cdots,2N-1\end{cases}$$

$$F(0)=\frac{1}{\sqrt{N}}\sum_{x=0}^{2N-1}f_e(x)$$

$$F(u)=\sqrt{\frac{2}{N}}\sum_{x=0}^{2N-1}f_e(x)\cos\frac{2(x+1)u\pi}{2N}=\sqrt{\frac{2}{N}}\mathrm{Re}\left\{\sum_{x=0}^{2N-1}f_e(x)e^{-j\frac{(2x+1)u\pi}{2N}}\right\}=\sqrt{\frac{2}{N}}\mathrm{Re}\left\{e^{-j\frac{u\pi}{2N}}\sum_{x=0}^{2N-1}f_e(x)e^{-j\frac{2\pi ux}{2N}}\right\}$$

where

$$\sum_{x=0}^{2N-1}f_e(x)e^{-j\frac{2\pi ux}{2N}}=\mathrm{DFT}[f_e(x)]$$

Calculate IDCT by IDFT

$$F_e(u) = \begin{cases} F(u), & u = 0,1,2,\cdots,N-1 \\ 0, & u = N, N+1, N+2,\cdots,2N-1 \end{cases}$$

$$f(x) = \frac{1}{\sqrt{N}}F(0) + \sqrt{\frac{2}{N}}\sum_{u=0}^{N-1}F(u)\cos\frac{(2x+1)u\pi}{2N} = \frac{1}{\sqrt{N}}F_e(0) + \sqrt{\frac{2}{N}}\sum_{u=1}^{2N-1}F_e(u)\cos\frac{(2x+1)u\pi}{2N}$$

$$= \frac{1}{\sqrt{N}}F_e(0) + \sqrt{\frac{2}{N}}\sum_{u=1}^{2N-1}\mathrm{Re}\left\{F_e(u)e^{j\frac{(2x+1)u\pi}{2N}}\right\} = \frac{1}{\sqrt{N}}F_e(0) + \sqrt{\frac{2}{N}}\sum_{u=1}^{2N-1}\mathrm{Re}\left\{F_e(u)e^{j\frac{u\pi}{2N}}e^{j\frac{2\pi ux}{2N}}\right\}$$

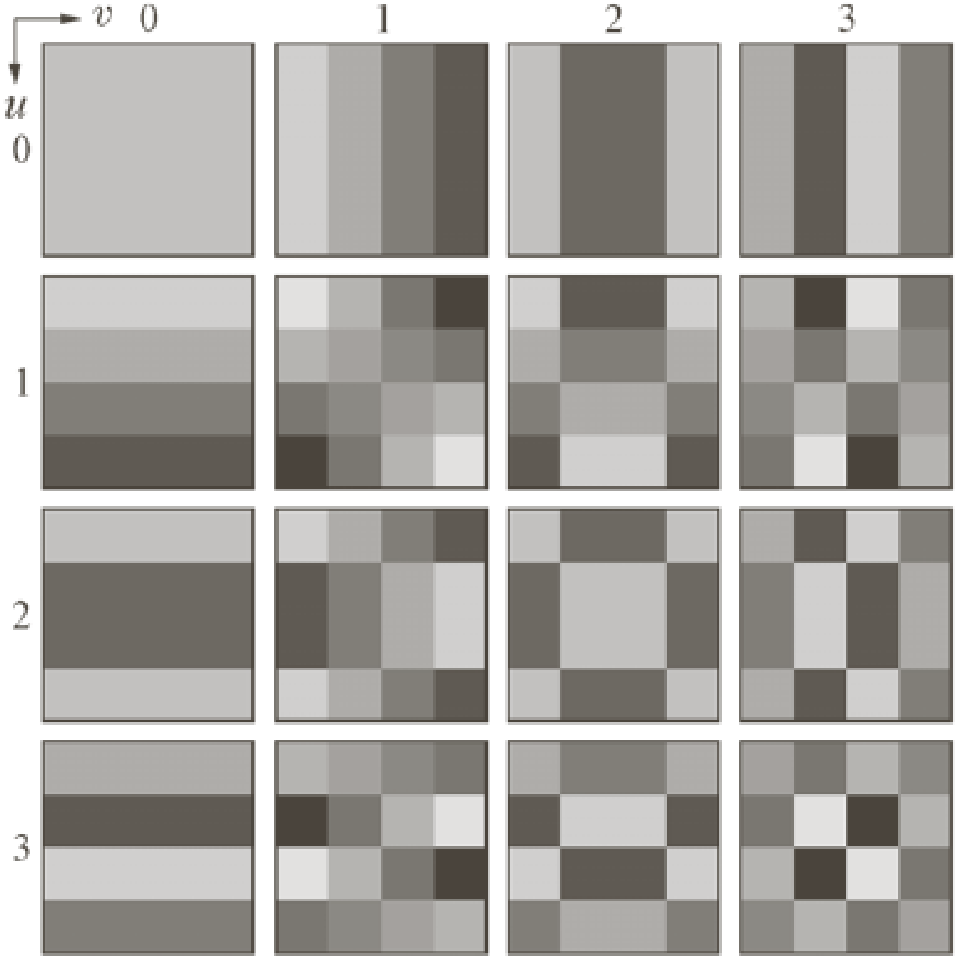
$$= \left(\frac{1}{\sqrt{N}} - \sqrt{\frac{2}{N}}\right)F_e(0) + \sqrt{\frac{2}{N}}\mathrm{Re}\left\{\sum_{u=0}^{2N-1}\left\{F_e(u)e^{j\frac{u\pi}{2N}}\right\}e^{j\frac{2\pi ux}{2N}}\right\}$$

where

$$\sum_{u=0}^{2N-1}\left\{F_e(u)e^{j\frac{u\pi}{2N}}\right\}e^{j\frac{2\pi ux}{2N}} = \mathrm{IDFT}\left[F_e(u)e^{j\frac{u\pi}{2N}}\right]$$

Basic Function for DCT

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v)$$



Walsh Transform

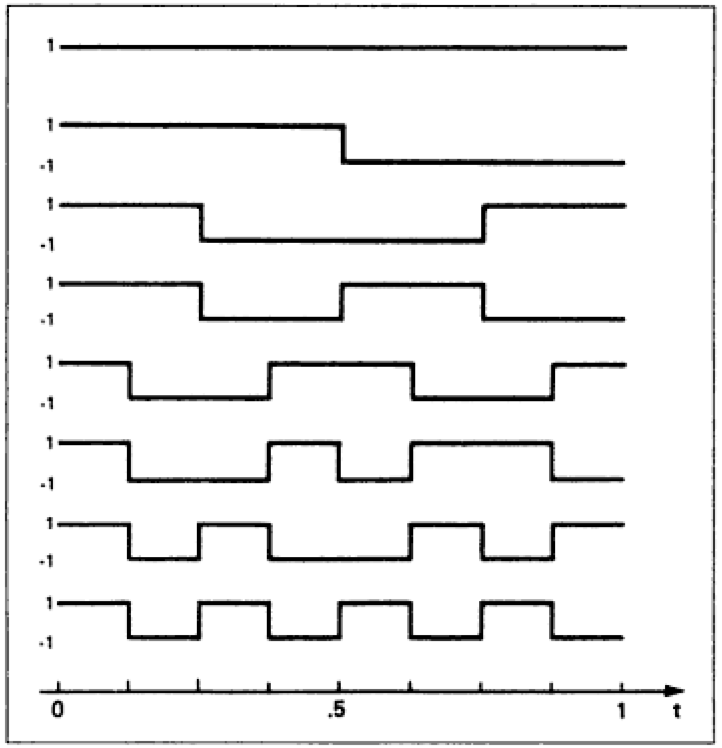
- **Consist of ±1 arranged in a checkerboard pattern**
- **Transform:**

$$W(i) = \frac{1}{N}\sum_{t=0}^{N-1} f(t)\cdot \mathrm{Wal}(i,t)$$

$$f(t) = \sum_{i=0}^{N-1} W(i)\cdot \mathrm{Wal}(i,t)$$

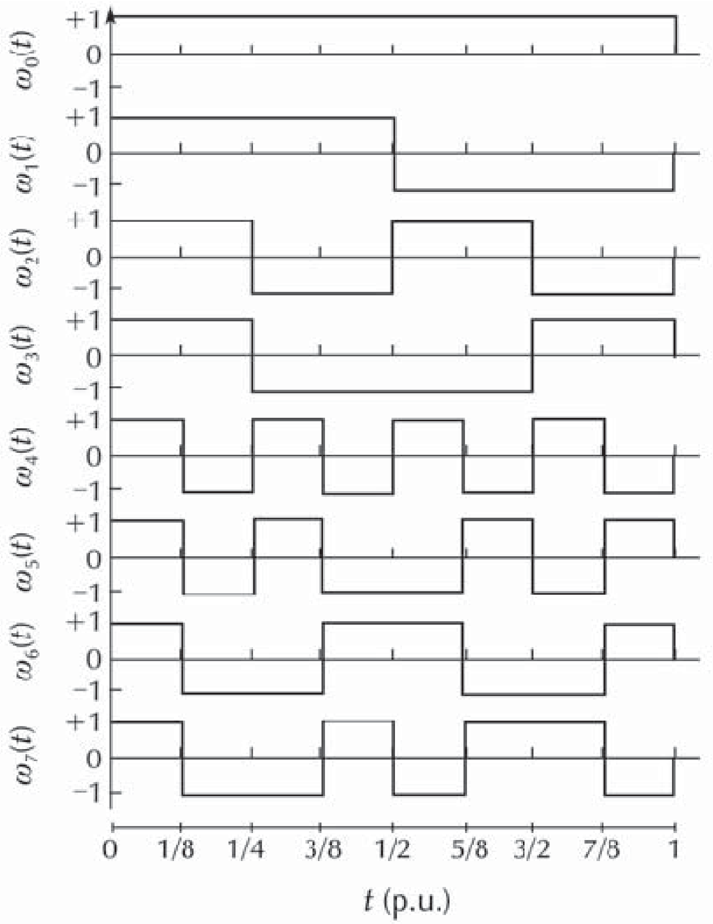
- **Types of** $\mathrm{Wal}(i,t)$
  - Walsh Ordering (沃尔什定序)
  - Paley Ordering （佩利定序）
  - Hadamard Matrix Ordering（哈达玛矩阵定序）

Walsh Ordering



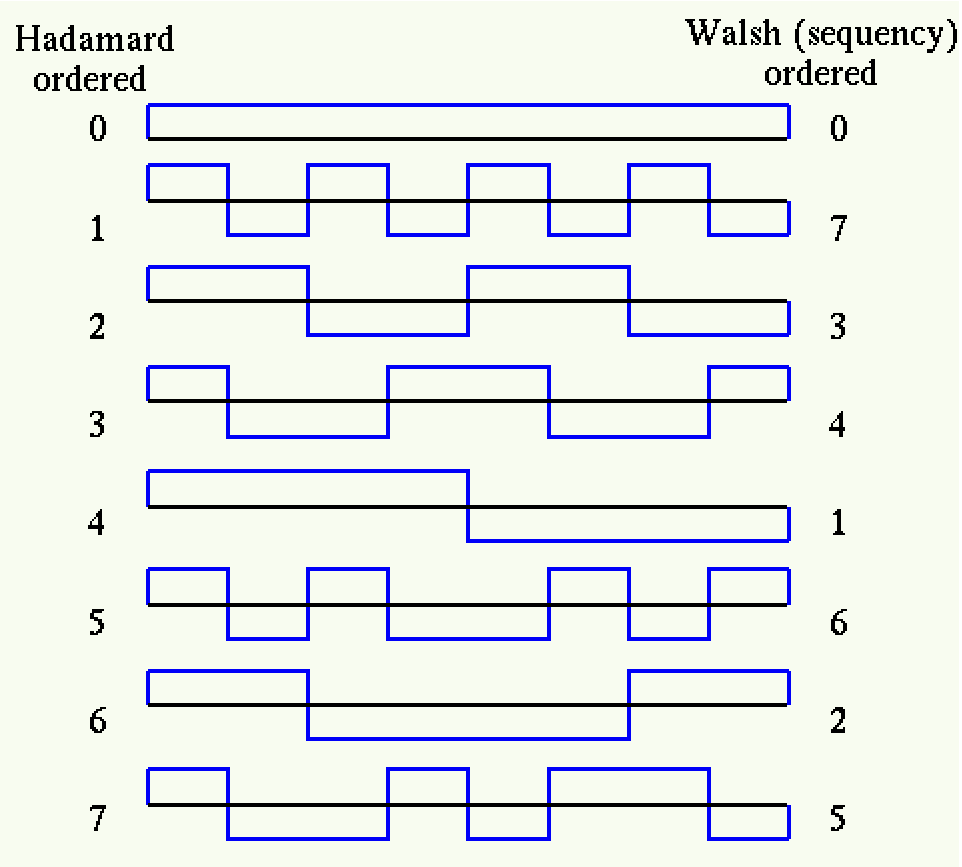
$$W_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

Paley Ordering



$$\boldsymbol{W_8} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Hadamard Matrix Ordering

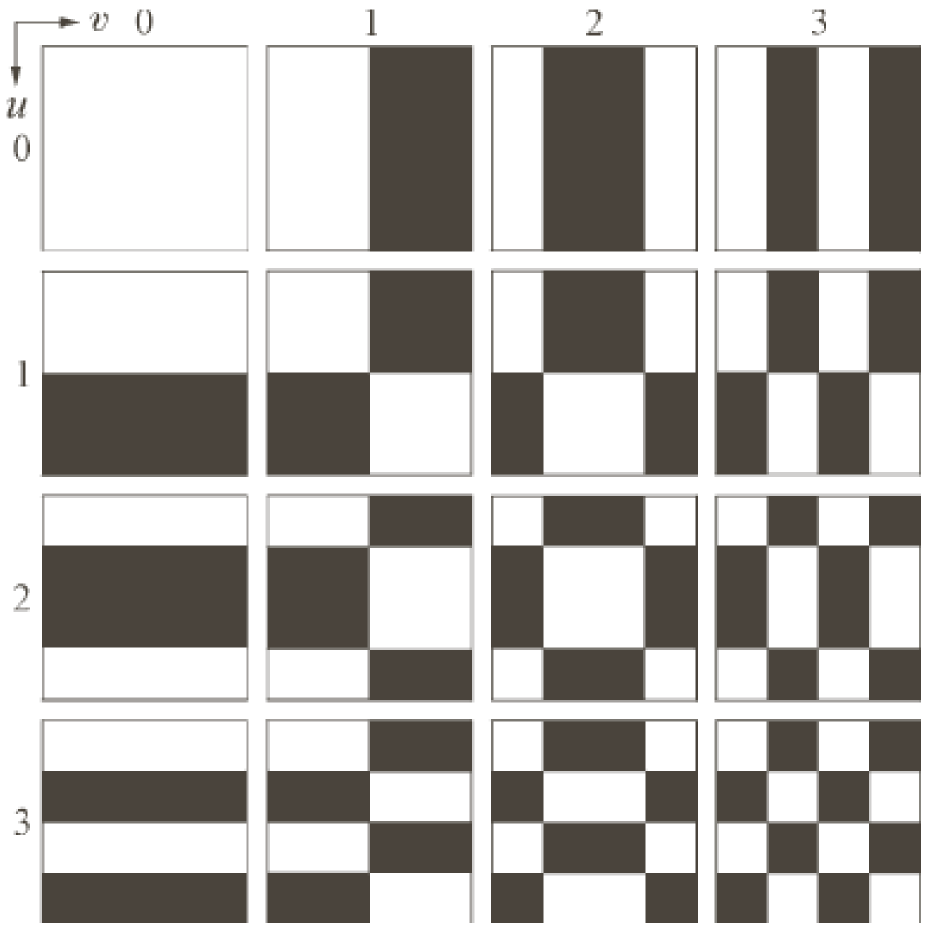


$$V_8 = \begin{pmatrix} W_4 & W_4 \\ W_4 & -W_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Relationship Between Ordering

Walsh ordering (*Signal processing*)	**Paley ordering** (*Control Engineering*)	**Hadamard ordering** (*Mathematics*)	**W(m,n)**
$Wal_W(0,t)$	$Wal_P(0,t)$	$Wal_H(0,t)$	[1 1 1 1 1 1 1 1]
$Wal_W(1,t)$	$Wal_P(1,t)$	$Wal_H(4,t)$	[1 1 1 1 -1 -1 -1 -1]
$Wal_W(2,t)$	$Wal_P(3,t)$	$Wal_H(6,t)$	[1 1 -1 -1 -1 -1 1 1]
$Wal_W(3,t)$	$Wal_P(2,t)$	$Wal_H(2,t)$	[1 1 -1 -1 1 1 -1 -1]
$Wal_W(4,t)$	$Wal_P(6,t)$	$Wal_H(3,t)$	[1 -1 -1 1 1 -1 -1 1]
$Wal_W(5,t)$	$Wal_P(7,t)$	$Wal_H(7,t)$	[1 -1 -1 1 -1 1 1 -1]
$Wal_W(6,t)$	$Wal_P(5,t)$	$Wal_H(5,t)$	[1 -1 1 -1 -1 1 -1 1]
$Wal_W(7,t)$	$Wal_P(4,t)$	$Wal_H(1,t)$	[1 -1 1 -1 1 -1 1 -1]

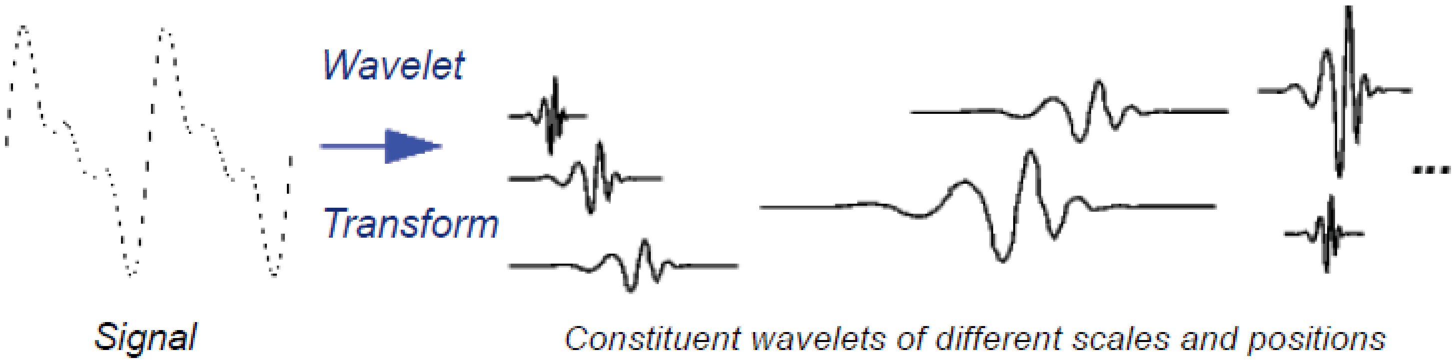
Basic Function for WHT



Block Transform

Wavelet Transform (小波变换)

- **Based on small waves called Wavelets –** 1) limited; 2) oscillation
- **Mother wavelet（母小波）:** Translation & Scaling
- **Varying frequency and limited duration**
- **localized in both time and frequency**



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Continuous Wavelet Transform (连续小波变换)

➢ **Continuous Wavelet Transform (CWT)**

$$W_\psi(s,\tau)=\int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x)dx$$

Where $\psi_{s,\tau}(x)=\frac{1}{\sqrt{s}}\psi\left(\frac{x-\tau}{s}\right)$

s : scale parameter (尺度参数) τ : translation parameter (平移参数)

➢ **Inverse Continuous Wavelet Transform (ICWT)**

$$f(x)=\frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} W_\psi(s,\tau)\frac{\psi_{s,\tau}(x)}{s}d\tau\, ds$$

Where $C_\psi=\int_{-\infty}^{\infty}\frac{|\Psi(\mu)|^2}{|\mu|}d\mu$, $\Psi(\mu)$ is Fourier transform of $\psi(x)$

Discrete Wavelet Transform (离散小波变换)

- **Discrete Wavelet Transform (DWT)**

$$W_\varphi(j_0,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\,\varphi_{j_0,k}(n)$$

$$W_\psi(j,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\,\psi_{j,k}(n)\qquad j\geq j_0$$

- **Inverse Continuous Wavelet Transform (ICWT)**

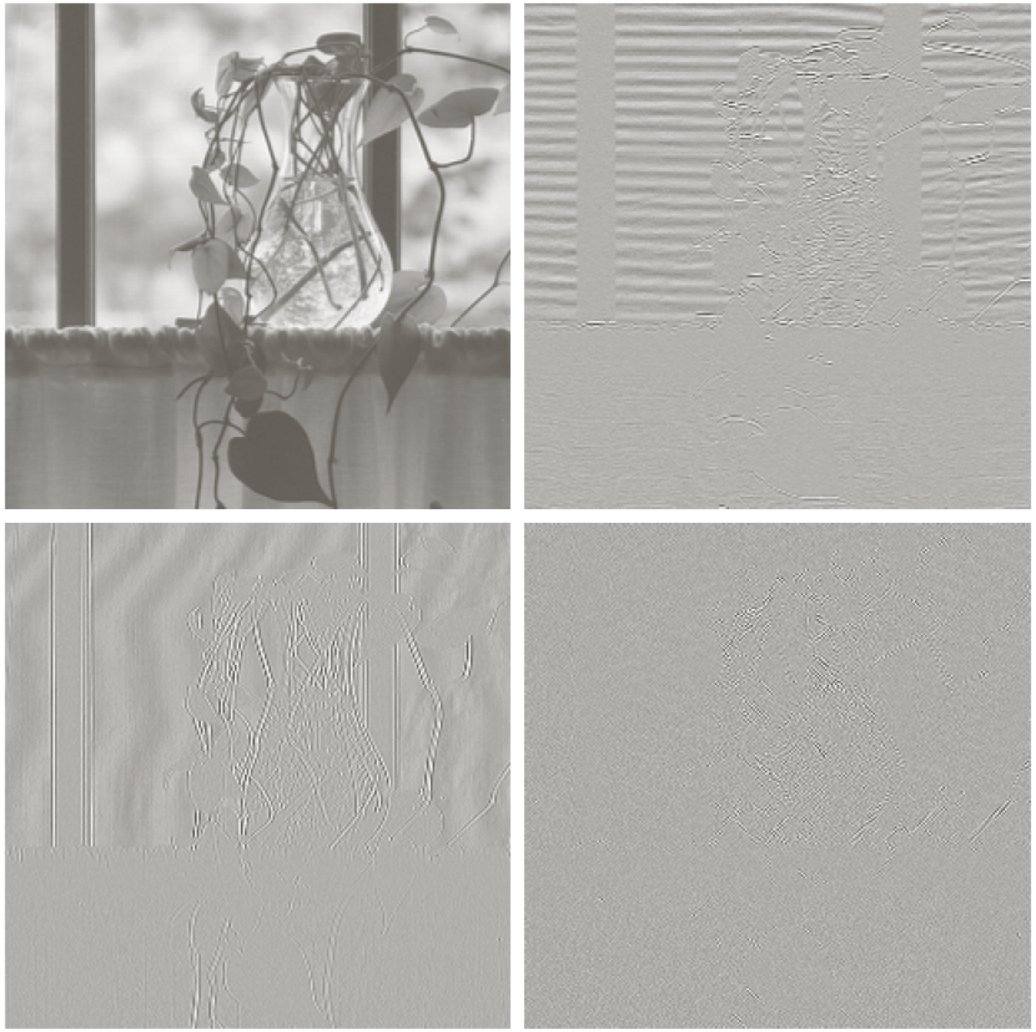
$$f(n)=\frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(n)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\,\psi_{j,k}(n)$$

Where

$\varphi_{j_0,k}(n)$: scaling function (尺度函数)　　$\psi_{j,k}(n)$: Wavelet (小波)

$W_\varphi(j_0,k)$: Approximation coefficients (近似系数)　　$W_\psi(j,k)$: detail coefficients (细节系数)

Scale & Wavelet



2D DWT

Define 2D scale function（二维尺度函数）:

$$\varphi(x,y) = \varphi(x)\varphi(y)$$

“Directionally sensitive” wavelet（“方向敏感” 小波）

$$\psi^H(x,y) = \psi(x)\varphi(y) \qquad \psi^V(x,y) = \varphi(x)\psi(y) \qquad \psi^D(x,y) = \psi(x)\psi(y)$$

2D DWT

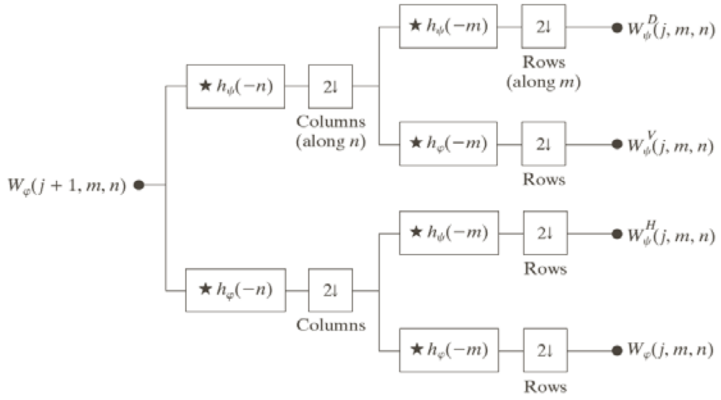
$$W_\varphi(j_0,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\varphi_{j_0,m,n}(x,y)$$

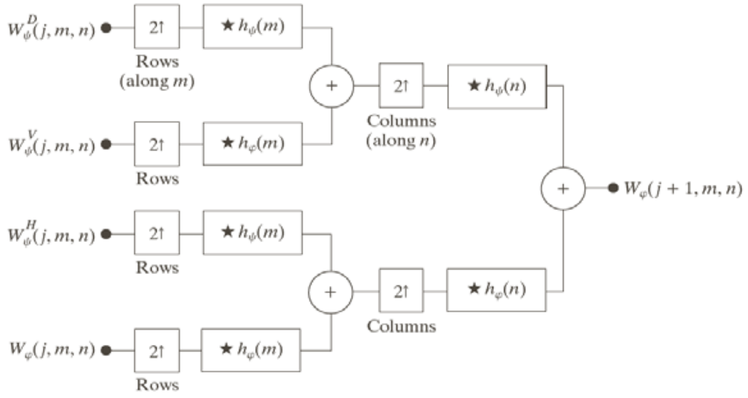
$$W_\psi(j,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\psi^i_{j,m,n}(x,y) \qquad i = \{H,V,D\}$$

2D IDWT

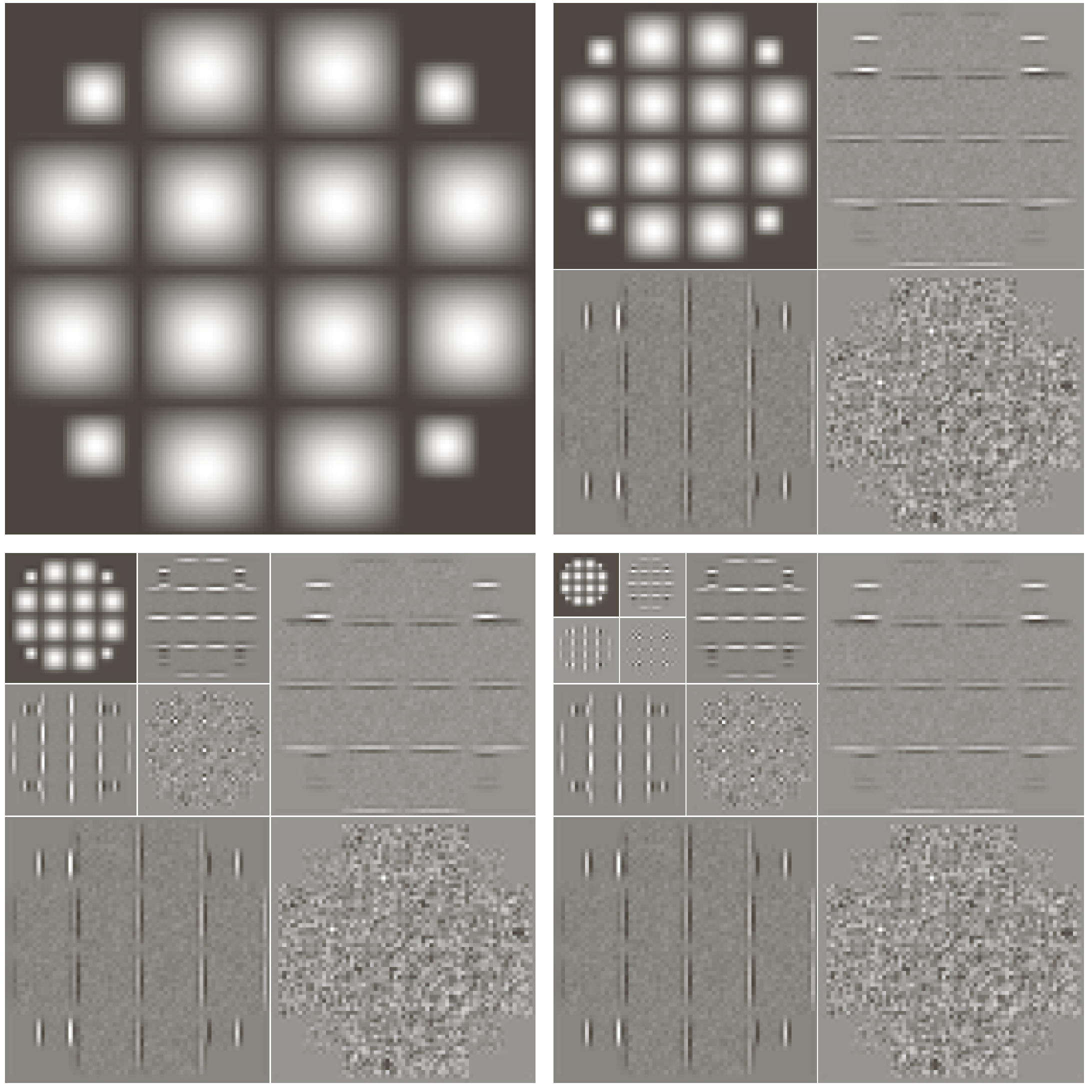
$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_m\sum_n W_\varphi(j_0,m,n)\,\varphi_{j_0,m,n}(x,y)$$

$$+\frac{1}{\sqrt{MN}}\sum_{i=\{H,V,D\}}\sum_{j=j_0}^{\infty}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} W_\psi(j,m,n)\,\psi^i_{j,m,n}(x,y)$$





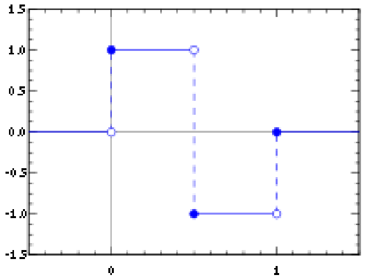
2D DWT



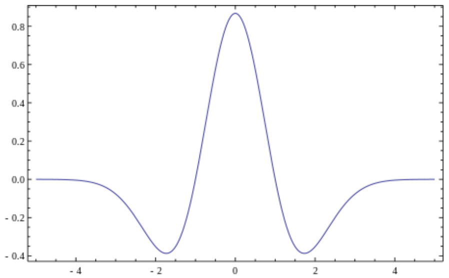
Mother Wavelet (母小波)

- **Mother Wavelet should satisfy**
  - $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$
  - $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$
  - $\int_{-\infty}^{\infty} \psi(t) dt = 0$

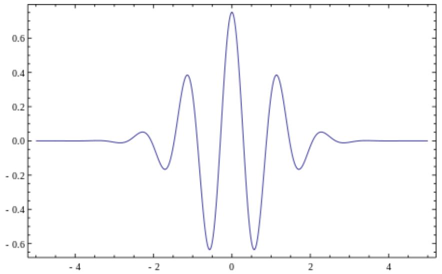
Haar



Mexican Hat



Morlet



Meyer