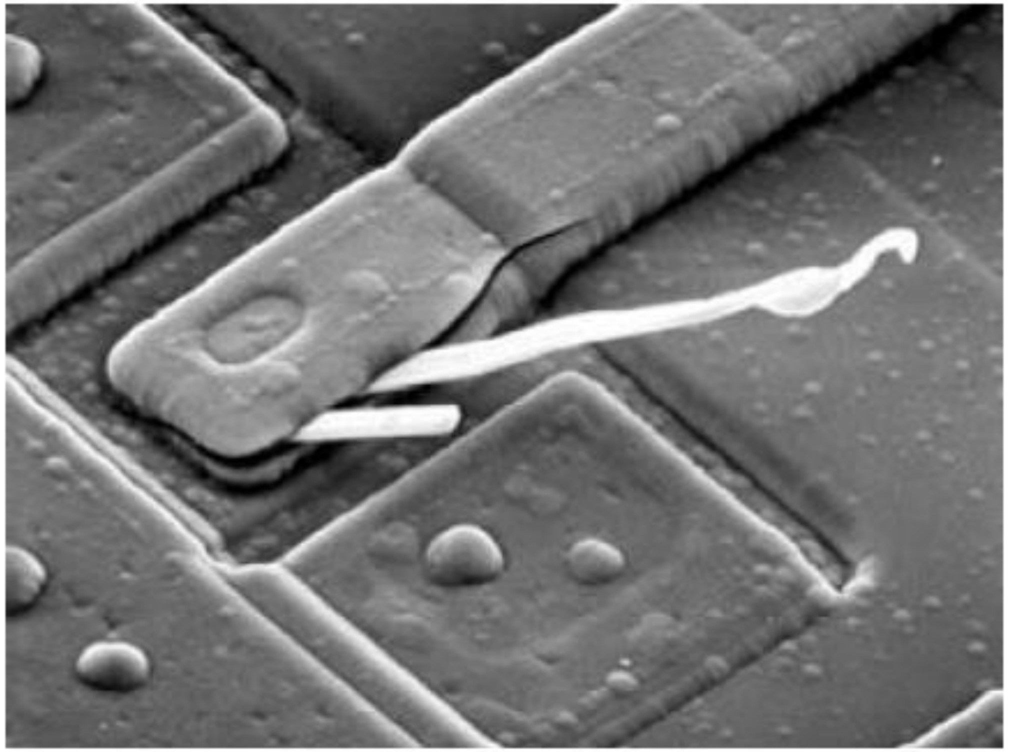
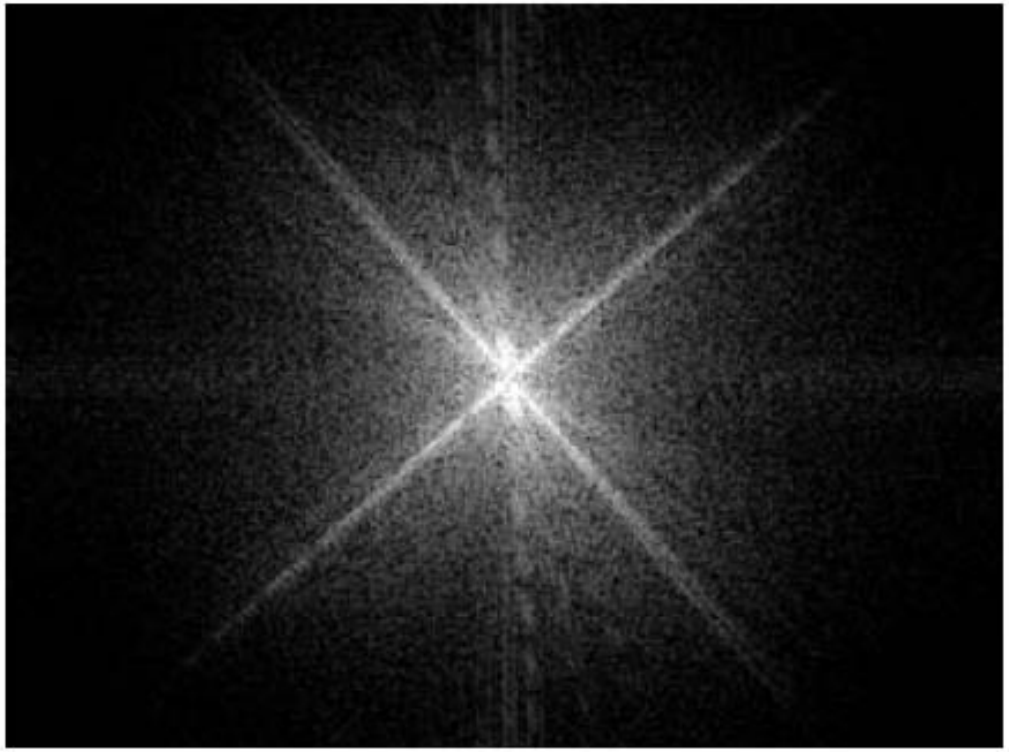
Lecture 4 – Frequency Domain Transform （频率域变换）

This lecture will cover:

- 2D Discrete Fourier Transform （傅里叶变换）
- **Frequency Domain Filtering（频率域滤波）**
  - **Lowpass Filtering（低通滤波器）**
  - **Highpass Filtering （高通滤波器）**
  - **Selective Filtering（选择性滤波）**
- Other Transform
  - Discrete Cosine Transform（余弦变换）
  - Walsh-Hadamard Transform
  - Discrete Wavelet Transform （小波变换）

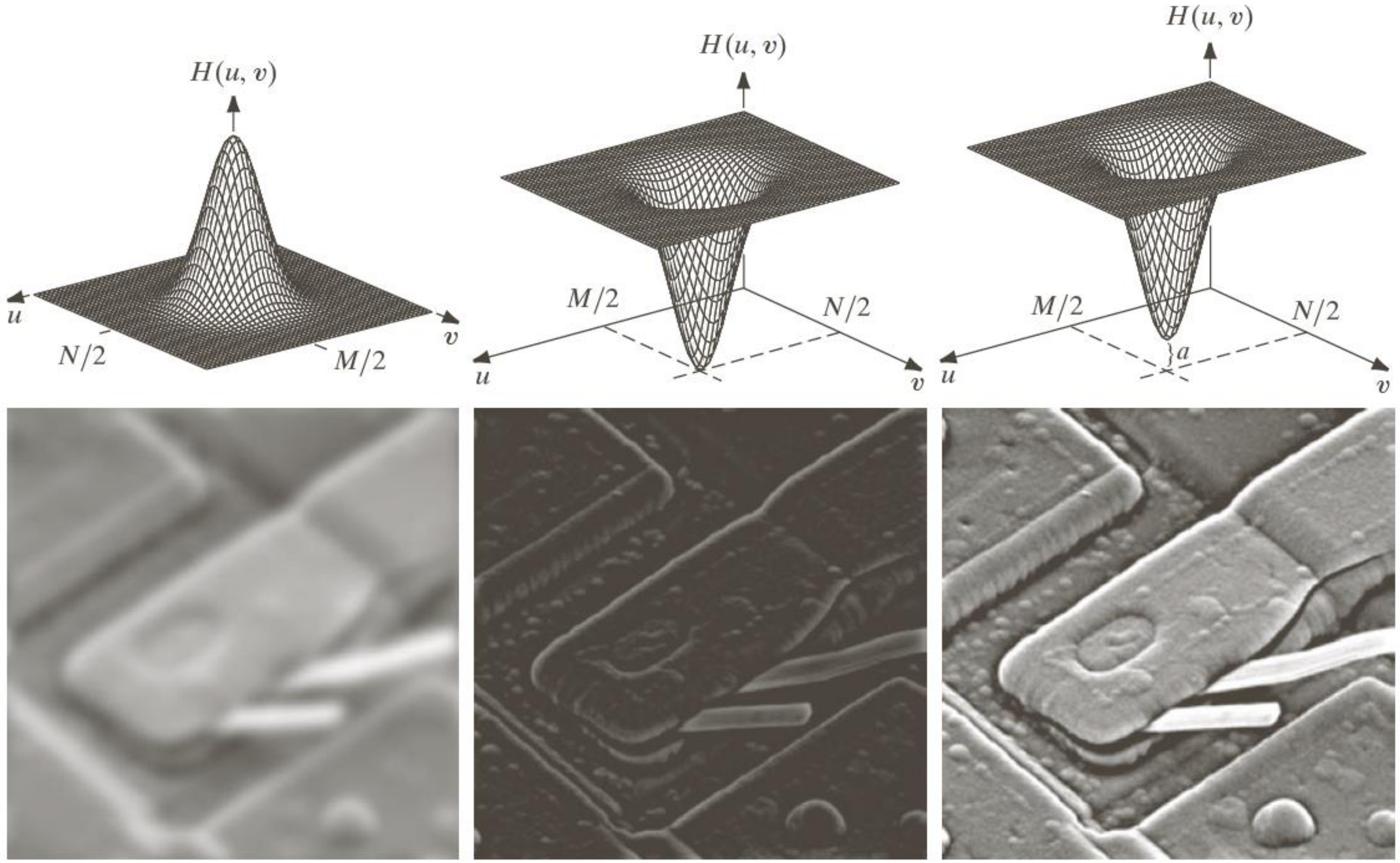
Fourier Spectrum





Frequency Domain Filtering

Basic Filtering form: $\boldsymbol{g}(x,y) = \mathcal{F}^{-1}[H(u,v)F(u,v)]$



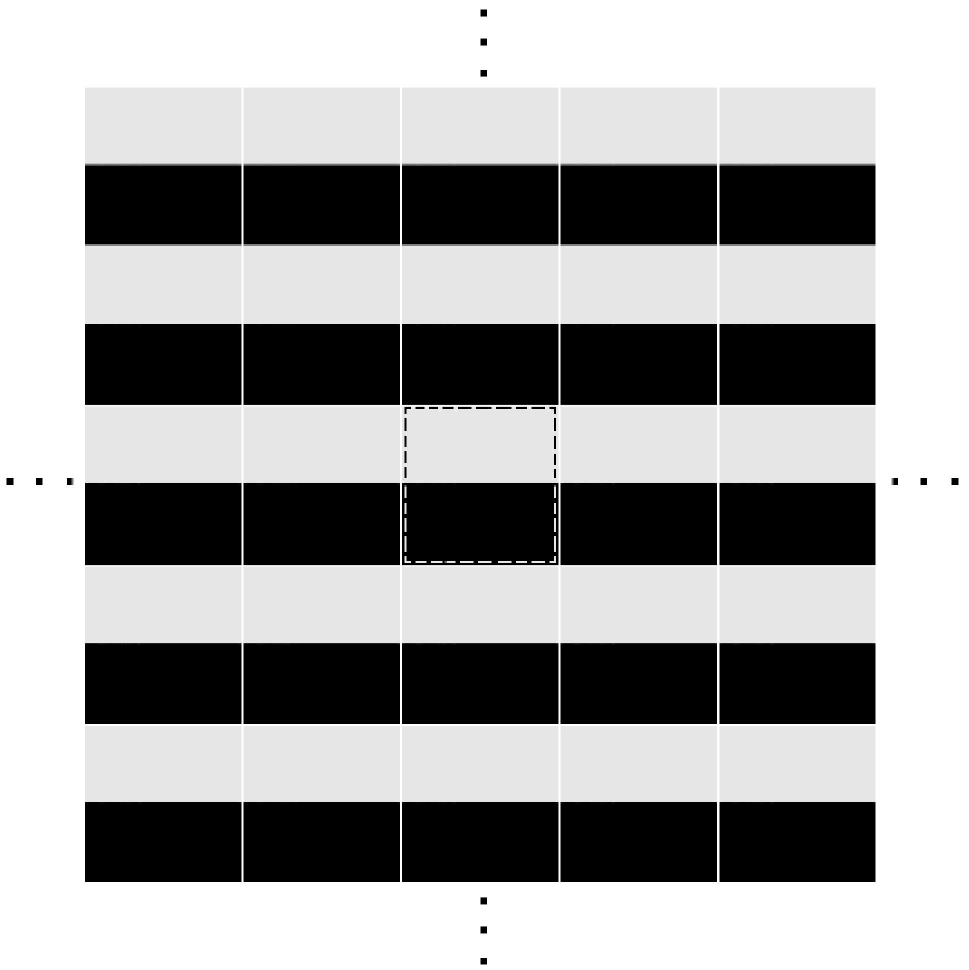
上海科技大学 ShanghaiTech University

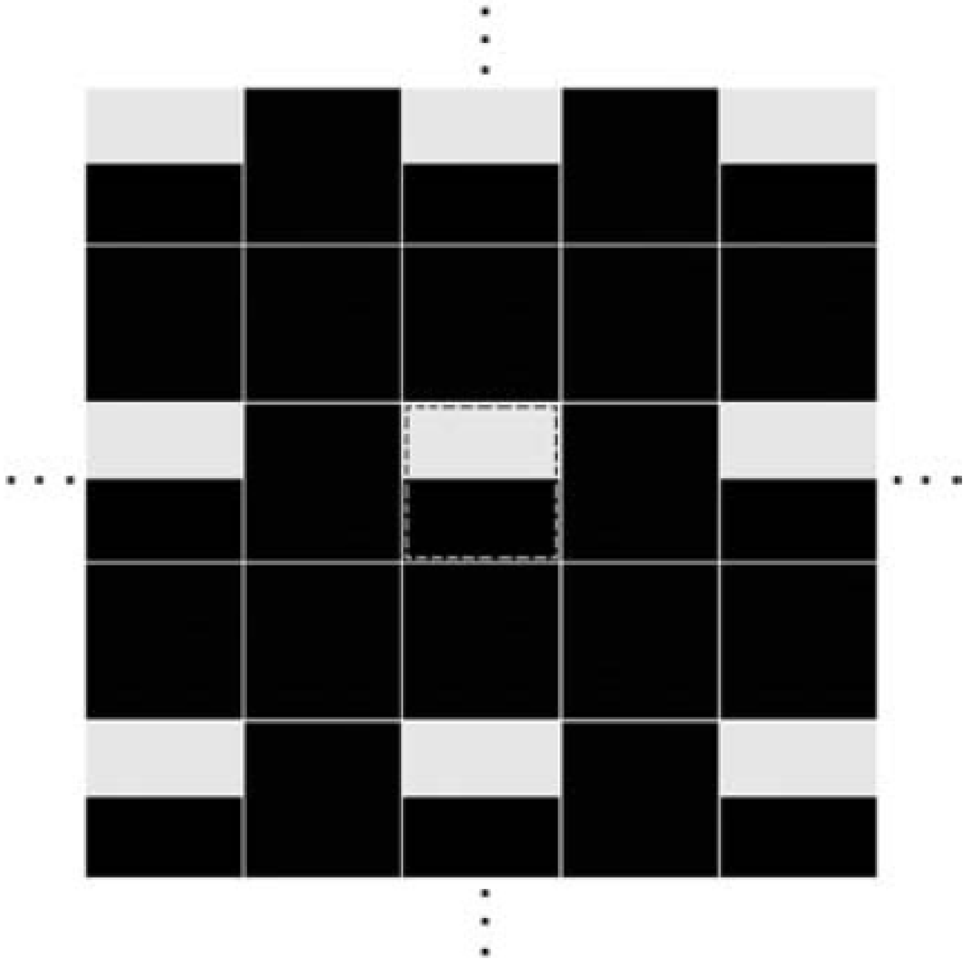
Padding (填充)

$$f_p(x,y) = \begin{cases} f(x,y), & 0 \le x \le A-1, 0 \le y \le B-1 \\ 0, & A \le x \le P, B \le y \le Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y), & 0 \le x \le C-1, 0 \le y \le D-1 \\ 0, & C \le x \le P, D \le y \le Q \end{cases}$$

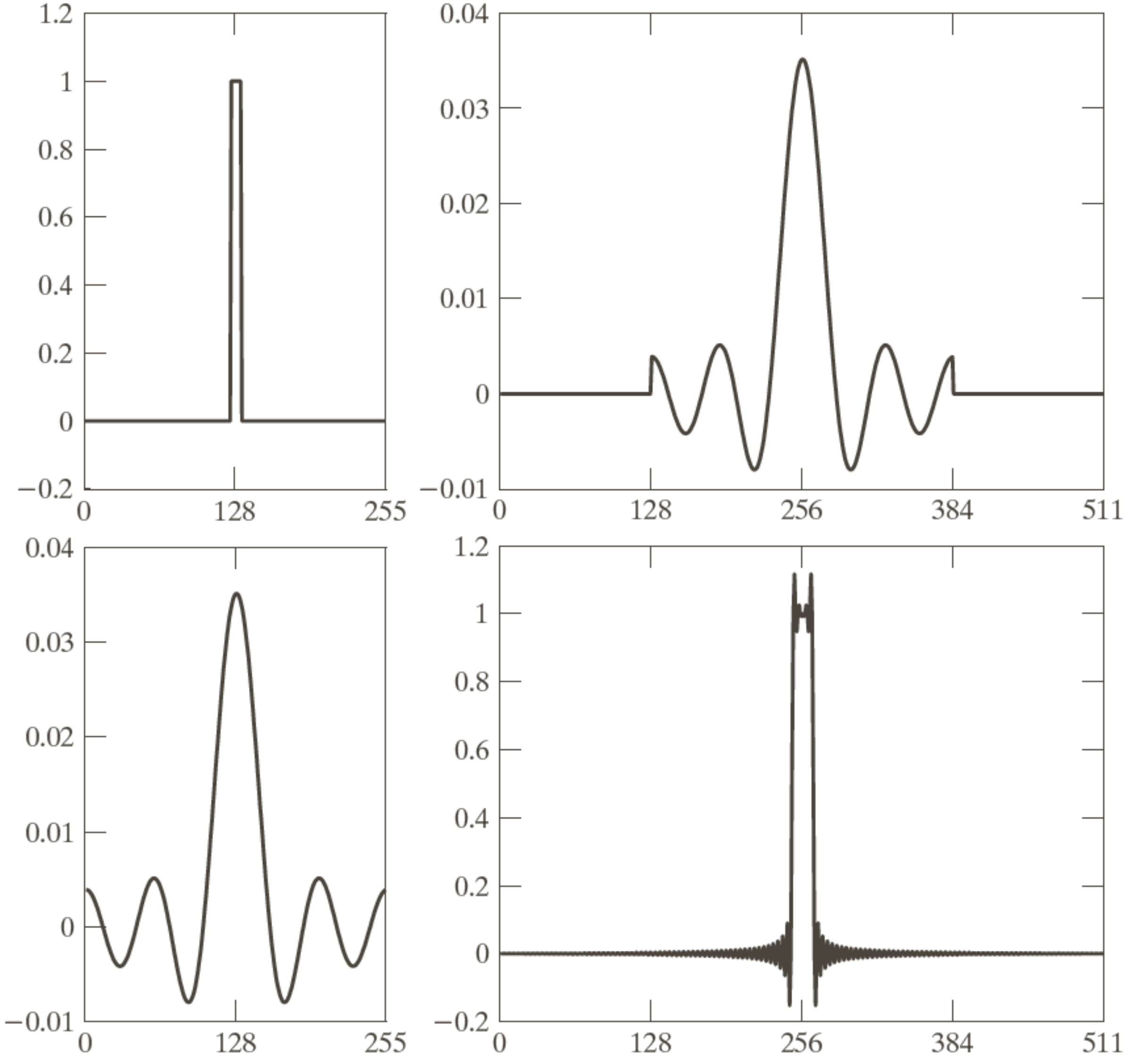
Padding (填充)





上海科技大学 ShanghaiTech University

Padding (填充)



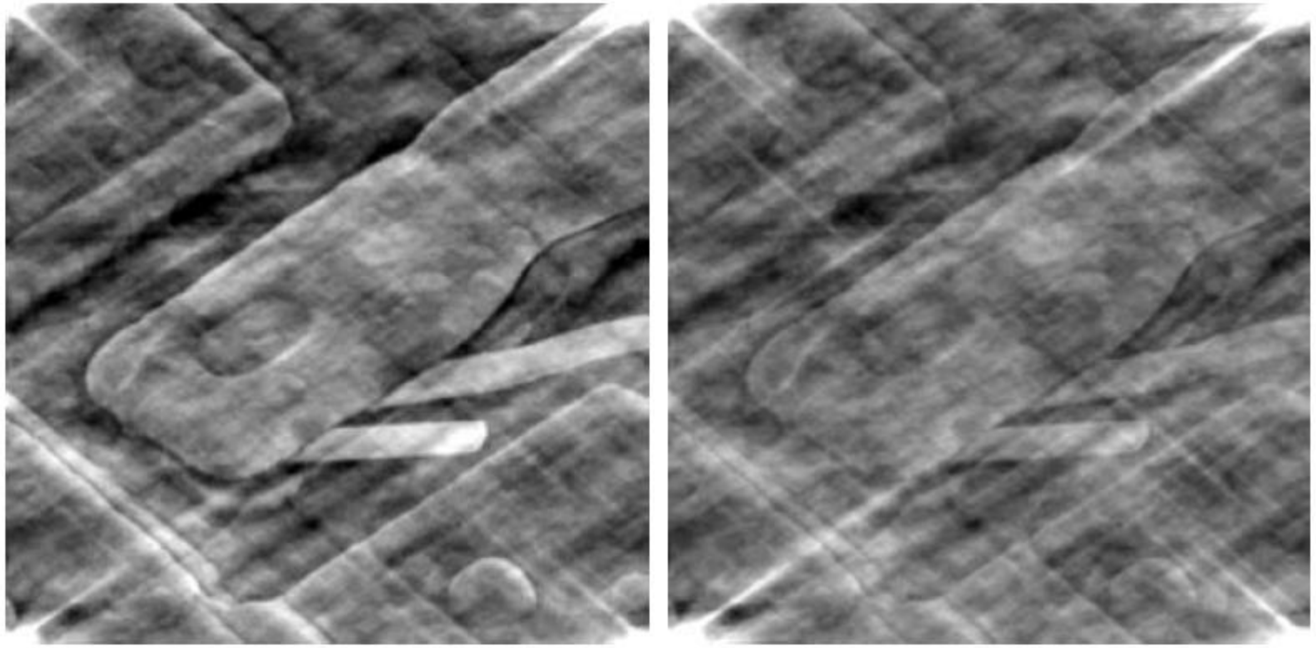
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
128
255
0.04
0.03
0.02
0.01
0
−0.01
0
128
256
384
511
0.04
0.03
0.02
0.01
0
−0.01
0
128
255
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
128
256
384
511

Phase Angle

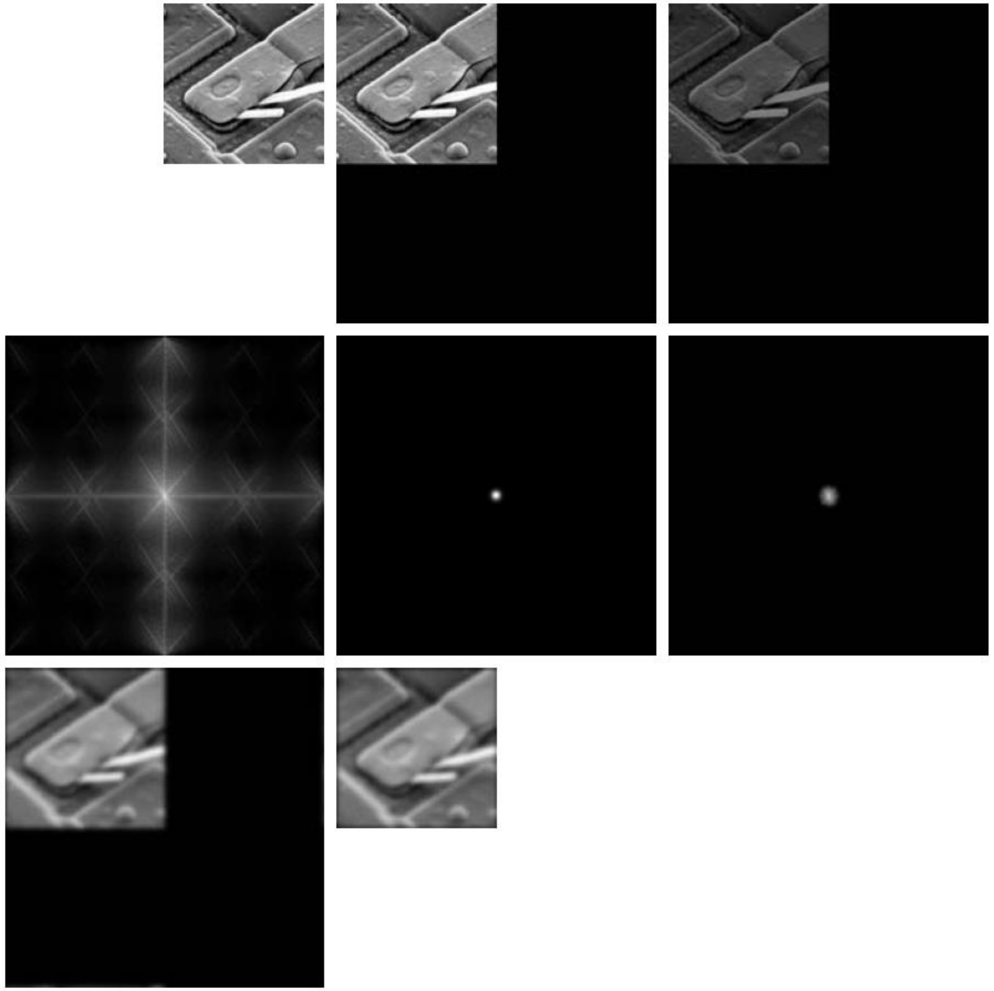
Let :$F(u,v) = R(u,v) + jI(u,v)$

$$\boldsymbol{g}(x,y) = \mathcal{F}^{-1}[H(u,v)R(u,v) + jH(u,v)I(u,v)]$$

$H(u,v)$**: zero-phase-shift filter（零相移滤波器）**



Steps of Frequency Domain Filtering



1. Zero-padding input image $f_p(x, y)$

2. $f_p(x, y)(-1)^{(x+y)}$ to center its transform

3. Compute DFT

4. $\boldsymbol{G}(u, v) = H(u, v)F(u, v)$

5. $\boldsymbol{g_p}(x, y) = Re[\mathcal{F}^{-1}[\left(G(u, v)\right)]]\ (-1)^{(x+y)}$

6. Obtain $g(x, y)$from top-left quadrant

Filtering in Spatial and Frequency Domains

- **Frequency filters ⇒ Spatial filter** $H(u,v) \Longrightarrow h(x,y)$

Design filter in frequency domain → Compute IDFT → Full size spatial filter → Smaller size spatial masks

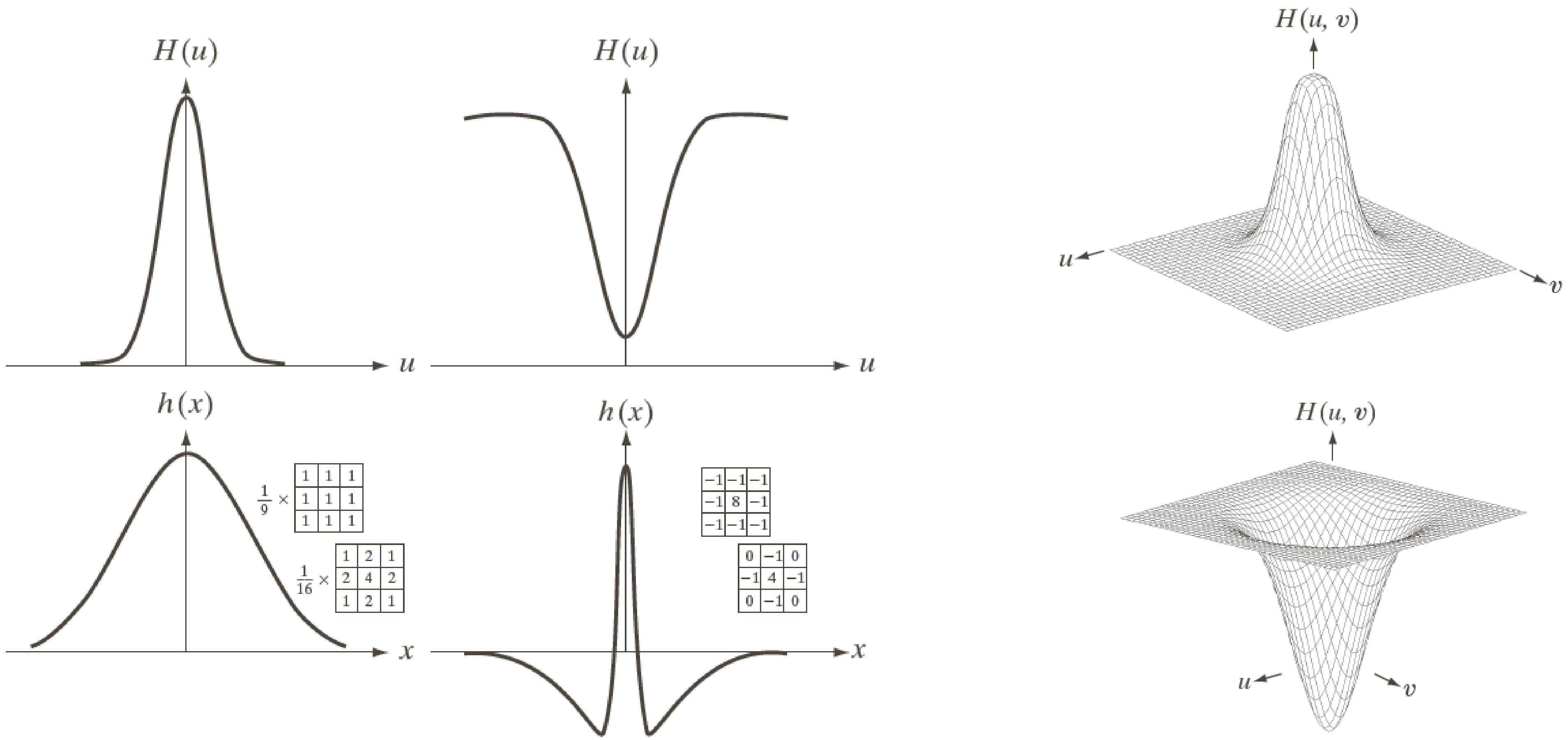
- **Gaussian Filter (高斯滤波器)**

$$H(u) = Ae^{-\frac{u^2}{2\sigma^2}} \iff h(x) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2x^2}$$

$$H(u) = Ae^{-\frac{u^2}{2\sigma_1^2}} - Be^{-\frac{u^2}{2\sigma_2^2}} \iff h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2x^2}$$

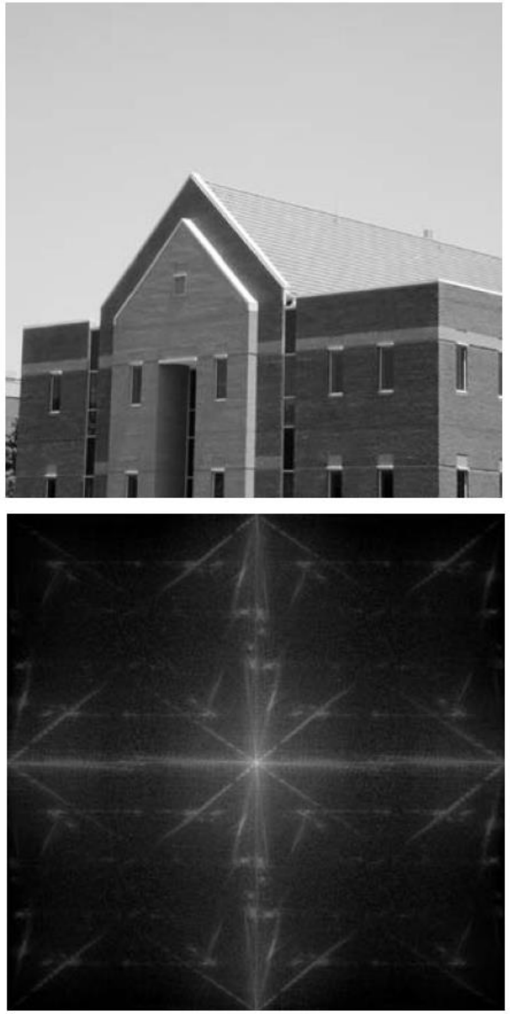
$$H(u,v) = Ae^{-\frac{u^2+v^2}{2\sigma^2}} \iff h(x,y) = A2\pi\sigma^2e^{-2\pi^2\sigma^2(x^2+y^2)}$$

Gaussian Filter (高斯滤波器)

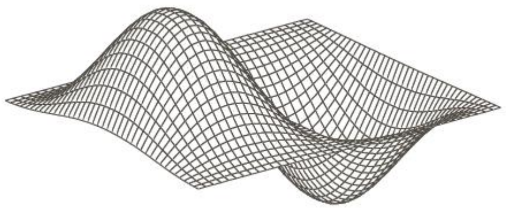


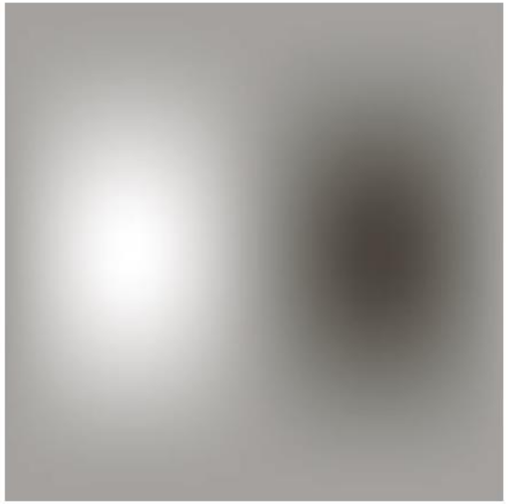
上海科技大学 ShanghaiTech University

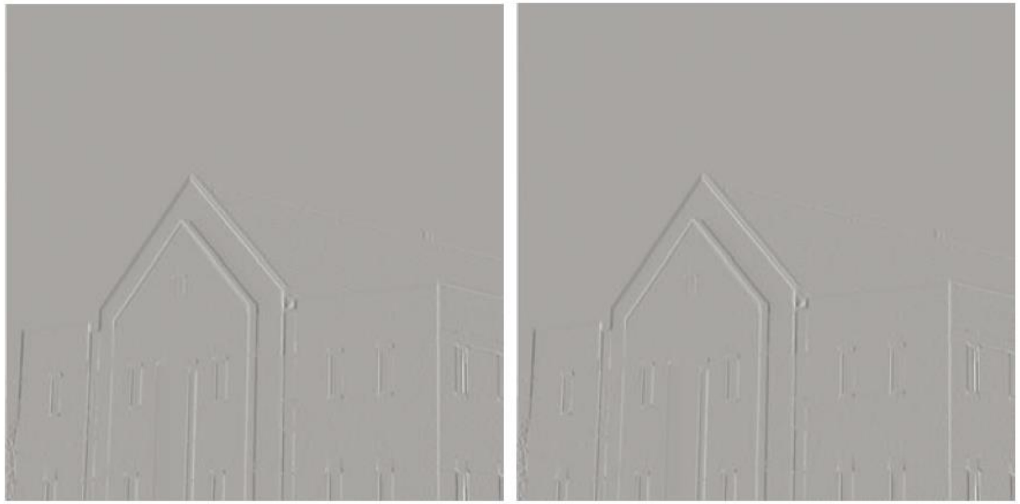
Spatial and Frequency Filtering



−1	0	1
−2	0	2
−1	0	1







Lowpass Filtering

- Ideal Lowpass Filter（理想低通滤波器）
- Butterworth Lowpass Filter（布特沃斯低通滤波器）
- Gaussian Lowpass Filter （高斯低通滤波器）

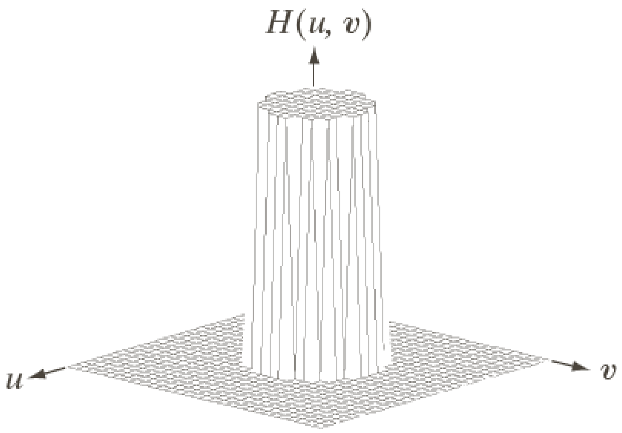
Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

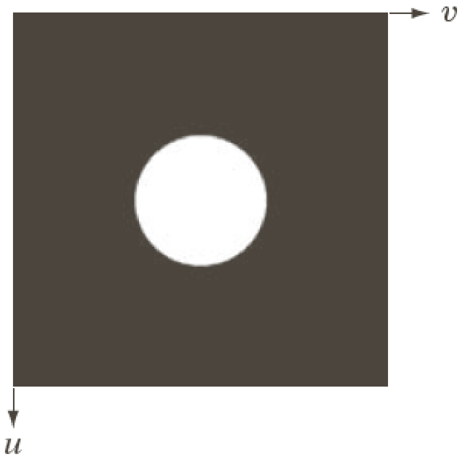
Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

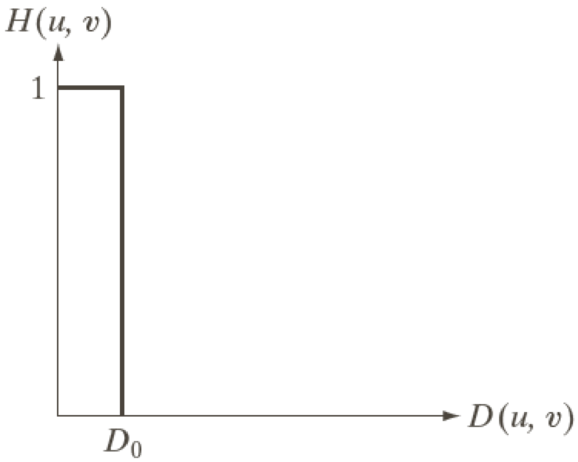
Ideal Lowpass Filter（理想低通滤波器）

Ideal Lowpass Filter (ILPF):

$$H(u,v)=\begin{cases}1, & D(u,v)\le D_0\\ 0, & D(u,v)>D_0\end{cases} \qquad D(u,v)=\left[(u-\frac{P}{2})^2+(v-\frac{Q}{2})^2\right]^{1/2}$$





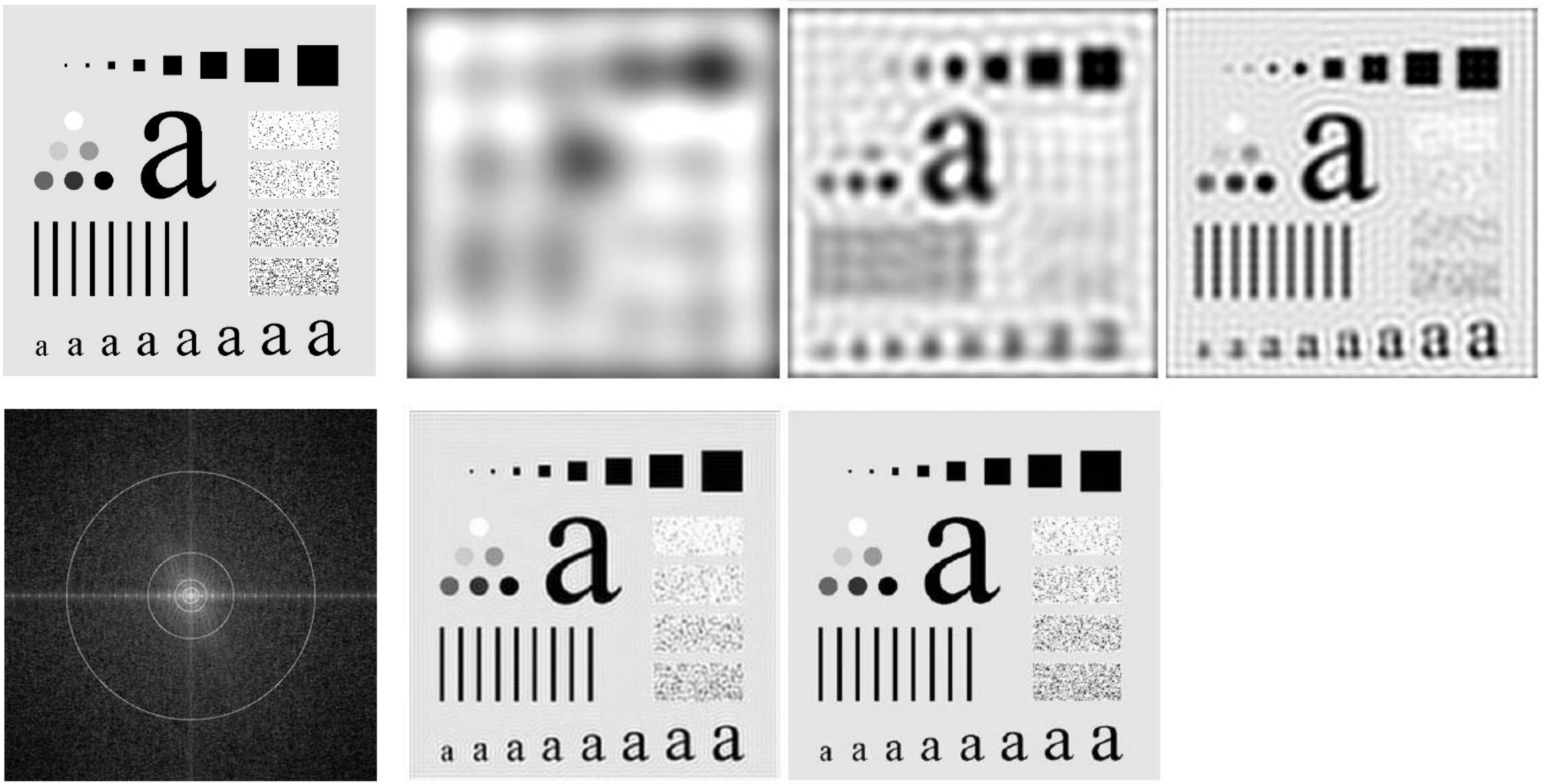


Cutoff Frequency（截止频率）

- **Power:** $P(u,v) = |F(u,v)|^2$
- **Total image Power:** $P_T = \sum_{u=0}^{P-1}\sum_{v=0}^{Q-1} P(u,v)$
- **The power in a circle of radius D_0:** $\alpha = 100\left[\frac{\sum_u \sum_v P(u,v)}{P_T}\right]$

上海科技大学 ShanghaiTech University

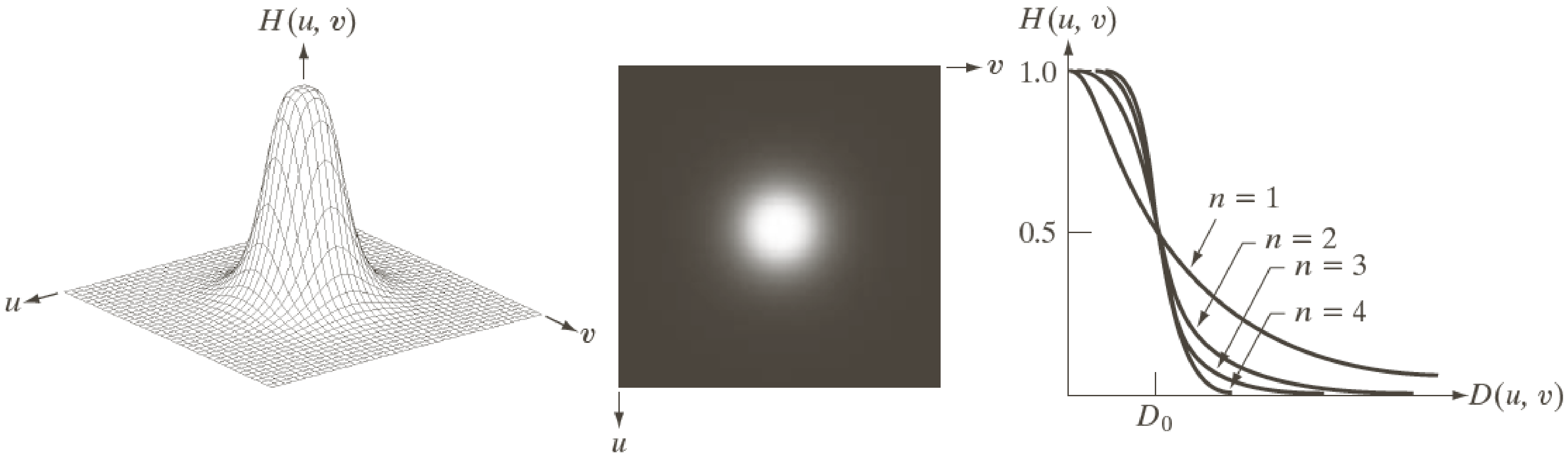
Cutoff Frequency（截止频率）



Butterworth Lowpass Filter（布特沃斯）

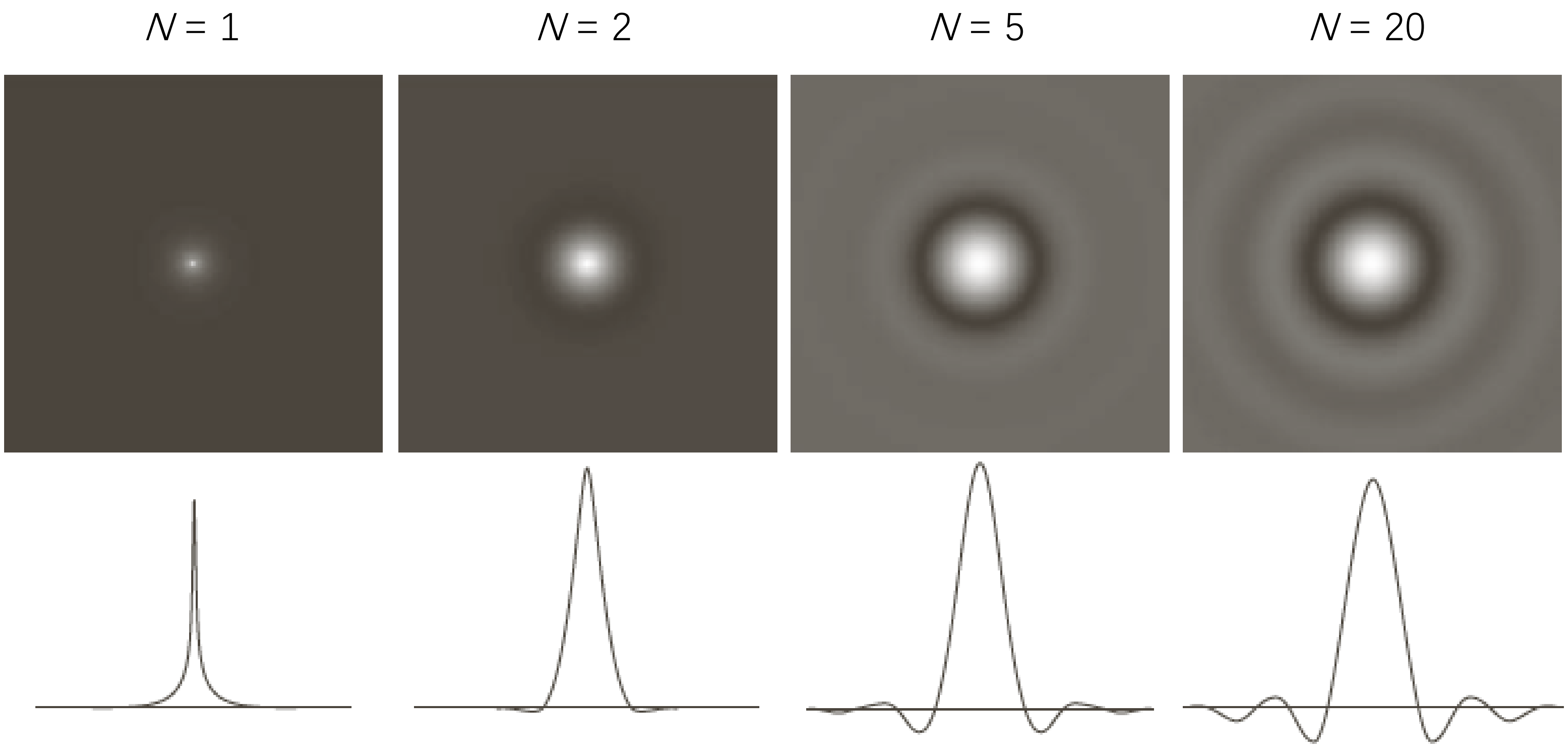
$$H(u,v)=\frac{1}{1+\left[\frac{D(u,v)}{D_0}\right]^{2n}}$$

Where $D(u,v)=\left[(u-\frac{P}{2})^2+(v-\frac{Q}{2})^2\right]^{1/2}$, and $H(u,v)=0.5$ when $D(u,v)=D_0$



Butterworth Lowpass Filter（布特沃斯）

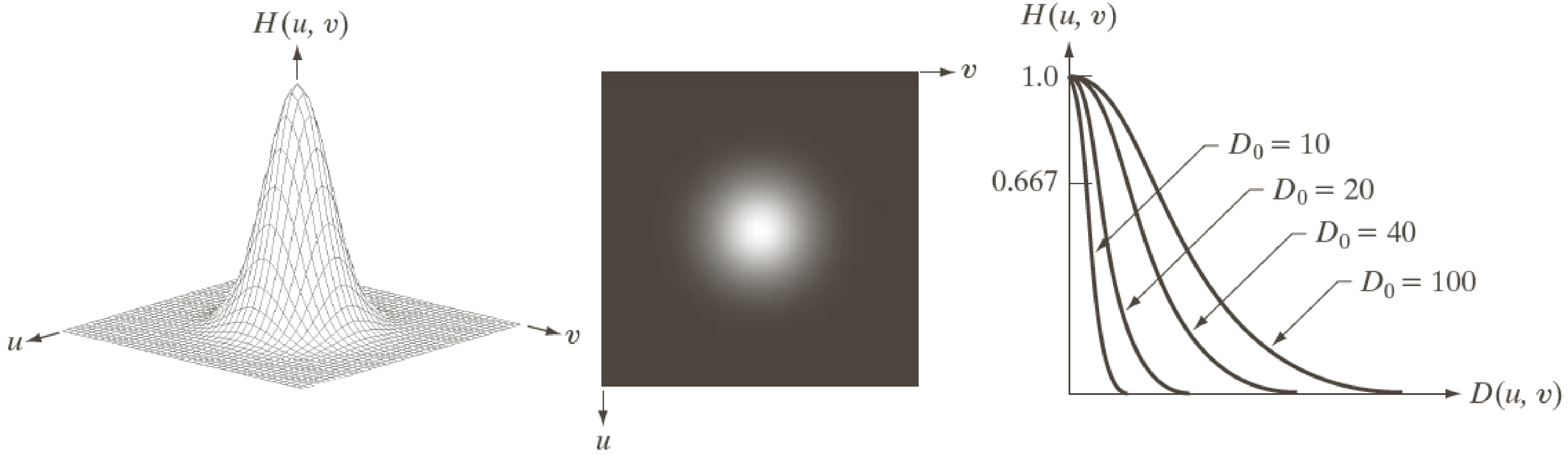
*n*th Butterworth Filter



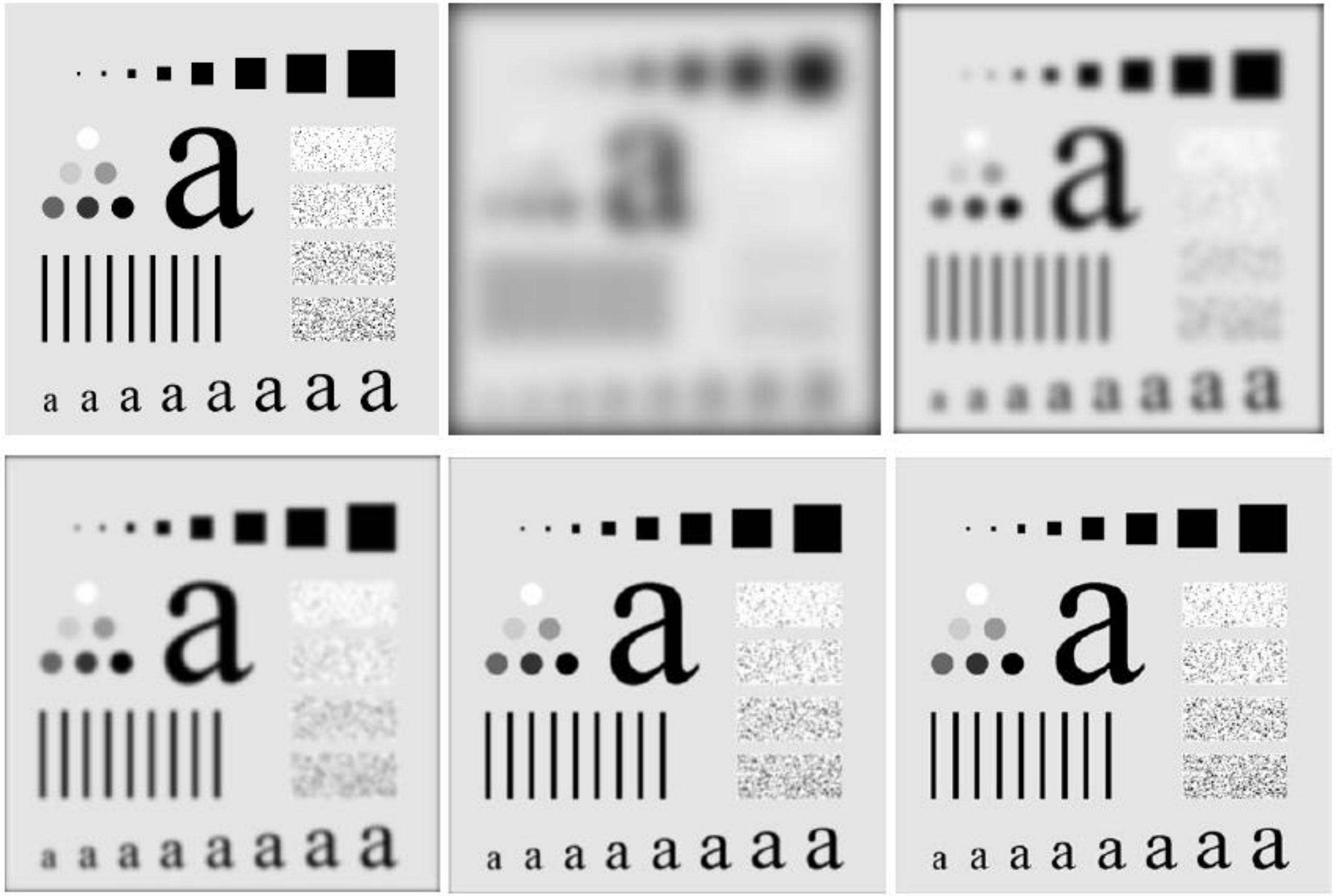
Gaussian Lowpass Filter (高斯滤波器)

$$H(u,v) = e^{-\frac{D(u,v)^2}{2D_0^2}}$$

Where $H(u,v) = 0.607$ when $D(u,v) = D_0$

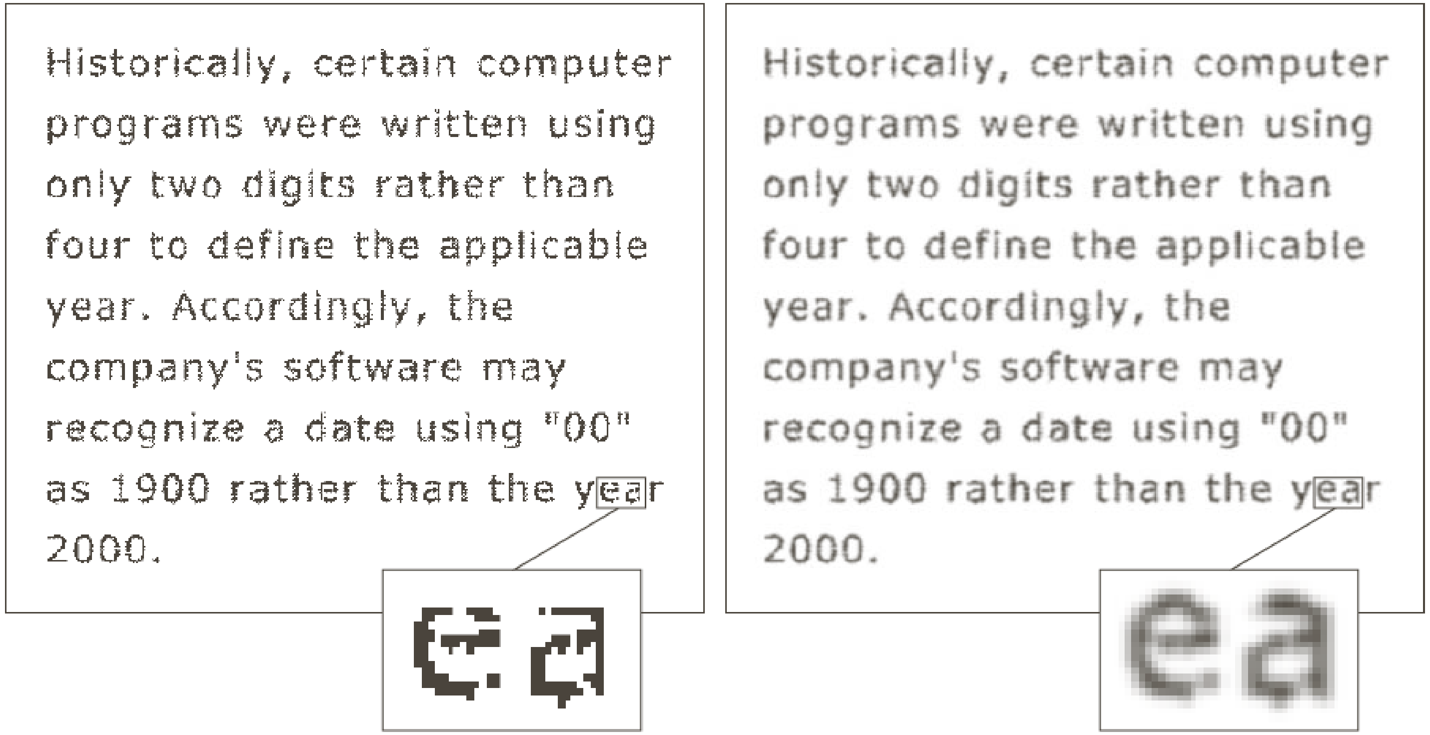


Gaussian Lowpass Filter (高斯滤波器)



Application of Lowpass Filters

- **Character Recognition**

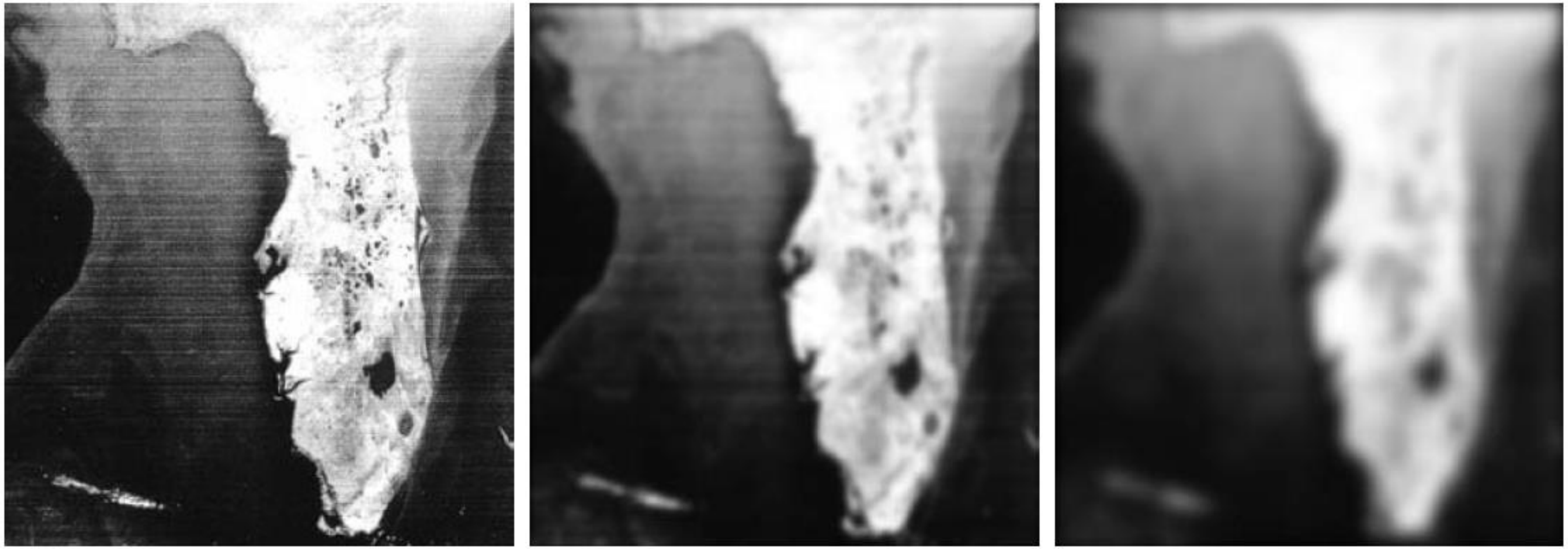


Application of Lowpass Filters

- “Cosmetic” processing

Application of Lowpass Filters

- **Satellite and Aerial Images**



Highpass Filtering

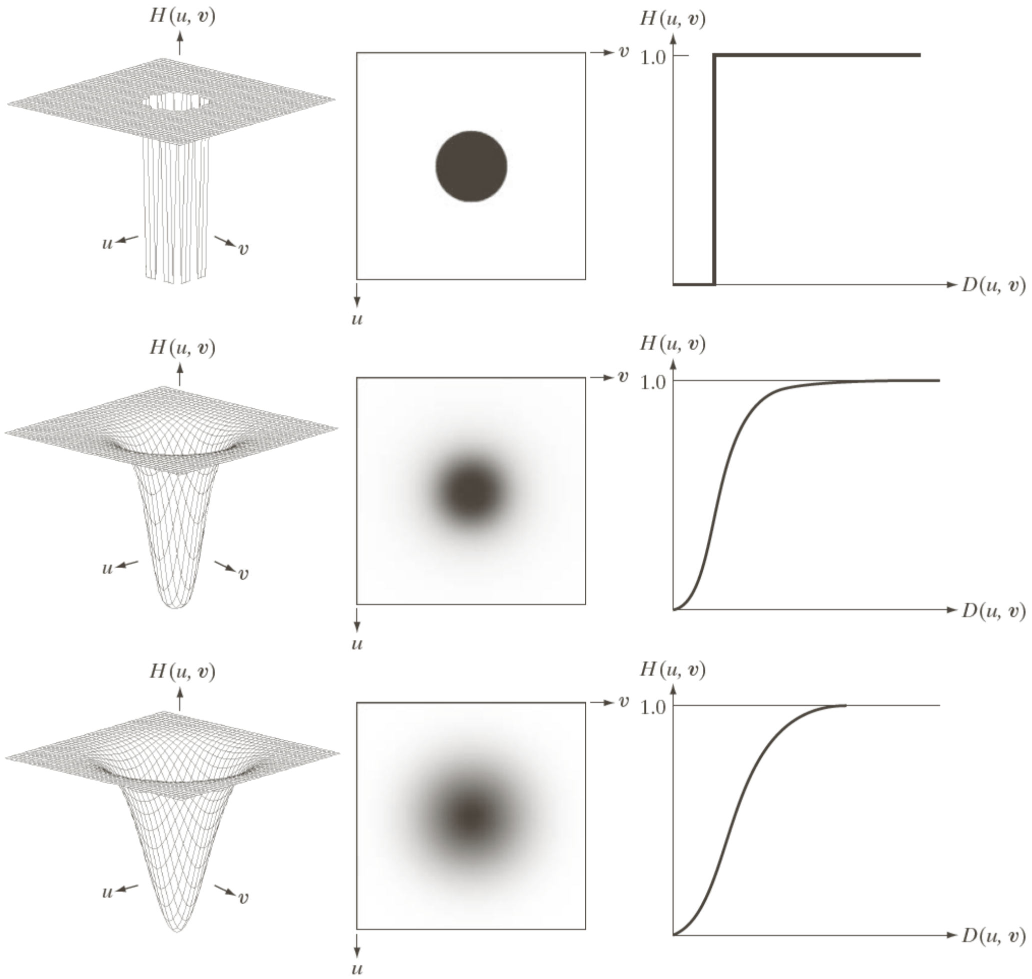
- Ideal Highpass Filter（理想高通滤波器）
- Butterworth Highpass Filter（布特沃斯高通滤波器）
- Gaussian Highpass Filter （高斯高通滤波器）

$$H_{\mathrm{HP}}(u,v) = 1 - H_{\mathrm{LP}}(u,v)$$

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

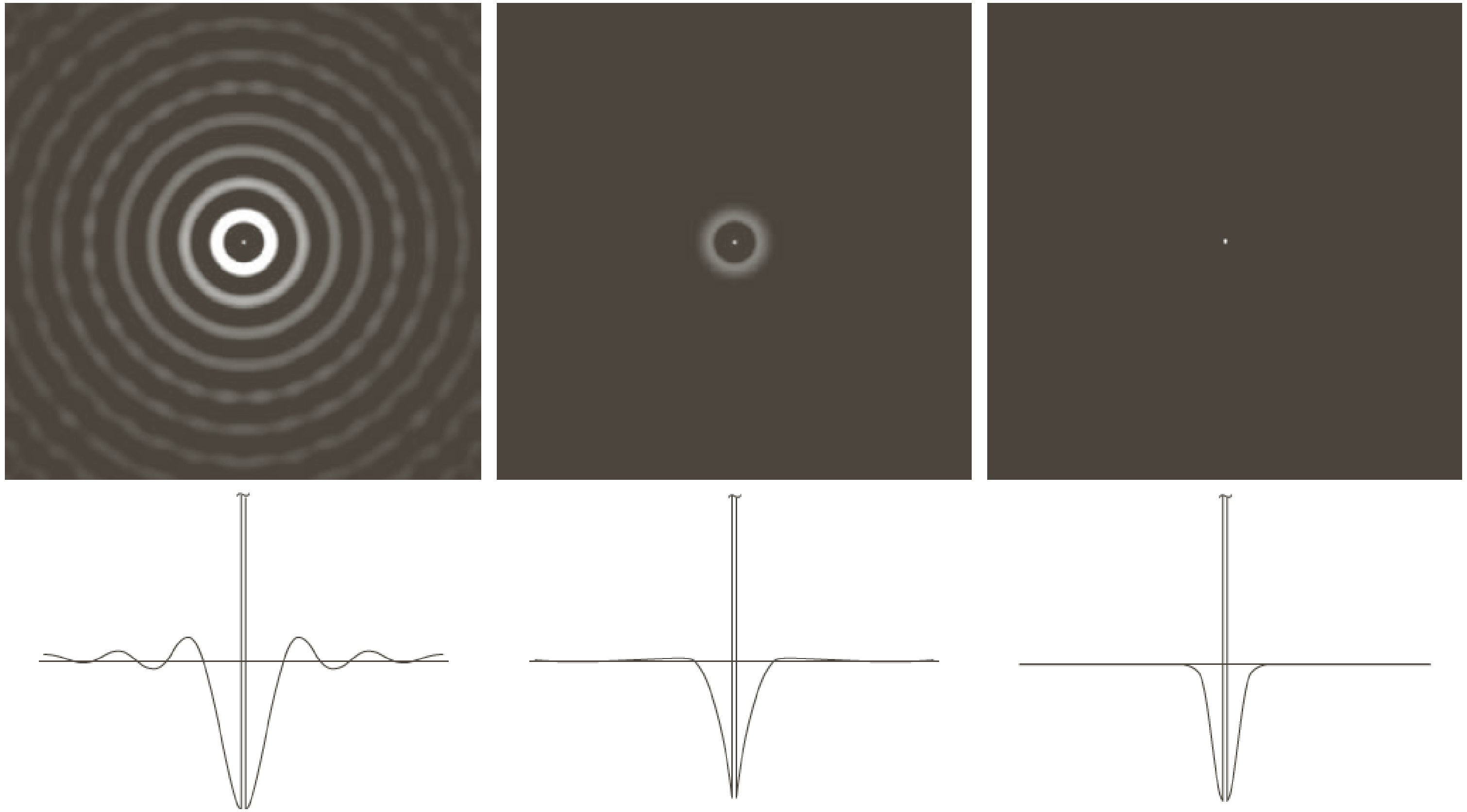
Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

Highpass Filtering



上海科技大学
ShanghaiTech University

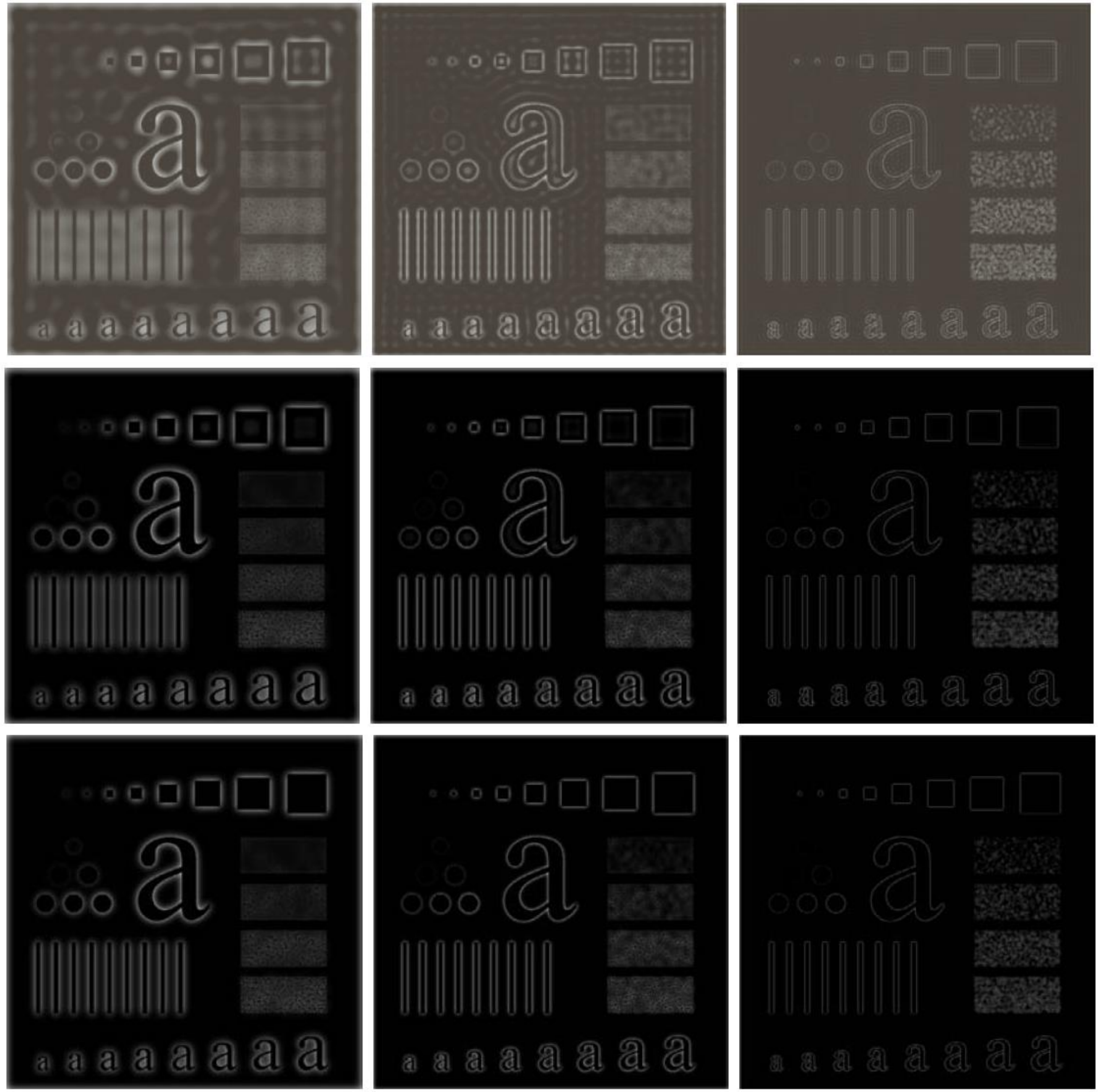
Highpass Filter in Spatial Domain



IHPF

BHPF

GHPF



Highpass Filtering and Thresholding

Highpass Filtering

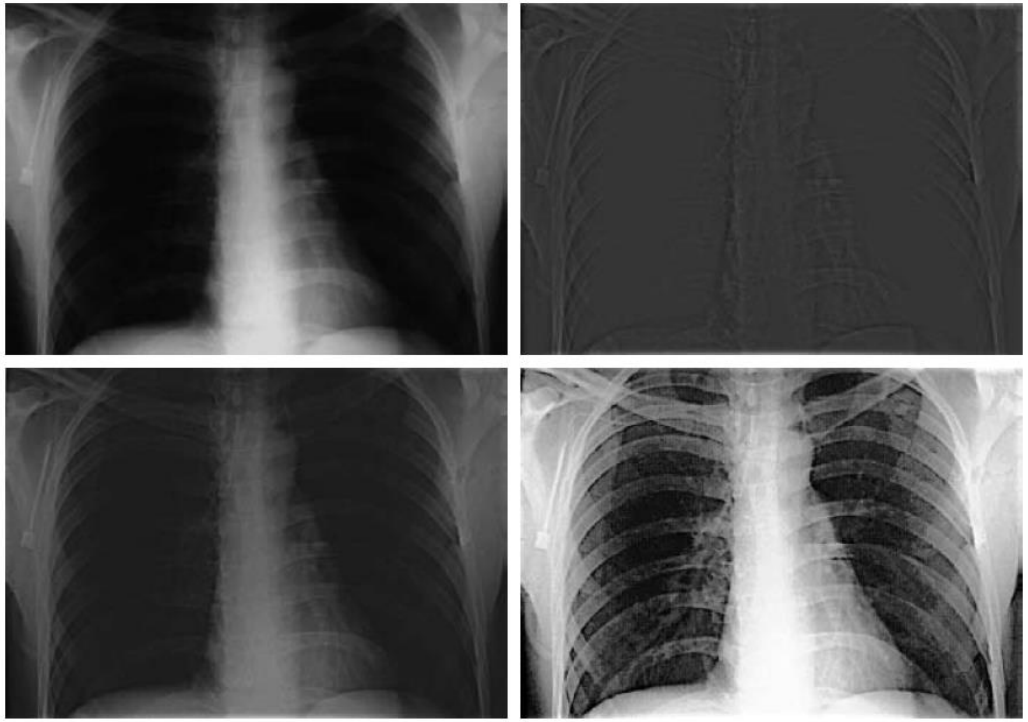
- Laplacian（拉普拉斯算子）
- Unsharp Mask（钝化模板）
- Homomorphic Filtering（同态滤波）

Laplacian（拉普拉斯算子）

$$\nabla^2 f(x,y) = \mathcal{F}^{-1}\{H(u,v)F(u,v)\} \quad \text{where } H(u,v) = -4\pi^2 D^2(u,v)$$

Unsharp Mask

$$g_{\text{mask}}(x,y) = f(x,y) - \overline{f(x,y)}$$
$$= f(x,y) - f_{LP}(x,y)$$

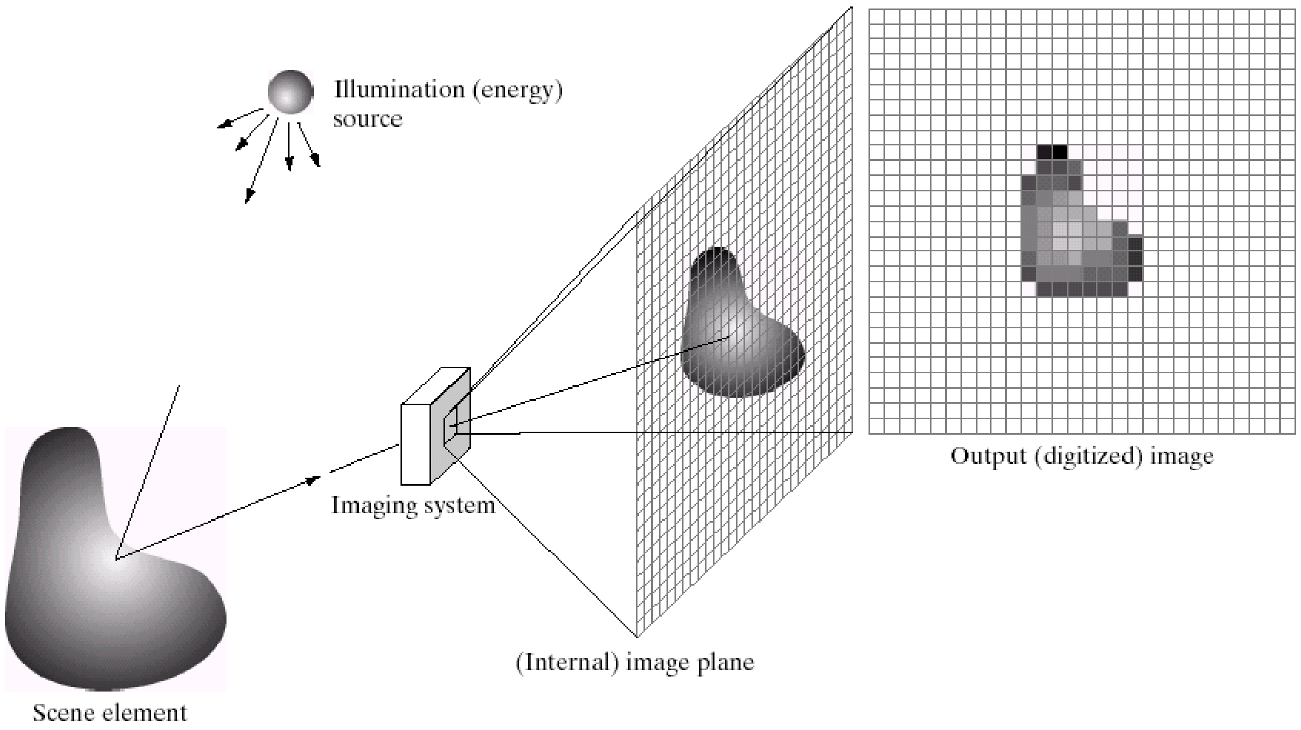


$$g(x,y) = f(x,y) + k * g_{\text{mask}}(x,y)$$
$$=\mathcal{F}^{-1}\{[1 + k * H_{HP}(u,v)]F(u,v)\}$$

-High Frequency Emphasis Filter （高频强调滤波器）

$$g(x,y)=\mathcal{F}^{-1}\{[k_1 + k_2 * H_{HP}(u,v)]F(u,v)\}$$

Image Acquisition



$$f(x,y) = i(x,y)r(x,y) \qquad 0 < i(x,y) < \infty, 0 \leq r(x,y) < 1$$

Homomorphic Filtering（同态滤波）

Let $z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y)$

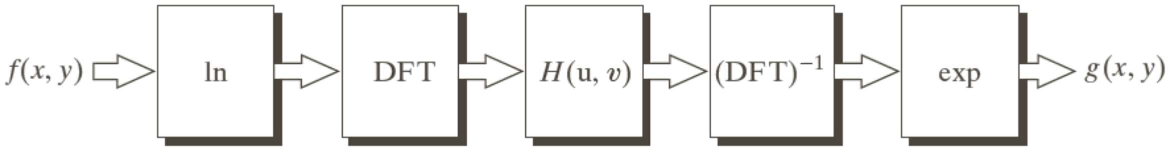
$$\boldsymbol{Z(u,v) = F_i(u,v) + F_r(u,v)}$$

Where
$Z(u,v) = \mathcal{F}[z(x,y)]$, $F_i(u,v) = \mathcal{F}[\ln i(x,y)]$, $F_r(u,v) = \mathcal{F}[\ln r(x,y)]$

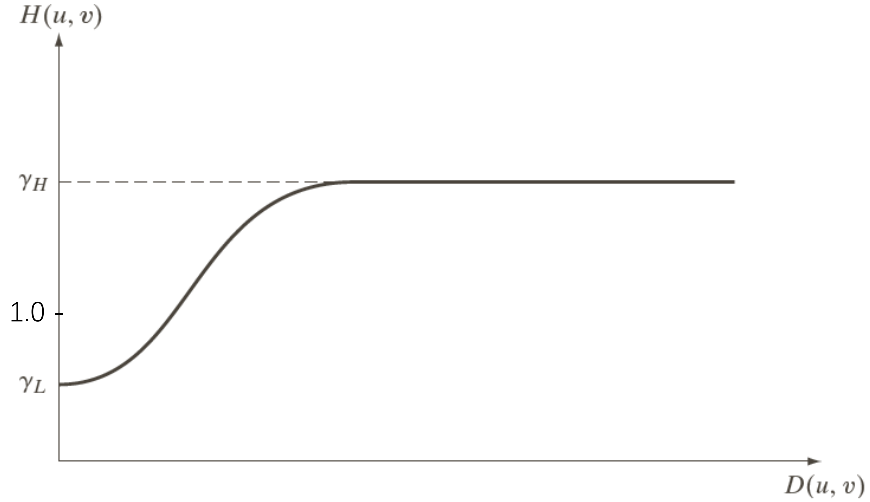
$$s(x,y) = \mathcal{F}^{-1}[H(u,v)Z(u,v)]$$
$$= \mathcal{F}^{-1}[H(u,v)F_i(u,v)] + \mathcal{F}^{-1}[H(u,v)F_r(u,v)]$$

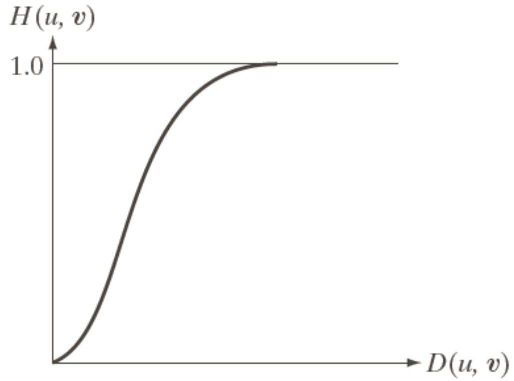
$$\boldsymbol{g(x,y) = e^{s(x,y)} = i_0(x,y)r_0(x,y)}$$

Where
$i_0(x,y) = \exp\{\mathcal{F}^{-1}[H(u,v)F_i(u,v)]\}$, $r_0(x,y) = \exp\{\mathcal{F}^{-1}[H(u,v)F_r(u,v)]\}$

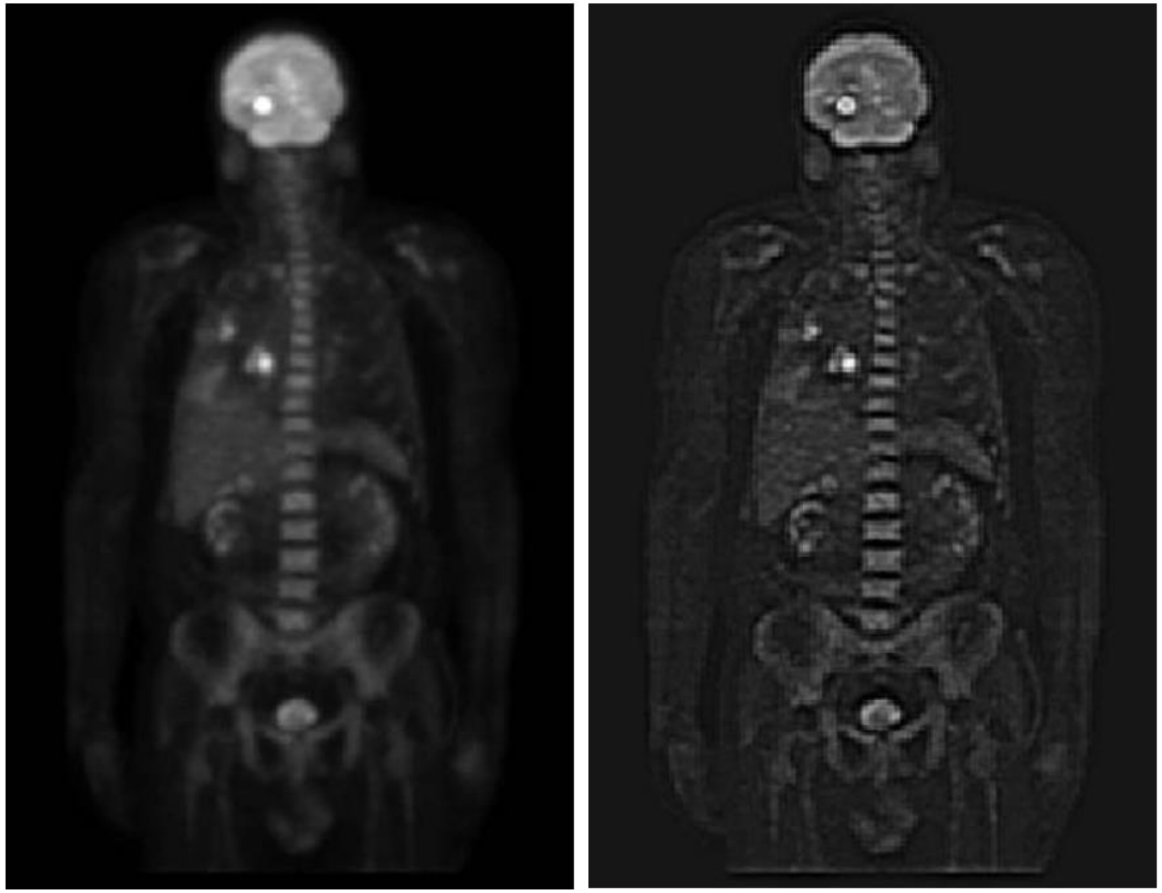


$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\left[\frac{D(u,v)}{D_0}\right]^2} + \gamma_L\right]$$





Homomorphic Filtering（同态滤波）

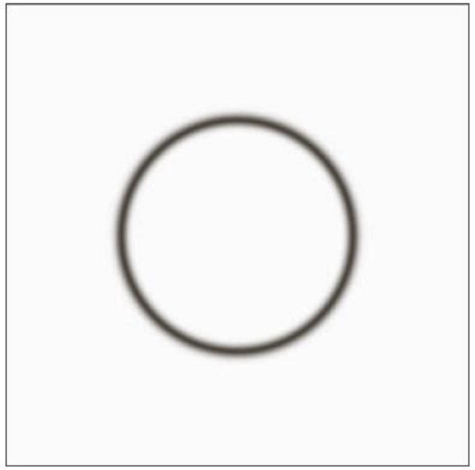


Selective Filtering

- **Bandreject(带阻) and Bandpass(带通) Filters**

$$H_{\mathrm{BP}}(u,v) = 1 - H_{\mathrm{BR}}(u,v)$$

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$



Selective Filtering

- **Notch Filter** (陷波滤波器)
  - Reject or pass frequencies in predefined neighborhood
  - Symmetric about the origin for a zero-phase shift filters
  - Selectively modify local regions of the DFT

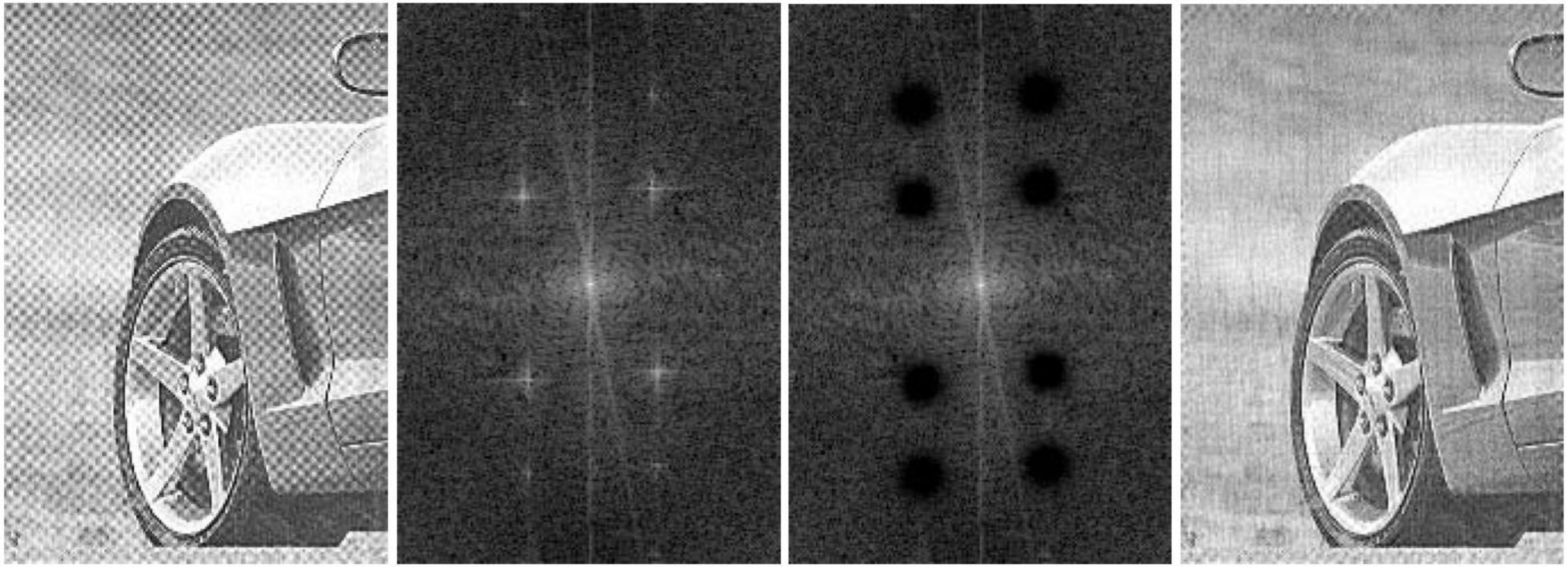
$$H_{\mathrm{NR}}(u,v)=\prod_{k=1}^{Q} H_k(u,v)H_{-k}(u,v)$$

$$H_{\mathrm{NP}}(u,v)=1-H_{\mathrm{NR}}(u,v)$$

Where $H_k(u,v)$, $H_{-k}(u,v)$ are Highpass filters with center at (u_k,v_k) and (u_{-k},v_{-k})

上海科技大学 ShanghaiTech University

Notch Filter (陷波滤波器)



上海科技大学
ShanghaiTech University

Notch Filter (陷波滤波器)