KL-Divergence, Trust-Region and Natural PG

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Recap

Policy Gradient Theorem (Infinite Setting)

The policy gradient theorem generalizes the likelihood ratio approach

$$
\begin{aligned}
\nabla_\theta J(\pi_\theta) &= \sum_{h=0}^{\infty} \gamma^h \mathbb{E}_{s,a\sim\mathbb{P}_h^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a\,|\,s) \cdot Q^{\pi_\theta}(s,a) \\
&= \frac{1}{1-\gamma} \mathbb{E}_{s,a\sim d_\mu^{\pi_\theta}} \left[\nabla_\theta \ln \pi_\theta(a\,|\,s) \cdot \boxed{Q^{\pi_\theta}(s,a)} \right]
\end{aligned}
$$

Policy Evaluation!

Policy Gradient Theorem (Infinite Setting)

The policy gradient theorem generalizes the likelihood ratio approach

$$\begin{aligned}\nabla_\theta J(\pi_\theta) &= \sum_{h=0}^{\infty} \gamma^h \mathbb{E}_{s,a \sim \mathbb{P}_h^{\pi_\theta}} \nabla_\theta \ln \pi_\theta(a \mid s) \cdot Q^{\pi_\theta}(s,a) \\ &= \frac{1}{1-\gamma} \mathbb{E}_{s,a \sim d_\mu^{\pi_\theta}} \left[\nabla_\theta \ln \pi_\theta(a \mid s) \cdot \boxed{Q^{\pi_\theta}(s,a)] } \right.\end{aligned}$$

We can use the return G as an unbiased estimate of Q (MC)

REINFORCE

Initialize policy parameters θ arbitrarily

for each episode $\{s_1, a_1, r_2, \cdots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ **do**

  for $t = 1$ to $T - 1$ **do**

    $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) G_t$

  endfor

endfor

return θ

VARIANCE!

Baseline

To reduce the variance we can introduce baselines (function of state)

$$\nabla_\theta J(\pi_\theta) = \frac{1}{1-\gamma} \boxed{\mathbb{E}_{s,a\sim d_\mu^{\pi_\theta}} \left[\nabla_\theta \ln \pi_\theta(a \mid s) \right.} \cdot \left(Q^{\pi_\theta}(s,a) - \boxed{b(s)} \right) \Big]$$

Is this term introducing a bias? NO!

Value Function as Baseline

As baselines have to be action-independent, a common choice for a baseline is

$$b(s) = V^{\pi_\theta}(s)$$

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_\theta}}\left[\nabla_\theta \ln \pi_\theta(a \mid s)\boxed{\left(Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)\right)}\right]$$

Called Advantage Function

$$\nabla_\theta J(\theta_t) = \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d^{\pi_{\theta_t}}}\left[\nabla_\theta \ln \pi_{\theta_t}(a \mid s)\boxed{\left(A^{\pi_{\theta_t}}(s,a)\right)}\right]$$

Advantage Function

Intuition: the advantage function tells us how good an action is compared to the average value of the state

Value of an action in the state

$$Q^{\pi_\theta}(s, a) - V^{\pi_\theta}(s)$$

Average value of the state

REINFORCE with Baseline

Initialize policy parameter θ, baseline b
for iteration$=1, 2, \cdots$ **do**
  Collect a set of trajectories by executing the current policy
  At each timestep t in each trajectory τ^i, compute
    Return $\boxed{G_t^i = \sum_{t'=t}^{T-1} r_{t'}^i}$, and
    Advantage estimate $\hat{A}_t^i = G_t^i - b(s_t)$.
  Re-fit the baseline, by minimizing $\sum_i \sum_t ||b(s_t) - G_t^i||^2$,
  Update the policy, using a policy gradient estimate $\hat{g}$,
    Which is a sum of terms $\nabla_\theta \log \pi(a_t|s_t, \theta)\hat{A}_t$.
    (Plug $\hat{g}$ into SGD or ADAM)
endfor

We're still using the return and collecting MC samples

Credits: Emma Brunskill

Advantage Function

$$Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)$$

If we can access the true value function, the TD error is an unbiased estimate of the advantage function

$$\begin{aligned}
\mathbb{E}_{\pi_\theta}\left[\delta^{\pi_\theta} | s, a\right] &= \mathbb{E}_{\pi_\theta}\left[r + \gamma V^{\pi_\theta}(s') | s, a\right] - V^{\pi_\theta}(s) \\
&= Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s) \\
&= A^{\pi_\theta}(s,a)
\end{aligned}$$

So we can use the TD error to compute the policy gradient

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ \delta^{\pi_\theta}\right]$$

$$\delta^{\pi_\theta} = r + \gamma V^{\pi_\theta}(s') - V^{\pi_\theta}(s)$$

Can be approximated!

Credits: David Silver

Reducing Variance with Critic

Motivation: Monte-Carlo policy gradient still has high variance!

We can estimate V/Q by using a *critic*

Such critic is also parameterized

$$Q_w(s, a) \approx Q^{\pi_\theta}(s, a)$$

MC vs TD Policy Gradient

- In MC policy gradient, the target is the return G

$$\Delta\theta = \alpha(G_t - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- In Actor-Critic the target is a TD target and relies on bootstrapping
  - Multiple timescales are possible (not only 1-step)
  - Also TD-lambda with forward/backward view

$$\Delta\theta = \alpha(r + \gamma V_v(s_{t+1}) - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

Credits: David Silver

Actor-Critic with LFA

Critic $Q_w(s,a) = \phi(s,a)^T w$ updates weights w by linear TD(0)
Actor updates weights by policy gradient

function QAC
  Initialise s, θ
  Sample $a \sim \pi_\theta$
  for each step **do**
    Sample reward $r = \mathcal{R}_s^a$; sample transition $s' \sim \mathcal{P}_{s,\cdot}^a$
    Sample action $a' \sim \pi_\theta(s', a')$
    $\delta = r + \gamma Q_w(s', a') - Q_w(s, a)$
    $\theta = \theta + \alpha \nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)$
    $w \leftarrow w + \beta \delta \phi(s, a)$
    $a \leftarrow a', s \leftarrow s'$
  end for
end function

Credits: David Silver

Policy Gradient Summary

$$\begin{aligned}
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) G_t\right] && \text{REINFORCE} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ Q^w(s,a)\right] && \text{Q Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ A^w(s,a)\right] && \text{Advantage Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ \delta\right] && \text{TD Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\ \delta e\right] && \text{TD}(\lambda)\text{ Actor-Critic}
\end{aligned}$$

Critic does policy evaluation to estimate Q, V or A using bootstrapping (*if it uses MC we do not call it a critic*)

Credits: David Silver

End Recap

Policy Iteration Recall

Procedure:

1. Start with a random guess π_0 (can be deterministic or stochastic)
2. For t=0,...,T:
    a. Do **policy evaluation** and compute $Q^{\pi t}$ for all s,a
    b. Do **policy improvement** as $\pi_{t+1}=\text{argmax}_a Q^{\pi t}(s,a)$ for all s

This algorithm only makes progress, and the performance progress of the policy is monotonic

Policy Iteration Recall

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    a. Do **policy evaluation** and compute $Q^{\pi t}$ for all s,a
    b. Do **policy improvement** as π_{t+1}=argmax$_a$ $\mathbf{A^{\pi t}(s,a)}$ for all s

$$\boxed{Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)}$$

We can also use the advantage function, it's equivalent: pick an action that has the largest advantage against π at every state s

Policy Iteration Recall

Procedure:

1. Start with a random guess π_0 (can be deterministic or stochastic)
2. For t=0,...,T:
   a. Do **policy evaluation** and compute $Q^{\pi t}$ for all s,a
   b. Do **policy improvement** as π_{t+1}=argmax$_a$ $\mathbf{A^{\pi t}(s,a)}$ for all s

$$\boxed{Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)}$$

$$\arg\max_a Q^\pi(s,a) = \arg\max_a A^\pi(s,a)$$

Performance Difference Lemma

We know that the new policy from PI is better than the old one, but what’s their performance difference?

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$$V^{\pi}(s_0) - V^{\pi'}(s_0) = ??$$

Performance Difference Lemma

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$$
\begin{aligned}
V^{\pi}(s_0) - V^{\pi'}(s_0) &= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)\right] \\
&:= \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]
\end{aligned}
$$

Performance Difference Lemma

We know that the new policy from PI is better than the old one, but what's their performance difference?

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)\right]$$

$$:= \frac{1}{1-\gamma}\boxed{\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]}$$

Average advantage value

Performance Difference Lemma

We know that the new policy from PI is better than the old one, but what's their performance difference?

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)\right]$$

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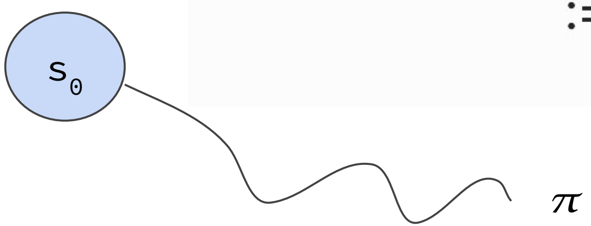
Average advantage value



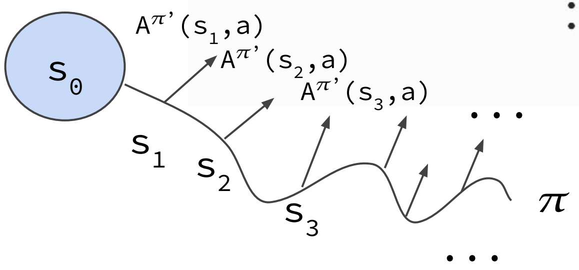
Performance Difference Lemma

We know that the new policy from PI is better than the old one, but what’s their performance difference?

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}Q^{\pi'}(s,a) - V^{\pi'}(s)\right]$$

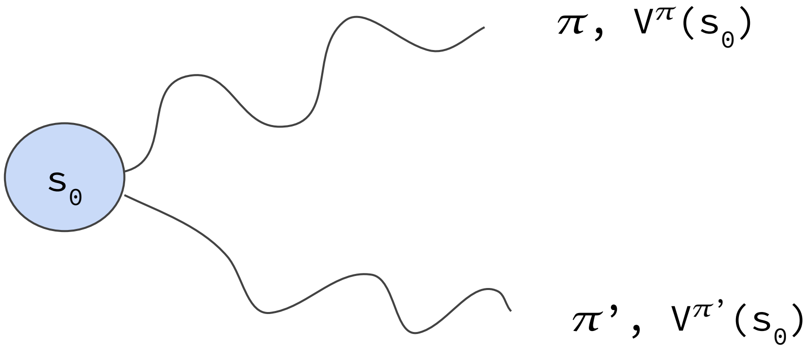
$$:= \frac{1}{1-\gamma}\mathbb{E}_{\boxed{s\sim d^{\pi}_{s_0}}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}\boxed{A^{\pi'}(s,a)}\right]$$

Average advantage value



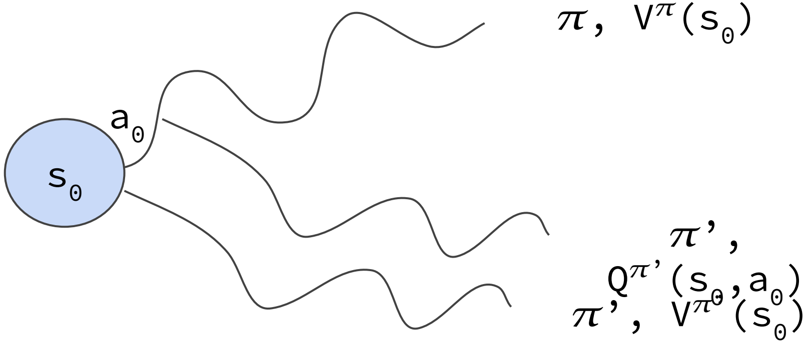
Performance Difference Lemma

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$



Performance Difference Lemma

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

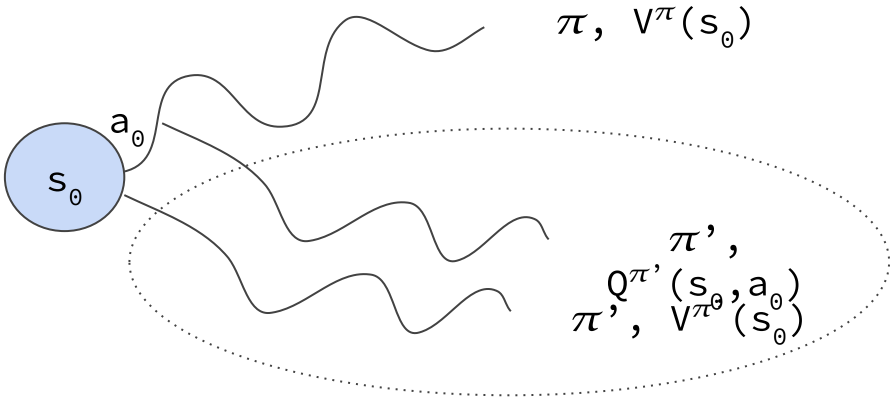


Performance Difference Lemma

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d^{\pi}_{s_0}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

This difference is exactly the definition of advantage

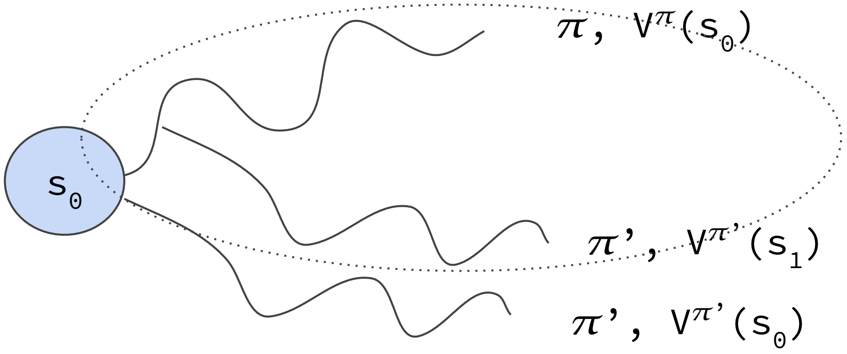
$$Q^{\pi'}(s,a) - V^{\pi'}(s)$$



Performance Difference Lemma

$$V^{\pi}(s_0) - V^{\pi'}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim d_{s_0}^{\pi}}\left[\mathbb{E}_{a\sim\pi(\cdot|s)}A^{\pi'}(s,a)\right]$$

We can do recursion and follow the same reasoning again



Performance Difference Lemma Proof Sketch

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

Performance Difference Lemma Proof Sketch

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$
$$= V^{\pi}(s_0) - \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

Performance Difference Lemma Proof Sketch

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= \boxed{V^{\pi}(s_0)} - \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[\cancel{r(s_0, a_0)} + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$\mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[\cancel{r(s_0, a_0)} + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi}(s')\right]$$

Performance Difference Lemma Proof Sketch

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= V^{\pi}(s_0) - \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[\cancel{r(s_0, a_0)} + \gamma \mathbb{E}_{s' \sim P(s_0, a_0)} \boxed{V^{\pi'}(s')}\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0, a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$\mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[\cancel{r(s_0, a_0)} + \gamma \mathbb{E}_{s' \sim P(s_0, a_0)} \boxed{V^{\pi}(s')}\right]$$

Performance Difference Lemma Proof Sketch

$$
\begin{aligned}
&V^{\pi}(s_0) - V^{\pi'}(s_0) \\
&= V^{\pi}(s_0) - \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0, a_0)} V^{\pi'}(s')\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0, a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0) \\
&= \gamma \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \mathbb{E}_{s_1 \sim P(s_0, a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0, a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)
\end{aligned}
$$

Performance Difference Lemma Proof Sketch

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= V^{\pi}(s_0) - \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$= \gamma \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \mathbb{E}_{s_1 \sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[\boxed{r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')}\right] - V^{\pi'}(s_0)$$

$$= \gamma \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \mathbb{E}_{s_1 \sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[\boxed{Q^{\pi'}(s_0, a_0)} - V^{\pi'}(s_0)\right]$$

Apply definition

Performance Difference Lemma Proof Sketch

$$V^{\pi}(s_0) - V^{\pi'}(s_0)$$

$$= V^{\pi}(s_0) - \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$= \gamma \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \mathbb{E}_{s_1 \sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)}\left[r(s_0, a_0) + \gamma \mathbb{E}_{s' \sim P(s_0,a_0)} V^{\pi'}(s')\right] - V^{\pi'}(s_0)$$

$$= \gamma \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \mathbb{E}_{s_1 \sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \boxed{\left[Q^{\pi'}(s_0, a_0) - V^{\pi'}(s_0)\right]}$$

$$= \gamma \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \mathbb{E}_{s_1 \sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0 \sim \pi(\cdot|s_0)} \boxed{\left[A^{\pi'}(s_0, a_0)\right]}$$

Apply definition

Performance Difference Lemma Proof Sketch

$$
\begin{aligned}
&V^{\pi}(s_0) - V^{\pi'}(s_0) \\
&= V^{\pi}(s_0) - \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0) \\
&= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[r(s_0,a_0) + \gamma\mathbb{E}_{s'\sim P(s_0,a_0)}V^{\pi'}(s')\right] - V^{\pi'}(s_0) \\
&= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right] + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[Q^{\pi'}(s_0,a_0) - V^{\pi'}(s_0)\right] \\
&= \gamma\mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\mathbb{E}_{s_1\sim P(s_0,a_0)}\boxed{\left[V^{\pi}(s_1) - V^{\pi'}(s_1)\right]} + \mathbb{E}_{a_0\sim\pi(\cdot|s_0)}\left[A^{\pi'}(s_0,a_0)\right]
\end{aligned}
$$

Recursion

Performance Difference Lemma

We know that the new policy from PI is better than the old one, but what's their performance difference?

$$V^{\pi_{new}}(s_0) - V^{\pi_{old}}(s_0) = \frac{1}{1-\gamma}\mathbb{E}_{s,a\sim d_{s_0}^{\pi_{new}}}\left[A^{\pi_{old}}(s,a)\right]$$

Advantage against old policy averaged over the new policy induced distribution

Approximate Policy Iteration Recall

Procedure:

1. Start with a random guess π_0 (can be deterministic or stochastic)
2. For t=0,...,T:
   a. Do **policy evaluation** and compute $\hat{A}^{\pi_t}$
   b. Do **policy improvement** as $\hat{\pi}_{t+1}=\text{argmax}_a \hat{A}^{\pi_t}(s,a)$ for all s

$$\boxed{Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)}$$

For example, estimate $\hat{A}$ directly through regression

Approximate Policy Iteration Recall

Procedure:

1. Start with a random guess π_0 (can be deterministic or stochastic)
2. For t=0,...,T:
   a. Do **policy evaluation** and compute $\mathbf{A}^{\wedge\pi t}$
   b. Do **policy improvement** as $\pi^{\wedge}_{t+1}$=argmax$_a$ $\mathbf{A}^{\wedge\pi t}\mathbf{(s,a)}$ for all s

$$\boxed{Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)}$$

$\pi^\wedge$ is an approximate greedy policy

$$\mathbb{E}_{s\sim d_\mu^{\pi^t}}\left[A^{\pi^t}(s,\hat{\pi}(s))\right] \approx \max_{\pi\in\Pi} \mathbb{E}_{s\sim d_\mu^{\pi^t}}\left[A^{\pi^t}(s,\pi(s))\right]$$

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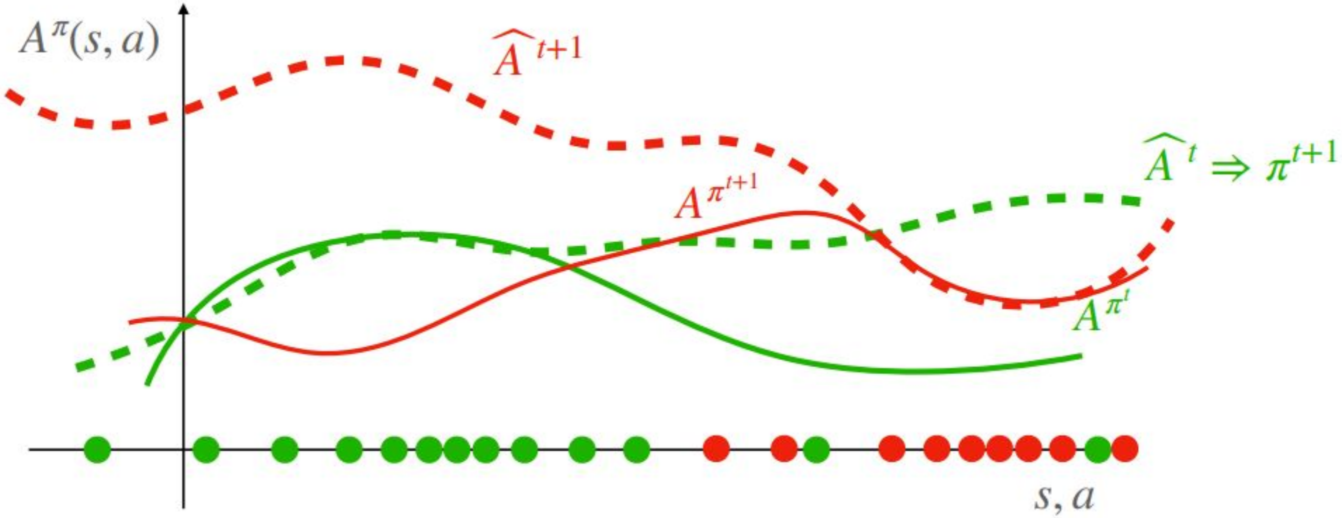
Approximate Policy Iteration Recall



No monotonic improvement

Credits: Wen Sun

Conservative Policy Iteration

Oscillations are due to the distribution change induced by the policy

Can we design an update rule that does not change the distribution so much?

$$d^{\pi^t} \approx d^{\pi^{t+1}}$$

$$\mathbb{E}_{s \sim d_\mu^{\pi^t}}\left[A^{\pi^t}(s, \pi^{t+1}(s))\right] \approx \mathbb{E}_{s \sim d_\mu^{\pi^{t+1}}}\left[A^{\pi^t}(s, \pi^{t+1}(s))\right]$$

Incremental Update of CPI

Oscillations are due to the distribution change induced by the policy

Can we design an update rule that does not change the distribution so much?

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\pi^{t}(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s$$

$$\|\pi^{t+1}(\cdot \mid s) - \pi^{t}(\cdot \mid s)\|_1 \leq 2\alpha \longrightarrow \|d_{\mu}^{\pi^{t+1}}(\cdot) - d_{\mu}^{\pi^{t}}(\cdot)\|_1 \leq \frac{2\gamma\alpha}{1-\gamma}$$

Incremental Update of CPI

If we set alpha appropriately we can get back monotonic improvement until termination

If $\max_{\pi \in \Pi} \mathbb{E}_{s \sim d_\mu^{\pi^t}}[A^{\pi^t}(s, \pi(s))] \leq \varepsilon$

Return π^t

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\pi^t(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s$$

$$\|\pi^{t+1}(\cdot \mid s) - \pi^t(\cdot \mid s)\|_1 \leq 2\alpha \longrightarrow \|d_\mu^{\pi^{t+1}}(\cdot) - d_\mu^{\pi^t}(\cdot)\|_1 \leq \frac{2\gamma\alpha}{1-\gamma}$$

Problem of CPI

I now need to retain all the old policies in memory: what if they are all large neural networks?

$$\pi^{t+1}(\cdot \mid s) = (1-\alpha)\pi^{t}(\cdot \mid s) + \alpha\pi'(\cdot \mid s), \forall s$$

Problem of CPI

I now need to retain all the old policies in memory: what if they are all large neural networks?

Let’s use KL-Divergence

KL-Divergence

Given two distributions Q and P, KL-Divergence is defined as

expected excess surprise from using Q as a model when the actual distribution is P

$$KL(P\,|\,Q) = \mathbb{E}_{x\sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

$$\boxed{KL(P\,|\,Q) \geq 0}$$

$\boxed{Q = P}$ $\quad KL(P\,|\,Q) = KL(Q\,|\,P) = 0$

$\boxed{P = \mathcal{N}(\mu_1, \sigma^2 I), Q = \mathcal{N}(\mu_2, \sigma^2 I)}$ $\quad KL(P\,|\,Q) = \|\mu_1 - \mu_2\|_2^2/\sigma^2$

Trust-Region Formulation for Policy Update

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right]$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta$$

$$\boxed{\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi_\theta(a_1 \mid s_1)\ldots}$$

KL-Divergence of State Distribution

$$KL(P \,|\, Q) = \mathbb{E}_{x \sim P}\left[\ln \frac{P(x)}{Q(x)}\right]$$

$$\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \,|\, s_0)P(s_1 \,|\, s_0, a_0)\pi_\theta(a_1 \,|\, s_1)\ldots$$

$$KL\left(\rho_{\pi_{\theta_t}} \,|\, \rho_{\pi_\theta}\right) = \mathbb{E}_{\tau \sim \rho_{\pi_{\theta_t}}} \ln \frac{\rho_{\pi_{\theta_t}}(\tau)}{\rho_{\pi_\theta}(\tau)}$$

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Initial state distribution, as well as next state distribution simplify, because they are the same. We are only left with the different policies.

KL-Divergence of State Distribution

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KL-Divergence of State Distribution

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$$= \frac{1}{1-\gamma}\mathbb{E}_{s_h,a_h\sim d_\mu^{\pi_{\theta_t}}}\left[\ln \frac{\pi_{\theta_t}(a_h\,|\,s_h)}{\pi_\theta(a_h\,|\,s_h)}\right] := \ell(\theta)$$

Trust-Region Formulation for Policy Update

$$\max_{\pi_\theta} \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \left[\mathbb{E}_{a \sim \pi_\theta(s)} A^{\pi_{\theta_t}}(s, a) \right]$$

$$\text{s.t., } \boxed{KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \leq \delta}$$

This is our trust-region, that maintains the distributions not so far

$$\boxed{\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \mid s_0)P(s_1 \mid s_0, a_0)\pi_\theta(a_1 \mid s_1)\ldots}$$

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How do we optimize this?

Trust-Region Formulation for Policy Update

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How do we optimize this?

Remember: the trajectory distribution is actually unknown and we do not know the transition function!

Trust-Region Formulation for Policy Update

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How do we optimize this?

1st or 2nd order Taylor expansion

Trust-Region Optimization: Objective Function

Let's first simplify and linearize the objective function using a 1st order Taylor Expansion

$$\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)}A^{\pi_{\theta_t}}(s,a)\right] \approx \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_{\theta_t}(s)}A^{\pi_{\theta_t}}(s,a)\right] + \underbrace{\mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_{\theta_t}(s)}\nabla_\theta \ln \pi_{\theta_t}(a\,|\,s)A^{\pi_{\theta_t}}(s,a)\right]}_{\nabla_\theta J(\pi_{\theta_t})} \cdot (\theta - \theta_t)$$

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Advantage of the policy against itself is 0

Inner product

Trust-Region Optimization: Objective Function

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$$= \boxed{\nabla_\theta J(\pi_{\theta_t})^\top(\theta-\theta_t)}$$

Trust-Region Optimization: Constraint

Let’s then simplify and linearize the constraint using a 2nd order Taylor Expansion

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta)$$

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$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right]$$

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ln(a/b) = ln a - ln b

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Does not depend on the variation of theta

$$\ell(\theta) \approx \boxed{\ell(\theta_t)} + \boxed{\nabla \ell(\theta_t)}^\top(\theta - \theta_t) + \frac{1}{2}(\theta - \theta_t)^\top \nabla_\theta^2 \ell(\theta_t)(\theta - \theta_t)$$

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$\boxed{\ln(a/b) = \ln a - \ln b}$

Expectation has nothing to do with gradient, so we bring gradient inside

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Bring sum inside:
this sums to 1

$$\nabla_\theta \ell(\theta)|_{\theta=\theta_t} = \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a|s)\left(-\nabla_\theta \ln \pi_\theta(a_h | s_h)|_{\theta=\theta_t}\right) = -\mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \cancel{\pi_{\theta_t}(a|s)} \frac{\boxed{\nabla_\theta \pi_{\theta_t}(a|s)}}{\cancel{\pi_{\theta_t}(a|s)}} = 0$$

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Expectation has nothing to do with gradient, so we bring gradient inside

$\boxed{\text{ln(a/b) = ln a - ln b}}$

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$$\nabla_\theta \ln \pi_\theta(a_h | s_h) |_{\theta=\theta_t} = \frac{\nabla_\theta \pi_{\theta_t}(a | s)}{\pi_{\theta_t}(a | s)} \qquad \nabla_\theta^2 \ln \pi_\theta(a | s) |_{\theta=\theta_t} \quad ?$$

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$$\nabla_\theta^2 \ln \pi_\theta(a | s)|_{\theta=\theta_t} \quad ?$$

We just have to compute the gradient of this now

Trust-Region Optimization: Constraint

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right]$$

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$$\nabla_\theta^2 \ell(\theta) |_{\theta=\theta_t} = \mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a | s) \left(- \nabla_\theta^2 \ln \pi_\theta(a | s) |_{\theta=\theta_t} \right)$$

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(f/g)’ = f’/g – fg’/g^2

Trust-Region Optimization: Constraint

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right]$$

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$$= -\mathbb{E}_{s \sim d_\mu^{\pi_{\theta_t}}} \sum_a \pi_{\theta_t}(a | s) \left(\frac{\nabla_\theta^2 \pi_{\theta_t}(a | s)}{\pi_{\theta_t}(a | s)} - \frac{\nabla_\theta \pi_{\theta_t}(a | s) \nabla_\theta \pi_{\theta_t}(a | s)^\top}{\pi_{\theta_t}^2(a | s)} \right)$$

(f/g)’ = f’/g - fg’/g^2

Trust-Region Optimization: Constraint

$$KL(\rho_{\theta_t} \mid \rho_\theta) := \ell(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h \mid s_h)}{\pi_\theta(a_h \mid s_h)} \right]$$

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Bring sum inside:
this sums to 1

(f/g)' = f'/g - fg'/g^2

Trust-Region Optimization: Constraint

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta) = \frac{1}{1-\gamma} \mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}} \left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)} \right]$$

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$$= \mathbb{E}_{s, a \sim d_\mu^{\pi_{\theta_t}}} \left[\boxed{\nabla_\theta \ln \pi_{\theta_t}(a | s) \left(\nabla_\theta \ln \pi_{\theta_t}(a | s) \right)^\top} \right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

This is the **Fisher Information Matrix**

Trust-Region Optimization: Constraint

$$KL(\rho_{\theta_t} | \rho_\theta) := \ell(\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s_h, a_h \sim d_\mu^{\pi_{\theta_t}}}\left[\ln \frac{\pi_{\theta_t}(a_h | s_h)}{\pi_\theta(a_h | s_h)}\right]$$

$$\ell(\theta) \approx \boxed{\ell(\theta_t)} + \boxed{\nabla \ell(\theta_t)}^\top(\theta - \theta_t) + \frac{1}{2}(\theta - \theta_t)^\top \boxed{\nabla_\theta^2 \ell(\theta_t)}(\theta - \theta_t)$$

Easy to compute, as we know how to compute the gradient of the log likelihood of the policy

$$= \mathbb{E}_{s,a \sim d_\mu^{\pi_{\theta_t}}}\left[\boxed{\nabla_\theta \ln \pi_{\theta_t}(a | s)\left(\nabla_\theta \ln \pi_{\theta_t}(a | s)\right)^\top}\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

This is the **Fisher Information Matrix**

Trust-Region Optimization: Simplified Constraint

$$KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \approx \frac{1}{2}(\theta - \theta_t)^\top F_{\theta_t}(\theta - \theta_t)$$

$$F_{\theta_t} := \mathbb{E}_{s,a\sim d_\mu^{\pi\theta_t}}\left[\nabla_\theta \ln \pi_{\theta_t}(a\,|\,s)\left(\nabla_\theta \ln \pi_{\theta_t}(a\,|\,s)\right)^\top\right] \in \mathbb{R}^{dim_\theta \times dim_\theta}$$

`F is always positive semi-definite`

Simplified Trust-Region Formulation

$$\max_{\pi_\theta} \mathbb{E}_{s\sim d_\mu^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_\theta(s)} A^{\pi_{\theta_t}}(s,a)\right]$$

$$\text{s.t., } KL\left(\rho_{\pi_{\theta_t}} | \rho_{\pi_\theta}\right) \le \delta$$

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \le \delta$$

Simplified Trust-Region Formulation

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$

$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

This looks very easy and we can compute the solution in closed form!

Simplified Trust-Region Solution

$$\max_{\theta} \boxed{\nabla_\theta J(\pi_{\theta_t})}^\top \boxed{(\theta - \theta_t)}$$

$$\text{s.t. } \boxed{(\theta - \theta_t)}^\top F_{\theta_t} \boxed{(\theta - \theta_t)} \leq \delta$$

Let's first simplify the notation:

$\theta - \theta_t = \Delta$

$\nabla_\theta J(\pi_{\theta t}) = \nabla$

Simplified Trust-Region Solution

$$\max_{\Delta} \nabla^\top \Delta,$$
$$\text{s.t. } \Delta^\top F \Delta \leq \delta$$

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Simplified Trust-Region Solution

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$$\text{s.t. } \Delta^\top F \Delta \leq \delta$$

Let's then introduce $F^{1/2}$

For a positive definite matrix this can be obtained from the Eigen Decomposition: $F = U\Sigma U^\top$, $F^{1/2} = U\sqrt{\Sigma}U^\top$

Simplified Trust-Region Solution

$$\max_{\Delta} \nabla^{\top} \Delta, \quad \text{s.t. } \Delta^{\top} F \Delta \leq \delta$$

$(F^{1/2})^2 = F$

$F^{1/2}F^{-1/2} = I$

$\longrightarrow$

$\max_{\Delta} \nabla^{T} F^{1/2} F^{-1/2} \Delta$

$\text{s.t.} (F^{1/2}\Delta)^{T}(F^{1/2}\Delta) \leq \delta$

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Simplified Trust-Region Solution

$$\max_{\widetilde{\Delta}} \left(F^{-1/2}\nabla\right)^\top \widetilde{\Delta},$$

$$\text{s.t. } \widetilde{\Delta}^\top \widetilde{\Delta} \leq \delta$$

$$\widetilde{\Delta} := F^{1/2}\Delta$$

$(F^{1/2})^2 = F$

$F^{1/2}F^{-1/2} = I$

$\longrightarrow$

$\max_\Delta \nabla^\top F^{1/2} F^{-1/2} \Delta$

$\text{s.t.} (F^{1/2}\Delta)^\top (F^{1/2}\Delta) \leq \delta$

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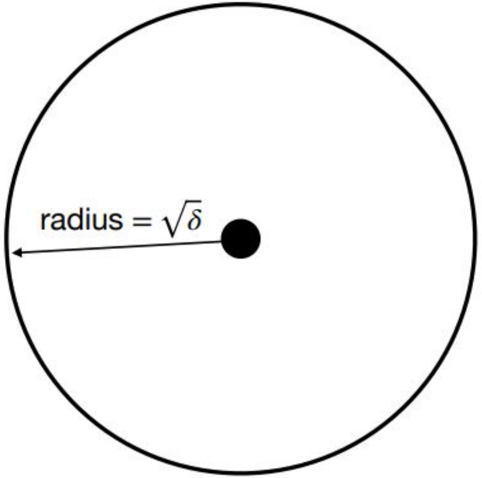
Simplified Trust-Region Solution

$$\max_{\tilde{\Delta}} \left(F^{-1/2}\nabla\right)^{\top}\tilde{\Delta},$$

$$\text{s.t. } \boxed{\tilde{\Delta}^{\top}\tilde{\Delta} \leq \delta}$$

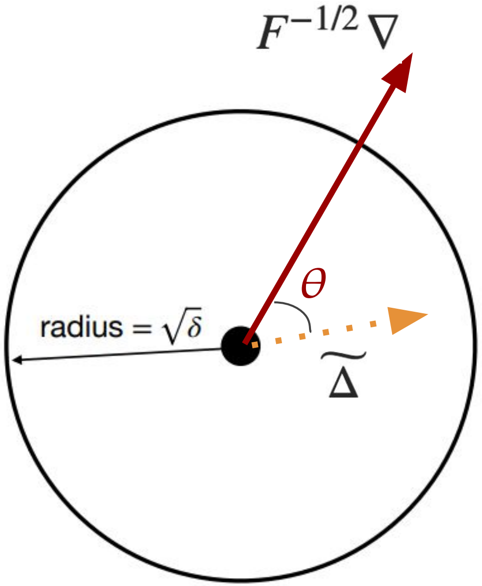
$$\tilde{\Delta} := F^{1/2}\Delta$$



This is my (ball) constraint: the norm of $\tilde{\Delta}$ has to be ≤ δ (so, any vector $\tilde{\Delta}$ falls in this ball)

Simplified Trust-Region Solution

This is the
direction of the
gradient



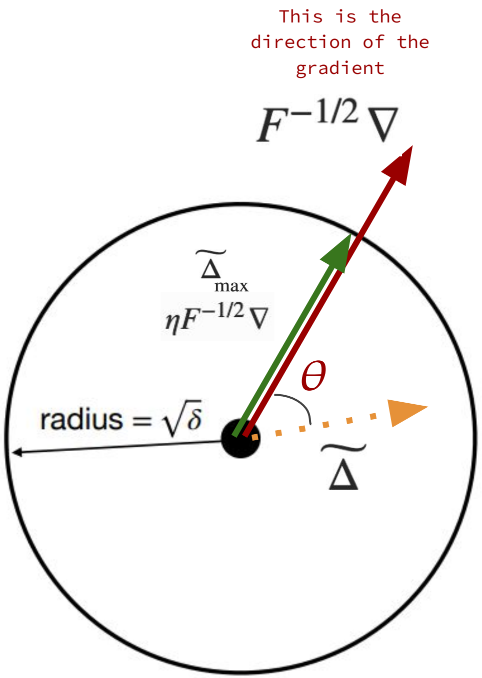
$$\max_{\widetilde{\Delta}} \left(F^{-1/2}\nabla\right)^{\top}\widetilde{\Delta},$$

$$\text{s.t. } \widetilde{\Delta}^{\top}\widetilde{\Delta} \leq \delta$$

$$\widetilde{\Delta} := F^{1/2}\Delta$$

What I do care about now is the inner product between the vector $F^{-1/2}\Delta$ and any vector $\Delta^{\sim}$ in this ball

Simplified Trust-Region Solution



$$\max_{\widetilde{\Delta}} \left(F^{-1/2}\nabla\right)^{\top}\widetilde{\Delta},$$

$$\text{s.t. } \widetilde{\Delta}^{\top}\widetilde{\Delta} \leq \delta$$

$$\widetilde{\Delta} := F^{1/2}\Delta$$

Which vector does maximize this inner product? The green one: minimum angle (same direction $F^{-1/2}\Delta$), maximum length (scaled by η)

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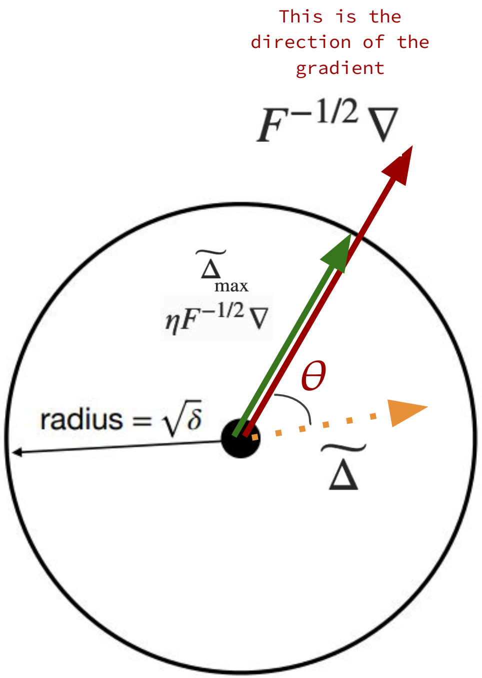
Simplified Trust-Region Solution



$$\max_{\widetilde{\Delta}} \left(F^{-1/2}\nabla\right)^{\top}\widetilde{\Delta},$$

$$\text{s.t. } \widetilde{\Delta}^{\top}\widetilde{\Delta} \leq \delta$$

$$\widetilde{\Delta} := F^{1/2}\Delta$$

$$\|\eta F^{-1/2}\nabla\|_2 = \sqrt{\delta} \quad \Rightarrow \eta = \sqrt{\frac{\delta}{\nabla^{\top}F^{-1}\nabla}}$$

$$\widetilde{\Delta}_{max} := \sqrt{\frac{\delta}{\nabla^{\top}F^{-1}\nabla}}F^{-1/2}\nabla$$

$$\boxed{\Delta_{max} := \sqrt{\frac{\delta}{\nabla^{\top}F^{-1}\nabla}}F^{-1}\nabla}$$

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Natural Policy Gradient

$$\boxed{\begin{aligned} &\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top \left(\theta - \theta_t\right) \\ &\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta \end{aligned}}$$

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t}) \qquad \eta = \sqrt{\frac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$$

The same solution can be obtained by applying Lagrange multipliers

$$\min_{\lambda \leq 0} \max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t) + \lambda \left((\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) - \delta \right)$$

Natural Policy Gradient

$$\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)$$
$$\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta$$

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t}) \qquad \eta = \sqrt{\frac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$$

F is generally invertible, but in case it is not you can use pseudo-inverse or add regularization (F = F + λI with λ very small)

Natural Policy Gradient

$$\boxed{\begin{aligned}&\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t) \\ &\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \leq \delta\end{aligned}}$$

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t}) \qquad \eta = \sqrt{\frac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$$

Step size (η) depends on the allowed trust region (δ is a hyper-parameter that we typically set to a small number like 1e-2 or 1e-3)

Natural Policy Gradient

$$\boxed{\begin{aligned}&\max_{\theta} \nabla_\theta J(\pi_{\theta_t})^\top (\theta - \theta_t)\\ &\text{s.t. } (\theta - \theta_t)^\top F_{\theta_t} (\theta - \theta_t) \le \delta\end{aligned}}$$

$$\theta_{t+1} = \theta_t + \eta F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t}) \qquad \eta = \sqrt{\frac{\delta}{\nabla_\theta J(\pi_{\theta_t})^\top F_{\theta_t}^{-1} \nabla_\theta J(\pi_{\theta_t})}}$$

F is pre-conditioning our gradient, instead of just fully going for it

TRPO: Line Search

Due to the quadratic approximation, the KL constraint might be violated: we solve this by doing a simple line search

for $j = 0, 1, 2, ..., L$ **do**
  Compute proposed update $\theta = \theta_k + \alpha^j \Delta_k$
  if $\mathcal{L}_{\theta_k}(\theta) \geq 0$ and $\bar{D}_{KL}(\theta||\theta_k) \leq \delta$ **then**
    accept the update and set $\theta_{k+1} = \theta_k + \alpha^j \Delta_k$
    break
  end if
end for

Natural Policy Gradient: Additional Comments

We want to keep two distributions close, but parameters can change a lot: learning rate (η) is very high if eigen-values of F are very small (as the matrix is inverted)

Generally, Natural PG moves faster than standard/plain PG

If we have many parameters, computing & inverting F is too heavy!

Extending TRPO: Proximal Policy Optimization

If we have many params, we can impose KL divergence as a regularization term and optimize (simply through SG Ascent)

$$\max_{\theta} \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_{\theta}(\cdot|s)} A^{\pi_{\theta_t}}(s,a)\right] \underbrace{-\lambda \mathbb{E}_{s\sim d_{\mu}^{\pi^t}}\left[\mathrm{KL}\left(\pi_{\theta_t}(a|s) \,|\, \pi_{\theta}(a|s)\right)\right]}_{\text{regularization}}$$

using importance weighting and expanding KL divergence through expectation

$$\ell(\theta) := \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}}\left[\mathbb{E}_{a\sim\pi_{\theta_t}(\cdot|s)} \frac{\pi_{\theta}(a|s)}{\pi_{\theta_t}(a|s)} A^{\pi_{\theta_t}}(s,a)\right] - \lambda \mathbb{E}_{s\sim d_{\mu}^{\pi_{\theta_t}}} \mathbb{E}_{a\sim\pi_{\theta_t}(\cdot|s)}\left[-\ln \pi_{\theta}(a|s)\right]$$