Approximate Policy Iteration

Reinforcement Learning

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Recap

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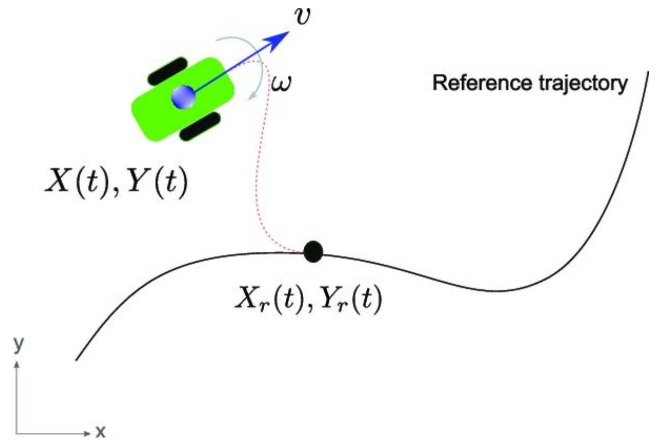
Finite-Horizon MDPs



Slightly different formulation:

$$(S, A, R, T, \mathbf{H}, \boldsymbol{\mu}_0)$$

(time-horizon) **$H \geq 0$ and $s_0 \sim \mu_0$** (initial state distribution)

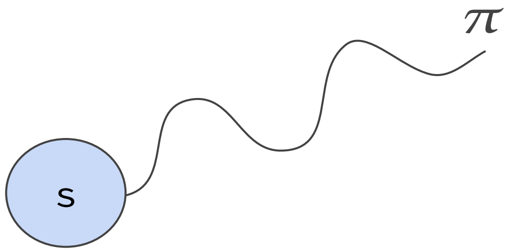
We consider time-dependent policies π

$$\pi = \{\pi_0, \pi_1, \pi_2 ... \pi_{H-1}\}$$

Actions might be different for the same state depending on t

Very common in control!

Finite-Horizon MDP: V & Q

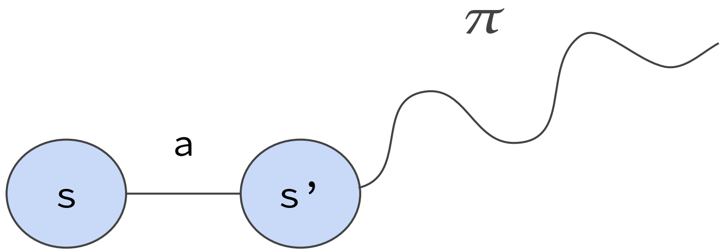


$$V_h^{\pi}(s) = \mathbb{E}_{\pi}[\sum_{t=h}^{H-1} r(s_t, a_t)]$$

where $s_h = s$, $a_t = \pi_t(s_t)$ and $s_{t+1} \sim P(.|s_t, a_t)$

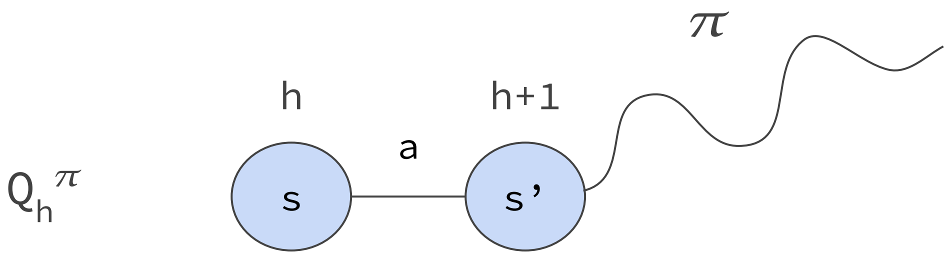
$$Q_h^{\pi}(s,a) = \mathbb{E}_{\pi}[\sum_{t=h}^{H-1} r(s_t, a_t)]$$

where $s_h = s$, $a_h = a$, $a_t = \pi_t(s_t)$ and $s_{t+1} \sim P(.|s_t, a_t)$



Finite-Horizon MDP: Bellman Equation

$$Q_h^{\pi}(s,a) = r(s,a) + \mathbb{E}_{s' \sim p(.|s,a)}[V_{h+1}^{\pi}(s')]$$



Finding the Optimal Policy

$$\pi^* = \{\pi_0^*, \pi_1^*, \pi_2^* \ldots \pi_{H-1}^*\}$$

Let’s reason backwards in time and apply dynamic programming:

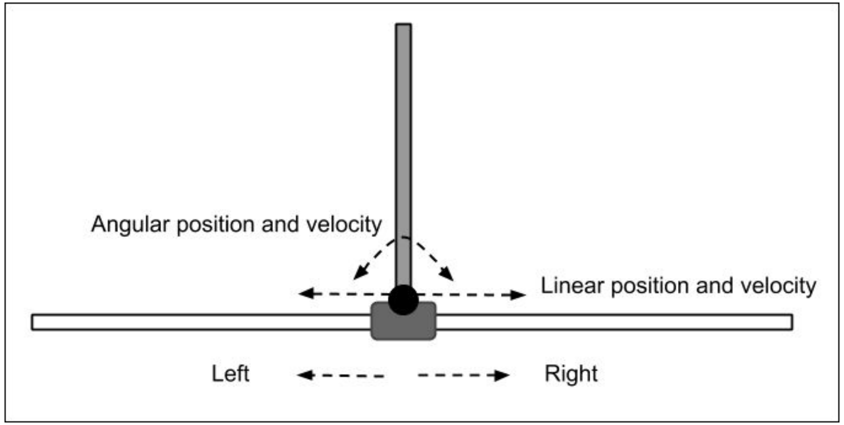
$$Q_{H-1}^*(s,a) = r(s,a)$$

$$\pi_{H-1}^*(s)=\text{argmax}_a Q_{H-1}^*(s,a)$$

$$V_{H-1}^*(s)=\max_a Q_{H-1}^*(s,a)=Q_{H-1}^*(s,\pi_{H-1}^*(s))$$

Control Problems

So far, we assumed discrete state and action spaces, but what about cartpole:



- **state:** angular pos & vel, linear pos & vel
- **action/control:** force applied on the cart
- **goal:** find the control policy which minimizes the long term cost c

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Optimal Control

Given a dynamical system with a non-linear transition function f, state x in $\mathbb{R}^d$ and control u in $\mathbb{R}^k$, we want to find a control policy π such that

$$\text{minimize } \mathbb{E}_{\pi}[c_H(x_H) + \sum_{h=0}^{H-1} c_h(x_h,u_h)]$$

$$\text{where } u_h=\pi(x_h) \text{ and } x_0 \sim \mu_0$$

Now this seems very familiar! Can we treat it as a Finite-Horizon MDP

Bellman's Curse of Dimensionality

- n-dimensional (discrete) state space
- The number of states grows exponentially in n

In practice discretization is useful, but it is only computationally feasible up to 5 or 6 dimensional state spaces

Let's try to work directly in continuous space, starting from simplified problems

Linear Systems

Consider a system of this kind:

$$\boxed{x_{t+1}=Ax_t+Bu_t}$$

This is our transition function!

- x_t state at time t
- u_t control (i.e., action) at time t

A in $\mathbb{R}^{dxd}$, B in $\mathbb{R}^{dxk}$

Quadratic Cost Function

Consider a cost function of this kind

$$c(x_t,u_t)=x_t^TQx_t+u_t^TRu_t$$

alternative notation
$g(x_t,u_t)$

- Q in $\mathbb{R}^{dxd}$ and R in $\mathbb{R}^{kxk}$ square matrices
- Q and R positive definite

As a result, there is a non-zero cost for any non-zero state with all-zero control

LQR algorithm

- Initialize P_H (at 0 or Q)
- Starting from h = H-1, backwards
  - Set $K_h^* = -(R+B^TP_{h+1}B)^{-1}B^TP_{h+1}A$
  - Compute $u = \pi_h^*(x_t) = K_h^*x_t$
  - Set $P_h = (Q + K_h^{*T}RK_h^* + (A + BK_h^*)^TP_{h+1}(A + BK_h^*))$ Riccati Equation
  - Set $J_h^* = x_t^TP_hx_t$

This is the Value Iteration update done in closed form: it is always the same and solves this particular continuous-state system with a quadratic cost

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LQR Extensions

Extensions to the LQR make it more generally applicable to:

- Affine systems
- Systems with stochasticity
- Regulation around non-zero fixed point for non-linear systems
- Trajectory following for non-linear systems
- ...

End Recap

Policy Iteration

- Outputs policies at every iteration: $\{\pi_0, \pi_1, \pi_2 ... \pi_T\}$
- Different from Value Iteration that was outputting values

Procedure:

1. Start with a random guess π_0 (can be deterministic or stochastic)
2. For t=0,...,T: $Q^{\pi}(s_t,a) = r_t+\gamma\mathbb{E}_{s'\sim p(.|s,a)}[V^{\pi}(s')]$
   a. Do **policy evaluation** and compute $Q^{\pi t}$ for all s,a
   b. Do **policy improvement** as $\pi_{t+1}=\text{argmax}_a Q^{\pi t}(s,a)$ for all s

This algorithm only makes progress, and the performance progress of the policy is monotonic

State Visitation Probability

What's the probability of visiting state s, a at time t according to π starting at s_0?

$$d^{\pi}_{s0}(s,a) = (1-\gamma)\sum^{\infty}_{h=0}\gamma^h \mathbb{P}^{\pi}_h(s,a;s_0)$$

$$\mathbb{P}^{\pi}_t(s,a;s_0)=\sum_{a0,s1,a1,\ldots st-1,at-1}\mathbb{P}^{\pi}(s_0,a_0,\ldots s_t=s,a_t=a)$$

$$\mathbb{P}^{\pi}(s_0,a_0,\ldots s_t,a_t)=\pi(a_0|s_0)p(s_1|s_0,a_0)\pi(a_1|s_1)p(s_2|s_1,a_1)\ldots p(s_t|s_{t-1},a_{t-1})\pi(a_t|s_t)$$

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State Visitation Probability

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$$d^{\pi}_{s0}(s,a) = (1-\gamma)\sum^{\infty}_{h=0}\gamma^{h}\mathbb{P}^{\pi}_{h}(s,a;s_0)$$

Note that d^{π}_{s0} is an infinite mixture

$$\mathbb{P}^{\pi}_{t}(s,a;s_0)=\sum_{a0,s1,a1,...st-1,at-1}\mathbb{P}^{\pi}(s_0,a_0,...s_t=s,a_t=a)$$

$$\mathbb{P}^{\pi}(s_0,a_0,...s_t,a_t)=\pi(a_0|s_0)p(s_1|s_0,a_0)\pi(a_1|s_1)p(s_2|s_1,a_1)...p(s_t|s_{t-1},a_{t-1})\pi(a_t|s_t)$$

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Approximate Policy Iteration

What if the state-space is large and we cannot do exact or iterative policy evaluation for all states?

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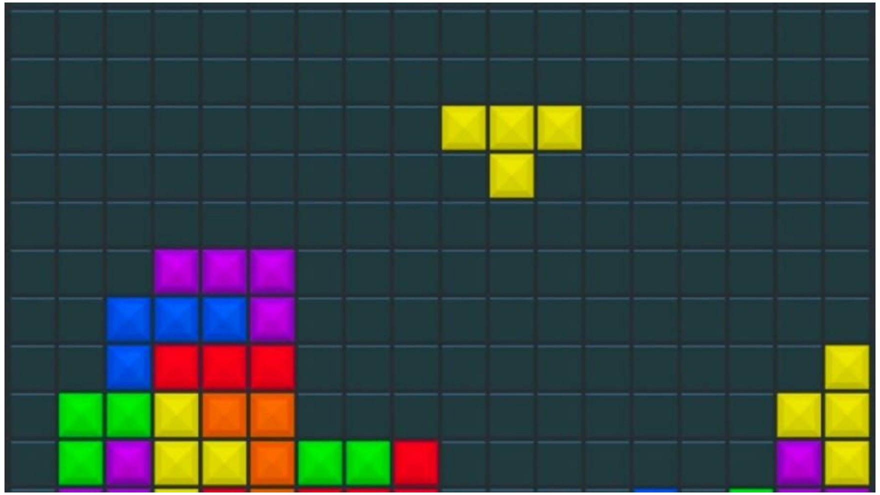

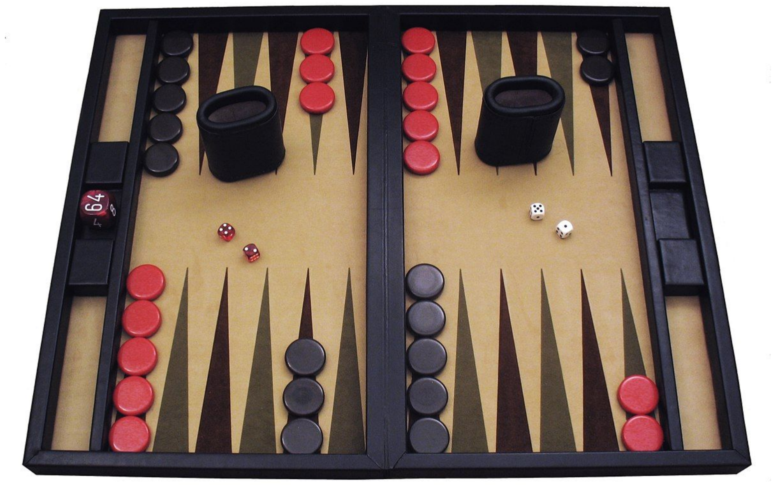

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Approximate Policy Iteration

What if the state-space is large **or continuous** and we cannot do exact or iterative policy evaluation for all states?

Approximate Policy Iteration

What if the state-space is large and we cannot do exact or iterative policy evaluation for all states?

Assumptions: the (infinite-horizon) MDP is still known, but the state-space is too large to just enumerate all states and compute $V^{\pi}(s)$

Approximate Policy Iteration

What if the state-space is large and we cannot do exact or iterative policy evaluation for all states?

Assumptions:

$(S, A, R, T, \gamma, \boldsymbol{\mu}_0)$ is given

Q is in $[0, 1/(1-\gamma)]$

Approximate Policy Iteration

- Outputs policies at every iteration: $\{\pi_0, \pi_1, \pi_2 ... \pi_T\}$

Procedure:

1. Start with a random guess π_0
2. For t=0,...,T:
    a. Do **policy evaluation** and compute $Q^{\pi t}$ ~~for all s,a~~

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    b. Do **policy improvement** as $\pi_{t+1} = \operatorname{argmax}_a Q^{\pi t}(s,a)$ ~~for all s~~

    argmax is still doable, we can still enumerate actions or discretize them

Approximate Policy Evaluation

We build an **approximation V^π** of the true value function V$^{\pi}$

If the approximation is close to the true value, then the optimal policy will be close-to-optimal

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Error bounds exist, but we will skip them for your happiness :)

Approximate Policy Evaluation

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If the approximation is close to the true value, then the optimal policy will be close-to-optimal

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We use a function approximator

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We use a function approximator

e.g., linear approximators, neural nets, non-parametric, etc.

Approximate Policy Evaluation

To be fair, we can directly approximate Q, so let’s do that

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Note that this also means that we can also get rid of the assumption of knowing the MDP

Approximate Policy Evaluation

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What do we need?

Data and Least Square Regression

To be fair, we can directly approximate Q, so let's do that

What do we need?

DATA $D = \{s_i, a_i, y_i\}_{i=1}^{N}$ with y being our label!

with those we can then use least-square regression to extract a function Q in the family of functions

$S \times A \to [0, 1/(1-\gamma)]$

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$Q: S \times A \rightarrow [0, 1/(1-\gamma)]$

$$\mathrm{argmin}_{Q \text{ in } \mathcal{Q}} \sum_{i=1}^{N} (Q(s_i, a_i) - y_i)^2$$

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Data and Least Square Regression

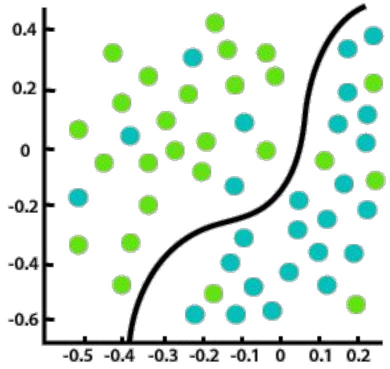
$$\mathrm{argmin}_{Q \text{ in } \mathcal{Q}} \sum_{i=1}^{N} (Q(s_i, a_i) - y_i)^2$$

This is just a regression problem, which

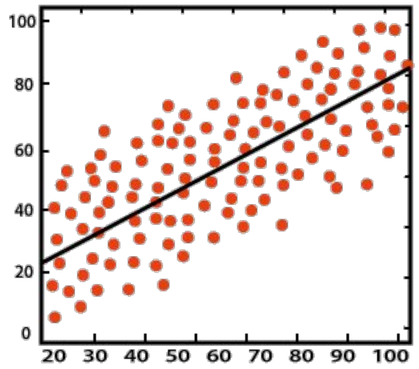
- is numerically tractable
- has generalization bounds

Supervised Learning Digression

Supervised Learning



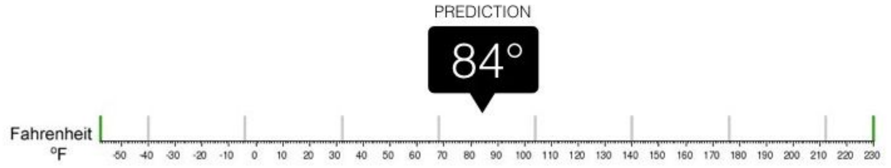
Classification



Regression

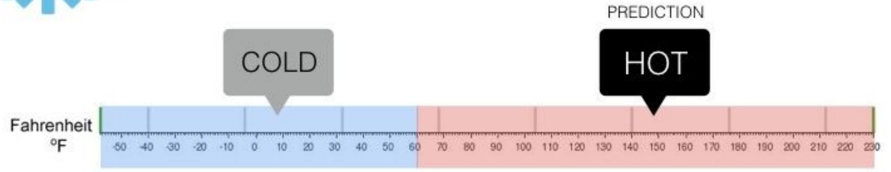
Regression

What is the temperature going to be tomorrow?



Classification

Will it be Cold or Hot tomorrow?



Supervised Learning: Regression

Given a ***data distribution*** from which we sample points x_i and labels $y_i=f(x_i)+\epsilon_i$, with $\mathbb{E}[\epsilon_i]=0$ and $|\epsilon_i|\leq c$, we want to approximate f using a finite set of data (dataset):

$$f\hat{} = \mathrm{argmin}_{f\hat{} \text{ in } F} \sum_{i=1}^{N} (f\hat{}(x_i)-y_i)^2$$

$$\text{with } F=\{f\hat{}: X \to \mathbb{R}\}$$

Supervised Learning: Regression

Given a ***data distribution D*** from which we sample points x_i and labels $y_i=f(x_i)+\epsilon_i$, with $\mathbb{E}[\epsilon_i]=0$ and $|\epsilon_i|\leq c$, we want to approximate f using a finite set of data (dataset):

Empirical Risk Minimizer

$$\hat{f} = \operatorname{argmin}_{\hat{f} \text{ in } F} \sum_{i=1}^{N} (\hat{f}(x_i)-y_i)^2$$

with $F=\{\hat{f}: X \to \mathbb{R}\}$

We can generalize under the same data distribution

$$\mathbb{E}_{x\sim D}(\hat{f}(x)-f(x))^2 \leq \delta \text{ with } \delta \text{ small}$$

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Supervised Learning: Distribution Mismatch

$\mathbb{E}_{x \sim D'}(\hat{f}(x) - f(x))^2$ can be huge!

If $D' \neq D$

End
Supervised Learning Digression

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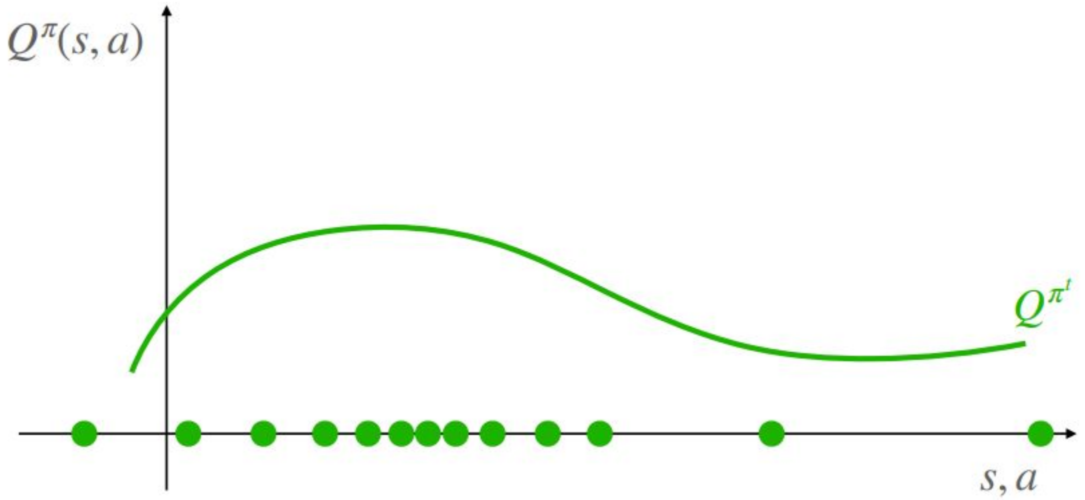
Oscillation from Distribution Change



Credits: Wen Sun

Green dots: (s, a) from π^t

Red dots: (s, a) from π^{t+1}

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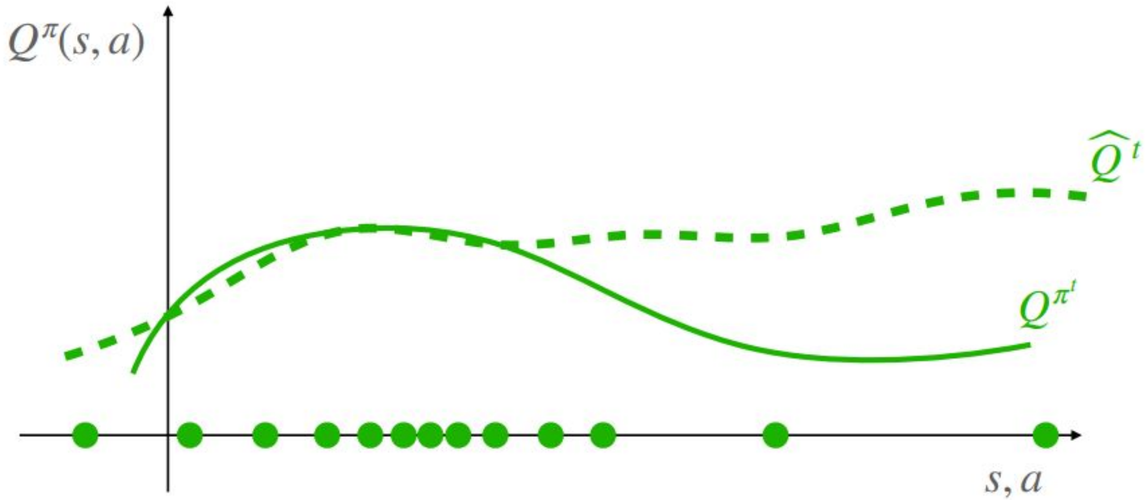

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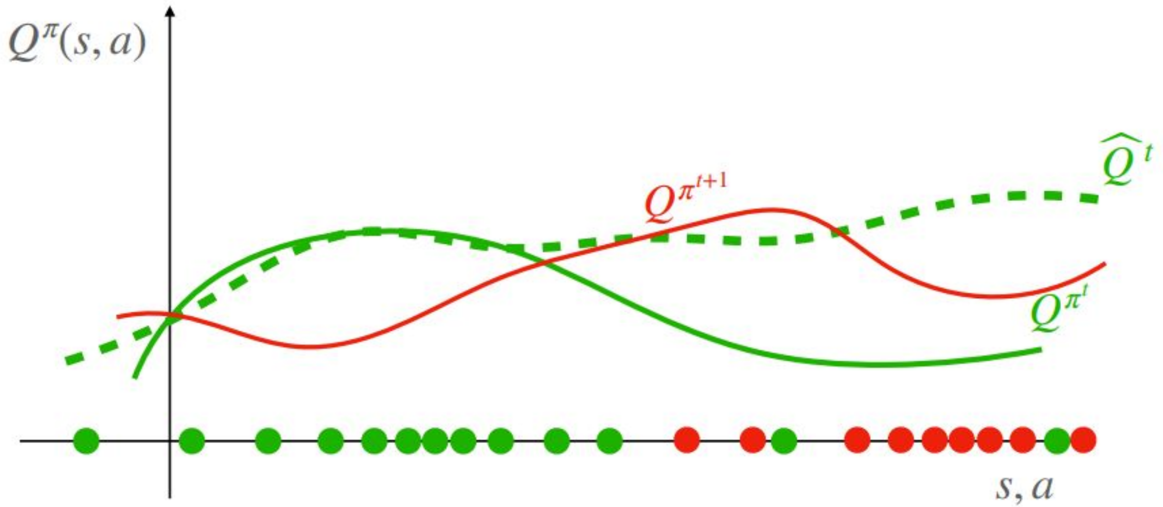

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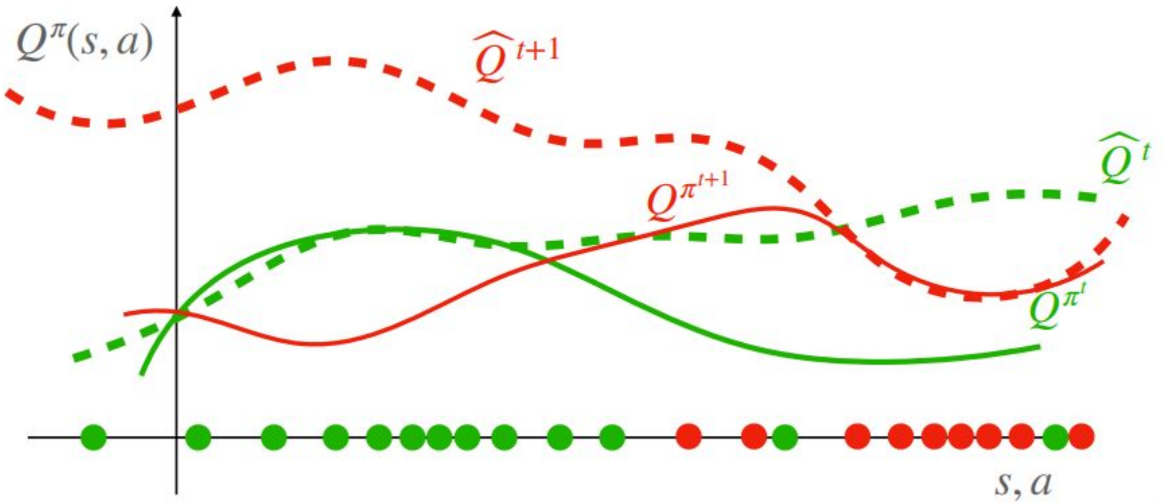

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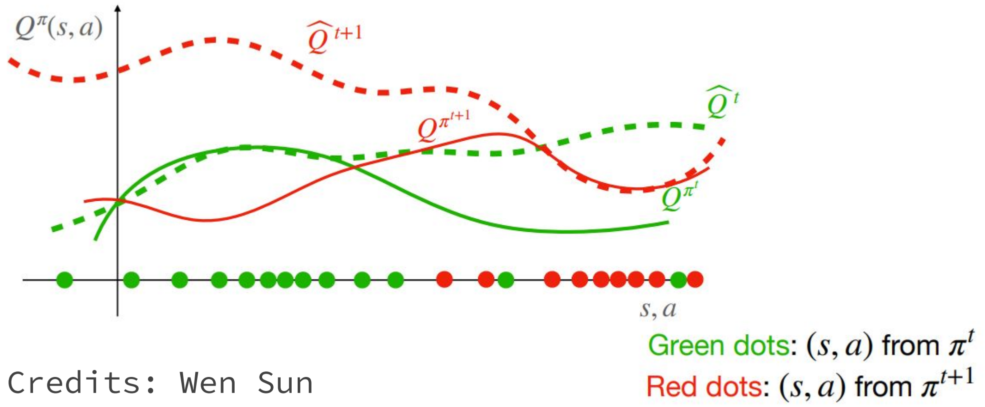

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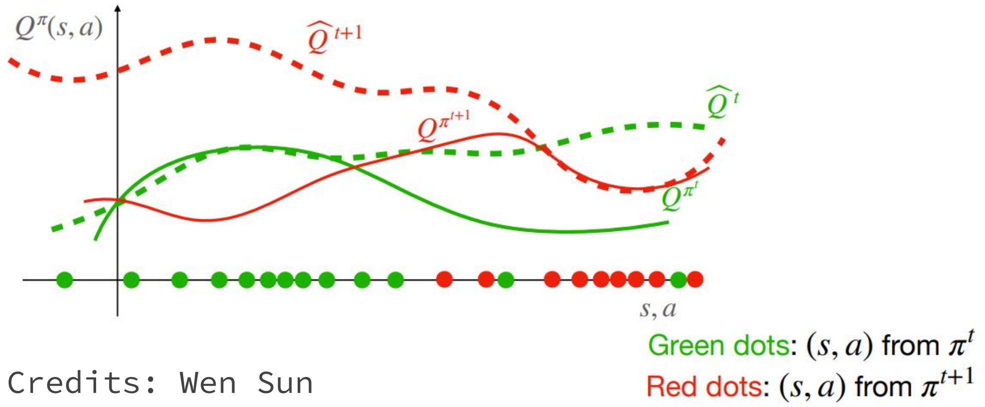
We cannot guarantee anymore monotonic improvement!

Oscillation from Distribution Change



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Our estimation is only good under $d_{\mu 0}^{\pi}$ and to make sure we have monotonic improvement we need a strong coverage assumption

Data Generation

---

2 steps:

1. Roll-in
2. Roll-out & compute supervision targets

Data Generation

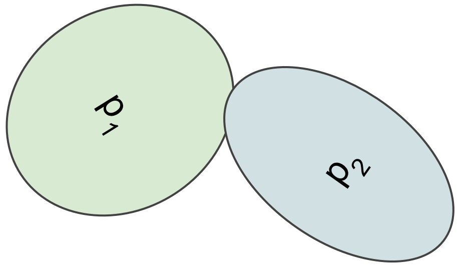
---

2 steps:

1. **Roll-in**
2. Roll-out & compute supervision targets

We want to sample our $(s,a) \sim d^{\pi}_{s0}(s,a) = (1-\gamma)\sum^{\infty}_{h=0}\gamma^{h}\mathbb{P}^{\pi}_{h}(s,a;s_{0})$

Sampling From Mixtures



$p = (1-\alpha)p_1 + \alpha p_2$

- Flip a coin with probability [α, 1-α]
- Commit to a specific p_i based on that and sample from p_i

Data Generation

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We want to sample our $(s,a) \sim d^{\pi}_{s0}(s,a) = (1-\gamma)\sum^{\infty}_{h=0}\gamma^{h}\mathbb{P}^{\pi}_{h}(s,a;s_{0})$

- Sample h from $\gamma^{h}(1-\gamma)$, thus committing to a specific $\mathbb{P}^{\pi}_{h}(s,a;s_{0})$

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- Sample h from $\gamma^h(1-\gamma)$, thus committing to a specific $\mathbb{P}^{\pi}_h(s,a;s_0)$
- Follow π for h timesteps starting from $s_0 \sim \mu_0$ and get s_h, a_h

Data Generation

2 steps:

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Given s, a, how do we estimate $Q^{\pi}(s,a)$?

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$$Q^{\pi}(s_t,a_t) = \mathbb{E}[\textstyle\sum^{\infty}_{h=0}\gamma^h r_h \,|\, (s_0,a_0)=(s_t,a_t), a_{h+1}=\pi(s_h), s_{h+1}\sim p(.|s_h,a_h)]$$

$$Q^{\pi}(s_t,a) = r_t+\gamma\mathbb{E}_{s'\sim p(.|s,a)}[V^{\pi}(s')]$$

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How do we get an unbiased
estimate of this?

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Sample many times and average!

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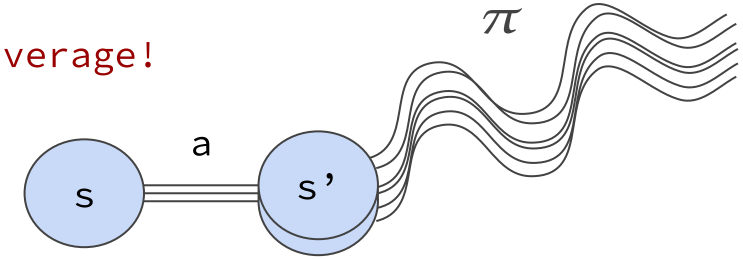

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Easier said than done :(

Infinite horizon!

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Use γ as a sampling factor
again to choose an horizon

Data Generation

2 steps:

1. Roll-in
2. **Roll-out & compute supervision targets**

Given s, a, how do we estimate $Q^{\pi}(s,a)$?

- Start at s,a
- Repeat:
  - Get r(s,a)
  - With probability 1-γ terminate and return $y=\sum \gamma^h r_h$
  - Execute action and get in s'

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$D = \{s_i, a_i, y_i\}_{i=1}^{N}$

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