On matrices and K-relations

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Matrix Databases

Schema: finite set of matrix names

Instance: assigns a matrix to each name

- complex numbers
- elements from some commutative semiring K

A query is a mapping from instances to matrices

Note: matrix dimensions are not fixed

MATLANG operations

$\mathbf{1}$

$$\mathbf{1}\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

MATLANG operations

$\mathbf{1}$

diag

$$\mathrm{diag}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$$

MATLANG operations

$\mathbf{1}$

diag

transpose

matrix multiplication

pointwise functions

- $A + B$ addition

- $A \circ B$ product

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$$

MATLANG operations

$\mathbf{1}$

diag

transposition

matrix multiplication

$A + B$ and $A \circ B$

Expressions are built from matrix names using these operations

K-relations [Green,Karvounarakis,Tannen]

Schema: finite set of relation names, with relation schemes (finite sets of attributes)

Instance $\mathcal{I}$:

- domain D

- assign to each name R a mapping

  $$\mathcal{I}(R) : D^X \to K$$

  where X is the relation scheme of R

  "K-relation"

A query is a mapping from instances to K-relations (of some fixed relation scheme)

ARA: Algebra of Annotated Relations

$\mathbf{1}_A : t \mapsto 1$

union $R \cup S : t \mapsto R(t) + S(t)$

natural join $R \bowtie S : t \mapsto R(t[X]) * S(t[Y])$,
where X and Y are relation schemes of R and S respectively

projection $\pi_Y(R) : t \mapsto \sum_{\substack{t' \in D^X \\ t'[Y]=t}} R(t')$

selection $\sigma_{A=B}(R) : t \mapsto \begin{cases} R(t) & \text{if } t(A) = t(B) \\ 0 & \text{otherwise} \end{cases}$

renaming $\rho_f(R) : t \mapsto R(t \circ f)$ with f a bijection from X to Y

K-relations can represent matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

row	col	value
1	1	1
1	2	2
1	3	3
2	1	4
2	2	5
2	3	6
3	1	7
3	2	8
3	3	9

MATLANG to ARA

$$
\begin{aligned}
\mathbf{1} &\equiv \mathbf{1}_{\mathsf{col}} \\
\mathsf{diag}(A) &\equiv \sigma_{\mathsf{row}=\mathsf{col}}(A \bowtie \rho_{\mathsf{row}\to\mathsf{col}}(A)) \\
A^{\mathsf{T}} &\equiv \rho_{\mathsf{row}\leftrightarrow\mathsf{col}}(A) \\
A \cdot B &\equiv \pi_{\mathsf{row},\mathsf{col}}(\rho_{\mathsf{col}\to\mathsf{col}'}(A) \bowtie \rho_{\mathsf{row}\to\mathsf{col}'}(B)) \\
A + B &\equiv A \cup B \\
A \circ B &\equiv A \bowtie B
\end{aligned}
$$

ARA(3) suffices

Does the converse hold?

binary ARA(3) to MATLANG

matrices can represent binary K-relations

E.g. $D = \{1, 2, 3\}$

A	B	value
1	1	a
1	2	b
1	3	c
2	1	d
2	2	e
2	3	f
3	1	g
3	2	h
3	3	i

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Theorem: Every binary ARA(3) query over binary K-relations can be expressed in MATLANG

Proof of the theorem

Define composition of K-relations

$$R(A,B);S(B,C) := \pi_{A,C}(R \bowtie S)$$

Step 1. Prove that every binary ARA(3) query over binary relations is expressible in ARA(2) + composition

Compare: Tarski&Givant: Every binary FO(3) query over binary relations can be expressed in the *algebra of binary relations*

(identity, union, complement, converse, composition)

Step 2. Translate ARA(2) + composition into MATLANG

ARA(2) + composition to MATLANG

$$\begin{aligned}
\mathbf{1}|_A &\equiv \mathbf{1} \\
R \cup S &\equiv R + S \\
R \bowtie S &\equiv R \circ S \\
R; S &\equiv R \cdot S \\
\pi_A(R) &\equiv R \cdot \mathbf{1}(R) \\
\sigma_{A=B}(R) &\equiv R \circ \mathsf{diag}(\mathbf{1}(R))
\end{aligned}$$

Indistinguishability

Immediate corollary is that, for matrices A and B,

$$A \equiv_{\mathsf{MATLANG}} B \quad \Leftrightarrow \quad A \equiv_{\mathsf{ARA(3)}} B$$

where a "sentence" is an expression returning a scalar

For symmetric 0-1 matrices A and B, this also follows from a recent result by Floris Geerts:

$$A \equiv_{\mathsf{MATLANG}} B \quad \Leftrightarrow \quad A \equiv_{C^3} B$$

Indeed,

$$\equiv_{C^3} \quad \Rightarrow \quad \equiv_{\mathsf{ARA(3)}} \quad \Rightarrow \quad \equiv_{\mathsf{MATLANG}} \quad \Rightarrow \quad \equiv_{C^3}$$

Further work

Extend MATLANG with more powerful operations

More pointwise functions, upper-triangulation, diagonalisation, . . .

Investigate expressive power

Non-deterministic operations

- ICDT 2018 full presentation of MATLANG
- ICDT 2019 Floris
- this result, not yet published