Neural Networks (2023/2024)
Practical Assignment I: Perceptron Training

The main topic of this assignment is the Rosenblatt perceptron algorithm. We apply it to randomized data and compare the results with our theoretical findings (capacity of a hyperplane) in computer experiments.

Rosenblatt Perceptron Algorithm

For this systematic study of linear separability, write a program which can be used to

a) ... generate artificial data sets containing P randomly generated N-dimensional feature vectors and binary labels: $\mathbb{D} = \{\boldsymbol{\xi}^\mu, S^\mu\}_{\mu=1}^{P}$. Here, the $\boldsymbol{\xi}^\mu \in I\!R^N$ are vectors of independent random components ξ_j^μ with mean *zero* and variance *one*. You can use, for instance, Gaussian components $\xi_j^\mu \sim \mathcal{N}(0,1)$. The labels S^μ are taken to be independent random numbers $S^\mu = \pm 1$ with equal probability $1/2$.

b) ... implement sequential perceptron training by repeated cyclic representation of the P examples. At *time step* $t = 1, 2, \ldots$ present example $\mu(t) = 1, 2, \ldots, P, 1, 2, \ldots$.

   This should be realized by using nested loops where the inner one runs from 1 to P and the outer loop counts the number n of *epochs*, i.e. *sweeps* through the data set $\mathbb{D}$. Limit the number of sweeps to $n \leq n_{max}$ so that the total number of individual update steps will be at most $n_{max}P$.

c) ... run the Rosenblatt algorithm for a generated data set $I\!D$ with updates of the form

$$\boldsymbol{w}(t+1) = \begin{cases} \boldsymbol{w}(t) + \frac{1}{N}\,\boldsymbol{\xi}^{\mu(t)}\,S^{\mu(t)} & \text{if } E^{\mu(t)} \leq 0, \\ \boldsymbol{w}(t) & \text{else,} \end{cases}$$

   where $E^{\mu(t)} = \boldsymbol{w}(t) \cdot \boldsymbol{\xi}^{\mu(t)}\,S^{\mu(t)}$. Initialize the weights as $\boldsymbol{w}(0) = 0$, but make sure that a training step is indeed performed for $E^{\mu(t)} = 0$.

   The training should be performed until either a solution is found such that $E^\nu > 0$ for all ν or the maximum number of *sweeps* n_{max} is reached.

d) ... repeat the training for several randomized data sets, e.g. by running (a–c) within yet another loop. For a given value of P, use a number n_D of independently generated sets $\mathbb{D}$. Determine the fraction $Q_{l.s.}$ of successful runs as a function of $\alpha = P/N$, by repeating the experiment for different values of P. The result should *resemble* the probability $P_{l.s.}(\alpha)$ that was derived in class, see lecture slides and full text lecture notes.

Computer experiments and report:

Rules and recommendations concerning the content, style and submission of the report and/or the code will be announced in good time.

To study percepton training, run your code **at least** for the following parameter settings:

$$N = 20 \textbf{ and } N = 40, \quad P = \alpha N \text{ with } \alpha = 0.75, 1.0, 1.25, \ldots 3.0, \quad n_D = 50, \quad n_{max} = 100.$$

As the key result, obtain $Q_{l.s.}$ as a function of α and display it as a graph in an appropriate fashion (caption text, axis labels, data points marked by symbols). Discuss your result in words, compare with the probability $P_{l.s.}(\alpha)$ that was derived in class. If your results differ, discuss potential reasons for the deviations and weaknesses of the computer experiments.

Remark:
Your actual choice of parameters will – of course – depend on your implementation and on available computing power. If (CPU-)time allows, improve the quality of your results by setting N, n_D, and/or n_{max} as large as possible.

Possible extensions ("bonus"):
The following points are only a few example suggestions, ideas of your own are very much encouraged (please discuss them with the TA in advance).

- Observe the behavior of $Q_{l.s.}(\alpha)$ for different system sizes N. Does it approach a step function with increasing N, as predicted by the theory? To this end, repeat the above experiments for several larger values of N. For this study, you might want to consider a limited range of α-values, e.g. $1.5 \leq \alpha \leq 2.5$. and perhaps consider a smaller increment of α in this interval.
- Determine the embedding strenghts x^μ (or formulate the algorithm in terms of the x^μ) to obtain a histogram which reflects their frequency in the successful cases upon convergence of the training.
- Consider a non-zero value of c as introduced in class for updates when $E^\mu \leq c$. Does the choice of c influence the results in terms of $Q_{l.s.}$?
- Modify the algorithm by adding a clamped input to all feature vectors in order to find and count inhomogeneous perceptron solutions as well. Does the number of successful training processes change significantly?