Integral trees of the form $T(a) \cdot T(c,4) \cdot T(d,1,3) \cdot T(e,3,1) \cdot T(1,f,1)$

Viola Campos

Consider the tree $T(a) \cdot T(c,4) \cdot T(d,1,3) \cdot T(e,3,1) \cdot T(1,f,1)$, in level order denoted as $01^a(12^4)^c(123^3)^d(1(23)^3)^e1(23)^f$.

When c and e are chosen as

$$c = 3a + 2d - 3$$
$$e = y - 4a - 3d$$

the trees characteristic polynomial becomes

$$P[T(a) \cdot T(c,4) \cdot T(d,1,3) \cdot T(e,3,1) \cdot T(1,f,1), x]$$
$$= (x^4 - (f + y + 2) \cdot x^2 + (fy + y + 4)) \cdot (x-1)^{-8a+f-6d+2y} \cdot (x-4)^{-a+y-4} \cdot x^{6a+6d+y-9}$$

In this case the tree has an integral spectrum when the term $(x^4 - (f+y+2) \cdot x^2 + (fy+y+4))$ can be factored into $(x^2 - s^2)(x^2 - t^2)$ for some integral values s and t. The nonnegative eigenvalues of the tree are $s, t, 2, 1, 0$ with multiplicities 1, 1, $-a+y-4$, $-8a+f-6d+2y$, $6a+6d+y-9$.

A parameter search for possible values for a, d, y, f, s, t yields for example the following solutions:

s	t	a	d	e	f	tree	spectrum	# of nodes
3	4	1	2	10	13	$T(1) \cdot T(4,4) \cdot T(2,1,3) \cdot T(1,13,1)$	$4,3,2^5,1^{13},0^{19}$	59
3	4	2	0	10	13	$T(2) \cdot T(3,4) \cdot T(2,3,1) \cdot T(1,13,1)$	$4,3,2^4,1^{17},0^{13}$	59
3	4	1	1	10	13	$T(1) \cdot T(2,4) \cdot T(1,1,3) \cdot T(3,3,1) \cdot T(1,13,1)$	$4,3,2^5,1^{19},0^{13}$	65
3	4	0	3	10	13	$T(3,4) \cdot T(3,1,3) \cdot T(1,3,1) \cdot T(1,13,1)$	$4,3,2^6,1^{15},0^{19}$	65
3	4	2	2	14	9	$T(2) \cdot T(7,4) \cdot T(2,1,3) \cdot T(1,9,1)$	$4,3,2^8,1^9,0^{29}$	67
3	4	3	0	14	9	$T(3) \cdot T(6,4) \cdot T(2,3,1) \cdot T(1,9,1)$	$4,3,2^7,1^{13},0^{23}$	67
3	4	1	0	10	13	$T(1) \cdot T(6,3,1) \cdot T(1,13,1)$	$4,3,2^5,1^{25},0^7$	71
3	4	0	2	10	13	$T(1,4) \cdot T(2,1,3) \cdot T(4,3,1) \cdot T(1,13,1)$	$4,3,2^6,1^{21},0^{13}$	71
3	4	2	1	14	9	$T(2) \cdot T(5,4) \cdot T(1,1,3) \cdot T(3,3,1) \cdot T(1,9,1)$	$4,3,2^8,1^{15},0^{23}$	73
3	4	1	3	14	9	$T(1) \cdot T(6,4) \cdot T(3,1,3) \cdot T(1,3,1) \cdot T(1,9,1)$	$4,3,2^9,1^{11},0^{29}$	73
3	4	2	0	14	9	$T(2) \cdot T(3,4) \cdot T(0,1,3) \cdot T(6,3,1) \cdot T(1,9,1)$	$4,3,2^8,1^{21},0^{17}$	79
3	4	1	2	14	9	$T(1) \cdot T(4,4) \cdot T(2,1,3) \cdot T(4,3,1) \cdot T(1,9,1)$	$4,3,2^9,1^{17},0^{23}$	79
3	4	0	4	14	9	$T(5,4) \cdot T(4,1,3) \cdot T(2,3,1) \cdot T(1,9,1)$	$4,3,2^{10},1^{13},0^{29}$	79
3	4	0	3	14	9	$T(3,4) \cdot T(3,1,3) \cdot T(5,3,1) \cdot T(1,9,1)$	$4,3,2^{10},1^{19},0^{23}$	85
3	4	1	0	14	9	$T(1) \cdot T(10,3,1) \cdot T(1,9,1)$	$4,3,2^9,1^{29},0^{11}$	91
3	4	0	2	14	9	$T(1,4) \cdot T(2,1,3) \cdot T(8,3,1) \cdot T(1,9,1)$	$4,3,2^{10},1^{25},0^{17}$	91
4	5	3	2	18	21	$T(3) \cdot T(10,4) \cdot T(2,1,3) \cdot T(1,21,1)$	$5,4,2^{11},1^{21},0^{39}$	107
...	...	...	...	...	...	...	...	...