Chapter 1. part 3.

§1.5. Conditional Probability

- We introduce one of the most important concepts in probability theory, that of conditional probability.
- We are often interested in calculating probabilities when some partial information concerning the result of the experiment is available; in such a situation the desired probabilities are conditional.
- Even when no partial information is available, conditional probabities can often be used to compute the desired probabilities more easily.

Example 1. Suppose we toss 2 dice, what's the probability that the sum of the two dice equals 8?

Ans: Let E = "sum of dice is 8". $\Omega = \left\{ \begin{matrix} (1,1), (1,2), \cdots, (1,6) \\ (2,1), (2,2), \cdots, (2,6) \\ \cdots\cdots \\ (6,1), (6,2), \cdots, (6,6) \end{matrix} \right\}$

$E = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$P(E) = \dfrac{n(E)}{n(\Omega)} = \dfrac{5}{36}$.

Expl. 2. Suppose we toss 2 dice and suppose further that we observe that the 1st die is 3. Then given this information, what is the prob. that the sum of the two dice equals 8?

Ans. E = "sum of dice is 8". F = "first die is 3".

$F = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$

given F occurs, the answer is $1/6$. denoted with $P(E|F) = \frac{1}{6}$.

note: $P(E|F) = \frac{1}{6} \neq \frac{5}{36} = P(E)$.

how to get $P(E|F)$? $P(F) = \frac{6}{36} = \frac{1}{6}$, $E \cap F = \{(3,5)\}$, $P(E \cap F) = \frac{1}{36}$

$$P(E|F) = \frac{1}{6} = \frac{1/36}{6/36} = \frac{P(E \cap F)}{P(F)}$$

$P(E|F)$ is called the conditional probability that E occurs given that F has occurred.

If the event F occurs, then in order for E to occur, it is necessary that the actual occurrence be a point in both E and in F; that is, it must in EF. Now, as we know that F has occured, it follows that F becomes our new or reduced sample space; hence the probability that the event EF occurs will equal the prob. of EF relative to $P(F)$.

Definition: If $P(F)>0$, then $$P(E|F)=\frac{P(E\cap F)}{P(F)}.$$

Multiplication Law: $P(E\cap F)=P(E|F)\cdot P(F)$, assume $P(F)\neq 0$

Expl. A. An urn contains three red balls and one blue ball. Two balls are selected without replacement. What is the prob. they are both red?

Ans. Let R_1 = "a red ball drawn on 1st trial"

R_2 = " . . . . 2nd . . "

$$P(R_1\cap R_2)=P(R_1)P(R_2|R_1)=\frac{3}{4}\cdot\frac{2}{3}=\frac{1}{2}.$$

$$\left(\text{or } P(R_1\cap R_2)=\frac{\binom{3}{2}}{\binom{4}{2}}=\frac{3}{\frac{4\times 3}{2}}=\frac{1}{2}\text{ too!}\right)$$

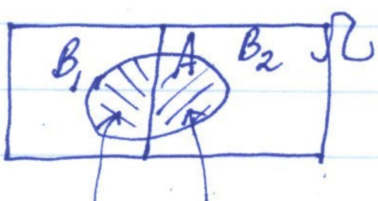
Expl. C. Referring to Expl. A, what is the prob. that a red ball is selected on the second draw?

Law of total probability

Let B_1, B_2 such that $B_1\cup B_2=\Omega$. $B_1\cap B_2=\phi$. Then, for any event A,

$$P(A)=P(A|B_1)\cdot P(B_1)+P(A|B_2)P(B_2).$$

proof.



$A\cap B_1$ $A\cap B_2$

$$A=(A\cap B_1)\cup(A\cap B_2)$$

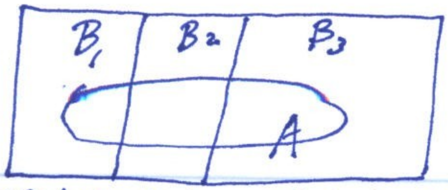
$$P(A)=P(A\cap B_1)+P(A\cap B_2)$$

$$=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)$$

Ans: $$P(R_2)=P(R_2|R_1)P(R_1)+P(R_2|R_1^c)P(R_1^c)$$ R_1 = "red" R_1^c = "blue"

$$=\frac{2}{3}\times\frac{3}{4}+\frac{3}{3}\times\frac{1}{4}=\frac{3}{4}$$

general case:



$B_1 \cup B_2 \cup B_3 = \Omega$

$B_i \cap B_j = \phi$, $i,j = 1,2,3.$

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$
$$= \sum_{i=1}^{3} P(A|B_i)\cdot P(B_i)$$

Expl. D Suppose that occupations are grouped into upper (U), middle (M), and lower (L) levels. U_1 = "a father's occupation is upper-level", U_2 = "a son's occupation is upper-level," etc. 1, 2, generations. Glass and Hall (1954) compiled the following statistics on occupational mobility in England and Wales:

	U_2	M_2	L_2
U_1	0.45	0.48	0.07
M_1	0.05	0.70	0.25
L_1	0.01	0.50	0.49

" matrix of transition probabilities"

$P(U_2|U_1) = 0.45$,

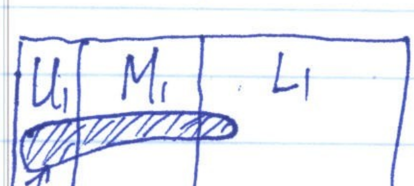
If a father is in U, prob. son is in U is 0.45.

similarly. $P(M_2|U_1) = 0.48$, etc.

Also, suppose that of the fathers generation, 10% are in U, 40% in M and 50% in L. What is the prob. that a son in the next generation is in U?

Ans: $$P(U_2) = P(U_2|U_1)P(U_1) + P(U_2|M_1)P(M_1) + P(U_2|L_1)P(L_1)$$
$$= 0.45 \times 0.10 + 0.05 \times 0.4 + 0.01 \times 0.5$$
$$= 0.07.$$

similarly, $P(M_2)$, $P(L_2)$.



$$U_2 = (U_2 \cap U_1) \cup (U_2 \cap M_1) \cup (U_2 \cap L_1)$$

A different question: If a son has occupational status U_2, what is prob. that his father had occupational status U_1?

"inverse" problem; given an "effect", find prob. of a particular "cause".

Bayes' rule:

$$P(U_1|U_2) = \frac{P(U_1 \cap U_2)}{P(U_2)}$$

multiplication law

$$= \frac{P(U_2|U_1)\cdot P(U_1)}{P(U_2|U_1)P(U_1) + P(U_2|M_1)P(M_1) + P(U_2|L_1)P(L_1)}$$

law of total probability

$$= \frac{0.45 \times 0.10}{0.07} = 0.64.$$

i.e., 64% of sons who are in upper-level occupations have fathers who were in upper-level occupations.

Bayes' rule: B_1, B_2, B_3, $B_i \cap B_j = \phi$, $B_1 \cup B_2 \cup B_3 = \Omega$.

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}, \quad j = 1, 2, 3.$$

Expl E. Diamond and Forrester (1979) applied Bayes' rule to the diagnosis of coronary artery disease. A procedure called cardiac fluoroscopy is used to determine whether there is calcification of coronary arteries and thereby to diagnose coronary artery disease. From the test, it can be determined if 0, 1, 2, or 3 coronary arteries are calcified. Let T_0, T_1, T_2, T_3 denote these events. Let $D+$ or $D-$ denote the event that disease is present or absent. following table based on medical studies:

i	$P(T_i\|D+)$	$P(T_i\|D-)$
0	0.42	0.96
1	0.24	0.02
2	0.20	0.02
3	0.15	0.00

Then,

$$P(D+|T_i) = \frac{P(T_i|D+)P(D+)}{P(T_i|D+)P(D+) + P(T_i|D-)P(D-)}$$

Thus, if initial $P(D^+), P(D^-)$ are known, the prob. a patient has coronary artery disease can be calculated.

e.g.1. a male between 30 and 39 ages suffers from nonanginal chest pain. For such a patient, $P(D^+) \approx 0.05$. Suppose test shows no arteries are calcified.

then $$P(D^+|T_0) = \frac{P(T_0|D^+)P(D^+)}{P(T_0|D^+)P(D^+) + P(T_0|D^-)P(D^-)}$$

$$= \frac{0.42 \times 0.05}{0.42 \times 0.05 + 0.96 \times 0.95} = 0.02.$$

it is unlikely patient has coronary artery disease.

e.g. 2. Suppose the test shows one artery is calcified. Then

$$P(D^+|T_1) = \frac{P(T_1|D^+)P(D^+)}{P(T_1|D^+)P(D^+) + P(T_1|D^-)P(D^-)}$$

$$= \frac{0.24 \times 0.05}{0.24 \times 0.05 + 0.02 \times 0.95} = 0.39$$

It is more likely that this patient has coronary artery disease, but by no means certain.

e.g. 3. a male between 50 and 59 ages, who suffers typical angina. For such a patient, $P(D^+) = 0.92$.

$$P(D^+|T_0) = \frac{P(T_0|D^+)P(D^+)}{P(T_0|D^+)P(D^+) + P(T_0|D^-)P(D^-)}$$

$$= \frac{0.42 \times 0.92}{0.42 \times 0.92 + 0.96 \times 0.08} = 0.83.$$

$$P(D^+|T_1) = \frac{P(T_1|D^+)P(D^+)}{P(T_1|D^+)P(D^+) + P(T_1|D^-)P(D^-)}$$

$$= \frac{0.24 \times 0.92}{0.24 \times 0.92 + 0.02 \times 0.08} = 0.99.$$

We see the strong influence of the prior probability, $P(D^+)$.

Bayes' rule: $B_1 \cup B_2 = \Omega$, $B_1 \cap B_2 = \phi$.

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)}$$

$P(B_1)$ prior probability, $P(B_1|A)$ posterior probability after observing A.

$$P(D^+|T_0) = \frac{P(T_0|D^+)P(D^+)}{P(T_0|D^+)P(D^+) + P(T_0|D^-)P(D^-)}.$$

$P(D^+)$ prior prob. $P(D^+|T_0)$ posterior probability.

Expl. Eddy (1982). regarding interpreting the results from mammogram screening.

- In the absence of any special information, prob. a woman (of the age and health status of this patient) has breast cancer is 1%.
- If the patient has breast cancer, prob. the radiologist will correctly diagnose it is 80%.
- If the patient has a benign lesion (no breast cancer), prob. the radiologist will incorrectly diagnose it as cancer is 10%.

question: What is prob. that a patient with positive mammogram actually has breast cancer?

Remark: 95% physicians estimated the prob. to be about 75%.

Ans: Let D^+ or D^- denote the event that breast cancer is present or absent. T^+ or T^- denote the event that the diagnose is positive or negative.

then
- $P(D+) = 0.01$, $P(D^-) = 0.99$.
- $P(T+|D+) = 0.8$, $P(T-|D+) = 0.2$.
- $P(T+|D^-) = 0.1$, $P(T-|D^-) = 0.9$.

$$P(D+|T+) = \frac{P(T+|D+)\,P(D^+)}{P(T+|D+)P(D^+) + P(T^+|D-)P(D^-)} = \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.1 \times 0.99}$$

$$= 0.075 = 7.5\%.$$