Class 6: Combinatorics 1

Monday, September 13, 2021 5:48 AM

Homework:

HW 2 pending (due Friday 9/17)

Chapter 2 (a short Chapter!): Combinatorics, Counting Methods

2.1.0 Finding Probabilities with Counting Methods

At the heart of combinatorics is the "so-called" **multiplication principle**.
All other definitions and rules are derived from the multiplication principle.

The methods of counting mainly arise in ***probability calculations where the sample space consists of equally likely outcomes:***

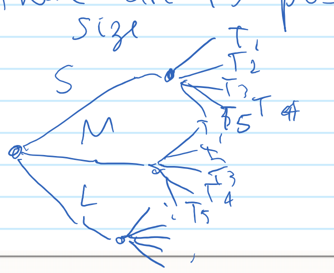
Remember, in that situation:

eg pizza 3 sizes
5 topping

$$P(A) = \frac{|A|}{|S|} = \frac{M}{N}$$

There are 15 possible "1-topping" pizzas

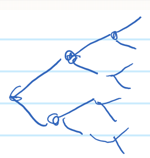
Read example 2.1



15 possible ways of ordering the pizza

Multiplication Principle
Suppose that we perform r experiments such that the kth experiment has n_k possible outcomes, for $k = 1, 2, \cdots, r$. Then there are a total of $n_1 \times n_2 \times n_3 \times \cdots \times n_r$ possible outcomes for the sequence of r experiments.

Example: Factorial Experiments:



$8 = 2^3$ possible experimental configuration /b

eg:

2^3 experiment: Factors with 2 levels: ; 3 factors

levels: outcomes of the factors (discrete)

levels: outcomes of the factors (discrete) configuration ⇓ # runs (1 replicate)

full factorial design

You have enough money to do 100 configurations.

You have 10 factors to consider.

How many factors can we use in a 1 replicate 2^k design?

— can use k factors ⇒ $2^k \le 100$

2^{10} very big! $k=6$ $(2^6 = 64)$ maximal # !!

Some Terminology: Sampling, Ordered / Unordered Sets:

randomly select from a set of (distinct) elements (outcomes)

Ordered / Unordered: ordered 3 elements $\{a_1, a_2, a_3\}$ → different from $\{a_3, a_1, a_2\}$ etc

unordered $\{a_3, a_1, a_2\}$ same as $\{a_1, a_2, a_3\}$

With / without replacement

→ repetitions are possible

eg 3 sampled from $\{a_1, \dots, a_5\}$

$\{a_1, a_2, a_2\}$

There are: four possibilities.

- ordered sampling without replacement

  eg: Top 3 (medal winners) in horse racing.

- ordered sampling with replacement

  eg: gaming with replacement in a poker game
  Passport numbers, lock code, identification...

- unordered sampling without replacement
  eg: select from applications
  membership in election
  lottery

- unordered sampling with replacement
  eg: (see text)

2.1.1 Ordered Sampling with Replacement

Select 3 from a set of 5

choices: 5 (1st select), 5 (2nd select), 5 (3rd selection)

Select k out of n

$= \underbrace{n \cdot n \cdot n \cdots n}_{k \text{ times}} = n^k$

(pad lock example)

Thus, when ordering matters and repetition is allowed, the total number of ways to choose k objects from a set with n elements is $= n^k$

2.1.2 Ordered Sampling without Replacement: Permutations

Example:
What is the # of different "displays" of n (distinct) books on a shelf when k ($\leq n$) are chosen.

Why does the order matter? → display differs when selected books are permuted

Equivalent examples?
- See above → called # of "permutations"
-
-

First: case $k = n$:

Permutations of *n* elements:

In how many ways can I permute my 5 keys ?

First draw 5 choices
2nd draw 4 choices
3 rd draw 3 choices
4 th draw 2 choices
5th draw 1 choice

multiplication principles

$\# \text{ permutations} = \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{5!} = 120$

Notation ${}^{n}_{k}P = n!$

First: case $k < n$:

5 keys, select 3 ⇒ # permutations of size 3

1st draw → 5
2nd draw → 4
3rd draw → 3

$= \underset{n}{5} \cdot \underset{n-1}{4} \cdot \underset{n-2}{3}$

$$P_k^n = n \times (n-1) \times \ldots \times (n-k+1).$$

$= n \cdot (n-1) \cdots\cdots 2 \cdot 1 \leftarrow n!$

$P^n_k = n \times (n-1) \times \ldots \times (n-k+1).$

$$= \frac{n \cdot (n-1) \cdots\cdots 2 \cdot 1 \leftarrow n!}{(n-k)(n-k-1) \cdots 2 \cdot 1 \leftarrow (n-k)!}$$

Written as a formula:

The number of k-permutations of n distinguishable objects is given by

$$P^n_k = \frac{n!}{(n-k)!}, \text{ for } 0 \le k \le n.$$

Note: There are several different common notations see text

2.1.3 Unordered Sampling without Replacement: Combinations

Examples: (Discuss) Selecting a committee of size *k* from a group of *n* people.

Notation: in text: $\binom{n}{k}$: "n choose k"

\# of combinations : other notations ${}^n_kC$ or $C(n,k)$ or $C_{n,k}$

Derivation: Select 3 to a committee out of 10

pres, vice pres, treasurer

answer : ${}^{10}_{3}P = 10 \cdot 9 \cdot 8 = 720$ ordered set

\# ways of orderings of the 3 selected people :

$= {}^3_3P = 3! = 6$

$\Rightarrow$ \# of

Task 1: select 3 people to the committee (unordered) \# ways $= \binom{10}{3}$

Task 2: now assign titles to the 3 selected \# ways $= 3!$

Combined ${}^{10}_{3}P = \binom{10}{3} \cdot 3!$ Multipl principle

solve for

720

Solve for ~

$$ {}^{10}_{3}P = \frac{10!}{(10-3)!} \Rightarrow \binom{10}{3} = \frac{10!}{(10-3)! \cdot 3!} = \frac{720}{6} = 120 $$

The number of k-combinations of an n-element set is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \le k \le n.$$

hand calc.

$$\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3}$$