§1.8 Problems

1. A coin is tossed three times and the sequence of heads and tails is recorded.

   a. List the sample space.

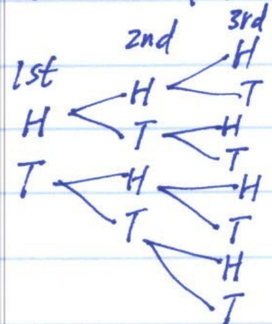
   b. List the elements that make up the following events.

   (1) A = at least two heads. (2) B = the first two toss are heads.

   (3) C = the last toss is a tail.

   c. List the elements of the following events: (1) A^c. (2) $A \cap B$ (3) $A \cup C$.

Ans: a.



$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

b. (1) $A = \{HHH, HHT, HTH, THH\}$

(2) $B = \{HHH, HHT\}$.

(3). $C = \{HHT, HTT, THT, TTT\}$

c. (1) $A^c = \{HTT, THT, TTH, TTT\}$

(2) $A \cap B = \{HHH, HHT\}$

(3) $A \cup C = \{HHH, HHT, HTH, THH, HTT, THT, TTT\}$.

6. Prove $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

proof. Let $A \cup B = D$, apply $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

one has $P(A \cup B \cup C) = P(D \cup C) = P(D) + P(C) - P(D \cap C)$, $(*)$

now $P(D \cap C) = P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C))$

$= P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))$

$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

thus, $(*)$ $P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) + P(C)$

$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

13. In a game of poker, what is the probability that a five-card hand will contain
(a). a straight (five cards in unbroken numerical sequence)
(b). four of a kind. (c) a full house (e.g. 666QQ).

Ans: (a) $A =$ " a straight ", e.g. $6\diamondsuit\ 7\clubsuit\ 8\diamondsuit\ 9\spadesuit\ 10\clubsuit$ 13 values, 4 suits

$$P(A) = \frac{n(A)}{n(\Omega)} \qquad n(\Omega) = \binom{52}{5}. \quad n(A) = 10(4^5 - 4)$$

$$= \frac{10(4^5-4)}{\binom{52}{5}} = 0.0039.$$

(b) $B =$ " four of a kind ", e.g. 8888Q.

$$P(B) = \frac{n(B)}{n(\Omega)} = \frac{\binom{13}{1}\cdot\binom{48}{1}}{\binom{52}{5}} = \frac{13\cdot 48}{\binom{52}{5}} = 0.00024.$$

(c) $C =$ " a full house "

$$P(C) = \frac{n(C)}{n(\Omega)} = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 0.0014.$$

more examples: (d) a flush? (all 5 cards are of the same suit)

$$P(D) = \frac{n(D)}{n(\Omega)} = \frac{\binom{13}{5}\binom{4}{1}}{\binom{52}{5}} = 0.002.$$

(e) one pair? (e.g. a. a, b, c, d)

$$P(E) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}} = 0.42257.$$

(f). two pairs? (a, a, b, b, c).   5588K

$$P(F) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0475.$$

(g) three of a kind? (a,a,a, b,c)

$$P(G) = \frac{n(G)}{n(\Omega)} = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0211$$

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15. How many different meals can be made from four kinds of meat, six vegetables, and three starches if a meal consists of one selection from each group?

Ans: $\binom{4}{1}\cdot\binom{6}{1}\cdot\binom{3}{1} = 4\cdot 6\cdot 3 = 72.$

17. In acceptance sampling, a purchaser samples 4 items from a lot of 100 and rejects the lot if 1 or more are defective. Graph the prob. that the lot is accepted as a function of the percentage of defective items in the lot.

Ans: Let k be the number of defective items in a lot.

$A=$ "lot is accepted"

then
$$P(A) = \frac{\binom{k}{0}\binom{100-k}{4} + \cancel{\binom{k}{1}\binom{100-k}{3}}}{\binom{100}{4}} \qquad \boxed{\begin{array}{l} k, \text{ defective} \\ 100-k, \text{ non-defective} \end{array}} \rightarrow 4 \text{ sampled.}$$

$$= \frac{\binom{100-k}{4} + \cancel{k\cdot\binom{100-k}{3}}}{\binom{100}{4}} = \frac{(100-k)(99-k)(98-k)(97-k)}{100\cdot 99\cdot 98\cdot 97}$$

thus, probability $P(A)$ is a function of k, or percentage of defective items $k/100$.

19. A committee consists of five Chicanos, two Asians, three African Americans and two Caucasians.

a. A subcommittee of four is chosen at random. What is prob. that all the ethnic groups are represented on the subcommittee?

b. answer the question for (a) if a subcommittee of five is chosen?

Ans. a.
$$\text{Prob.} = \frac{\binom{5}{1}\binom{3}{1}\binom{2}{1}\binom{2}{1}}{\binom{12}{4}} = \frac{5\cdot 3\cdot 2\cdot 2}{\binom{12}{4}} = 60/\binom{12}{4}$$

b.
$$\text{prob.} = \frac{\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{2}{1} + \binom{5}{1}\binom{3}{2}\binom{2}{1}\binom{2}{1} + \binom{5}{1}\binom{3}{1}\binom{2}{2}\binom{2}{1} + \binom{5}{1}\binom{3}{1}\binom{2}{1}\binom{2}{2}}{\binom{12}{5}}$$

$$= 240/\binom{12}{5}$$

21. A fair coin is tossed five times. What's the prob. of getting a sequence of three heads? $H_1T_2H_3H_4T_5$

Ans: denoted event with A. $P(A) = \frac{n(A)}{n(\Omega)} = \frac{\binom{5}{3}}{2^5} = \frac{\frac{5!}{3!\,2!}}{2^5} = 10/32.$

23. How many ways are there to place n indistinguishable balls in n urns so that exactly one urn is empty?

Ans:

balls

$|\bullet|\bullet|\cdots| \;|\bullet|$

n urns

one urn has two balls

$\binom{n}{1}\cdot\binom{n-1}{1} = n\cdot(n-1).$

empty urn

27. If a five-letter word is formed at random, what's the prob. that no letter occurs more than once?

Ans. $P(A) = \frac{n(A)}{n(\Omega)} = \frac{\cancel{\binom{26}{5}}}{26^5} = \frac{26\times25\times24\times23\times22}{26^5}$ $\left(\text{or. } \frac{\binom{26}{5}}{26^5/5!}\right)$

31. Six male and six female dancers perform the Virginia reel. This dance requires that they form a line consisting of six male/female pairs. How many such arrangements are there?

Ans: $\underline{MF}\,\underline{MF}\,\underline{MF}\,\underline{MF}\,\underline{MF}\,\underline{MF}$ $6^2\cdot5^2\cdot4^2\cdot3^2\cdot2^2\cdot1^2 = (6!)^2 = 518400.$

or $\underbrace{\binom{6}{1}\binom{6}{1}}_{M/F}\underbrace{\binom{5}{1}\binom{5}{1}}_{M/F}\binom{4}{1}\binom{4}{1}\binom{3}{1}\binom{3}{1}\binom{2}{1}\binom{2}{1}\binom{1}{1}\binom{1}{1}$ ......

37. What is the coefficient of $x^2y^2z^3$ in the expansion of $(x+y+z)^7$?

Ans: $\binom{7}{2\;2\;3} = \frac{7!}{2!\,2!\,3!}$

39. A monkey at a typewriter types each of the 26 letters of the alphabet exactly once, the order being random.

a. What is the prob. that the word "Hamlet" appears somewhere in the string of letters?

6. How many independent monkey typists would you need in order that the prob. that the word appears is at least 0.90? 5

Ans: a. Event is denoted with A. $P(A) = \frac{n(A)}{n(\Omega)} = \frac{20!\cdot 21}{26!} = \frac{21!}{26!} = 1.27\times 10^{-7}$.

$26 = 6 + 20$

Hamlet together!

b. Assum n typists. E = "word appears in n types"

$$P(E) = 1 - P(E^c) = 1 - P(E_1^c)P(E_2^c)\cdots P(E_n^c)$$

doesn't appear in n typings.

$$= 1 - (1 - 1.27\times 10^{-7})\cdot(1 - 1.27\times 10^{-7})\cdots(1 - 1.27\times 10^{-7})$$

$$= 1 - (1 - 1.27\times 10^{-7})^n.$$

determine n, such that $P(E) \geq 0.90$.

i.e.

$$1 - (1 - 1.27\times 10^{-7})^n \geq 0.90, \quad (1 - 1.27\times 10^{-7})^n \leq 0.1$$

$$n\cdot\log(1 - 1.27\times 10^{-7}) \leq \log(0.1), \quad n \geq \frac{\log(0.1)}{\log(1 - 1.27\times 10^{-7})} \approx 1.813\times 10^{7}.$$

41. A drawer of ~~silk~~ socks contains seven black socks, eight blue socks, and nine green socks. Two socks are chosen in the dark.

a. what is the prob. that they match?

b. what is the prob. that a black pair is chosen?

Ans: a. $P(A) = \frac{\binom{7}{2} + \binom{8}{2} + \binom{9}{2}}{\binom{24}{2}}$.

b. $P(B) = \binom{7}{2} \Big/ \binom{24}{2}$.