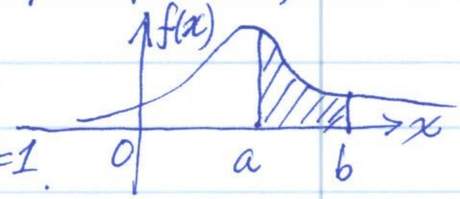
Chapter2 part3

Continuous Random Variables

random variables take on a continuum of values rather than a finite or countably infinite number. e.g. a model for the life time of an electronic component, it can be any positive real number. length, width, weight, etc.

For continuous r.v, frequency function / prob. mass. function $p(x)=P(X=x)$, $x=0,1,2,\cdots$

density function, $f(x)$. $1^\circ.\ f(x)\geq 0$
$2^\circ.\ \int_{-\infty}^{\infty} f(x)dx=1$.

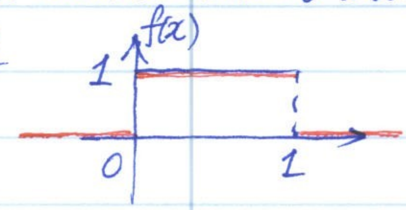


typically, $P(a<X<b)=\int_a^b f(x)dx$, $\quad P(X=c)=\int_c^c f(x)dx=0$.

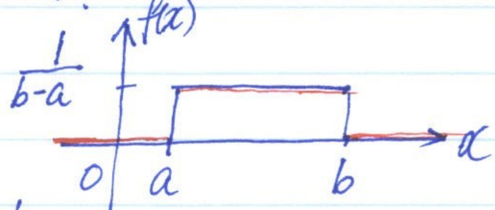
Expl. A. A uniform r.v. on $[0,1]$. "choose a number at random between 0 and 1".

the p.d.f for such X, $f(x)=\begin{cases}1, & 0\leq x\leq 1\\ 0, & \text{else}\end{cases}$

denoted with $X\sim U[0,1]$.



sometime. $X\sim U[a,b]$.



$f(x)=\begin{cases}\frac{1}{b-a}, & a\leq x\leq b\\ 0, & \text{else}.\end{cases}$

$1^\circ.\ f(x)\geq 0$
$2^\circ.\ \int_{-\infty}^{\infty} f(x)dx=\int_a^b \frac{1}{b-a}dx=1$

Cumulative distribution function $F(x)=P(X\leq x)$.

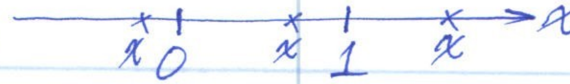
$F(x)=P(-\infty\leq X\leq x)=\int_{-\infty}^{x} f(t)dt$. $\quad f(x)=F'(x)$.

also. $P(a\leq X\leq b)=\int_a^b f(x)dx=\int_{-\infty}^{b} f(x)dx-\int_{-\infty}^{a} f(x)dx=F(b)-F(a)$.

Expl B. $X\sim U[0,1]$, find $F(x)$.

Ans: $F(x)=\int_{-\infty}^{x} f(t)dt$, here $f(t)=\begin{cases}1, & t\in[0,1]\\ 0, & \text{else}\end{cases}$

thus, if $x\le 0$, $F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{x} 0\,dt=0$.



if $0<x<1$, $F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{0} f(t)dt+\int_{0}^{x} f(t)dt$

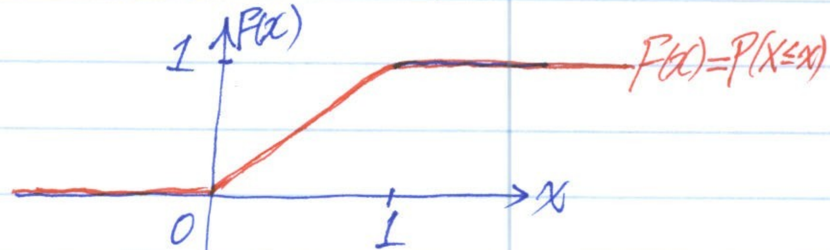
$$=\int_{-\infty}^{0} 0\,dt+\int_{0}^{x} 1\,dt=0+x.$$

if $1\le x<\infty$, $F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{0} f(t)dt+\int_{0}^{1} f(t)dt+\int_{1}^{x} f(t)dt$

$$=\int_{-\infty}^{0} 0\,dt+\int_{0}^{1} 1\,dt+\int_{1}^{x} 0\,dt$$

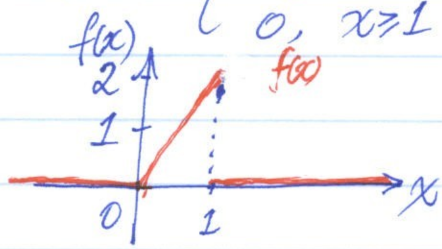
$$=0+1+0=1.$$

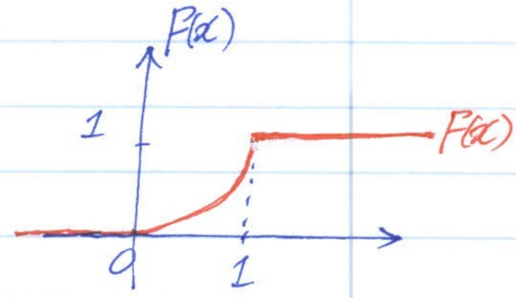
i.e., $F(x)=\begin{cases}0, & x\le 0\\ x, & 0<x<1\\ 1, & x\ge 1\end{cases}$



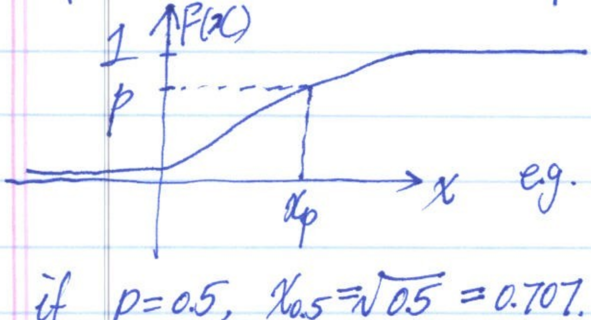
Expl C. $F(x)=x^2$, $0\le x\le 1$, find $f(x)$. or $F(x)=\begin{cases}0, & x\le 0\\ x^2, & 0\le x\le 1\\ 1, & x\ge 1\end{cases}$

Ans: $f(x)=F'(x)=\begin{cases}0, & x\le 0\\ 2x, & 0\le x\le 1\\ 0, & x\ge 1\end{cases}$





p-th quantile of $X\sim F(x)$, denoted with x_p, such that $F(x_p)=p$. or $P(X\le x_p)=p$.



$F(x_p)=p \Rightarrow x_p=F^{-1}(p)$

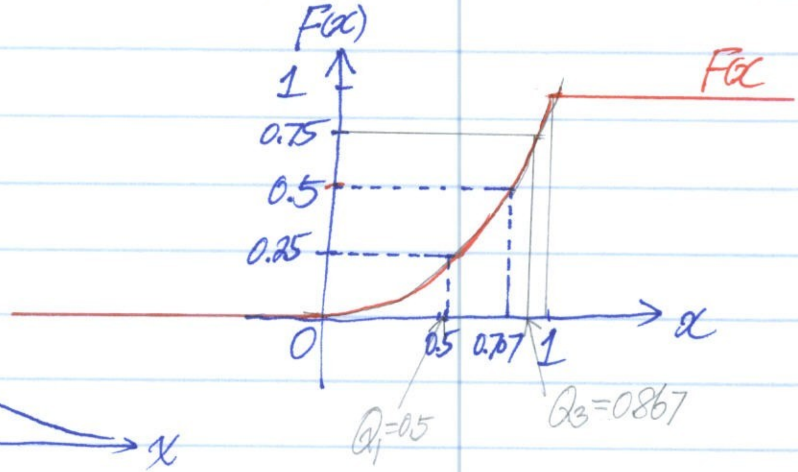
e.g. $F(x)=x^2$, $F(x_p)=p$, $x_p^2=p$, $x_p=\sqrt{p}$.

if $p=0.5$, $x_{0.5}=\sqrt{0.5}=0.707$. $x_{0.5}$ is called median.

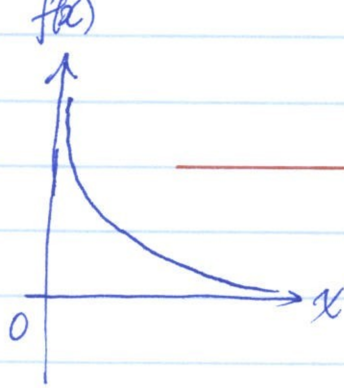
$p=0.25$, $x_{0.25}$ is called lower quartile of F.

from $x_p=\sqrt{p}$, one has $x_{0.25}=\sqrt{0.25}=0.5$

$p=0.75$, $x_{0.75}$ is called upper quartile of F. $x_{0.75}=\sqrt{0.75}=0.867$.



§2.2.1. Exponential density

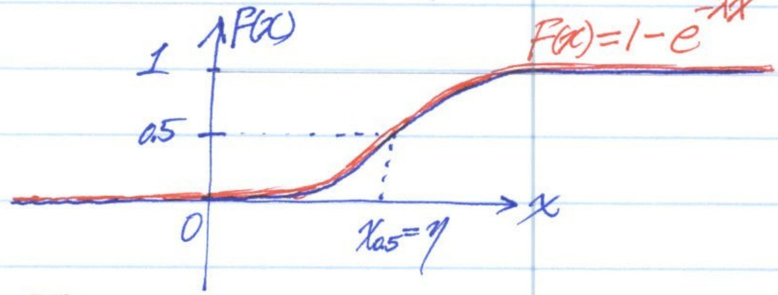


$$X \sim f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0, & x<0.\end{cases}$$

$x>0$

$$F(x)=P(X\leq x)$$
$$=\int_{-\infty}^{x} f(t)\,dt$$
$$=\int_0^x \lambda e^{-\lambda t}\,dt = \left(-e^{-\lambda t}\right)\Big|_0^x = \left(-e^{-\lambda x}\right)-\left(-e^{-\lambda\cdot 0}\right)=1-e^{-\lambda x}$$

i.e. $F(x)=\begin{cases}0, & x\leq 0\\ 1-e^{-\lambda x}, & x\geq 0.\end{cases}$



What's the median? $x_{0.5}=\eta$.

$F(\eta)=\frac{1}{2}$, $1-e^{-\lambda\eta}=\frac{1}{2}$, $e^{-\lambda\eta}=\frac{1}{2}$, $-\lambda\eta\ln e=\ln\frac{1}{2}$

$-\lambda\eta=-\ln 2$ $\quad \eta=\ln 2/\lambda$.

X is used to model lifetimes or waiting times.

memoryless property: $P(T>t+s \mid T>s) = P(T>t)$

it will last t more time units   component has lasted a length of time s

proof: $$\text{left}=\frac{P(T>t+s \cap T>s)}{P(T>s)}=\frac{P(T>t+s)}{P(T>s)}=\frac{1-P(T\leq t+s)}{1-P(T\leq s)}$$
$$=\frac{1-\left(1-e^{-\lambda(t+s)}\right)}{1-\left(1-e^{-\lambda s}\right)}=\frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}=\frac{e^{-\lambda t}\cdot e^{-\lambda s}}{e^{-\lambda s}}=e^{-\lambda t}=P(T>t)$$

§2.2.2. The Gamma Density

(4)

$X \sim Gamma(\alpha, \lambda)$, if $X \sim f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$

$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du. \quad x > 0.$

$$\int_a^b u(x) dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x) du(x)$$
$$= [u(b)v(b) - u(a)v(a)] - \int_a^b v(x) du(x)$$

$\Gamma(1) = \int_0^\infty u^{1-1} e^{-u} du = \int_0^\infty e^{-u} du = 1.$

$\Gamma(2) = \int_0^\infty u^{2-1} e^{-u} du = \int_0^\infty u e^{-u} du$

$= \int_0^\infty u \cdot d(-e^{-u}) = u \cdot (-e^{-u})\Big|_0^\infty - \int_0^\infty (-e^{-u}) du$

$= [0 - 0] + \int_0^\infty e^{-u} du = 1. \quad$ i.e. $\Gamma(1) = \Gamma(2) = 1.$

Note: $\underline{\Gamma(n+1) = n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.}$

$\underline{\Gamma(x+1) = x\Gamma(x)}$

proof: $\Gamma(x+1) = \int_0^\infty u^{(x+1)-1} e^{-u} du = \int_0^\infty u^x \cdot e^{-u} du$

$= \int_0^\infty u^x d(-e^{-u}) = u^x \cdot (-e^{-u})\Big|_0^\infty - \int_0^\infty (-e^{-u}) d(u^x)$

$= [0 - 0] + \int_0^\infty e^{-u} \cdot x u^{x-1} du$

$= x \int_0^\infty u^{x-1} e^{-u} du = x\Gamma(x).$

thus $\Gamma(n+1) = n\Gamma(n) = n \cdot \Gamma((n-1)+1) = n \cdot (n-1)\Gamma(n-1)$

$= n \cdot (n-1) \cdots 2\Gamma(2) = n \cdot (n-1) \cdots 2 \cdot 1 \cdot \Gamma(1) = n!$

$\Gamma(\frac{1}{2}) = \int_0^\infty u^{\frac{1}{2}-1} e^{-u} du = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du \quad$ let $u^{\frac{1}{2}} = t$, $u = t^2$

$= \int_0^\infty t^{-1} e^{-t^2} \cdot 2t dt = 2\int_0^\infty e^{-t^2} dt.$

from $\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 1$

$\Rightarrow \int_0^\infty e^{-\frac{t^2}{2}} dt = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$

$= 2(\sqrt{\pi}/2) = \sqrt{\pi}.$

let $t = \sqrt{2}y$

$\int_0^\infty e^{-y^2} \sqrt{2} dy = \sqrt{\frac{\pi}{2}}$

$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$

i.e. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \Gamma(1 + \frac{1}{2}) = \frac{1}{2} \Gamma(\frac{1}{2})$

$= \sqrt{\pi}/2.$

comparsions of density functions.

Gamma(0.5, 1) v.s. Gamma(1, 1). $\exp(\lambda) = \text{Gamm}(1, \lambda)$.

Gamma(5, 1) v.s. Gamma(10, 1).


```
t <- seq(0.01, 2, 0.05)
gt1 <- dgamma(t, 0.5, 1); gt2 <- dgamma(t, 1, 1)
plot(t, gt1, type='l')
lines(t, gt2, col="red", lty=2)
```

Some examples:

Problem 33. Let $F(x) = 1 - e^{-\alpha x^{\beta}}$ for $x \geq 0$, $\alpha > 0$, $\beta > 0$, $F(x) = 0$ for $x < 0$.
show that $F(x)$ is a cdf, and find p.d.f. $f(x)$.

Ans: Above $F(x)$ satisfies conditions:

1°. $\lim_{x \to -\infty} F(x) = 0$. 2°. $\lim_{x \to \infty} F(x) = 1$, and 3° $F(x)$ is not decreasing function.

$$f(x) = F'(x) = [1 - e^{-\alpha \cdot x^{\beta}}]' = 0 - [e^{-\alpha \cdot x^{\beta}}]'$$

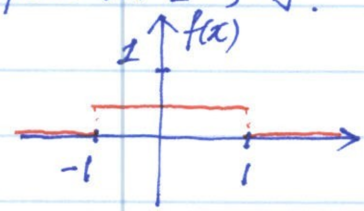
$$= -e^{-\alpha \cdot x^{\beta}} \cdot [-\alpha \cdot x^{\beta}]' = -e^{-\alpha \cdot x^{\beta}} \cdot (-\alpha \cdot \beta x^{\beta - 1})$$

$$= \alpha \beta x^{\beta - 1} e^{-\alpha \cdot x^{\beta}}, \quad x \geq 0.$$

i.e., $f(x) = \begin{cases} \alpha \beta x^{\beta-1} e^{-\alpha \cdot x^{\beta}}, & x \geq 0 \\ 0, & x < 0. \end{cases}$ Weibull distribution.

35. Sketch the pdf and cdf of a r.v that is uniform on $[-1, 1]$.

Ans: $X \sim U[-1, 1]$, $f(x) = \begin{cases} \frac{1}{2}, & x \in [-1, 1] \\ 0, & \text{else}. \end{cases}$



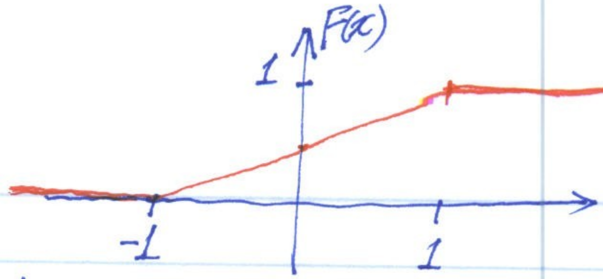
$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$,

if $x \leq -1$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$.

if $-1 \leq x \leq 1$, $F(x) = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{x} f(t)dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{1}{2} dt = \frac{1}{2}(x+1)$.

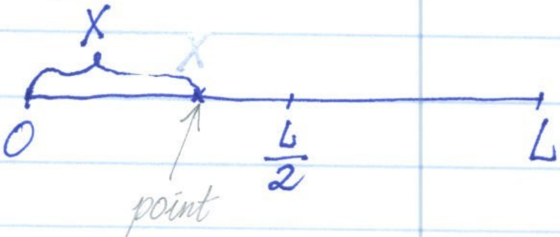
if $x \geq 1$. $F(x) = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{1} f(t)dt + \int_{1}^{x} f(t)dt = 0 + \int_{-1}^{1} \frac{1}{2} dt + 0 = 1$.

thus, $F(x)=\begin{cases} 0, & x\le -1 \\ \frac{1}{2}(x+1), & -1\le x\le 1 \\ 1, & x\ge 1. \end{cases}$



Expl. A point is chosen at random on line segment of length L. Find the prob. that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

Ans: Let X be the length from one left end of the segment to the point randomly selected.



then $X \sim U[0, L]$, i.e., X is a r.v. uniformly distributed over $[0, L]$.

also. $X \sim f(x)=\begin{cases} \frac{1}{L}, & x\in[0, L] \\ 0, & \text{else}. \end{cases}$

Let R denote the ratio of the shorter to the longer segment, then

$$R=\begin{cases} \frac{X}{L-X}, & \text{if } X\le \frac{L}{2} \\ \frac{L-X}{X}, & \text{if } X\ge \frac{L}{2}. \end{cases}$$

law of total probability

$$P(R<\tfrac{1}{4}) = P(R<\tfrac{1}{4} \mid X\le\tfrac{L}{2})P(X\le\tfrac{L}{2}) + P(R<\tfrac{1}{4} \mid X\ge\tfrac{L}{2})P(X\ge\tfrac{L}{2})$$

$$= P(\tfrac{X}{L-X}<\tfrac{1}{4} \mid X\le\tfrac{L}{2})P(X\le\tfrac{L}{2}) + P(\tfrac{L-X}{X}<\tfrac{1}{4} \mid X\ge\tfrac{L}{2})P(X\ge\tfrac{L}{2})$$

$4X < L-X$

$X<\frac{L}{5}$

$$= P(X<\tfrac{L}{5} \mid X\le\tfrac{L}{2})\cdot P(X\le\tfrac{L}{2}) + P(X>\tfrac{4L}{5} \mid X\ge\tfrac{L}{2})P(X\ge\tfrac{L}{2})$$

$$= \frac{P(X<\frac{L}{5}, X\le\frac{L}{2})}{P(X\le\frac{L}{2})}\cdot P(X\le\tfrac{L}{2}) + \frac{P(X>\frac{4L}{5}, X\ge\frac{L}{2})}{P(X\ge\frac{L}{2})}\cdot P(X\ge\tfrac{L}{2})$$

$$= P(X<\tfrac{L}{5}) + P(X>\tfrac{4L}{5})$$

$$= \int_0^{\frac{L}{5}} \frac{1}{L}dx + \int_{\frac{4L}{5}}^{L} \frac{1}{L}dx = \frac{L}{5}\cdot\frac{1}{L} + (L-\frac{4L}{5})\frac{1}{L}$$

$$= \frac{1}{5}+\frac{1}{5} = \frac{2}{5}.$$

Expl. Given p.d.f. $f(x)$. find c and c.d.f $F(x)$.

$$f(x)=\begin{cases} c\cdot(1-x^2), & -1<x<1 \\ 0, & \text{otherwise.} \end{cases}$$

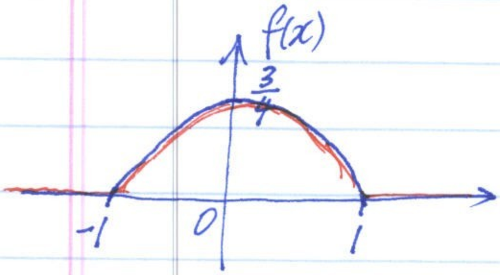
Ans: $\begin{cases} f(x)\geq 0 \\ \int_{-\infty}^{\infty} f(x)dx=1. \end{cases}$ now $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} c(1-x^2)dx + \int_{1}^{\infty} 0\,dx$$

$$= 0 + c\int_{-1}^{1}(1-x^2)dx + 0 = c\cdot \left[x-\tfrac{1}{3}x^3\right]\Big|_{-1}^{1}$$

$$= c\left((1-\tfrac{1}{3}) - (-1-\tfrac{1}{3}(-1))\right) = c\left(1-\tfrac{1}{3}+1-\tfrac{1}{3}\right) = c\cdot\tfrac{4}{3}.$$

Since $\int_{-\infty}^{\infty} f(x)dx=1, \Rightarrow c\cdot\frac{4}{3}=1, \Longrightarrow c=\frac{3}{4}.$



$$f(x)=\begin{cases} \frac{3}{4}(1-x^2), & -1<x<1 \\ 0, & \text{else} \end{cases}$$

$$F(x)=P(X\leq x)=\begin{cases} 0, & x\leq -1 \\ \frac{1}{2}+\frac{3}{4}x-\frac{1}{4}x^3, & -1\leq x\leq 1 \\ 1, & x\geq 1. \end{cases}$$

Problem 44. Let T be an r.v. $Exp(\lambda)$, i.e., $T\sim f(t)=\begin{cases} \lambda e^{-\lambda t}, & t\geq 0 \\ 0, & t<0. \end{cases}$

Let X be a discrete r.v. defined as

$$X=k, \text{ if } k\leq T<k+1, \; k=0,1,\cdots$$

Find the frequency function of X.

Ans: $X=0$, if $0\leq T<1$. $\quad P(X=0)=P(0\leq T<1)=F(1)-F(0)$,

$=1$, if $1\leq T<2$, $\quad$ here $F(b)=\int_0^b \lambda e^{-\lambda t}dt = 1-e^{-\lambda b}, \; \forall b\geq 0$

$=2$, if $2\leq T<3$, $\quad$ thus $P(X=0)=(1-e^{-\lambda\cdot 1})-(1-e^{-\lambda\cdot 0})=1-e^{-\lambda}.$

$\vdots$

$$P(X=1)=P(1\leq T<2)=F(2)-F(1)=(1-e^{-\lambda\cdot 2})-(1-e^{-\lambda\cdot 1})$$
$$= e^{-\lambda}-e^{-2\lambda} = e^{-\lambda}(1-e^{-\lambda})$$

$P(X=2)=P(2\le T<3)=F(3)-F(2)$

$= (1-e^{-\lambda\cdot 3})-(1-e^{-\lambda\cdot 2}) = e^{-2\lambda}-e^{-3\lambda} = e^{-2\lambda}(1-e^{-\lambda})$

$\cdots$

$P(X=k)=e^{-k\lambda}(1-e^{-\lambda}), \quad k=0,1,2,\cdots$

$X\sim$ Geometric dist. if let $p=1-e^{-\lambda}$, $1-p=e^{-\lambda}$,

then $P(X=k)=(1-p)^k\cdot p, \quad k=0,1,2,\cdots$

Problem 47. If $\alpha>1$, show that the gamma density has a maximum at $(\alpha-1)/\lambda$.

proof: $X\sim Gamm(\alpha,\lambda)$, $\quad f(x)=\dfrac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, \quad x>0.$

$$f'(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\left[x^{\alpha-1}\cdot e^{-\lambda x}\right]' \qquad \left(u(x)\cdot v(x)\right)'=u'(x)v(x)+u(x)\cdot v'(x)$$

$$=\frac{\lambda^\alpha}{\Gamma(\alpha)}\left(\left[x^{\alpha-1}\right]'e^{-\lambda x}+x^{\alpha-1}\cdot\left[e^{-\lambda x}\right]'\right)$$

$$=\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\left((\alpha-1)x^{\alpha-1-1}\cdot e^{-\lambda x}+x^{\alpha-1}\cdot e^{-\lambda x}\cdot(-\lambda x)'\right)$$

$$=\frac{\lambda^\alpha}{\Gamma(\alpha)}\left((\alpha-1)x^{\alpha-2}\cdot e^{-\lambda x}+x^{\alpha-1}e^{-\lambda x}(-\lambda)\right)$$

$$=\frac{\lambda^\alpha}{\Gamma(\alpha)}\left(x^{\alpha-2}\cdot e^{-\lambda x}\left((\alpha-1)-\lambda\cdot x\right)\right)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot x^{\alpha-2}\cdot e^{-\lambda x}\left(\alpha-1-\lambda x\right)$$

Let $f'(x)=0 \Longrightarrow \alpha-1-\lambda x=0 \Longrightarrow x=\dfrac{\alpha-1}{\lambda}$.