Class 9

Monday, September 20, 2021 9:32 AM

2. When working with real numbers, our universal set is $\mathbb{R}$. Find each of the following sets.
   (a) $[6,8] \cup [2,7)$
   (b) $[6,8] \cap [2,7)$
   (c) $[0,1]^c$
   (d) $[6,8] - (2,7)$

(a) $[2,8]$

$[6,8]$ "closed interval"
includes endpoints

(b) $[6,7)$

(c) $[0,1]^c$ = all in $\mathbb{R}$ except $[0,1)$
$(-\infty, 0) \cup (1, \infty)$

(d) $[7,8]$

P4) (Do it just for practice)

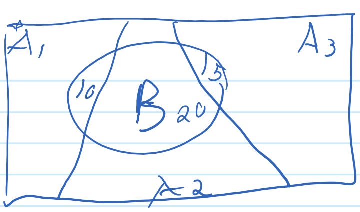
4. A coin is tossed twice. Let S be the set of all possible pairs that can be observed, i.e., $S = \{H,T\} \times \{H,T\} = \{(H,H),(H,T),(T,H),(T,T)\}$. Write the following sets by listing their elements.
   (a) A: The first coin toss results in head. → $\{(H,H),(H,T)\}$
   (b) B: At least one tail is observed. → $\{(H,T)(T,H)(T,T)\}$
   (c) C: The two coin tosses result in different outcomes.

↳ $\{HT, TH\}$

6. Suppose that A_1, A_2, A_3 form a partition of the universal set S. Let B be an arbitrary set. Assume that we know:

$$|B \cap A_1| = 10$$
$$|B \cap A_2| = 20$$
$$|B \cap A_3| = 15$$

Find $|B|$.



$|B| = 45$

12. Recall that

$$\{H,T\}^3 = \{H,T\} \times \{H,T\} \times \{H,T\}$$
$$= \{(H,H,H),(H,H,T),\cdots,(T,T,T)\}.$$

Consider the following function

$$\boxed{f : \{H,T\}^3 \longrightarrow \mathbb{N} \cup \{0\}.}$$ — nonnegative integers

Defined as

$$f(x) = \text{the number of H's in x.}$$

For example:

$$f(HTH) = 2.$$

(a) Determine the domain and co-domain for f. → $\{H,T\}^3$ domain

(b) Find range of f:Range(f).

(c) If we know $f(x) = 2$, what can we say about x? → co domain — defined by the function definition

→ range of f: $= \{0,1,2,3\}$

(c) HHT HTH THH

14. * Let A and B be two events such that:

$$P(A) = 0.4, P(B) = 0.7, P(A \cup B) = 0.9$$

(a) Find $P(A \cap B)$. → .2 → $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

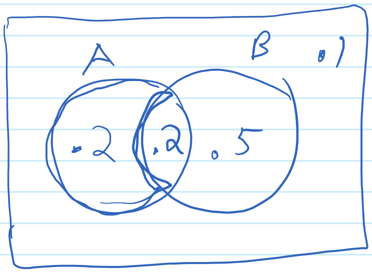
(b) Find $P(A^c \cap B)$.

(c) Find $P(A - B)$.

(d) Find $P(A^c - B)$.

(e) Find $P(A^c \cup B)$.

(f) Find $P(A \cap (B \cup A^c))$.



(b) $P(A^c \cap B) = P(B \setminus A)$

(c) $P(A \setminus B) = P(A \cap B^c)$

(d) $P(A^c - B) = P(A^c \cap B^c) = P(A \cup B)^c$
$= 1 - .9 = .1$

(e) $P(A^c \cup B) = .8$

(f) $P(A \cap (B \cup A^c)) = A \cap B = .2$

$= P((A \cap B) \cup (A \cap A^c)) = .2$

dist. Law $\quad = \phi$

P16)

16. Consider a random experiment with a sample space.

$$S = \{1, 2, 3, \cdots\}. \longrightarrow \text{countable}$$

Suppose that we know:

$$P(k) = P(\{k\}) = \frac{c}{3^k} \quad \text{for} \quad k = 1, 2, \cdots$$

where c is a constant number.
(a) Find c.
(b) Find $P(\{2, 4, 6\})$.
(c) Find $P(\{3, 4, 5, \cdots\})$.

$$P(1) = \frac{c}{3}, \quad P(2) = \frac{c}{3^2}, \quad P(3) = \frac{c}{3^3}, \ldots \text{ etc}$$

c : any constant > 0

(a) $P(S) = 1$

$$1 = \{1, 2, 3, \ldots\} = P(1) + P(2) + \cdots + P(k) + \cdots \to \infty$$

$$= \sum_{k=1}^{\infty} \frac{c}{3^k} = c \cdot \underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k}$$

geometric series

$0 < q < 1$

I know: $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$

$q = \frac{1}{3}$

$$c \cdot \left[\frac{1}{1-q} - 1\right] = c\left(\underbrace{\frac{1}{2/3} - 1}_{0.5}\right)$$

$$\Rightarrow 1 = c \cdot 0.5 \Rightarrow c = \underline{\underline{2}}$$

(b) $P(2, 4, 6) = P(2) + P(4) + P(6)$

$$= \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} =$$

(c) $P(3, 4, 5, \ldots\ldots) = 1 - (1, 2)$ ✓ $\boxed{\text{complement}}$

$$= 1 - \frac{2}{3} - \frac{2}{3^2} =$$

18. Let T be the time needed to complete a job at a certain factory. By using the historical data, we know that

$$P(T \le t) = \begin{cases} \frac{1}{16}t^2 & \text{for} \quad 0 \le t \le 4 \\ 1 & \text{for} \quad t \ge 4 \end{cases}$$

(a) Find the probability that the job is completed in less than one hour, i.e., find $P(T \le 1)$.
(b) Find the probability that the job needs more than 2 hours.
(c) Find the probability that $1 \le T \le 3$.

Events are time intervals $[0, t] = (T \le t)$

$p(T \le t) = \frac{1}{16} t^2$

(a) $p(T \le 1) = \frac{1}{16} \cdot 1^2 = 1/16$

(b) More than 2 hours $\Longleftrightarrow$ [less or equal 2]^c

$P(T > 2) = 1 - p[T \le 2]$

$1 - 1/16 \cdot 2^2 = 3/4$

(c) $p(1 \le T \le 3)$

$= p(T \le 3) - p(T \le 1)$

$\frac{1}{16} \cdot 3^2 - \frac{1}{16} = 1/2$

HW 2

P22:

Suppose that of all the customers at a coffee shop:

-70% purchase a cup of coffee.
-40% purchase a piece of cake.
-20% purchase both a cup of coffee and a piece of cake.

Given that a randomly chosen customer has purchased a piece of cake, what is the probability that he/she has also purchased a cup of coffee?

$p(\text{coffee} \mid \text{cake}) = \frac{p(\text{coffee} \cap \text{cake})}{p(\text{cake})} = \frac{.2}{.4} = .5$

P24:

A **real** number X is selected $\boxed{\text{uniformly}}$ at random in the continuous interval $[0, 10]$. (For example, X could be 3.87.)
(a) Find $P(2 \le X \le 5)$.
(b) Find $P(X \le 2 | X \le 5)$.
(c) Find $P(3 \le X \le 8 | X \ge 4)$.

(a) $P(2 \le X \le 5) = .3$

(b) $P(X \le 2 | X \le 5) = \dfrac{\overbrace{P(X \le 2) \cap P(X \le 5)}^{P(X \le 2)}}{P(X \le 5)}$

$= \dfrac{.2}{.5} = \underline{\underline{.4}}$

(c) $P(3 \le X \le 8 | X \ge 4)$

$= \dfrac{P(4 \le X \le 8)}{P(X \ge 4)} = \dfrac{.4}{.6} = \underline{\underline{2/3}}$

P26

I roll a dice n times, $n \in \mathbb{N}$. Find the probability that numbers 1 and 6 are both observed at least once.

A_1 : at least one 1
A_6 : at least one 6

$P(A_1 \cap A_6) = P(A_1) + P(A_6) - P(A_1 \cup A_6)$

$(1 - P(\text{no } 1)) + (1 - P(\text{no } 6)) + 1 - P(\text{no } 1 \,\&\, \text{no } 6)$

$A_1^c =$ "no 1"
$A_6^c =$ "no 6"

$(A_1 \cup A_6)^c = (A_1^c \cap A_6^c)$ De Morgans

$Pr(\text{no } 1) = \left(\frac{5}{6}\right)^n$ $\quad Pr(\text{no } 6) = \left(\frac{5}{6}\right)^n$

$Pr(\text{no } 1 \,\&\, \text{no } 6) = \left(\frac{4}{6}\right)^n$

$= \left(1 - \left(\frac{5}{6}\right)^n\right) + \left(1 - \left(\frac{5}{6}\right)^n\right) - \left(1 - \left(\frac{4}{6}\right)^n\right)$

simplify $=$ $= 2\left[1 - \left(\frac{5}{6}\right)^n\right] - 1 + \left(\frac{2}{3}\right)^n.$

simplify =

$$= 2\left[1-\left(\frac{5}{6}\right)^n\right] - 1 + \left(\frac{2}{3}\right)^n.$$

P34:

P34:

I toss a fair die twice, and obtain two numbers X and Y . Let A be the event that X = 2, B be the event that X + Y = 7, and C be the event that Y = 3.

(a) Are A and B independent?

(b) Are A and C independent?

(c) Are B and C independent?

(d) Are A, B, and C are independent?

$A \cap B = \{2, 5\}$

(a) $A \cap B = \{2,5\}$ $\quad P(A) = 1/6$ $\quad P(X+Y=7) = 1/6$

$P(A \cap B) = 1/36 = P(A) \cdot P(B)$ Indepd!

b) $A \cap C = \{2,3\}$

$P(A \cap C) = \frac{1}{36} = P(X=2) \cdot P(Y=3) = 1/6 \cdot 1/6$

$\Rightarrow$ indept.

c) $(B \cap C) = \{4, 3\}$

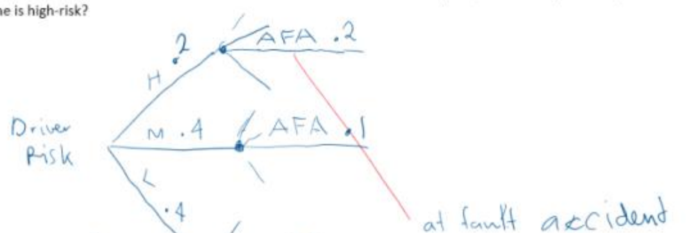
$P(B \cap C) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(X+Y=7) \cdot P(Y=3)$

d) $A \cap B \cap C = \emptyset$ (impossible that each occurs

$P(\emptyset) = 0 \neq P(A) \cdot P(B) \cdot P(C) > 0$

not independent.

Problem : An insurance company writes policies for a large number of newly-licensed drivers each year. Suppose 40% of these are low-risk drivers, 40% are moderate risk, and 20% are high risk. The company has no way to know which group any individual driver falls in when it writes the policies. None of the low-risk drivers will have an at-fault accident in the next year, but 10% of the moderate-risk and 20% of the high-risk drivers will have such an accident. If a driver has an at-fault accident in the next year, what is the probability that he or she is high-risk?



Bayes' Rule:

$$P(H \mid AFA) = \frac{P(AFA \mid H)\, P(H)}{P(AFA \mid H) \cdot P(H) + P(AFA \mid M)\, P(M) + P(AFA \mid L)\, P(L)}$$

$$= \frac{.2 * .2}{.2 * .2 + .1 * .4 + 0 * .4} = \frac{.04}{.04 + .04}$$

$P(\text{high risk} \mid AFA) = 0.5 = \frac{1}{2}$

36. * A box contains two coins: a regular coin and one fake two headed coin (P(H)=1). I choose a coin at random and toss it n times. If the first n coin tosses result in heads, what is the probability that the $(n+1)^{th}$ coin toss will also result in heads?

A_n : 1^{st} n tosses are heads

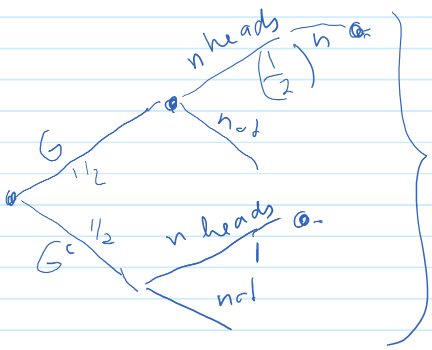
H_{n+1} head in $(n+1)^{st}$ toss

G = Coin is Fair

G^c = Coin is fake

$$P(H_{n+1} \mid G) = 1/2$$

$$P(H_{n+1} \mid G^c) = 1$$



$$Pr(G \mid n \text{ heads}) \overset{\text{Bayes}}{=} \frac{\overset{(1/2)^n}{P(n \text{ heads} \mid G)}\,\overset{1/2}{P(G)}}{\underset{(1/2)^n}{P(n \text{ heads} \mid G)}\,\underset{1/2}{P(G)} + \underset{1}{P(n \text{ heads} \mid G^c)} \cdot \underset{1/2}{P(G^c)}}$$

$$= \frac{(1/2)^{n+1}}{(1/2)^{n+1} + 1/2}$$

$$P(G^c \mid n \text{ heads}) = 1 - P(G \mid n \text{ heads})$$

$$= \frac{1/2}{\left(\frac{1}{2}\right)^{n+1} + \frac{1}{2}}$$

$$P(H_{n+1} \mid A_n) = \overset{1/2}{P(H_{n+1} \mid G, n \text{ heads})} \cdot P(G \mid n \text{ heads})$$

$$+ \underset{=1}{P(H_{n+1} \mid G^c, n \text{ heads})} \cdot P(G^c \mid n \text{ heads})$$

$$= \frac{(1/2)^{n+2}}{(1/2)^{n+1} + \frac{1}{2}} + \frac{\frac{1}{2}}{(1/2)^{n+1} + \frac{1}{2}}$$

$$= \frac{\frac{1}{2}\left[\left(\frac{1}{2}\right)^{n+1} + 1\right]}{\frac{1}{2}\left[\left(\frac{1}{2}\right)^{n} + 1\right]} = \frac{1 + 2^{n+1}}{2 + 2^{n+1}}$$