Midterm Exam 1

MATH 755/855, October 16, 2020

Name Solution Score

1. (10pts) Three cards are chosen at random from a deck of 52 playing cards. What is the probability that they

(a) (5pts) are all in the same suit?

(b) (5pts) have the same value?

(a) $A =$ "all in the same suit"

$$P(A) = \frac{\binom{4}{1}\binom{13}{3}}{\binom{52}{3}} = \frac{22}{425} = 0.052.$$

(b) $B =$ " same value "

$$P(B) = \frac{\binom{13}{1}\binom{4}{3}}{\binom{52}{3}} = \frac{1}{425} = 0.0024.$$

2. (10pts) Suppose that A and B are two events for which $P(A) = 0.5$ and $P(B) = 0.4$.

(a) (5pts) If A and B are mutually exclusive, what is the probability that either A or B occurs;

(b) (5pts) If A and B are independent, what is the probability that either A or B occurs;

(a) A, B are mutually exclusive, then $A \cap B = \phi$, $P(A \cap B) = 0$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - 0 = 0.9.$

(b). A, B are independent, then $P(A \cap B) = P(A) \cdot P(B)$. Thus

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - (0.5)(0.4) = 0.7.$

3. (10pts) A pair of fair dice are rolled.

(a) (5pts) Given the sum of the face values is 5, what is the probability that at least one of the dice came up a three?

(b) (5pts) What is the conditional probability that the sum of two dice is bigger than 8, given the one of the two dice is 5?

(a) Let $A =$ "sum of the face values is 5". $A = \{(1,4), (2,3), (3,2), (4,1)\}$.

$B =$ " at least one of dice came up a 3".

Using reduced sample space method: Given A, the new sample space is A.

i.e. $\Omega_{new} = \{(1,4), (2,3), (3,2), (4,1)\}$. Thus, $P(B|A) = \frac{2}{4} = \frac{1}{2}$.

(b). $C =$ "the one of two dice is 5". $C = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5)\}$

$n(C) = 11$. $D =$ "sum > 8". Again, using reduced sample space method,

$P(D|C) = \frac{5}{11} = 0.4545.$

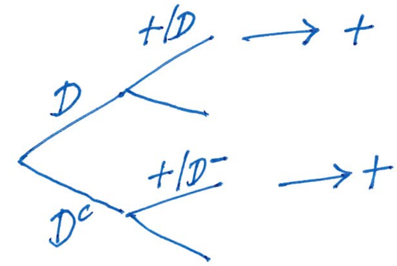
4. (15pts) A laboratory blood test is 98 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 5 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability 0.05, the test will imply he or she has the disease.) Assume 0.1 percent of the population actually has the disease.

(a) (8pts) One person is randomly chosen from population and tested, what is the probability that the test is positive?

(b) (7pts) What is the conditional probability that the person is actually healthy, given that the test is positive?

(a) Let $+$ = "test positive", D = "person with disease". D^c = "healthy person".

$$P(+) = P(D)\cdot P(+|D) + P(D^c)P(+|D^c)$$
$$= (0.001)\cdot(0.98) + (0.999)\cdot(0.05) = 0.051.$$



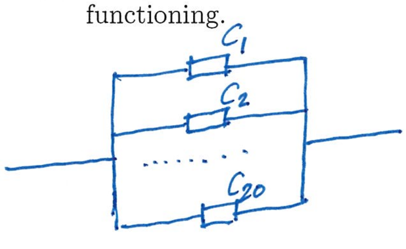
(b). $$P(D^c|+) = \frac{P(D^c \cap +)}{P(+)} = \frac{P(D^c)P(+|D^c)}{P(+)}$$
$$= \frac{(0.999)(0.05)}{0.051} = 0.982$$

5. (10pts) A parallel system functions when at least one of its components works. Consider a parallel system of 20 components and suppose that each component independently works with probability 0.8.

(a) (5pts) What is the probability that the system functions?

(b) (5pts) Find the conditional probability that components 1 and 2 work, given that the system is functioning.

(a).



Let C_i = "i-th Component functions", $i = 1, 2, 3, \cdots, 20$.

$P(C_i) = 0.8$.

Let A = "system functions".

$P(A) = P(C_1 \cup C_2 \cup \cdots \cup C_{20})$. A^c = "system failures". $A^c = C_1^c \cap C_2^c \cap \cdots \cap C_{20}^c$.

$$P(A) = 1 - P(A^c) = 1 - P(C_1^c \cap C_2^c \cap \cdots \cap C_{20}^c)$$
$$= 1 - P(C_1^c)\cdot P(C_2^c)\cdots P(C_{20}^c) = 1 - (1-0.8)^{20} = 1 - 0.2^{20} \approx 1.$$

(b). $$P(C_1 \cap C_2 | A) = \frac{P(C_1 \cap C_2 \cap A)}{P(A)} = \frac{P(C_1 \cap C_2)}{P(A)} = \frac{P(C_1)P(C_2)}{P(A)} \approx \frac{0.8^2}{1} = 0.64.$$

since $C_1 \cap C_2 \subset A$.

6. (15pts) Suppose that each child born to a couple is equally likely to be a boy or a girl independent of the sex distribution of the other children in the family. For a couple having 5 children, compute the probabilities of the following events:

(a) (5pts) All children are of the same sex;

(b) (5pts) The 3 eldest are boys and the others are girls;

(c) (5pts) Exactly 3 are boys;

(a). Let X be the number of boys in 5 children, then $X \sim B(5, \frac{1}{2})$.

$P\{\text{same sex}\} = P\{X=0 \text{ or } 5\} = P(X=0) + P(X=5) = \binom{5}{0}(\frac{1}{2})^0(\frac{1}{2})^5 + \binom{5}{5}(\frac{1}{2})^5(\frac{1}{2})^0 = 0.0625$.

(b). $P(B_1B_2B_3G_4G_5) = P(B_1)P(B_2)P(B_3)P(G_4)P(G_5) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = 0.03125$

(c). $P(3 \text{ boys}) = P(X=3) = \binom{5}{3}(\frac{1}{2})^3(\frac{1}{2})^2 = 0.3125$

7. (15pts) Suppose the counts recorded by a geiger counter follow a Poisson process with an average of two counts per minute.

(a) (8pts) What is the probability that there are exactly 2 counts in one minute interval?

(b) (7pts) What is the probability that there are at least one count in two minutes interval?

(a). Let X be the number of counts in one minute interval. Then $X \sim P(2)$.
i.e., X follows Poisson distribution with $\lambda = 2$.
Then, $P(X=2) = \frac{2^2}{2!}e^{-2} = 2e^{-2} = 0.271$.

(b). Let Y be the number of counts in two-minute interval.
Then $Y \sim P(4)$, here new paramter or new rate is $2*2=4$.

$P(Y \geq 1) = P(Y = 1 \text{ or } 2 \text{ or } 3 \text{ or } \cdots) = 1 - P(Y=0)$

$= 1 - \frac{4^0}{0!}e^{-4} = 1 - e^{-4} = 0.982$.

8. (15pts) A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first.

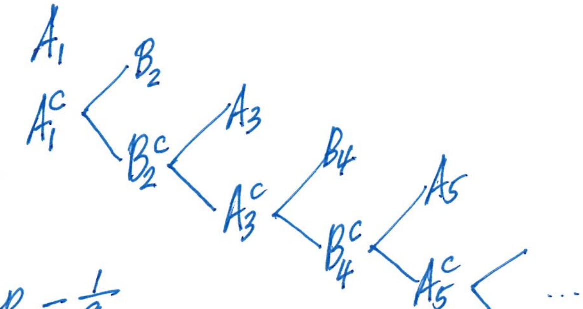
(a) (8pts) Find the frequency function for the total number of rolls.

(b) (7pts) Find the probability that the final rolls is made by A.

(a). Let A_i = "i-th roll, A gets 9". then $P(A_i) = \frac{4}{36} = \frac{1}{9} =: P_A$

B_i = "i-th roll, B gets 6". then $P(B_i) = \frac{5}{36} =: P_B$.

Let X be the total number of rolls.



Sample space

$\Omega = \{A_1, A_1^cB_2, A_1^cB_2^cA_3, A_1^cB_2^cA_3^cB_4, \cdots\}$

Values of $X = \{1, 2, 3, 4, \cdots\}$

$\{X=1\} = \{A_1\}$, $\quad P\{X=1\} = P(A_1) = P_A = \frac{1}{9}$.

$\{X=2\} = \{A_1^cB_2\}$, $\quad P\{X=2\} = P(A_1^cB_2) = (1-P_A)P_B = \frac{8}{9}\cdot\frac{5}{36}$

$\{X=3\} = \{A_1^cB_2^cA_3\}$ $\quad \cdots$

$\{X=4\} = \{A_1^cB_2^cA_3^cB_4\}$ $\quad P\{X=4\} = (1-P_A)(1-P_B)(1-P_A)P_B = (1-P_A)^2(1-P_B)P_B$.

$\cdots\cdots$

Thus, when $n = 2m+1$, odd numbers for n; $m = 0, 1, 2, \cdots$; $n = 1, 3, 5, 7, \cdots$

$$P(X=n) = P(X=2m+1) = (1-P_A)^m\cdot(1-P_B)^m\cdot P_A.$$

when $n = 2m$, even numbers, $n = 2, 4, 6, 8, \cdots$, $m = 1, 2, 3, \cdots$

$$P(X=n) = P(X=2m) = (1-P_A)^m(1-P_B)^{m-1}\cdot P_B.$$

i.e. $P(X=n) = \begin{cases} (1-P_A)^m(1-P_B)^m P_A; & n = 2m+1, \\ (1-P_A)^m\cdot(1-P_B)^{m-1}\cdot P_B; & n = 2m. \end{cases}$

(b) $P\{\text{final rolls by } A\} = P\{X = 1 \text{ or } 3 \text{ or } 5 \text{ or } 7 \text{ or} ..\}$

$$= \sum_{n \text{ odd}} P(X=n) = \sum_{m=0}^{\infty} P(X=2m+1) = \sum_{m=0}^{\infty}(1-P_A)^m(1-P_B)^m\cdot P_A = P_A\cdot\frac{1}{1-(1-P_A)(1-P_B)}$$

$$= \frac{1}{9}\cdot\frac{1}{1-(1-\frac{1}{9})(1-\frac{5}{36})} = 0.4737. \ \left(=\frac{9}{19}\right)$$