1.2 +

Tuesday, August 31, 2021 1:13 PM

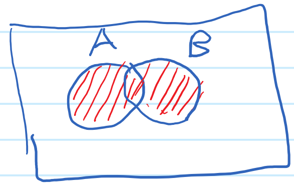
1.2.2 Set Operations

Union, Intersection, Complement, Difference,

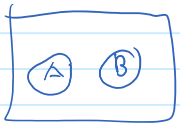
$\boxed{C} = A \cup B$

in words A "or" B

↓ event condition ↓ event



Mutially Exclusuive (or Disjoint)



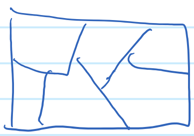
$\boxed{C} = A \cap B$

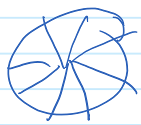
wording A and B

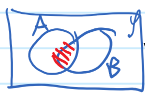
↓ restriction

Partition

Pizza

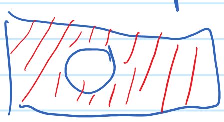




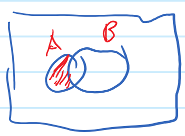


Complement:

$\boxed{A^c}$ = : everything except A



Difference : $A \setminus B$ "only A"



$\|$

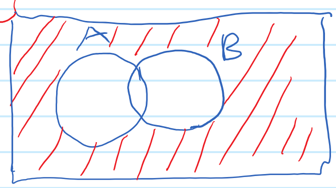
Note : $A \cap B^c$!

Theorem 1.1: De Morgan's law
For any sets $A_1, A_2, \cdots, A_n$, we have

- $(A_1 \cup A_2 \cup A_3 \cup \cdots A_n)^c = A_1^c \cap A_2^c \cap A_3^c \cdots \cap A_n^c$;
- $(A_1 \cap A_2 \cap A_3 \cap \cdots A_n)^c = A_1^c \cup A_2^c \cup A_3^c \cdots \cup A_n^c$.

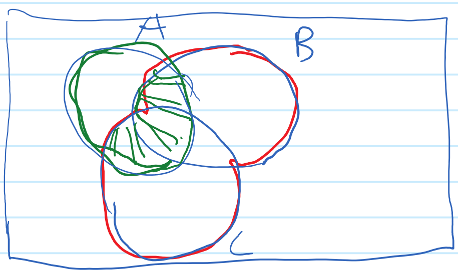
Simpler 2 events

$$\boxed{(A \cup B)^c} = A^c \cap B^c$$



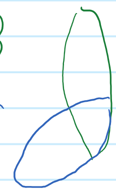
Theorem 1.2: Distributive law
For any sets A, B, and C we have

- $\boxed{A \cap (B \cup C)} = (A \cap B) \cup (A \cap C)$;
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



$A \cap B$

$A \cap C$



Cartesian Product, Multiplication Principle

Def: A **Cartesian product** of two sets A and B, written as $A \times B$, is the set containing **ordered** pairs from A and B.

That is, if $C = A \times B$, then each element of C is of the form (x, y), where $x \in A$ and $y \in B$:

Note that here the pairs are ordered, so for example, $(1, H) \neq (H, 1)$.
Thus $A \times B$ is **not** the same as $B \times A$.

Read this

1.2.3 Cardinality: Countable and Uncountable

Sets

Here we need to talk about **cardinality** of a set, which is basically the size of the set. The cardinality of a set is denoted by |*A*|

We first discuss cardinality for finite sets and then talk about infinite sets.

Finite Sets:

Consider a set *A*. If *A* has only a finite number of elements, its cardinality is simply the number of elements in *A*.

For example, if *A*={2,4,6,8,10}, then |*A*| =

First:

Inclusion-exclusion principle:

1. $|A \cup B| = |A| + |B| - |A \cap B|$,
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

1. See Figure 1.16 Problem 2

Generally, for n finite sets $A_1, A_2, A_3, \cdots, A_n$, we can write

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j|$$
$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

Do Example 1.5 (Party)

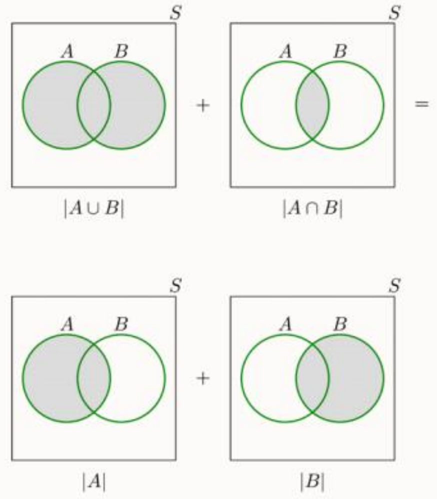


Fig.1.16 - Venn diagrams for some identities.

Countable Sets:

Definition 1.1
Set A is called countable if one of the following is true

a. if it is a finite set, $|A| < \infty$; or
b. it can be put in one-to-one correspondence with natural numbers $\mathbb{N}$, in which case the set is said to be countably infinite.
   A set is called uncountable if it is not countable.

Some Useful Results:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and any of their subsets are countable.
- Any set containing an interval on the real line such as $[a, b], (a, b], [a, b)$, or (a, b), where $a < b$ is uncountable.

Theorem 1.3
Any subset of a countable set is countable.
Any superset of an uncountable set is uncountable.

Theorem 1.4
If $A_1, A_2, \cdots$ is a list of countable sets, then the set $\bigcup_i A_i = A_1 \cup A_2 \cup A_3 \cdots$ is also countable.

Proof:

- $A_1 = \{a_{11}, a_{12}, \cdots\}$,
- $A_2 = \{a_{21}, a_{22}, \cdots\}$,
- $A_3 = \{a_{31}, a_{32}, \cdots\}$,
- ...

Now we need to make a list that contains all the above lists. This can be done in different ways. One way to do this is to use the ordering shown in Figure 1.12 to make a list. Here, we can write

$$\bigcup_i A_i = \{a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, \cdots\} \tag{1.1}$$

$$\begin{array}{ccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & \cdots \\
 & \swarrow & & \nearrow & & & & &
\end{array}$$

Similarly

Theorem 1.5
If A and B are countable, then $A \times B$ is also countable.

$$\begin{array}{cccccccc}
(a_1,b_1) & \to & (a_1,b_2) & & (a_1,b_3) & \to & (a_1,b_4) & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \\
(a_2,b_1) & & (a_2,b_2) & & (a_2,b_3) & & (a_2,b_4) & \cdots \\
\downarrow & \nearrow & & \swarrow & & & & \\
(a_3,b_1) & & (a_3,b_2) & & (a_3,b_3) & & (a_3,b_4) & \cdots \\
 & \swarrow & & & & & & \\
(a_4,b_1) & & (a_4,b_2) & & (a_4,b_3) & & (a_4,b_4) & \cdots
\end{array}$$

Consequence:

Rational Numbers

$$\mathbb{Q} = \bigcup_{i\in\mathbb{Z}} \bigcup_{j\in\mathbb{N}} \{\frac{i}{j}\}.$$

Uncountable Sets:

However, as we mentioned, intervals in R are uncountable. Thus, you can never provide a list in the form of $\{a_1,a_2,a_3,\cdots\}$ *that contains all the elements* in, say, [0,1]. *This fact can be proved using a so-called diagonal argument, and we omit the proof here as it is not instrumental for the rest of the book.*

1.2.4 Functions

Please read about domain, co-domain and range of a function.

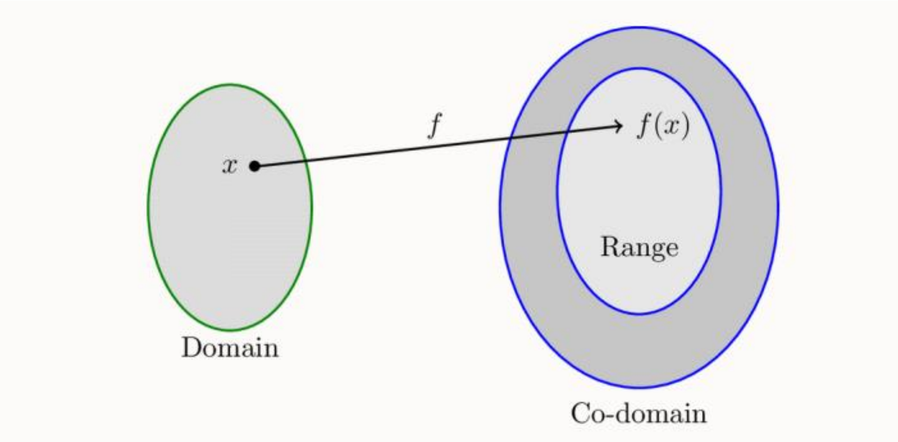


Fig.1.14 - Function $f : A \to B$, the range is always a subset of the co-domain.

RECOMMENDATION: Solve all 5 problems of

1.2.5 Solved Problems: Review of Set Theory