Homework 2 Math 756/856

Problem 1 (10 points). Let r.v's $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \exp(1)$; $n=20$.

$$X_1 \sim f_X(x) = \begin{cases} e^{-x}, & x>0, \\ 0, & x \le 0. \end{cases}$$ exponential distribution with $\lambda = 1$.

(a). Find $P(X_1 < 1)$. note, X_1 is not order statistic, $X_1 \sim f_X(x) = \exp(1)$.

(b). Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be their order statistics.
Find $P(X_{(1)} < 1)$.

(c) Also, find $P(X_{(20)} < 1)$. note, $n=20$. $X_{(20)} = \max\{X_1, X_2, \cdots X_{20}\}$.

(d). Let $V = X_{(1)}$, find $f_V(v) = ?$, i.e. p.d.f of r.v. $V = X_{(1)}$.
From $f_V(v)$, find $E(V)$.

(e). Let $U = X_{(n)} = X_{(20)}$, find $f_U(u) = ?$ i.e. p.d.f of r.v. $U = X_{(20)}$.
From $f_U(u)$, find $E(U)$.

Note: In parts (b) and (c), it is not necessary that one has to find the p.d.f's of $X_{(1)}$ and $X_{(20)}$ in order to find $P(X_{(1)} < 1)$ and $P(X_{(20)} < 1)$.
for (e), one could use R-function "integrate" to do the calculation!

Problem 2. (10 points). Let a random sample $X_1, X_2, \cdots, X_n \sim U[0, 1]$, i.e. uniform distribution over $[0, 1]$.

(a). Consider $n=10$. Let $Y = X_{(5)}$, i.e. 5-th smallest of $\{X_1, X_2, \cdots X_{10}\}$.
What distribution does r.v. Y follow? Also from Y's distribution,

find $P(0.4 < Y < 0.6) = ?$

(b). Similar to part (a), but consider $n=100$ in this part (b).
Let $W = X_{(50)}$; find what distribution does r.v. W follow?
Also, find $P(0.4 < W < 0.6)$ from above W's distribution.

(C) In part (b) with $n=100$. Let's assume one is not able to derive W's distribution. In this case, using statistical simulation method (or Bootstrapping method) to approximate probability $P(0.4 < W < 0.6)$.
Comment your simulation result with your theoretical result in part (b).

Problem 3 (10 points). Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(100, 10^2)$; $\mu=100$, $\sigma^2=10^2$, $\sigma=10$.

(a) Find $P(X_1 > 120)$; note $X_1 \sim N(100, 10^2)$, X_1 is not an order statistic

(b). Consider $n=30$, $X_{(30)} = \max\{X_1, X_2, \cdots X_{30}\}$. Find $P(X_{(30)} > 120)$.

(C). Using simulation method to approximate above probability $P(X_{(30)} > 120)$.

In specific; simulate $K=10{,}000$'s $X_{(30)}$ using "for" loop in R, calculate the proportion of $X_{(30)}$ which is greater than 120 among above 10,000's $X_{(30)}$. Compare your result from your simulation study with your theoretical value of $P(X_{(30)} > 120)$ in part (b).

(d). Repeat parts (b) and (c) with $n=100$. i.e. find $P(X_{(100)} > 120)$ using both anlytical and simulation methods. Comment your results.