5. Suppose that X is a discrete rv w/

$P(X=1)=\theta$ and $P(X=2)=1-\theta$.

Three independent obs are made $x_1=1, x_2=2, x_3=2$

a) find the method of moments estimate of θ

for a discrete r.v., $EX = \Sigma\left(k \cdot P(X=k), \cdots, n \cdot P(X=n)\right)$

for this r.v., $EX = 1 \cdot P(X=1) + 2 \cdot P(X=2) = \frac{5}{3}$

$\frac{5}{3} = \theta + 2 \cdot (1-\theta)$

$\theta = \frac{1}{3}$

b) what is the likelihood function?

$lik(\theta) = f(x_1|\theta) \cdot f(x_2|\theta) \cdot f(x_3|\theta)$

$= \theta \cdot (1-\theta)^2$

c) what is the maximum likelihood estimate of θ?

mle $\rightarrow$ $\log lik(\theta) = \log\left(\theta(1-\theta)^2\right)$

$= \log\theta + 2\log(1-\theta)$

$\frac{\partial \log lik}{\partial\theta} = \frac{1}{\theta} - \frac{2}{1-\theta}$

look for critical point

$0 = \frac{1}{\theta} - \frac{2}{1-\theta}$

$1-\theta = 2\theta$

$\theta = \frac{1}{3}$

7. Suppose that X follows a geometric distribution:

$P(X=k) = p(1-p)^{k-1}$

assume an iid sample size of n

a) find the method of moments estimate of p

EX of geometric $= \sum_{k=1}^{\infty} kp(1-p)^{k-1} = \frac{1}{p}$

$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$\frac{1}{\hat{p}} = \bar{X}$, $\hat{p} = \frac{1}{\bar{X}}$

b) find the mle of p

$lik(p) = \prod_{i=1}^{n}\left[p(1-p)^{x_i-1}\right] = p^n(1-p)^{\left(\sum_{i=1}^{n} x_i\right)-n}$

$\log lik(p) = n\log(p) + \left[\left(\sum_{i=1}^{n} x_i\right)-n\right]\log(1-p)$

$\frac{\partial \log lik}{\partial p} = \frac{n}{p} - \frac{(\Sigma x_i - n)}{1-p} = 0$

$\frac{1-p}{p} = \frac{\Sigma x_i - n}{n}$

$\frac{1}{p} - \cancel{1} = \frac{\Sigma x_i}{n} - \cancel{1}$ $\quad \frac{\Sigma x_i}{n} = \bar{X}$

$\hat{p} = 1/\bar{X}$

21. Suppose that X_1, X_2, X_n are i.i.d. w/ density function

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

a. find the method of moments estimate of θ

a) $EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\theta}^{\infty} x e^{-(x-\theta)}\,dx$

$y = x-\theta \quad x = y+\theta$

$dy = dx$

$\rightarrow \theta + \int_0^{\infty} y e^{-y}\,dy$

$= \theta + \left[(y)(-e^{-y})\right]_0^{\infty} + \int_0^{\infty} e^{-y}\,dy$

$= \theta + 1$

$\mu_1 = \hat{\mu}_1$

$\theta + 1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$\hat{\theta} = \bar{X} - 1$

b) find the mle of θ (for what values of θ is the likelihood positive?)

$lik(\theta) = \prod^{n} e^{-(x-\theta)} = e^{-\left(\sum^n x_i - \theta n\right)} = e^{-\sum^n x_i} \cdot e^{\theta n}$ $\quad x \geq \theta$

The likelihood function is maximized when x is minimized as larger x values will lead to smaller outputs for $x \geq \theta$.

47. The Pareto dist has been used in economics as a model for a density func w/ a slowly decaying tail.

$f(x|x_0,\theta) = \theta x_0^{\theta} x^{-\theta-1}, \quad x > x_0, \ \theta > 1$

a) find the method of moments estimate of θ

$EX = \bar{X} = \frac{x_0\theta}{\theta-1} \rightarrow x_0 = \frac{\bar{X}(\theta-1)}{\theta}$

$s^2 = \frac{x_0^2\theta}{(\theta-1)^2(\theta-2)} = \frac{x_0\bar{X}}{(\theta-1)(\theta-2)} \rightarrow x_0 = \frac{s^2(\theta-1)(\theta-2)}{\bar{X}}$

$x_0 = \frac{\bar{X}(\theta-1)}{\theta} = \frac{s^2\cancel{(\theta-1)}(\theta-2)}{\bar{x}}$

$\frac{\bar{X}^2}{s^2} = \theta(\theta-2)$

$0 = \theta^2 - 2\theta - \left(\frac{\bar{x}}{s}\right)^2$

$\theta = \frac{2 \pm \sqrt{4+4\left(\frac{\bar{x}}{s}\right)^2}}{2} = 1 \pm \sqrt{1+\left(\frac{\bar{x}}{s}\right)^2}$

$\theta > 1$, so $\theta = 1 + \sqrt{1+\left(\frac{\bar{x}}{s}\right)^2}$

$x_0 = \frac{\bar{x}(\theta-1)}{\theta}$

b) find the mle of θ

$lik(\theta, x_0, x) = \prod_{i=1}^{n} \theta x_0^{\theta} x^{-\theta-1} = \theta^n x_0^{\theta n} \cdot \prod_{i=1}^{n} x_i^{-\theta-1}$

$\log lik = n\log(\theta) + \theta n\log(x_0) - (\theta+1) \cdot \sum_{i=1}^{n}\log(x)$

$\frac{\partial \log lik(\theta, x_0)}{\partial\theta} = \frac{n}{\theta} + n\log(x_0) - \sum_{i=1}^{n}\log(x) = 0$

$\frac{n}{\theta} = \sum_{i=1}^{n}\log(x) - n\log(x_0)$

$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}\log(x) - n\log(x_0)}$

50. Let $X_1 \ldots X_n$ be an i.i.d. sample from a Rayleigh distribution w/ param $\theta > 0$

$f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad x \geq 0$

(Alt param of ex. in 3.6.2)

a) find the method of moments estimate for θ

$EX = \bar{x} = \theta\sqrt{\frac{\pi}{2}}, \quad \theta = \frac{\bar{X}}{\sqrt{\pi/2}}$

$s^2 = \frac{4-\pi}{2}\theta^2, \quad \theta^2 = s^2\left(\frac{2}{4-\pi}\right)$

$\theta = s\sqrt{\frac{2}{4-\pi}}$

$\theta = \bar{X}\sqrt{\frac{2}{\pi}} = s\sqrt{\frac{2}{4-\pi}}$

b) find the mle of θ

$lik(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i^2/2\theta^2}$

$\log lik = \Sigma_i^n \log(x_i) - 2n\log(\theta) - \frac{1}{\theta^2}\left(\Sigma_i^n x_i^2/2\right)$

$\frac{\partial \log lik(\theta)}{\partial\theta} = 0 = -\frac{2n}{\theta} + \frac{2}{\theta^3}\left(\Sigma_i^n x_i^2/2\right)$

$\frac{2n}{\theta} = \frac{2}{\theta^3}\left(\Sigma_i^n x_i^2/2\right)$

$\theta^2 = \frac{1}{n}\left(\Sigma_i^n x_i^2/2\right)$

$\theta = \sqrt{\frac{1}{n}\left(\Sigma_i^n x_i^2/2\right)}, \quad \theta > 0$