Chapter 4 part 4. §4.5 The Moment-Generating Function

Definition. m.g.f. of a r.v. X $\quad M(t)=E(e^{tX})$.

if X is a discrete r.v.

X	x_1	x_2	$\cdots$	x_n	$\cdots$
$p(x)$	$p(x_1)$	$p(x_2)$	$\cdots$	$p(x_n)$	$\cdots$

$g(X)=e^{tX}$, $E[g(X)]=E(e^{tX})$

$$M(t)=E(e^{tX})= e^{tx_1}p(x_1)+e^{tx_2}\cdot p(x_2)+\cdots+e^{tx_n}p(x_n)+\cdots$$
$$=\sum_{i=1}^{\infty} e^{tx_i}\cdot p(x_i) \quad \left(=\sum_x e^{tx}p(x) \text{ in textbook}\right)$$

if $X\sim f(x)$. $\quad M(t)=E(e^{tX})=\int_{-\infty}^{\infty} e^{tx}\cdot f(x)\,dx$.

Property A. If the moment-generating function exists for t in an open interval containing zero, it uniquely determines the probability distribution.

$$M(t)=\int_{-\infty}^{\infty} e^{tx}f(x)dx,\quad M'(t)=\int_{-\infty}^{\infty}[e^{tx}f(x)]'_t\,dx$$
$$=\int_{-\infty}^{\infty} xe^{tx}f(x)dx,\quad M''(t)=\int_{-\infty}^{\infty}x^2e^{tx}f(x)dx.$$

$$M'(0)=M'(t)\big|_{t=0}=\int_{-\infty}^{\infty}xf(x)dx=E(X).$$

$$M''(0)=M''(t)\big|_{t=0}=\int_{-\infty}^{\infty}x^2\cdot f(x)dx=E(X^2).$$

Property B. If $M_X(t)$ exists in an open interval containing zero.
then $M^{(r)}(0)=M^{(r)}(t)\big|_{t=0}=E(X^r)$.

Expl. A. Poisson dist. $X\sim P(\lambda)$, $P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$, $k=0,1,2,\cdots$

$$M_X(t)=E(e^{tX})=\sum_{k=0}^{\infty}e^{tk}\cdot\frac{\lambda^k}{k!}e^{-\lambda}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}$$

$\left\{\text{using } e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}\right\}$ $\quad =e^{-\lambda}\cdot e^{\lambda e^t}=e^{\lambda(e^t-1)}$.

i.e., $M(t)=e^{\lambda(e^t-1)}$. $M'(t)=e^{\lambda(e^t-1)}\cdot[\lambda(e^t-1)]'_t=\lambda e^t\cdot e^{\lambda(e^t-1)}$.

$M''(t)=(\lambda e^t)'_t e^{\lambda(e^t-1)}+\lambda e^t[e^{\lambda(e^t-1)}]'_t=\lambda e^t e^{\lambda(e^t-1)}+\lambda^2 e^{2t}\cdot e^{\lambda(e^t-1)}$

thus $M'(0)=\lambda\cdot e^0\cdot e^{\lambda(e^0-1)}=\lambda$. $E(X)=M'(0)=\lambda$.

$M''(0)=\lambda e^0 e^{\lambda(e^0-1)}+\lambda^2\cdot e^0\cdot e^{\lambda(e^0-1)}=\lambda+\lambda^2$. $E(X^2)=M''(0)=\lambda+\lambda^2$.

$Var(X)=E(X^2)-[E(X)]^2=\lambda+\lambda^2-\lambda^2=\lambda$.

Expl. B. $X\sim\Gamma(\alpha,\lambda)$. $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$, $x>0$.

$M(t)=E(e^{tX})=\int_0^\infty e^{tx}\cdot\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}dx=\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\lambda-t)x}dx$,

one needs $t<\lambda$, $x^{\alpha-1}e^{-(\lambda-t)x}$ relates to $\Gamma(\alpha,\lambda-t)$'s density.

$\left\{X\sim\Gamma(\alpha,\lambda-t),\ f(x)=\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-(\lambda-t)x},\ \int_0^\infty f(x)dx=1,\ \int_0^\infty x^{\alpha-1}e^{-(\lambda-t)x}dx=\frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}\right\}$

thus $M(t)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}=\left(\frac{\lambda}{\lambda-t}\right)^\alpha$.

$M'(t)=[\lambda^\alpha\cdot(\lambda-t)^{-\alpha}]'_t=\lambda^\alpha\cdot(-\alpha)(\lambda-t)^{-\alpha-1}\cdot(-1)=\frac{\alpha\cdot\lambda^\alpha}{(\lambda-t)^{\alpha+1}}$

$M'(0)=\frac{\alpha\cdot\lambda^\alpha}{(\lambda-0)^{\alpha+1}}=\frac{\alpha}{\lambda}$, $E(X)=M'(0)=\frac{\alpha}{\lambda}$.

$M''(t)=\alpha\cdot\lambda^\alpha\cdot[-(\alpha+1)(\lambda-t)^{-(\alpha+1)-1}\cdot(-1)]=\frac{\alpha\cdot(\alpha+1)\cdot\lambda^\alpha}{(\lambda-t)^{\alpha+2}}$

$M''(0)=\frac{\alpha\cdot(\alpha+1)\cdot\lambda^\alpha}{\lambda^{\alpha+2}}=\frac{\alpha\cdot(\alpha+1)}{\lambda^2}$. $E(X^2)=M''(0)=\frac{\alpha\cdot(\alpha+1)}{\lambda^2}$.

$Var(X)=E(X^2)-(EX)^2=\frac{\alpha\cdot(\alpha+1)}{\lambda^2}-\left(\frac{\alpha}{\lambda}\right)^2=\frac{\alpha}{\lambda^2}$.

Expl. $Z\sim N(0,1)$.

$M(t)=E(e^{tZ})=\int_{-\infty}^\infty e^{tx}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{x^2}{2}+tx}dx$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{1}{2}(x^2-2tx)}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{1}{2}[x^2-2tx+t^2-t^2]}dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2 + \frac{t^2}{2}} dx = e^{\frac{t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx = e^{t^2/2}.$$

$$\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx \overset{x-t=u}{=\!=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = 1 \right\}$$

$M(t) = e^{t^2/2}$. $M'(t) = e^{t^2/2} \cdot \left(\frac{t^2}{2}\right)'_t = t e^{t^2/2}$. $M'(0) = 0$. $E(Z) = 0$.

$M''(t) = (t \cdot e^{t^2/2})'_t = e^{t^2/2} + t^2 \cdot e^{t^2/2}$, $M''(0) = 1 + 0 = 1$. $E(Z^2) = 1$. $Var(Z) = 1$.

Property C. $X \sim M_X(t) = E(e^{tX})$. $Y = a + bX$, $M_Y(t) = E(e^{tY})$,

then $M_Y(t) = e^{at} \cdot M_X(bt)$.

Expl. $X \sim N(\mu, \sigma^2)$, $Z \sim N(0,1)$. $M_Z(t) = e^{t^2/2}$

then $X = \mu + \sigma Z$, (or $Z = \frac{X-\mu}{\sigma}$)

thus $M_X(t) = e^{\mu t} \cdot M_Z(\sigma t) = e^{\mu t} \cdot e^{\sigma^2 t^2/2} = e^{\mu t + \frac{\sigma^2}{2} t^2}$

Property D. $X \sim M_X(t)$, $Y \sim M_Y(t)$, $X \perp\!\!\!\perp Y$. $Z = X + Y$.

$M_Z(t) = M_X(t) \cdot M_Y(t)$.

Expl E. $X \sim P(\lambda)$, $Y \sim P(\mu)$, $\overset{X \perp\!\!\!\perp Y}{}$ $X + Y \sim$?

Ans. $M_X(t) = e^{\lambda(e^t - 1)}$, $M_Y(t) = e^{\mu(e^t - 1)}$.

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}$.

thus $X + Y \sim P(\lambda + \mu)$.

Expl. F. $X \sim \Gamma(\alpha_1, \lambda)$, $Y \sim \Gamma(\alpha_2, \lambda)$. $\overset{X \perp\!\!\!\perp Y}{}$ $X + Y \sim$?

$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1}$, $M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_2}$.

$M_{X+Y}(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1} \cdot \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_2} = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_1+\alpha_2}$,

thus $X + Y \sim \Gamma(\alpha_1 + \alpha_2, \lambda)$.

Expl G. $X \sim N(\mu, \sigma^2)$. $Y \sim N(\nu, \tau^2)$. $X \perp\!\!\!\perp Y$.

$M_X(t) = e^{\mu t + \frac{\sigma^2}{2}t^2}$, $M_Y(t) = e^{\nu t + \frac{\tau^2}{2}t^2}$, $M_{X+Y}(t) = e^{\mu t + \frac{\sigma^2}{2}t^2 + \nu t + \frac{\tau^2}{2}t^2}$

$= e^{(\mu+\nu)t + \frac{(\sigma^2+\tau^2)}{2}t^2}$, thus $X+Y \sim N(\mu+\nu, \sigma^2+\tau^2)$.

§4.6. Approximate Method

r.v. X, know only $\underset{\mu_X}{\underset{\shortparallel}{E(X)}}$, $\underset{\sigma_X^2}{\underset{\shortparallel}{Var(X)}}$. $Y = g(X)$, $E(Y) = ?$ $Var(Y) = ?$

If $g(X) = a + bX$, i.e. $Y = a + bX$, then $E(Y) = a + b \cdot E(X)$.
$Var(Y) = b^2 \cdot Var(X)$.

if not linear

$Y = g(X) \approx g(\mu_X) + g'(\mu_X) \cdot (X - \mu_X)$

$\left\{ \begin{array}{l} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots \\ \text{Taylor series expansion} \end{array} \right\}$

$\mu_Y = E(Y) \approx g(\mu_X)$.

$\sigma_Y^2 = Var(Y) \approx [g'(\mu_X)]^2 \cdot \sigma_X^2$.

Expl A. $V = I \cdot R$ — V: voltage, I: current, R: resistence

suppose V is hold constant at V_0.
R fluctuates randomly. is a r.v.
I is a r.v. too! know $\mu_I = E(I) \neq 0$.
$\sigma_I^2 = Var(I)$.

We wish to find $\mu_R = E(R)$, $\sigma_R^2 = Var(R)$.

Ans. $R = \frac{V_0}{I}$

$$\left\{ \begin{array}{l} f(x) = \frac{V_0}{x}, \quad f'(x) = -\frac{V_0}{x^2}, \quad f''(x) = \frac{2V_0}{x^3} \\ f(x) \approx \frac{V_0}{x_0} + \left(-\frac{V_0}{x_0^2}\right)(x - x_0) + \frac{2V_0}{x_0^3}(x - x_0)^2 \\ \frac{V_0}{x} \approx \frac{V_0}{x_0} - \frac{V_0}{x_0^2}(x - x_0) + \frac{2V_0}{x_0^3}(x - x_0)^2 \end{array} \right.$$

$$\frac{V_0}{I} \approx \frac{V_0}{\mu_I} - \frac{V_0}{\mu_I^2}(I - \mu_I) + \frac{2V_0}{\mu_I^3}(I - \mu_I)^2$$

i.e. $R = \frac{V_0}{I} \approx \frac{V_0}{\mu_I} - \frac{V_0}{\mu_I^2}(I - \mu_I) + \frac{2V_0}{\mu_I^3}(I - \mu_I)^2$.

$$E(R) \approx \frac{V_0}{\mu_I} - \frac{V_0}{\mu_I^2} \cdot 0 + \frac{2V_0}{\mu_I^3} \cdot \sigma_I^2 = \frac{V_0}{\mu_I} + \frac{2V_0}{\mu_I^3}\sigma_I^2.$$

$\not\approx \frac{V_0^2}{\mu_I^4} \cdot \sigma_I^2$

$$Var(R) = E(R - E(R))^2 \approx E\left(R - \frac{V_0}{\mu_I}\right)^2 \approx E\left\{ -\frac{V_0}{\mu_I^2}(I - \mu_I) + \frac{2V_0}{\mu_I^3}(I - \mu_I)^2 \right\}^2$$

Again, $Y=g(X)=g(\mu_X)+g'(\mu_X)(X-\mu_X)+\frac{g''(\mu_X)}{2}(X-\mu_X)^2+\frac{g'''(\mu_X)}{3!}(X-\mu_X)^3+\cdots$

$$\approx g(\mu_X)+g'(\mu_X)(X-\mu_X)+\frac{g''(\mu_X)}{2}(X-\mu_X)^2$$

thus, $E(Y)=E[g(X)]\approx g(\mu_X)+g'(\mu_X)\cdot E(X-\mu_X)+\frac{g''(\mu_X)}{2}E(X-\mu_X)^2$

$$=g(\mu_X)+0+\frac{g''(\mu_X)}{2}\cdot Var(X).$$

i.e. $\underline{E(Y)=E[g(X)]\approx g(\mu_X)+\frac{g''(\mu_X)}{2}\sigma_X^2}.$

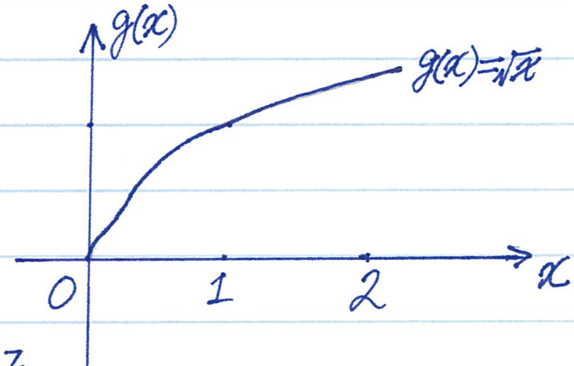
also $\underline{Var(Y)\approx[g'(\mu_X)]^2\cdot\sigma_X^2}.$

How good such approximations are depends on how nonlinear g is in a neighborhood of μ_X and on the size of σ_X. If g can be reasonably well approximated by a linear function, the approximations for the moments will be reasonable as well.

Expl. B. Consider $g(x)=\sqrt{x}$.

two cases: $X\sim U[0,1]$. less linear

$X\sim U[1,2]$. more linear



<u>Case 1</u> $Y=g(X)=\sqrt{X}$, $E(Y)=\int_0^1\sqrt{x}\,dx=\frac{2}{3}=0.667.$

$E(Y^2)=\int_0^1 x\,dx=\frac{1}{2}$, $Var(Y)=\frac{1}{2}-\left(\frac{2}{3}\right)^2=1/18=0.05556.$ $\sigma_Y=\sqrt{Var(Y)}=0.236.$

i.e., actual $\mu_Y=E(Y)=0.667.$ $\sigma_Y^2=0.05556.$ $\sigma_Y=0.236.$

from approximation: $E(Y)\approx g(\mu_X)+\frac{g''(\mu_X)}{2}\sigma_X^2.$ $\mu_X=E(X)=\int_0^1 x\,dx=\frac{1}{2}.$

$\sigma_X^2=Var(X)=1/12.$

$g'(x)=\frac{1}{2\sqrt{x}}$, $g''(x)=-\frac{1}{4}x^{-\frac{3}{2}}.$

thus $g(\mu_X)=\sqrt{1/2}$, $g''(\mu_X)=-\frac{1}{4}\cdot(1/2)^{-3/2}=-\frac{1}{4}\cdot 2^{\frac{3}{2}}=-\frac{1}{4}\sqrt{8}=-\frac{\sqrt{2}}{2}$

therefore, $E(Y)\approx\sqrt{1/2}+\frac{1}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{12}=0.678$ (compare 0.667)

$Var(Y)\approx[g'(\mu_X)]^2\cdot\sigma_X^2=\left(\frac{1}{2\sqrt{1/2}}\right)^2\cdot\frac{1}{12}=\frac{1}{4}\cdot2\cdot\frac{1}{12}=0.042$ (compare 0.05556)

$\sigma_Y=\sqrt{Var(Y)}\approx\sqrt{0.042}=0.204.$ (compare 0.236).

Case 2. $X \sim U[1,2]$. $\mu_X = E(X) = 1.5$, $\sigma_X^2 = Var(X) = 1/12$.

Exact value of $\mu_Y = E(Y) = \int_1^2 \sqrt{x}\,dx = \int_1^2 x^{\frac{1}{2}}\,dx = \left.\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^2 = 1.219.$

$E(Y^2) = \int_1^2 x\,dx = 1.5.$ $Var(Y) = 1.5 - 1.219^2 = 0.0142.$ $\sigma_Y = \sqrt{0.0142} = 0.119.$

by approximation. $E(Y) \approx g(\mu_X) + \dfrac{g''(\mu_X)}{2}\sigma_X^2$ $\quad \left\{ g''(x) = -\frac{1}{4}x^{-3/2} \right\}$

$= \sqrt{3/2} - \frac{1}{4}\left(\frac{3}{2}\right)^{-3/2} \cdot \frac{1}{12} = \sqrt{3/2} - \frac{1}{4}\sqrt{\frac{8}{27}} \cdot \frac{1}{12}$

$= 1.219$ (compare to 1.219)

$Var(Y) \approx [g'(\mu_X)]^2 \cdot \sigma_X^2$ $\quad \left\{ g'(x) = \frac{1}{2\sqrt{x}} \right\}$

$= \left(\dfrac{1}{2\cdot\sqrt{3/2}}\right)^2 \cdot \dfrac{1}{12} = \dfrac{1}{4} \cdot \dfrac{2}{3} \cdot \dfrac{1}{12} = 0.0138$ (compare 0.0142)

$\sigma_Y = \sqrt{Var(Y)} \approx \sqrt{0.0138} = 0.118$ (compare 0.119).