Chapter 1. part 2.

§1.4.2 Permutations and Combinations

A permutation is an ordered arrangement of objects.

1° how many different ordered arrangements of the letters a, b and c are possible?

An: abc, acb, bac, bca, cab, cba. $3\times2\times1=6$

from basic principle, 1st can be any of the 3, 2nd can be chosen from any of the remaining 2, etc.

2°. Similary, for n objects, how many different ordered arrangments?

An: $n\times(n-1)\times(n-2)\times\cdots\times3\times2\times1=n!$

3°. Choose r elements from the set $C=\{c_1, c_2, c_3, \cdots c_n\}$ and list them in order. How many ways can we do this?

Ans: $n\times(n-1)\times(n-2)\times\cdots\times(n-r+1)$.

Example A. How many ways can five children be lined up?

Ans. $5\times4\times3\times2\times1=5!=120$.

Example B: Suppose that from ten children, five are to be chosen and lined up. How many different lines are possible?

Ans: $10\times9\times8\times7\times6=30{,}240$ different lines.

Expl C: License plates have six characters: three letters followed by three numbers. How many distinct such plates are possible?

Ans: There are $26\times26\times26=26^3=17{,}576$ different ways for letters.

$10\times10\times10=10^3=1000$ ways to choose numbers.

Thus, there are $26^3\times10^3=17{,}576{,}000$ different plates!

Expl D: If all sequences of 6 characters are equally likely, what is the probability that license plate for a new car will contain no duplicate letters or numbers?

Ans: Denote that event with A. $P(A) = \frac{n(A)}{n(\Omega)}$.

$n(\Omega) = 26^3 \times 10^3 = 17,576,000$.

$n(A) = 26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$.

thus, $P(A) = \frac{n(A)}{n(\Omega)} = \frac{11,232,000}{17576000} = 0.64$.

Expl. E. Birthday Problem.

A room contains n people. What is the probability that at least two of them have a common birthday?

Ans: Denote that event with A. $P(A) = \frac{n(A)}{n(\Omega)}$. $n(\Omega) = 365 \times 365 \times \cdots \times 365 = 365^n$.

$n(A)$ more complicated!

Consider A^c = none of them have the same birthday.

$$P(A) = 1 - P(A^c) = 1 - \frac{n(A^c)}{n(\Omega)} = 1 - \frac{365 \times (365-1) \times \cdots \times (365-n+1)}{365^n}$$

Some numbers:

n	4	16	23	32	40	56
$P(A)$	0.016	0.284	0.507	0.753	0.891	0.988

The probabilitys are larger than one might guess! every pair prob $= \frac{365}{365^2} = \frac{1}{365}$

$n = 23$, $P(A) = 0.507$. the reason is: there are $\binom{23}{2} = 253$ different pairs.

Expl F. How many people must you ask to have a 50:50 chance of finding someone who shares your birthday?

Ans: Suppose you ask n people. A = "Someone's birthday is the same as yours"

A^c is easier. determine n such $P(A) = 0.5$,

$P(A^c) = 1 - P(A) = 0.5$. $P(A^c) = \frac{364^n}{365^n}$

let $1 - \frac{364^n}{365^n} = P(A) = 0.5$, $n \doteq 253$.

Combination: If r objects are taken from a set of n objects without replacement and disregarding order, how many different samples are possible?

e.g. $n=3$, A, B, C. $r=2$, AB, AC, BC. Ans: 3 different samples.
$(=BA)$ $(=CA)$. $(=CB)$

$\frac{3 \times 2}{2!} = 3$. denoted $\binom{3}{2} = \frac{3 \times 2}{2!} = \frac{3 \times 2 \times 1}{2!\,1!} = 3$

generally, r objects from n objects without replacement and order

$$\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-r+1)}{r!}$$

$\leftarrow$ number of ordered sample

$r! \leftarrow$ a sample of size r can be ordered in $r!$ ways

$$= \frac{n\cdot(n-1)\cdots(n-r+1)\cdot(n-r)(n-r-1)\cdots 2\cdot 1}{r!\cdot(n-r)(n-r-1)\cdots 2\cdot 1}$$

$$= \frac{n!}{r!\,(n-r)!} = \binom{n}{r}.$$

Proposition B (P.12)

The number of unordered samples of r objects selected from n objects without replacement is $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

Also, $$(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \cdots + \binom{n}{n}a^nb^0$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}.$$

if $a=b=1$, $\quad 2^n = \sum_{k=0}^{n}\binom{n}{k}.$

if $a+b=1$. $\quad 1 = \sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}.$

1°. How many different groups of 3 could be selected from 5 items, A, B, C, D, E ?

Ans: $\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\,2!} = 10.$ $\left(\text{or } \frac{5\times4\times3}{3!} \right)$

$\leftarrow$ # of ordered

$\leftarrow$ a sample of 3 can be ordered 3! ways

2°. A committee of 3 is to be formed from a group of 20 people. How many different committee are possible?

Ans: $\binom{20}{3} = \frac{20!}{3!\,(20-3)!} = \frac{20\cdot19\cdot18\cdot\cancel{(17!)}}{3!\,\cancel{17!}} = \frac{20\cdot19\cdot18}{3\cdot2\cdot1} = 1140.$

Expl. G. A player of CA state lottery could win the jackpot prize by choosing the 6 numbers from 1 to 49 that were subsequently chosen at random by the lottery officials. There are $\binom{49}{6} = 13{,}983{,}816$ ways to choose 6 numbers from 49. prob. of winning is $1/\binom{49}{6}$, about 1 in 14 million.
In 1991, rules were changed. Select 6 number from 1 to 53.
prob of winning $1/\binom{53}{6} = 1/22{,}957{,}480$, about 1 in 23 million!

Expl. H. quality control. n items are in a lot and a sample of size r is taken. Suppose that the lot contains k defective items. What is the prob. that the sample contains exactly m defective?

k defective $\to r$
n-k nondefective

Ans: $$P(A) = \frac{n(A)}{n(\Omega)} = \frac{\binom{k}{m}\binom{n-k}{r-m}}{\binom{n}{r}}$$

$\binom{n}{r} \leftarrow$ total # of samples

$\binom{k}{m}$ ways to choose the m defective items in the sample from k defectives in lot.

Expls: $\underline{A}\,\underline{BC}$ $\binom{3}{1,2}$
$\underline{AB}\,\underline{CD}$ $\binom{4}{2\,2}$ list

Extension of proposition B:
The number of ways that n objects can be grouped into r classes with n_i in the i-th class, $i = 1, 2, \cdots, r$, and $n_1 + n_2 + \cdots + n_r = n$ is

$$\binom{n}{n_1\ n_2\ \cdots\ n_r} = \frac{n!}{(n_1!)\cdot(n_2!)\cdots(n_r!)}.$$

proof: for the 1st class, $\binom{n}{n_1}$ ways. 2nd class $\binom{n-n_1}{n_2}$ ways.

3rd class $\binom{n-n_1-n_2}{n_3}$ ways. . . . .

thus $$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdots\binom{n-n_1\cdots n_{r-1}}{n_r} = \frac{n!}{n_1!\,\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\cdot\frac{\cancel{(n-n_1-n_2)!}}{(n_3!)(n-n_1-n_2-n_3)!}$$
$$= \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Expl. J. A committee of seven members is to be divided into three subcommittees of size three, two, and two. How many ways?

Ans: $$\binom{7}{3\ 2\ 2} = \frac{7!}{3!\,2!\,2!} = 210.$$

Expl K. In how many ways can the set of nucleotides $\{A, A, G, G, G, G, C, C, C\}$ be arranged in a sequence of nine letters?

Ans: $$\frac{9!}{2!\,4!\,3!} = \binom{9}{2\ 4\ 3} = 1260.$$

↑ 9 positions divided into subgroups of 2. 4. 3.

Expl L. In how many ways can $n = 2m$ people be paired and assigned to m courts for the first round of a tennis tournament?

Ans. $$\binom{n}{2\ 2 \cdots 2} = \frac{(2m)!}{2!\,2! \cdots 2!} = \frac{(2m)!}{2^m} \text{ pairs.}$$

since there are $m!$ ways to assign the m pairs to m courts, the final answer is: $$\frac{(2m)!}{2^m\,m!}$$

ABCD. $\binom{4}{2\ 2} = 6$. $\frac{6}{2!} = 3$

court 1	court 2
AB	CD
AC	BD
AD	BC
BC	AD ×
BD	AC ×
CD	AB ×

don't care about different courts.

$$\binom{n}{n_1\ n_2 \cdots n_r} \quad \text{multinomial coefficients.}$$

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{n_1, \ldots, n_r \\ n_1 + n_2 + \cdots + n_r = n}} \binom{n}{n_1\ n_2 \cdots n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$