Class 5: 1.4-Law o TP and Bayes

Friday, September 10, 2021 5:48 AM

Homework 2 : coming soon

Formulas & Theorems Covered Today:

- Law of Total Probability
- Bayes' Theorem
- Conditional Independence
-

Problem 1

You purchase a certain product. The manual states that the lifetime T of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{-\frac{t}{5}}, \text{ for all } t \geq 0.$$

Prob of "surviving" time t

For example, the probability that the product lasts more than (or equal to) 2 years is $P(T \geq 2) = e^{-\frac{2}{5}} = 0.6703$. I purchase the product and use it for two years without any problems. What is the probability that it breaks down in the third year?

Solution

Note event $\underbrace{\{T \geq 5\}}_{A}$ & $\underbrace{\{T \geq 6.3\}}_{B}$

$\Rightarrow$ $B \subset A$!!

$P(\underbrace{2 \leq T \leq 3}_{M}) \Rightarrow$ A: $T \geq 3$ B: $T \geq 2$

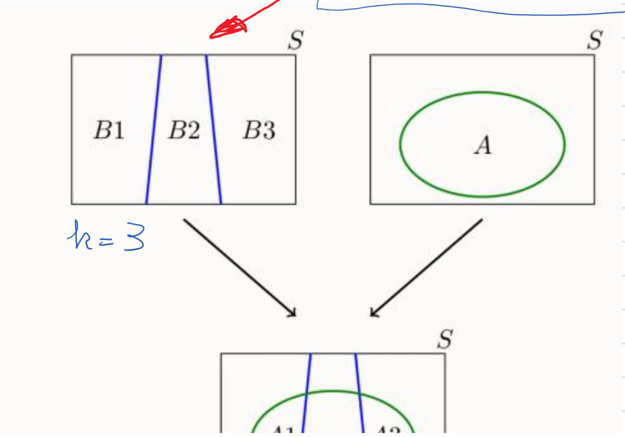
$M \subset B$ $M \not\subset A$

$P(B \cap A^c) = P(B \setminus A \cap B)$
$= P(B) - P(A)$
$= e^{-2/5} - e^{-3/5}$

Today's Discussion

1.4.2 Law of Total Probability

Discussion: What is a **Partition of the Sample Space**:



$k=3$

Partition: $B_1, B_2, \ldots B_k$

$B_1 \cup B_2 \cup \cdots \cup B = S$

"exhaustive"

$B_i \cap B_j = \phi$ $P(B_i) \cap (B_j) \cap (B_k) = \phi$

mutually exclusive

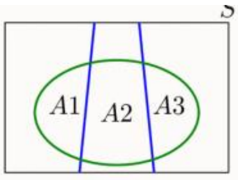


Fig.1.24 - Law of total probability.

↳ mutually exclusive $= \phi^{(B_k)}$ and exhaustive.

another event A (of interest)

$P(A) = ?$ Assume we know $P(A|B_i)$ for every i

↳ A_1, A_2, A_3 are a partition of A

$A = A_1 \cup A_2 \cup A_3$ also A_i are disjoint

$$P(A) = \underbrace{P(A_1)}_{} + P(A_2) + P(A_3)$$

$$P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$

$$= P(A|B_1)\cdot P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)\cdot P(B_3)$$

↓ Multipl. Rule (chain)

> Law of Total Probability:
> If $B_1, B_2, B_3, \cdots$ is a partition of the sample space S, then for any event A we have
> $$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i).$$

also called the → "stratified sampling rule"

Typical scenario: in which we use the law of total probability:

- We are interested in finding the probability of an event *A*, but we don't know how to find *P*(*A*)directly.
- Instead, we know the conditional probability of *A* given some events B_i, where the B_i's form a partition of the sample space.
- Thus, we will be able to find *P*(*A*)using the law of total probability,

Solve: Example 1.24 (easy!)

Example 1.24
I have three bags that each contain 100 marbles:

- Bag 1 has 75 red and 25 blue marbles;
- Bag 2 has 60 red and 40 blue marbles;
- Bag 3 has 45 red and 55 blue marbles.

I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Solution

A : Marble is Red

Partition: Bag 1, Bag 2, Bag 3
B_1 B_2 B_3

$P(B_1) = 1/3 = P(B_2) = P(B_3)$

$P(A \mid B_1) = \frac{75}{100}$ $P(A \mid B_2) = \frac{60}{100}$ $P(A \mid B_3) = \frac{45}{100}$

$P(A) = P(A \mid B_1) \cdot P(B_1) + \cdots$ etc

$= \frac{3}{4} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{1}{3} + \frac{9}{20} \cdot \frac{1}{3} = \frac{1}{3}(.75 + .6 + .45)$

$= 1.8/3 = \underline{0.6}$

1.4.3 Bayes' Rule

→ Thomas Bayes 17… –1761 ?

In the situation of a partition of a sample space consisting of B_i's, and an event A for which we know the conditional probabilities $P(A \mid B_i)$ for all B_i. , also know $P(B_i)$, $\forall i$

The law of total probability gives us the (unconditional) probability P(A).

Now assuming A has occurred, we are interested in the (inverse) conditional probability $P(B_i \mid A)$.

Clearly this is using the definition of conditional probability and the Multiplication (Chain) rule:

Solution: $P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} \overset{\text{mult Rule}}{=} \frac{P(A \mid B_i) P(B_i)}{P(A)}$

use law of total probability

$P(B_i)$: prior prob. of partition

$P(B_i \mid A)$: posterior prob. of partition
↳ after A has occured

Writing this result using the original setup of partition {Bi} and event A gives us:

Bayes' Rule

- For any two events A and B, where $P(A) \neq 0$, we have
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$
- If $B_1, B_2, B_3, \cdots$ form a partition of the sample space S, and A is any event with $P(A) \neq 0$, we have
$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}.$$

Example 1.25
In Example 1.24, suppose we observe that the chosen marble is red. What is the probability that Bag 1 was chosen?

Solution

" red

Solution

$$P(B_1 \mid A^{\text{"red"}}) = \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)\cdot P(B_3)}$$

$$= \frac{.75 \cdot 1/3}{.6} = \frac{.25}{.6} = \frac{1}{4 \cdot .6}^{10} = \frac{10}{24}$$

A more interesting example:

Example 1.26 (False positive paradox [5])
A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. The test is quite accurate. In particular, we know that

- the probability that the test result is positive (suggesting the person has the disease), given that the person does not have the disease, is only 2 percent; False positive rate
- the probability that the test result is negative (suggesting the person does not have the disease), given that the person has the disease, is only 1 percent. false negative rate

A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?

Solution

Partition: $B_1 =$ Disease $B_2 =$ Healthy

Event: Test is positive $A \Rightarrow A^c$ (test is negative)

$$P(B_1 \mid A) = \frac{P(A \mid B_1)\cdot P(B_1)}{P(A \mid B_1)\cdot P(B_1) + P(A \mid B_2)\cdot P(B_2)}$$

$$= \frac{.99 \cdot 1/10000}{.99 * 1/10000 + .02 * \frac{9999}{10000}} = \frac{.99}{.99 + .02 * 9999}$$

$=$

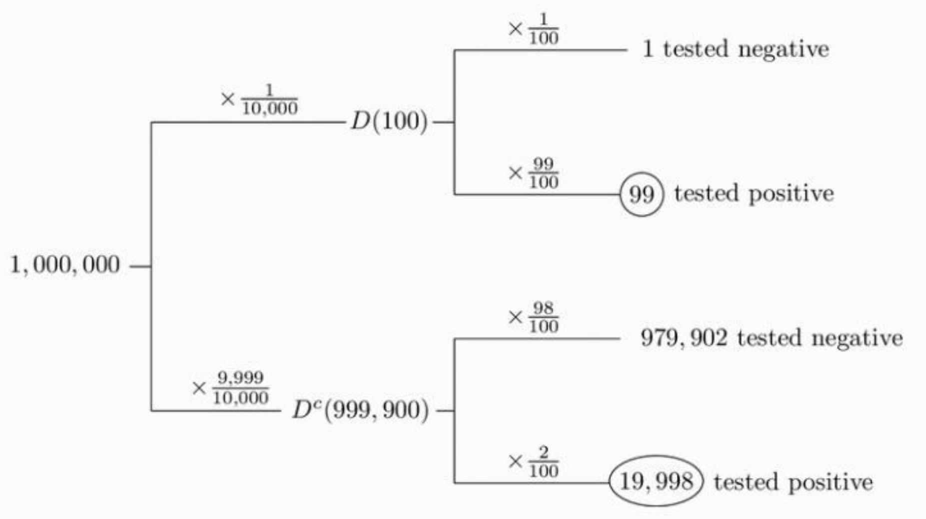


Fig.1.25 - Tree diagram for Example 1.26.

1.4.4 Conditional Independence

Really just a remark:

Definition 1.2
Two events A and B are **conditionally independent** given an event C with $P(C) > 0$ if

$$P(A \cap B|C) = P(A|C)P(B|C) \tag{1.8}$$

Some derviations here……read!

Thus, if A and B are conditionally independent given C, then

$$P(A|B,C) = P(A|C) \tag{1.9}$$

Thus, Equations 1.8 and 1.9 are equivalent statements of the definition of conditional independence. Now let's look at an example.

Let's look at this example:

Example 1.27

A box contains two coins: a regular coin and one fake two-headed coin ($P(H) = 1$). I choose a coin at random and toss it twice. Define the following events.

- A= First coin toss results in an H.
- B= Second coin toss results in an H.
- C= Coin 1 (regular) has been selected.

Find $P(A|C), P(B|C), P(A \cap B|C), P(A), P(B)$, and $P(A \cap B)$. Note that A and B are NOT independent, but they are *conditionally* independent given C.

Solution

See solutions in text.

One **important lesson** here is that

- Conditional independence neither implies (nor is it implied by) independence.
- Thus, we can have two events that are conditionally independent but they are not unconditionally independent (such as A and B above).
- We can also have two events that are independent but not conditionally independent, given an event C.