2. $P(X=k) = \frac{1}{n}$, $k \in [1,n]$

$$E(X) = \sum_{i=1}^{n} k \cdot P(X=k),\ k \in [1,n]$$
$$= \sum_{i=1}^{n} k \cdot \frac{1}{n} = \left(\frac{1}{n}\right)\left(\frac{n \times (n+1)}{2}\right) = \frac{n+1}{2}$$

$$Var(X) = E(X^2) - E(X)^2$$
$$= \sum k^2 \cdot P(X=k^2)\ k \in [1,n] - \left(\frac{n+1}{2}\right)^2$$
$$= \frac{1}{n}\left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n^2+2n+1}{4}\right)$$
$$= \frac{n(n^2+2n+1)}{4} \quad \frac{-n^2-2n-1}{4}$$
$$= \left(\frac{1}{4}\right)\left(n^3+n^2-n-1\right)$$

5

$$f(x) = \frac{1+\alpha x}{2} = \frac{1}{2} + \frac{\alpha x}{2},\ -1 \le x \le 1,\ -1 \le \alpha \le 1$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{1} \frac{x}{2}\,dx + \int_{-1}^{1} \frac{\alpha x^2}{2}\,dx$$
$$= \left(\frac{x^2}{4}\Big|_{-1}^{1}\right) + \left(\frac{\alpha x^3}{6}\Big|_{-1}^{1}\right)$$
$$= \frac{1}{2} + \frac{\alpha}{3}$$

$$Var = E(X^2) - E(X)^2 =$$

$$E(X^2) = \int_{-1}^{1} x^2 f(x)\,dx = \int_{-1}^{1} \frac{x^2}{2}\,dx + \int_{-1}^{1} \frac{\alpha x^3}{2}\,dx = \left(\frac{x^3}{6}\Big|_{-1}^{1}\right) + \left(\frac{\alpha x^4}{8}\Big|_{-1}^{1}\right)$$
$$= \frac{1}{3} + \frac{\alpha}{4}$$

$$Var(X) = \frac{1}{3} + \frac{\alpha}{4} - \left(\frac{1}{2} + \frac{\alpha}{3}\right)^2$$
$$= \frac{1}{3} + \frac{\alpha}{4} - \frac{1}{4} - \frac{\alpha}{3} - \frac{\alpha^2}{9}$$
$$= \frac{3 - 3\alpha - 4\alpha^2}{36}$$

6

a) $f(x) = 2x, \quad 0 \le x \le 1$

$$E(X) = \int x \cdot 2x \, dx, \quad 0 \le x \le 1$$

$$= \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

b) $Y = X^2$

$$E(Y) = E(X^2) = \int x^2 f(x)\, dx, \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

$$= \int x^2 \cdot 2x \, dx = \frac{x^4}{2}\Big|_0^1 = \frac{1}{2}$$

c) $$E(Y) = \int_{-\infty}^{\infty} g(x) f(x)\, dx = \int_0^1 x^2 \cdot 2x \, dx = \frac{x^4}{2}\Big|_0^1 = \frac{1}{2} = \frac{1}{2}$$

d)

$$Var(X) = E\{[X - E(X)]^2\}$$

$$= E(X^2) - E(X)^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{18}$$

If $Var(X)$ exists and $Y = a + bX$, $Var(Y) = b^2 Var(X)$

$$Y = X^2 = 0 + 1 \cdot X$$

$$Var(Y) = (1)^2 Var(X)$$

$$= Var(X) = \frac{1}{18}$$

7

$$X = \begin{cases} \frac{1}{2}, & x=0 \\ \frac{3}{8}, & x=1 \\ 1/8, & x=2 \end{cases}$$

a) $E(x) = \sum k \cdot P(x=k) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{8} = \frac{5}{8}$

b) $Y = X^2$, $Y = \begin{cases} \frac{1}{2}, & y=0 \\ 3/8, & y=1 \\ 1/8, & y=4 \end{cases}$

$E(Y) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{3}{8} + 4 \cdot \frac{1}{8} = \frac{7}{8}$

c) $E(Y) = \sum_x g(x)p(x) = 0^2 \cdot \frac{1}{2} + 1^2\left(\frac{3}{8}\right) + (2^2)\frac{1}{8}$

$= \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$

d)

$Var(x) = \sum_i (x_i - \mu)^2 p(x_i)$

$= \left(0 - \frac{5}{8}\right)^2\left(\frac{1}{2}\right) + \left(1 - \frac{5}{8}\right)^2\left(\frac{3}{8}\right) + \left(2 - \frac{5}{8}\right)^2\left(\frac{1}{8}\right)$

$= \frac{25}{32} + \frac{27}{512} + \frac{121}{512} = \frac{348}{512} = \frac{174}{256} = \frac{87}{128}$

$Var(x) = E(x^2) - E(x)^2 = \frac{7}{8} - \frac{25}{64} = \frac{31}{64} = \frac{62}{128}$

I probably made a minor arithmetic error somewhere.

(16) $E(X) = \mu \quad Var(X) = \sigma^2 \quad Z = \left(\frac{X - \mu}{\sigma}\right)$

$X = \mu + Z\sigma$

$E(\mu + Z\sigma) = \mu$

$\sigma \cdot E(Z) + \mu = \mu$

$E(Z) = 0$

$Var(X) = \sigma^2$

$Var(\mu + Z\sigma) = \sigma^2$

$\sigma^2 Var(Z) = \sigma^2$

$Var(Z) = 1$

(21) $\text{Area}_{Sq} = A = (\text{side length})^2$

$$E(\text{Area}) = E(\text{side length}^2)$$
$$= E(\sim U[0,1]^2)$$
$$= \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

(23)

$X_1 \sim \text{gama}(\alpha_1, \lambda_1)$
$X_2 \sim \text{gama}(\alpha_2, \lambda_2)$ $R^2 = X_1^2 + X_2^2$

$$E(R^2) = E(X_1^2 + X_2^2)$$
$$= E(X_1^2) + E(X_2^2)$$

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}\lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E(X_1^2) = \frac{1}{\Gamma(\alpha_1)}\int x^2(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}\,dx$$
$$\frac{1}{\lambda\Gamma(\alpha_1)}\int_0^\infty (\lambda x)^{\alpha+1}\lambda e^{-\lambda x}\,dx$$
$$= \frac{\Gamma(\alpha_1+2)}{\lambda^2\Gamma(\alpha_1)} = \frac{\alpha_1(\alpha_1+1)}{\lambda_1^2}$$

so, $E(R^2) = E(X_1^2) + E(X_2^2) = \frac{\alpha_1(\alpha_1+1)}{\lambda_1^2} + \frac{\alpha_2(\alpha_2+1)}{\lambda_2^2}$

(31) $E(X) = \int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$

$E(\frac{1}{X}) = \int_1^2 \frac{1}{x}\,dx = \ln(2) - \ln(1) = \ln(2)$

$\frac{3}{2} \neq \ln(2)$, so $E(X) \neq E(\frac{1}{X})$

(49) $E(X) = E(Y) = \mu \qquad \sigma_x \neq \sigma_y$

$Z = \alpha X + (1-\alpha)Y$ where $0 \leq \alpha \leq 1$

$E(Z) = E(\alpha X + (1-\alpha)Y)$

(a) $= \alpha E(X) + (1-\alpha)E(Y)$

$= \mu(\alpha + 1 - \alpha) = \mu$

(b) $Var(Z) = Var(\alpha X + (1-\alpha)Y)$

$= \alpha^2 \sigma_x^2 + (1-\alpha)^2 \sigma_y^2 = f(\alpha)$

min using $f''(\alpha) \rightarrow \qquad f(\alpha)' = 2\alpha\sigma_x^2 - 2(1-\alpha)\sigma_y^2$

$f(\alpha)'' = 2\sigma_x^2 + 2\sigma_y^2$

min@ $f(\alpha)'' > 0$, $f(\alpha)' = 0$

$f(\alpha)''$ always ≥ 0 b/c $\sigma_x \neq \sigma_y$

$\cancel{2}(1-\alpha)\sigma_y^2 = \cancel{2}\alpha\sigma_x^2$

$1 = \alpha(\sigma_y^2 + \sigma_x^2)$

Min@ $\alpha = \frac{1}{(\sigma_x^2 + \sigma_y^2)}$

(c) $\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{X+Y}{2}\right) = \frac{\sigma_x^2 + \sigma_y^2}{4}$

better when

So, $\frac{\sigma_x^2 + \sigma_y^2}{4} < \sigma_x^2$

$\sigma_x^2 + \sigma_y^2 < 4\sigma_x^2$

$\sigma_y^2 < 3\sigma_x^2$

$\frac{1}{3} < \frac{\sigma_x^2}{\sigma_y^2}$ or $3 > \frac{\sigma_y^2}{\sigma_x^2}$

$\frac{X+Y}{2} \rightarrow \frac{1}{3} < \frac{\sigma_x^2}{\sigma_y^2} < 3$

better when