Midterm Exam 2

MATH 755/855, November 18, 2020

Name Solution Score

1. (15pts) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1; \\ 0, & \text{otherwise;} \end{cases}$$

(a) (10pts) Find $P(X < 0.5)$ and $P(0.25 \le X \le 0.75)$.

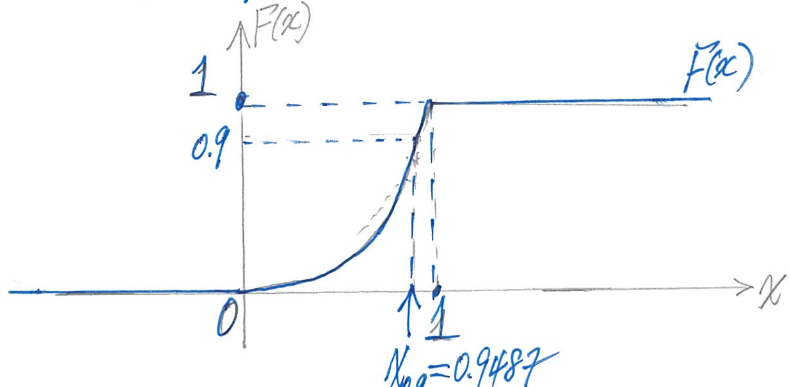
(b) (5pts) Find the 90th percentile of the distribution of X, i.e., find the constant $x_{0.9}$ such that $P(X \le x_{0.9}) = 0.9$.

(a) $$P(X<0.5) = \int_{-\infty}^{0.5} f(x)\,dx = \int_0^{0.5} 2x\,dx = x^2\Big|_0^{0.5} = (0.5)^2 - 0^2 = 0.25.$$

$$P(0.25 \le X \le 0.75) = \int_{0.25}^{0.75} 2x\,dx = x^2\Big|_{0.25}^{0.75} = 0.75^2 - 0.25^2 = 0.5.$$

(b) $$P(X \le x_{0.9}) = \int_{-\infty}^{x_{0.9}} f(x)\,dx = \int_0^{x_{0.9}} 2x\,dx = x^2\Big|_0^{x_{0.9}} = x_{0.9}^2 - 0^2 = x_{0.9}^2$$

From $x_{0.9}^2 = 0.9$, one has $x_{0.9} = \sqrt{0.9} = 0.9487$.



Note: $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$,

if $x \le 0$, $F(x) = 0$.

$x \ge 1$, $F(x) = 1$,

$0 \le x \le 1$, $F(x) = \int_0^x 2t\,dt = x^2$.

i.e. $$F(x) = \begin{cases} 0, & x \le 0, \\ x^2, & 0 \le x \le 1, \\ 1, & x \ge 1. \end{cases}$$

2. (15pts) If X is a Gamma random variable with $\alpha = 2$ and $\lambda = 3$, i.e., X has the following *p.d.f.*

$$X \sim f(x) = \begin{cases} 9xe^{-3x}, & \text{if } x \geq 0; \\ 0, & \text{otherwise}; \end{cases}$$

Let $Y = 2X$. Find the probability density function of random variable Y, what distribution does it follow?

Solution 1. Formulae method. $Y = g(X)$, $f_Y(y) = f_X(h(y)) \cdot |h'(y)|$,

where $y = g(x)$, solve for x, $x = g^{-1}(y) = h(y)$.

Now, $Y = 2X$, or $y = 2x$, one has $x = \frac{y}{2} = h(y)$, $h'(y) = \left(\frac{y}{2}\right)' = \frac{1}{2}$.

Apply above formulae, one has $f_Y(y) = f_X\left(\frac{y}{2}\right) \left|\frac{1}{2}\right| = 9 \cdot \left(\frac{y}{2}\right) \cdot e^{-3\left(\frac{y}{2}\right)} \cdot \frac{1}{2}$, for $x = \frac{y}{2} > 0$.

i.e. $Y \sim f_Y(y) = \begin{cases} \frac{9}{4} y e^{-\frac{3}{2}y}, & y > 0. \\ 0, & \text{otherwise}, \end{cases}$ which is Gamma$\left(2, \frac{3}{2}\right)$ distribution.

Solution 2. Two steps method.

step 1 From $Y = 2X$, $X > 0$, one has $F_Y(y) = P(Y \leq y) = 0$, if $y \leq 0$.

Thus, for $y > 0$. $F_Y(y) = P(Y \leq y) = P(2X \leq y) = P\left(X \leq \frac{y}{2}\right)$

$$= \int_{-\infty}^{\frac{y}{2}} f(x)dx = \int_0^{\frac{y}{2}} 9xe^{-3x} dx.$$

step 2. $f_Y(y) = F_Y'(y) = \left(\int_0^{\frac{y}{2}} 9xe^{-3x}dx\right)_y' = \left(9 \cdot \frac{y}{2} \cdot e^{-3 \cdot \frac{y}{2}}\right)\left(\frac{y}{2}\right)_y' = \frac{9}{4} y e^{-\frac{3}{2}y}$, $y > 0$.

Alternative for step 1. $\int_0^{\frac{y}{2}} 9x e^{-3x} dx = \int_0^{\frac{y}{2}} \underbrace{(-3x)}_{u(x)} \underbrace{(-3e^{-3x})}_{v'(x)} dx = \underbrace{(-3x) \cdot (e^{-3x})}_{u(x) \cdot v(x)}\Big|_0^{\frac{y}{2}} - \int_0^{\frac{y}{2}} u'(x) \cdot v(x) dx$

$= \left(-3 \cdot \frac{y}{2}\right) e^{-3 \cdot \frac{y}{2}} - 0 - \int_0^{\frac{y}{2}} (-3) e^{-3x} dx = -\frac{3}{2} y e^{-\frac{3}{2}y} - e^{-3x}\Big|_0^{\frac{y}{2}} = -\frac{3}{2} y e^{-\frac{3}{2}y} - \left(e^{-\frac{3}{2}y} - e^{0}\right)$

$= -\frac{3}{2} y e^{-\frac{3}{2}y} - e^{-\frac{3}{2}y} + 1$. Now $f_Y(y) = F'(y) = \frac{9}{4} y e^{-\frac{3}{2}y}$, $y > 0$.

3. (30pts) A fair coin is tossed three times. Let X denote the number of heads on the first toss and Y the total number of heads. The joint frequency function of (X, Y) is derived and given in the following table:

X \Y	0	1	2	3	$P(X=x)$
0	1/8	1/4	1/8	0	$\frac{1}{2}$
1	0	1/8	1/4	1/8	$\frac{1}{2}$
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$3/8$	$1/8$	

(a) (10pts) Find $P(X+Y>2)$ and $P(Y-X\geq 1)$.

(b) (5pts) Compute the marginal distributions of X and Y.

(c) (5pts) Are X and Y independent? Why?

(d) (5pts) Compute the conditional frequency function of X given $Y=1$.

(e) (5pts) Find $P_{Y|X}(Y\geq 2|X=1)$.

(a) $P(X+Y>2) = P(X=0, Y=3) + P(X=1, Y=2) + P(X=1, Y=3)$
$= 0 + 1/4 + 1/8 = 3/8.$

$P(Y-X\geq 1) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$
$+ P(X=1, Y=2) + P(X=1, Y=3) = 1/4 + 1/8 + 0 + 1/4 + 1/8 = 3/4.$

(b) See above table.

X	0	1
$p(x)$	1/2	1/2

,

Y	0	1	2	3
$p(y)$	1/8	3/8	3/8	1/8

(c) $P(X=1)=\frac{1}{2}$, $P(Y=1)=3/8$, $P(X=1, Y=1)=1/8$. $P(X=1, Y=1)=\frac{1}{8}\neq P(X=1)\cdot P(Y=1)$.
Thus, X and Y are not independent.

(d) $P(X=0|Y=1) = \dfrac{P(X=0, Y=1)}{P(Y=1)} = \dfrac{1/4}{3/8} = \dfrac{2}{3}$, $P(X=1|Y=1) = \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{1/8}{3/8} = 1/3.$

Thus,

X|Y=1	0	1
$p(x\|1)$	2/3	1/3

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(e) $P(Y\geq 2|X=1) = P(Y=2|X=1) + P(Y=3|X=1)$

$= \dfrac{P(X=1, Y=2)}{P(X=1)} + \dfrac{P(X=1, Y=3)}{P(X=1)} = \dfrac{1/4}{1/2} + \dfrac{1/8}{1/2} = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}.$

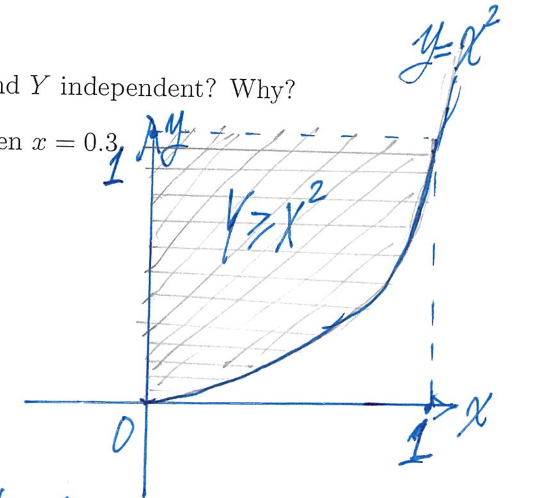
4. (25pts) The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} x+y, & \text{if } 0<x<1,\ 0<y<1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10pts) Find $P(Y \geq X^2)$.

(b) (10pts) Find the marginal density functions of X and Y. Are X and Y independent? Why?

(c) (5pts) Find the conditional density $f_{Y|X}(y|0.3)$? i.e., $f_{Y|X}(y|x)$ when $x = 0.3$.



(a) $P(Y \geq X^2) = \iint_A f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^1 (x+y)\,dy\,dx$

$= \int_0^1 \left[\int_{x^2}^1 x\,dy + \int_{x^2}^1 y\,dy \right] dx$

$= \int_0^1 \left[x(1-x^2) + \frac{y^2}{2}\Big|_{x^2}^1 \right] dx = \int_0^1 \left[x - x^3 + \frac{1}{2} - \frac{x^4}{2} \right] dx$

$= \frac{x^2}{2}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 + \frac{1}{2} - \frac{1}{2}\cdot\frac{x^5}{5}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{10} = 13/20.$

(b) For $0<x<1$, $f_X(x) = \int f(x,y)\,dy = \int_0^1 (x+y)\,dy = \int_0^1 x\,dy + \int_0^1 y\,dy = x + \frac{1}{2}$,

i.e. $f_X(x) = \begin{cases} x+\frac{1}{2}, & x \in (0,1) \\ 0, & \text{else.} \end{cases}$ Similarly, $f_Y(y) = \begin{cases} y+\frac{1}{2}, & y \in (0,1) \\ 0, & \text{else.} \end{cases}$

Since $f(x,y) = x+y \neq (x+\frac{1}{2})(y+\frac{1}{2}) = f_X(x) f_Y(y)$.

X and Y are not independent!

(c) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{x+y}{x+\frac{1}{2}}, \ 0<y<1.$

In particular. $f_{Y|X}(y|0.3) = \dfrac{0.3+y}{0.3+\frac{1}{2}} = \dfrac{0.3+y}{0.8} = \frac{1}{8}(3+10\cdot y), \ 0<y<1.$

i.e. $Y|X=0.3 \sim f_{Y|X}(y|0.3) = \begin{cases} (3+10\cdot y)/8, & 0<y<1. \\ 0, & \text{otherwise.} \end{cases}$

5. (15pts) Let random variable X follow a uniform distribution on $[0,\ 1]$. Conditional on $X = x$, a random variable Y has a uniform distribution on $[x,\ 1]$, i.e., $Y|_{X=x} \sim U[x,\ 1]$.

(a) (5pts) Find the joint density function of $(X,\ Y)$ and the marginal density function of Y. .

(b) (5pts) Find the conditional density $f_{X|Y}(x|0.6)$, i.e., $f_{X|Y}(x|y)$ when $Y = 0.6$.

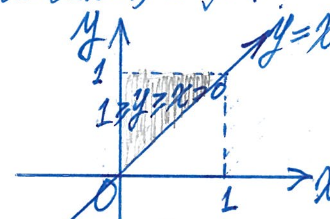
(c) (5pts) Find $P(X \geq 0.5)$ and $P(X \geq 0.5|Y = 0.6)$.

(a). $X \sim U[0,1]$, thus, $X \sim f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{else}. \end{cases}$

Given $Y|X=x \sim U[x,1]$, thus $Y|X=x \sim f_{Y|X}(y|x) = \begin{cases} \frac{1}{1-x}, & x \leq y \leq 1 \\ 0, & \text{else}. \end{cases}$

Therefore, $(X,Y) \sim f_{(X,Y)}(x,y) = f_X(x) \cdot f_{Y|X}(y|x) = 1 * \frac{1}{1-x}$, when $0 \leq x \leq 1$, $x \leq y \leq 1$.

Hence, one has
$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{1-x}, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$



Marginal for Y. for $0 \leq y \leq 1$, (or $0 < y < 1$).
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \frac{1}{1-x}\,dx = -\ln(1-x)\Big|_0^y$$
$$= -\ln(1-y) - (-\ln(1-0)) = -\ln(1-y). \quad (\because \ln 1 = 0)$$

Hence, $Y \sim f_Y(y) = \begin{cases} -\ln(1-y), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$ (See remark below).

(b). $X|Y=y \sim f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$.

for $0 < y < 1$, one has $f_{X|Y}(x|y) = \dfrac{\frac{1}{1-x}}{-\ln(1-y)}$, $0 \leq x \leq y$.

In particular, when $y = 0.6$, one has $f_{X|Y}(x|0.6) = \begin{cases} \dfrac{\frac{1}{1-x}}{-\ln(1-0.6)}, & 0 \leq x \leq 0.6 \\ 0, & \text{otherwise}. \end{cases}$

i.e., $f_{X|Y}(x|0.6) = \begin{cases} \dfrac{-1}{\ln(0.4)(1-x)}, & 0 \leq x \leq 0.6. \\ 0, & \text{otherwise}. \end{cases}$

(c). $P(X \geq 0.5) = \int_{0.5}^{1} 1 \cdot dx = 1 - 0.5 = 0.5.$

$$P(X \geq 0.5 \mid Y = 0.6) = \int_{0.5}^{\infty} f_{X|Y}(x|0.6)\, dx = \int_{0.5}^{0.6} \frac{-1}{\ln(0.4) \cdot (1-x)}\, dx$$

$$= \frac{-1}{\ln 0.4} \int_{0.5}^{0.6} \frac{1}{1-x}\, dx$$

$$= \frac{-1}{\ln 0.4}\left[-\ln(1-x)\Big|_{0.5}^{0.6}\right] = \frac{-1}{\ln 0.4}\left[-\ln(1-0.6) - (-\ln(1-0.5))\right]$$

$$= \frac{-1}{\ln 0.4}\left[-\ln(0.4) + \ln(0.5)\right] = 1 - \frac{\ln 0.5}{\ln 0.4} = 0.243.$$

Remark: How to calculate $\int_0^y \frac{1}{1-x}\, dx$ in part (a).

One also can use change of variable

Let $t = 1-x$, then $x = 1-t$, $dx = -dt$.

then, $\int_0^y \frac{1}{1-x}\, dx = \int_{1-0}^{1-y} \frac{1}{t}(-dt) = -\int_1^{1-y} \frac{1}{t}\, dt = \int_{1-y}^{1} \frac{1}{t}\, dt$

$$= \ln t\Big|_{1-y}^{1} = \ln 1 - \ln(1-y) = -\ln(1-y).$$