Class 7: Combinatorics 2

Monday, September 13, 2021 5:48 AM

Homework:

pending, due Friday 9/17

First some review problems on Multiplication Principle, and on Permutations:
(Solved Problems 1 and 2)

2.1.3 Unordered Sampling without Replacement:

Combinations

review from last class.

Then:
Example 2.6:
I choose 3 cards from the standard deck of cards. What is the probability that these cards contain at least one ace?

SOLU: Prob (at least one ace) $= 1 - P(\text{no ace})$ $= 1 - \frac{\binom{48}{3}}{\binom{52}{3}}$

$$P(\text{no ace}) = \frac{|\text{no ace}|}{|S|} = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{\frac{48\cdot 47\cdot 46}{\cancel{1\cdot 2\cdot 3}}}{\frac{52\cdot 51\cdot 50}{\cancel{1\cdot 2\cdot 3}}} = \frac{48\cdot 47\cdot 46}{52\cdot 51\cdot 50}$$

Note the **coefficient** in the Binomial Expansion:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

$$(a+b)^2 = \binom{2}{0} a^0 b^2 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^2 b^0$$
$$= b^2 + 2ab + a^2$$

$$\binom{n}{0} = 1 = \binom{n}{n}$$

$$\frac{5\cdot 4}{1\cdot 2} = \binom{5}{2} = 10 = \binom{5}{3} = \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = \boxed{\frac{5!}{2!\,3!}} = \frac{5\cdot 4\cdot 3\cdot \cancel{2\cdot 1}}{(1\cdot 2)(\cancel{1\cdot 2\cdot 3})}$$

$$\binom{n}{k} = \binom{n}{n-k} \;!!$$

Equivalence of # combinations and # of partitions.

 # Partitions into two groups = # Combinations of selecting one of the groups (the other one "falls in place" by being the "leftovers"

> The total number of ways to divide n distinct objects into two groups A and B such that group A consists of k objects and group B consists of $n-k$ objects is $\binom{n}{k}$.

A new situation: $\longrightarrow$ 2 types of outcomes

Two Types of Objects: Binary Sampling (with Replacement)

here: ***replacement is different from sampling from n individual different objects)***

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See Example 2.7

How many distinct sequences can we make using 3 letter "A"s and 5 letter "B"s? (AAABBBBB, AABABBBB, etc.)

Think of tasks ("experiments") or trials

A A A — assign an outcome to position $i = 1, \dots, n$ // 8

1 2 3 … 8

n: distinct positions in the sequence
select $k (= 3)$ to assign the A's
The rest is filled out as B's
} # partitions

Result $= \binom{8}{3}$ (or $\binom{n}{k}$)

Bernoulli Trials - Binomial Probabilities

→ Tossing a coin → 2 types – $\{H, T\}$

– n trials each with same 2 possible outcomes < S success → 1, F Failure → 0 (numeric coding)

trials are independent, (randomness)

– $P(\text{Success}) = P(S) = p$ is constant across trials

note: $S^c = F$!!

Interest: n trials, k = # successes ⇒ $k \in \{0, 1, \dots, n\}$

$n = 5$, $k = 3$ eg: $SSSFF = P(S) \cdot P(S) \cdot P(S) \cdot P(F) \cdot P(F) = p^k (1-p)^{n-k}$ (indept !)

k successes, n-k failures $p^3 (1-p)^2$

how about SFSFS $P(SFSFS) = p^3 (1-p)^2$ same !!

$P(3 \text{ successes in } 5 \text{ trials}) = (\#\text{ sequences } 3 S\text{'s } 2 F\text{'s}) \cdot p^3 (1-p)^2 = \binom{5}{3} p^3 (1-p)^2$

Binomial Formula: (Probability)
For n independent Bernoulli trials where each trial has success probability p, the probability of k successes is given by

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

See Prob 3, 4 of Solved Problems:

P3:

An urn contains 30 red balls and 70 green balls. What is the probability of getting exactly k red balls in a sample of size 20 if the sampling is done **with** replacement (repetition allowed)?

Assume $0 \le k \le 20$.

$P(\text{red in one trial (draw)}) = 0.3$ → constant across trials

the sampling is done **with** replacement (repetition allowed)?
Assume $0 \le k \le 20$.

$P(\text{red in one trial draw}) = 0.3 \to$ constant across trials b/c replacement

$$P(k) = \binom{20}{k}(.3)^k(\underbrace{1-.3}_{.7})^{20-k}$$

P4:An urn consists of 30 red balls and 70 green balls. What is the probability of getting exactly $k$red balls in a sample of size 20 if the sampling is done **without** replacement (repetition not allowed)?

Binary Sampling Without Replacement:
(Hypergeometric Probabilities)

Want $P(5 \text{ red balls} \cap 15 \text{ green balls})$

S : {all possible size 20 selections of 100 balls}

$|S| = \binom{100}{20}$ Event: R=Select 5 red balls from 30 red balls

$|R| = \binom{30}{5}$

G: event: 15 green balls $|G| = \binom{70}{15}$

$$= \frac{|\text{event}|}{|S|} = \frac{\binom{30}{5}\cdot\binom{70}{15}}{\binom{100}{20}}$$

$k = 5$
$M = 20$
↳ # selected

$$= \frac{\binom{30}{k}\cdot\binom{70}{M-k}}{\binom{100}{M}}$$

Multinomial Coefficients:

Discuss Ex: 2:10
Ten people have a potluck. Five people will be selected to bring a main dish, three people will bring drinks, and two people will bring dessert. How many ways can they be divided into these three groups?

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Solution

We can solve this problem in the following way. First, we can choose 5 people for the main dish. This can be done in $\binom{10}{5}$ ways. From the remaining 5 people, we then choose 3 people for drinks, and finally the remaining 2 people will bring desert. Thus, by the multiplication principle, the total number of ways is given by

$$\binom{10}{5}\binom{5}{3}\binom{2}{2} = \frac{10!}{5!5!}\cdot\frac{5!}{3!2!}\cdot\frac{2!}{2!0!} = \frac{10!}{5!3!2!}.$$

generalizes! total n objects r different kinds
select n_1 of first
n_2 of 2^{nd}
n_3 etc.

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}.$$

n_1 of first kind
n_2 of 2^{nd}
etc.

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r} \qquad (2.2)$$

P(n_1 1^{st} kind, n_2 of 2^{nd} kind, .. etc.)

$$\binom{n}{n_1, n_2, \ldots, n_r} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} = \frac{n!}{n_1!n_2!\ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}.$$

2.1.4 Unordered Sampling with Replacement

$n = 3$ elements select k. $(= 2)$

- 1,1 $\to (x_1, x_2, x_3) = (2, 0, 0)$;
- 1,2 $\to (x_1, x_2, x_3) = (1, 1, 0)$;
- 1,3 $\to (x_1, x_2, x_3) = (1, 0, 1)$;
- 2,2 $\to (x_1, x_2, x_3) = (0, 2, 0)$;
- 2,3 $\to (x_1, x_2, x_3) = (0, 1, 1)$;
- 3,3 $\to (x_1, x_2, x_3) = (0, 0, 2)$.

Lemma 2.1
The total number of distinct k samples from an n-element set such that repetition is allowed and ordering does not matter is the same as the number of distinct solutions to the equation

$$x_1 + x_2 + \ldots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \ldots\}.$$

So far we have seen the number of unordered k-samples from an n element set is the same as the number of solutions to the above equation. But how do we find the number of solutions to that equation?

Theorem 2.1
The number of distinct solutions to the equation

$$x_1 + x_2 + \ldots + x_n = k, \text{ where } x_i \in \{0, 1, 2, 3, \ldots\} \quad (2.3)$$

is equal to

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

read this (proof).

Solved Examples for practice

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