Alex Beckwith Math 855 HW4

(34) Let $f(x) = \frac{1+\alpha x}{2}$ for $-1 \le x \le 1$ and $f(x) = 0$ otherwise

if $f(x)$ is a density, then

$$F(x) = \int_{-1}^{1} \frac{1+\alpha x}{2}\,dx \overset{\text{(should)}}{=} 1$$

$$= \underbrace{\frac{1}{2}x + \frac{\alpha x^2}{4}}_{\text{cdf}}\Big|_{-1}^{1} = \frac{1}{2} + \frac{\alpha}{4} - \left(\frac{-1}{2} + \frac{\alpha}{4}\right) = 1$$

(37) line segment. uniform $P(\text{longer}/2 > \text{shorter})$

$$P(x < \tfrac{1}{3}) + P(x > \tfrac{2}{3}) = 1 - P(\tfrac{1}{3} \le x \le \tfrac{2}{3})$$

$$\text{cdf} = 1 - \int_{1/3}^{2/3} \frac{1}{b-a}\,dx = 1 - \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{2}{3}$$

(Also solved experimentally in Python)

(40) $f(x) = cx^2$ for $0 \le x \le 1$ and 0 otherwise.

$$1 = c\int_0^1 x^2\,dx = c\left(\frac{x^3}{3}\Big|_0^1\right) = \frac{c}{3}$$

a) $c = 3$

b) $\text{cdf} = x^3$

c) $P(.1 \le x \le .5) = \frac{1}{8} - \frac{1}{1000} = 0.124$

(45) (calculated in Python)

a. 0.6321

b. 0.3834

c. $t = 46.0518$

(46) T is exp. $P(T < 1) = 0.05$ what is λ?

$$1 - e^{-\lambda x}\Big|_0^1 = 0.05$$

$$0.95 = e^{-\lambda}$$

$$\ln(.95) = -\lambda$$

$$\lambda = -\ln(0.95)$$

(52) Please see Python for calcs.

a) 0.2525

b) $N(177.8, 7.62)$

$N(1.778, 0.0762)$

(55) $X \sim N(\mu, \sigma^2)$ find c in terms of σ

such that $P(\mu - c \le X \le \mu + c) = 0.95$

$0.95 = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$ where $a = \mu - c$, $b = \mu + c$

$= \Phi\left(\frac{\mu + c - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - c - \mu}{\sigma}\right)$

$= \Phi\left(\frac{c}{\sigma}\right) - \Phi\left(\frac{-c}{\sigma}\right)$ ← integral so can remove constant.

$0.95 = 2c\,\Phi\left(\frac{1}{\sigma}\right)$

$c = \frac{0.95}{2\Phi\left(\frac{1}{\sigma}\right)}$

(59) U is uniform on $[-1, 1]$, find density of U^2

$F_{U^2}(u) = P(U^2 \le u) \le P(-\sqrt{u} \le U \le \sqrt{u})$ when $u \ge 0$

$$\int_{-\sqrt{u}}^{\sqrt{u}} \frac{1}{b-a}\,dx = \frac{1}{2}(x)\Big|_{-\sqrt{u}}^{\sqrt{u}}$$

$$= \frac{1}{2}(\sqrt{u} + \sqrt{u})$$

$$= \sqrt{u}$$

to find density, derivative

$\frac{\partial \sqrt{u}}{\partial u} = \frac{1}{2} u^{-1/2}$

60. Lognormal find the density function of $y = e^z$
where $Z \sim N(\mu, \sigma^2)$

density function of $z = g(x)$
(for this z)

$P(Y \leq y)$
$= P(e^z \leq y)$
$= P(Z \leq \ln y)$
$= \int_{-\infty}^{\ln y} g(x)\,dx$, when $y \leq 0$, $f_Y = 0$

for $y > 0$, $f_Y = \dfrac{\partial g(\ln y)}{\partial y} = \dfrac{1}{y} g(\ln y)$

68. radius of circle is Exp()
find density function of area.

$x \in (0, \infty)$ $F(x) \int_0^{e^{-x}} 4\pi r^2\,dr$, $F_r(0) = 0$

so $F_r(x) = \frac{4\pi}{3}(e^{-x})^3$

and $f_r(x) = \dfrac{dF_r}{dx} = 4\pi (e^{-x})^2 e^{-x} = 4\pi e^{-3x}$