Chapter 1 part 4. §1.6 Independence.

$P(A|B)=P(A)$. $P(B|A)=P(B)$.

knowing B had occured gave us no info about whether A had occured.

$$P(A|B)=\frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B)=P(A)P(B) \quad \text{if } P(A|B)=P(A).$$

Definition: A and B are said to be independent events, if $P(A\cap B)=P(A)\cdot P(B)$.

Expl. A. A card is selected randomly from a deck. A=" it is an ace". D="it is a diamond". A and D are independent?

Ans: $P(A)=\frac{4}{52}=\frac{1}{13}$. $P(D)=\frac{13}{52}=\frac{1}{4}$. $A\cap D=\{A\diamondsuit\}$. $P(A\cap D)=\frac{1}{52}$

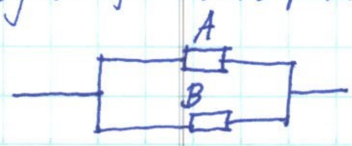
one has $P(A\cap D)=P(A)P(D)$, they are independent!

Expl. B. Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If A is the event "1st coin lands H", B="2nd lands T". A and B independent?

Ans: $\Omega=\{HH, HT, TH, TT\}$. $A=\{HH, HT\}$, $B=\{HT, TT\}$. $AB=\{HT\}$.

$P(A)=\frac{1}{2}$, $P(B)=\frac{1}{2}$, $P(A\cap B)=1/4$. since $P(A\cap B)=P(A)P(B)$. they are independent!

Expl. C. A system is designed so that it fails only if a unit and a backup unit both fail. Assuming that these failures are independent and that each unit fails with prob. p, the system fails with prob. p^2.



$P(A)=0.1$, prob. unit A fails.
$P(B)=0.1$. . . . . B.
Prob. system fails $=P(A\cap B)=P(A)P(B)=0.01$.

For more than two events: A, B, C.

pairwise independent (any two are independent) doesn't guarantee mutual independence!

i.e. $P(A\cap B\cap C)\neq P(A)P(B)P(C)$.

e.g.1. A fair coin is tossed twice. A="1st is H", B="2nd is H", C="exactly one H"

$A=\{HH, HT\}$, $B=\{HH, TH\}$, $C=\{HT, TH\}$. $AB=\{HH\}$, A and B indep.

$AC=\{HT\}$, A, C indep. $B\cap C=\{TH\}$, B and C indep. they are pairwise independent.

However, $A\cap B\cap C=\phi$, $P(A\cap B\cap C)=0\neq P(A)P(B)P(C)$. they are not mutual independence.

Definition: A, B, C are mutually independent, if $P(A\cap B)=P(A)P(B)$, $P(A\cap C)=P(A)P(C)$, $P(B\cap C)=P(B)P(C)$
$P(A\cap B\cap C)=P(A)P(B)P(C)$.

more general: $A_1, A_2, \cdots, A_n$ mutually independent, if any subcollection, $A_{i_1}, A_{i_2}, \cdots, A_{i_m}$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_m}).$$

Expl D. (infectivity of AIDS).

Suppose that virus transmissions in 500 acts of intercourse are mutually independent events and that the prob of transmission in any one act is $\frac{1}{500}$. Under this model, what is the prob. of infection?

Ans: Let A = "infection in this model". B_i = "virus transmissions in i-th act", $i=1,2,\cdots 50$.

$A = B_1 \cup B_2 \cup \cdots \cup B_{500}$. $P(B_i) = \frac{1}{500}$. B_i's are mutually independent.

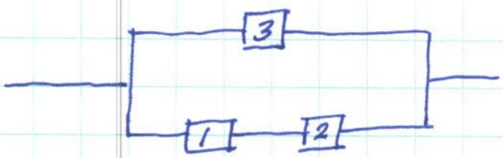
$A^c = B_1^c \cap B_2^c \cap \cdots \cap B_{500}^c$, $P(B_i^c) = 1 - \frac{1}{500}$. B_i^c's are mutually independent too!

$$P(A) = 1 - P(A^c) = 1 - P(B_1^c \cap B_2^c \cap \cdots \cap B_{500}^c) = 1 - P(B_1^c) P(B_2^c) \cdots P(B_{500}^c)$$

$$= 1 - \left(1 - \frac{1}{500}\right)^{500} = 1 - 0.37 = 0.63.$$

the prob. of infection is 0.63.

Expl. E.



A circuit with three relays.

A_i = "i-th relay works", $P(A_i) = p$.

A_1, A_2, A_3. mutually independent.

Let F = "system works", then $F = A_3 \cup (A_1 \cap A_2)$.

$$P(F) = P(A_3 \cup (A_1 \cap A_2)) = P(A_3) + P(A_1 \cap A_2) - P(A_3 \cap (A_1 \cap A_2))$$

$$= P(A_3) + P(A_1)P(A_2) - P(A_1)P(A_2)P(A_3) = p + p^2 - p^3.$$

Expl. F.

A system consists of components connected in a series, so the system fails if any one component fails.

if $P(A_i) = p$, each fails with prob. p, they are mutually independent.

then what's prob system will fail?

Ans: Let B = "system will fail", $B = A_1 \cup A_2 \cup \cdots \cup A_n$.

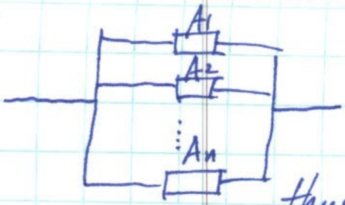
$P(B) = P(A_1 \cup A_2 \cup \cdots \cup A_n)$, one consider $\underset{\text{system works}}{B^c} = (A_1 \cup \cdots \cup A_n)^c = \underset{\text{all compontents work}}{A_1^c \cap A_2^c \cap \cdots \cap A_n^c}$

$$P(B^c) = P(A_1^c) P(A_2^c) \cdots P(A_n^c) = (1-p) \cdot (1-p) \cdots (1-p) = (1-p)^n.$$

thus $P(B) = 1 - P(B^c) = 1 - (1-p)^n$.

if $n=10$, $p=0.05$, $P(\text{system works}) = (1-p)^n = (1-0.05)^{10} = 0.95^{10} = 0.60$.

$P(\text{system fails}) = 1 - 0.60 = 0.40$.



components are connected in parallel, the system fails only when all components fail.

thus $B = A_1 \cap A_2 \cap \cdots \cap A_n$, $P(B) = P(A_1) \cdots P(A_n) = p^n$.

if $n=10$, $p=0.05$, $P(B) = 0.05^{10} = 9.8 \times 10^{-14} \approx 0$.

"reliability studies for system". "crucial assumption on independence".

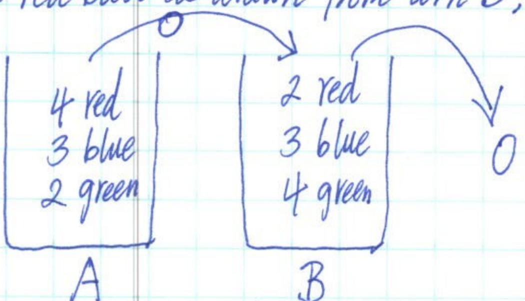
More examples:

Problem 47. Urn A has four red, three blue, and two green balls. Urn B has two red, three blue, and four green balls. A ball is drawn from urn A and put into urn B, and then a ball is drawn from urn B.

a. What. prob. that a red ball is drawn from urn B?

b. If a red ball is drawn from urn B, what. prob. that a red ball was drawn from urn A?

Ans:



a. Let R = "a red ball was draw from urn A"

B = "a blue . . . "

G = "a green . . . "

A = "a red ball is drawn from urn B"

From law of total probability, $P(A) = P(A|R)P(R) + P(A|B)P(B) + P(A|G)P(G)$

$$= \frac{3}{10}\cdot\frac{4}{9} + \frac{2}{10}\cdot\frac{3}{9} + \frac{2}{10}\cdot\frac{2}{9} = \frac{11}{45}.$$

b. $$P(R|A) = \frac{P(A|R)P(R)}{P(A)} = \frac{\frac{3}{10}\cdot\frac{4}{9}}{\frac{11}{45}} = \frac{6}{11}$$

49. A fair coin is tossed three times.

a. What's prob. of two or more heads given that there was at least one head?

b. what's prob . . . there was at least one tail?

Ans: a. Let A = "two or more heads". B = "there was at least one head"

$P(A|B) = \frac{P(AB)}{P(B)}$. $P(B) = 1 - P(B^c) = 1 - \frac{1}{8} = \frac{7}{8}$.

$P(AB) = P(A) = P(\text{two or } 3) = \binom{3}{2}\frac{1}{8} + \frac{1}{8} = \frac{3}{8} + \frac{1}{8} = \frac{4}{8}$, thus $P(A|B) = \frac{4}{7}$.

another approach is to use reduced sample space method.

$B = \{hhh, hhT, hth, htt, thh, tht, tth\}$. $P(A|B) = \frac{4}{7}$.

b. Let C = "there was at least one tail".

$C = \{hht, hth, htt, thh, tht, tth, ttt\}$. $P(A|C) = \frac{3}{7}$. reduced sample space.

also, one can apply $P(A|C) = P(AC)/P(C) = P(2h\text{'s})/P(C) = \binom{3}{2}\frac{1}{8} \Big/ \frac{7}{8} = \frac{3}{7}$.

59. A box has three coins. One has two heads, one has two tails, and the other is a fair coin with one head and one tail. A coin is chosen at random, is flipped, and come up heads.

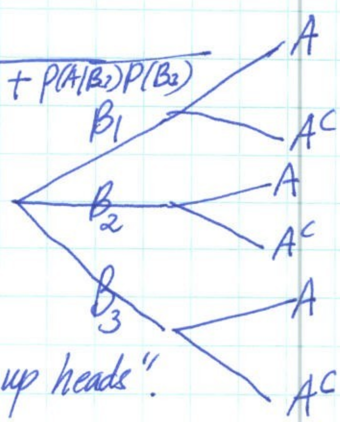
a. What's prob. that the coin chosen is the two headed coin?
b. . . . . if it is thrown another time it will come up heads?
c. Answer part (a) again, supposing that the coin is thrown a second time and comes up heads again.

Ans:
a. Let B_1 = "the coin chosen is the two headed coin".
B_2 = " . . . . tailed "
B_2 = "the coin chosen is one head and one tail"
A = "A coin is chosen at random, flipped and come up heads".

$$P(B_1|A) = \frac{P(B_1A)}{P(A)} = \frac{P(B_1)P(A|B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}$$

$$= \frac{\frac{1}{3}\cdot 1}{1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}}$$

$$= 2/3.$$



b. Let C = "it is thrown another time, it will come up heads".

$$P(C|A) = \frac{P(CA)}{P(A)},$$ Let $D = CA$ = "the ~~chos~~ coin is flipped twice with two heads"

$$= \frac{P(D)}{P(A)} = \frac{P(D|B_1)P(B_1) + P(D|B_2)P(B_2) + P(D|B_3)P(B_3)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}$$

$$= \frac{1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \frac{1}{4}\frac{1}{3}}{1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{1}{3}+\frac{1}{12}}{\frac{1}{3}+\frac{1}{6}} = \frac{5}{6}$$

c. $$P(B_1|D) = \frac{P(B_1D)}{P(D)} = \frac{P(D|B_1)P(B_1)}{P(D)} = \frac{1\cdot\frac{1}{3}}{1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{12}} = \frac{4}{5}$$

Note: in (a) $P(B_1|A) = \frac{2}{3}$. in (c) $P(B_1|D) = \frac{4}{5}$. $P(B_1|A) < P(B_1|D)$.

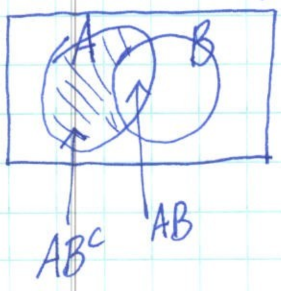
63. Suppose that the prob. of living to be order than 70 is 0.6 and the prob. of living to be older than 80 is 0.2. If a person reaches her 70th birth day, what's prob. that she will celebrate her 80th?

Ans: Let B = "living to be order than 70" $P(B)=0.6$, $P(A)=0.2$.
A = " . . . . 80"

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{0.2}{0.6} = 1/3.$$

65. If A and B are independent, then A and B^c, A^c and B^c are independent.

Proof.



$A = AB^c \cup AB$, thus $P(A) = P(AB^c) + P(AB)$

then $P(AB^c) = P(A) - P(AB) \overset{\text{indep.}}{=} P(A) - P(A)P(B)$
$= P(A)[1 - P(B)]$
$= P(A)P(B^c)$. thus A and B^c are independent.

69. If A and B are disjoint, can they be independent?

Ans: No. since $P(AB) \equiv 0$, if they are disjoint. thus $P(AB) = 0 \neq P(A)P(B)$

71. Show that if A, B and C are mutually independent, then $A \cap B$ and C are independent so does $A \cup B$ and C.

proof. one wants to show $P((A \cap B) \cap C) = P(A \cap B)P(C)$, which is from $P(ABC) = P(A)P(B)P(C)$

Next, show $P((A \cup B) \cap C) = P(A \cup B) \cdot P(C)$. (*)

left side of (*) is $P((A \cap C) \cup (B \cap C)) = P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))$
$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$.

right side of (*) is $[P(A) + P(B) - P(AB)]P(C) = P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$.

73. A system has n independent units, each of which fails with prob. p. The system fails only if k or more of the units fail. What's prob. that the system fails?

Ans. Let A_i = "there are i units fail". E = "system fails"

$E = A_k \cup A_{k+1} \cup \cdots \cup A_n$, $P(E) = \sum_{j=k}^{n} P(A_j)$, where $P(A_j) = \binom{n}{j} p^j (1-p)^{n-j}$.

77. A player throws darts at a target. On each trial, independently of the other trials, he hits the bull's-eye with prob. 0.05. How many times should he throw so that his prob. of hitting the bull's eye at least once is 0.5?

Ans: Let A_i = "i-th trial, he hits the bull's eye".
n be the times of trials. A = "hitting the eye at least once in n trials"

$A = A_1 \cup A_2 \cup \cdots \cup A_n$. $A^c = A_1^c \cap A_2^c \cap A_3^c \cdots \cap A_n^c$

$P(A^c) = P(A_1^c)P(A_2^c) \cdots P(A_n^c) = (1-0.05)(1-0.05)\cdots(1-0.05) = 0.95^n$.

$P(A) = 1 - P(A^c) = 1 - 0.95^n$. determine n such that $P(A) \geq 0.5$

or $0.95^n \leq 0.5$, $n \ln(0.95) \leq \ln(0.5)$, $n \geq \ln(0.5)/\ln(0.95) = 13.5$, thus $n = 14$.