Class 4 Cond. Prob

Wednesday, September 8, 2021 5:48 AM

1.4.0 Conditional Probability

As you obtain additional information, how should you update probabilities of events?

Example: (from text)

For example, suppose that in a certain city, 23 percent of the days are rainy. Thus, if you pick a random day, the probability that it rains that day is 23 percent. Now suppose that I pick a random day, but I also tell you that it is cloudy on the chosen day. If C is the event that it is cloudy, then we write this as

$P(R|C)$, the conditional probability of R given that C has occurred.

It is reasonable to assume that in this example, $P(R|C)$ should be greaer than the original $P(R)$, which is called the **prior probability** of R. But what exactly should $P(R|C)$ be?

Another Example:

You are drawing one card at random from a deck and it is a King. If you draw a second card, what is the chance it is also a King? Discuss!

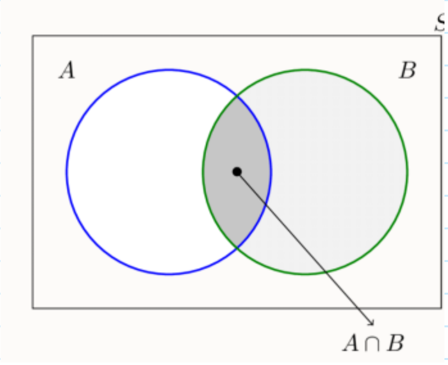
Let B: the first card is a king $P(B) = 4/52 = 1/13$

Let A: the second card is a king

Then $P(A \mid B) =$ Prob(2nd card is a King, given that the first card is a King)

$= 3/51 \rightarrow$ based on remaining deck of cards

Discussion:



equally likely events:

$$P(A|B) = \frac{|A \cap B|}{|B|}$$

In general : Def:

If A and B are two events in a sample space S, then the **conditional probability of A given B** is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.$$

Card example:

$$P(A|B) = \frac{3}{51} = \longrightarrow$$

later combinatorics!

Simpler Roll a die

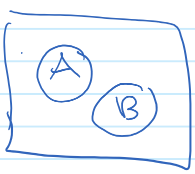
A: # dots is even

B: # dots is ≥ 4

$$P(A|B) = \frac{P\{4,6\}}{P\{4,5,6\}} = \frac{2}{3}$$

$$= \frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}$$

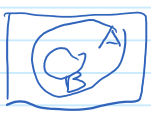
When A and B are disjoint:



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

When B is a subset of A:

$A \cap B = B$!!



$$P(A|B) = \frac{P(B)}{P(B)} = 1$$

When A is a subset of B:

$A \cap B = A$

$$P(A|B) = \frac{P(A)}{P(B)}$$

It is important to note that conditional probability itself is a probability measure, so it satisfies probability axioms. In particular,

- Axiom 1: For any event A, $P(A|B) \geq 0$.
- Axiom 2: Conditional probability of B given B is 1, i.e., $P(B|B) = 1$.
- Axiom 3: If $A_1, A_2, A_3, \cdots$ are disjoint events, then $P(A_1 \cup A_2 \cup A_3 \cdots |B) = P(A_1|B) + P(A_2|B) + P(A_3|B) + \cdots$.

A famous probability problem, called the two-child problem. Many versions of this problem have been discussed [1] in the literature and we will review a few of them in this chapter. We suggest that you **try to guess the answers before solving the problem** using probability formulas.

Example
Consider a family that has two children. We are interested in the children's genders. Our sample space is $S=\{(G,G),(G,B),(B,G),(B,B)\}$
. Also assume that all four possible outcomes are equally likely. (by genetics)

1. What is the probability that both children are girls given that the first child is a girl?
2. We ask the father: "Do you have at least one daughter?" He responds "Yes!" Given this extra information, what is the probability that both children are girls? In other words, what is the probability that both children are girls given that we know at least one of them is a girl?

1.) Guess: 0.5 Equ. $P[\text{both } G \mid 1^{st} \text{ is } G]$

$$= \frac{|\text{both } G|}{|1^{st}\ G|} = \frac{|\{GG\}|}{\{GG, GB\}} = \underline{\underline{1/2}}$$

Note $(\text{both } G) \cap (1^{st}\ G) = \text{both } G$

$$P = \frac{P(\text{both } G)}{P(1^{st}\ G)} = \frac{1/4}{2/4} = \frac{1}{2}$$

2. Is it one third? 1/3 !

At least one girl: $\{(GG), (GB), (BG)\} = B$

Both Girls $\{GG\} := A$

Cardinality: $\frac{|GG|}{|B|} = \frac{1}{3}$

or $A \cap B = \{GG\} \Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \underline{\underline{1/3}}$

Chain rule (also called **"Multiplication Rule") for conditional probability:I**

It just turning the equation around: Obain the *P*(intersection) given the conditional probability and the "prior" probability:

$$\mathbf{P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)}$$

Three events: (Discuss!)

$$P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A) \cdot \underbrace{P(B \cap C \mid A)}_{P(B|A)\ P(C|B,A)}$$

apply chain rule twice

A, B, C jointly happening

for n events

Chain rule for conditional probability:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1) \cdots P(A_n|A_{n-1}, A_{n-2}, \cdots, A_1)$$

Chain rule for conditional probability:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1)\cdots P(A_n|A_{n-1}, A_{n-2}, \cdots, A_1)$$

1.4.1 Independence

If two events A and B are independent and $P(B)\neq 0$, then $\boxed{P(A|B) = P(A)}$

In other words: The occurrence of B does not change the chance of A happening.

Now: $\underbrace{P(A|B)}_{\frac{P(A\cap B)}{P(B)}} = P(A)$ $\rightarrow$ rearrange $P(A\cap B) = P(A)\cdot P(B)$

We use this as a **Definition:**

Two events A and B are independent if and only if $P(A\cap B) = P(A)P(B)$.

Ex: Roll 2 dice 1st shows a 6, 2nd " a 4 $\Big\}\rightarrow P(6,4) = P(6)\cdot P(4)$ in 2 dice $= 1/6 \cdot 1/6 = \underline{1/36}$

Sample Space of equally likely events when rolling 2 dice

1st die \ 2nd die	1	2	3	4	5	6
1	1,1	1,2	1,3			
2						
3						
4						
5						
6				6,4		

outcome of 1st die is independent of outcome of 2nd die

Three or more events: (discuss) : A, B, C are "mutually" independent.

$$\boxed{\begin{aligned} P(A\cap B) &= P(A)\cdot P(B) \\ P(A\cap C) &= P(A)\cdot P(C) \\ P(B\cap C) &= P(B)\cdot P(C) \end{aligned}\Bigg\} \text{ pair wise} \\ \text{but also } P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)}$$

Lemma 1.1
If A and B are independent then

- A and B^c are independent,
- A^c and B are independent,
- A^c and B^c are independent.

Assume $A_1, \dots, A_n$ independent

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - P(A_1^c \cap A_2^c \cap \cdots A_n^c)$$

De Morgan's law

indep

$$1 - P\left((A_1 \cup A_2 \cup \cdots A_n)^c\right) \qquad 1 - (1 - P(A_1))(1 - P(A_2)) \cdots (1 - P(A_n))$$

Lemma 1.1

If $A_1, A_2, \cdots, A_n$ are independent then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - (1 - P(A_1))(1 - P(A_2)) \cdots (1 - P(A_n)).$$

Warning! One common mistake is to confuse independence and being disjoint. These are completely different concepts. When two events A and B are disjoint it means that if one of them occurs, the other one cannot occur, i.e., $A \cap B = \emptyset$. Thus, event A usually gives a lot of information about event B which means that they cannot be independent.

Concept	Meaning	Formulas
Disjoint	A and B cannot occur at the same time	$A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
Independent	A does not give any information about B	$P(A\|B) = P(A), P(B\|A) = P(B)$ $P(A \cap B) = P(A)P(B)$

Table 1.1: Differences between disjointness and independence.

From the End of Chapter Problems:

Problem 25
A professor thinks students who live on campus are more likely to get As in the probability course. To check this theory, the professor combines the data from the past few years:

a. 600 students have taken the course,
b. 120 students have gotten As,
c. 200 students lived on campus,
d. 80 students lived off campus and got As.

Does this data suggest that "getting an A" and "living on campus" are dependent or independent?

From the solved problems:

Problem 3
For three events A, B, and C, we know that

- A and C are independent,
- B and C are independent,
- A and B are disjoint,
- $P(A \cup C) = \frac{2}{3}, P(B \cup C) = \frac{3}{4}, P(A \cup B \cup C) = \frac{11}{12}$

Find $P(A), P(B)$, and $P(C)$.

For a more difficult example: See Ex 1.23 (basketball playing)

Summary
After the lecture, summarize the main points of this lecture topic.

- Conditional Probability
- Independence