3 4 6 7 9 11

3

$\bar{X} \approx$ the average of a sample of 16 independent normal r.v.s.

average $\longrightarrow$ 16 $\longrightarrow$ $N(0,1)$

Determine c such that $P(|\bar{X}|<c) = 0.5$

$$0.5 = P(|\bar{X}| < c)$$
$$= P(\bar{X} < c) - P(\bar{X} < -c)$$
$$= P(\bar{X} < c) - [1 - P(\bar{X} < c)]$$
$$= 2 \cdot P(\bar{X} < c) - 1$$
$$1.5 = 2 \cdot F_{\bar{x}}(c) \qquad \bar{X} \approx N(0, \tfrac{1}{16})$$

normalize

$$0.75 = \Phi\left(\frac{c-0}{\sqrt{\frac{1}{16}}}\right)$$
$$\Phi^{-1}(0.75) = 4c$$
$$c = \frac{\Phi^{-1}(0.75)}{4} =$$

```python
# Question 3
from scipy.stats import norm

# use inverse of norm cdf to calc c
c = norm.ppf(0.75) / 4
print(f"c = {c}")
```

✓ 0.2s

```
c = 0.16862243754902043
```

4 If T follows a t_7 dist., find t_0 such that

a) $P(|T| < t_0) = 0.9$

$$0.9 = P(-t_0 < t_7 < t_0)$$
$$0.9 = 1 - 2 \cdot P(t_7 < -t_0) \quad \text{(symmetry)}$$
$$0.05 = P(t_7 < -t_0)$$
$$t_0 = t_{inv}(\overset{p}{0.05}, \overset{dof}{7})$$

b) $P(T > t_0) = 0.05$

$$P(T > t_0) = 0.05$$
$$P(t_7 < t_0) = 0.95$$
$$t_{inv}(\overset{p}{0.95}, \overset{dof}{7})$$

```python
# Question 4
from scipy.stats import t
dof = 7                         # degrees of freedom
a = t.ppf(0.05, dof)
b = t.ppf(0.95, dof)
print(f"4a = {a}")
print(f"4b = {b}")
```

✓ 0.2s

```
4a = -1.8945786050613054
4b = 1.894578605061305
```

6 Show that if $T \sim t_n$, then $T^2 \sim F_{1,n}$.

Student's t pdf

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{x^2}{n}+1\right)^{(n+1)/2}} \qquad -\infty < x < \infty$$

$$Y = g(x) = x^2 \qquad \Rightarrow \qquad \frac{\partial x}{\partial y} = \frac{1}{2\sqrt{y}}$$
$$X = g^{-1}(y) = \sqrt{y}$$

w/ transformation $f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$

so

$$f_Y(y) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{\sqrt{y}^2}{n}+1\right)^{\left(\frac{n+1}{2}\right)}}\left|\frac{1}{2\sqrt{y}}\right|$$

$$= \underbrace{\frac{\Gamma\left(\frac{n+1}{2}\right)\left(\frac{1}{n}\right)^{1/2} y^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)\left(\frac{y}{n}+1\right)^{\left(\frac{n+1}{2}\right)}}}_{\text{pdf of } F_{1,n} \text{ rand var.}} \qquad \text{where } y > 0$$

7 Show that the Cauchy dist & the t dist w/ 1 dof. are the same.

Student's t pdf

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{x^2}{n}+1\right)^{(n+1)/2}} \qquad -\infty < x < \infty$$

when $n=1$, $f(x)\ \dfrac{\Gamma(1)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)(x^2+1)}$ $\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(1) = 1$

so $f(x) = \dfrac{1}{\pi(x^2+1)}$ $\quad -\infty < x < \infty$

The Cauchy dist.

9 Find the mean + variance, when

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$E(\chi^2_{n-1}) = E\left(\frac{(n-1)S^2}{\sigma^2}\right)$$

$$n-1 = \frac{(n-1)}{\sigma^2}\cdot E(S^2)$$

$$E(S^2) = \frac{\sigma^2\cancel{(n-1)}}{\cancel{(n-1)}} = \sigma^2$$

$$Var(\chi^2_{n-1}) = Var\left(\frac{n-1\, S^2}{\sigma^2}\right)$$

$$2(n-1) = \frac{(n-1)^2}{\sigma^4}Var(S^2)$$

$$Var(S^2) = \frac{2\sigma^4}{(n-1)}$$

11 Let $X_1 \cdots X_n$ be a sample from an $N(\mu_x, \sigma^2)$ dist.

Let $Y_1 \cdots Y_m$ be a sample from an $N(\mu_y, \sigma^2)$ dist.

Show how to use the F dist to find $P(S_x^2/S_y^2 > c)$

First is ratio of two χ^2 distributions.
Variance of a Normal dist can be represented by a χ^2 dist.
We showed in a previous problem that sample std dev is an unbiased estimator for population variance.

Therefore, $F_{n,m} \sim \frac{S_x^2/n}{S_y^2/m}$ and $\frac{S_x^2}{S_y^2} \sim \left(\frac{m}{n}\right) F_{n,m}$

$$P\left(\frac{S_x^2}{S_y^2} > c\right) = 1 - P\left(\frac{S_x^2}{S_y^2} < c\right)$$
$$= 1 - \left(\frac{m}{n}\right) F_{n,m}(c)$$

where $F_{n,m}(c)$ is an expression of the F cdf applied @ c with degrees of freedom n and m, respectively.