Chapter 1. Probability

§1.1 Introduction

Probability theory originated from: games of chance, randomness.

1°. used in genetics as a model for mutations and ensuing natural variability, bioinformatic

2°. In designing and analyzing computer operating systems, the lengths of various queues in the system are modeled as random phenomena.

3°. In operations research, the demands on inventories of goods are often modeled as random.

4°. Actuarial science, which is used by insurance companies, relies heavily on the tools of probability theory.

5°. Probability theory is a cornerstone of the theory of finance.

part one: theory of probability, as a mathematical model for chance phenomena.

part two: statistics; procedures for analyzing data.

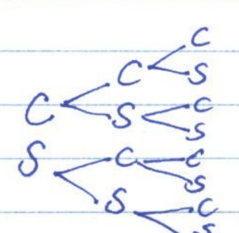
§1.2. Sample Spaces.

Probability theory is concerned with: "outcomes occur randomly".
such situations are called experiments.
samplespace: the set of all possible outcomes. Ω, $\Omega = \{\omega_1, \omega_2, \dots, \}$

Example A. Driving to work, a commuter passes through a sequence of three intersections with traffic lights. At each light, she either stops, S, or continues, C. The sample space is:

Answer:

intersection 1,	intersection 2	3
C or S	C or S	C or S

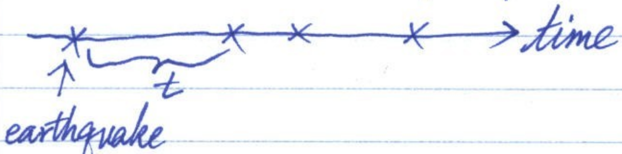


$\Omega = \{ccc, ccs, csc, css, scc, scs, ssc, sss\}$.

Example B. The number of jobs in a print queue of a mainframe computer may be modeled as random. sample space $\Omega = \{0, 1, 2, \cdots\}$.

If there is an upper limit, N, then $\Omega = \{0, 1, 2, \cdots, N\}$.

Example C. Earthquakes exhibit very erratic behavior, which is sometimes modeled as random. The length of time between successive earthquakes may be regarded as an experiment. sample space $\Omega = \{t \mid t \geq 0\}$.



random events: subsets of Ω.

Example A. event = "stops at the first light". $A = \{scc, scs, ssc, sss\}$

B. event is that there are fewer than five jobs in the print queue.
$A = \{0, 1, 2, 3, 4\}$.

C. event is that length of times is less than 3 years.
$A = \{t \mid 0 \leq t < 3\}$.

Operations on events.

1°. union of two events. $C = A \cup B$, event C is that either A occurs or B occurs or both occur.

In Example A; let A = "stops at the first light", B = "stops at the third light". then $C = A \cup B$ = "stops at the first light or third light".

$A = \{scc, scs, ssc, sss\}$, $B = \{ccs, css, scs, sss\}$

thus, $C = A \cup B = \{scc, scs, ssc, sss, ccs, css\}$.

2°. intersection of two events. $C = A \cap B$, is the event that both A and B occur.

In example A, C = "commuter stops at the first light and stops at the third light".
$C = \{scs, sss\}$.

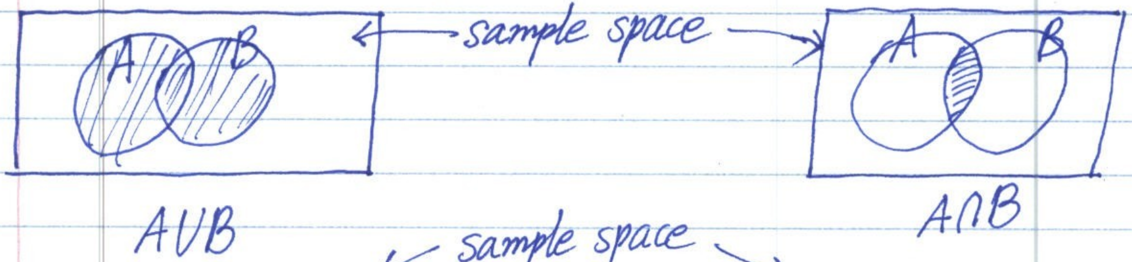
3°. Complement of an event. A^c = the event that A doesn't occur.
A^c consists of all those elements in Ω that are not in A.
In example A. $A^c = \{ccc, ccs, csc, css\}$.

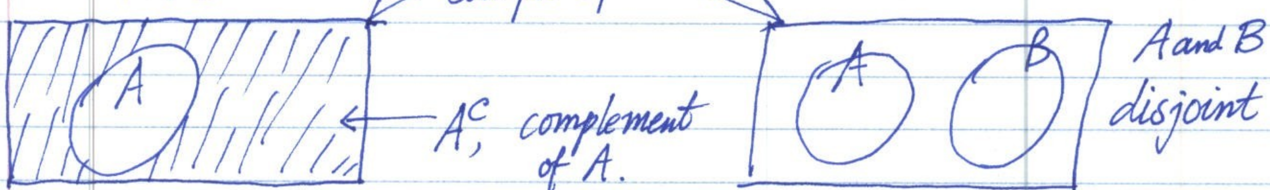
empty set: ϕ, is the set with no elements, with no outcomes.

e.g. A = "stops at 1st light", C = "continues through all three lights" $= \{ccc\}$.

then $A \cap C = \phi$, the set $A \cap C$ is an empty set.

disjoint: call event A and event C are disjoint.





Properties: Commutative Laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$.

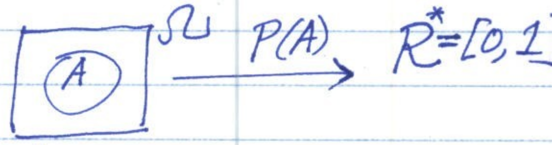
Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$.

Distributive Laws: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

§1.3. Probability Measures.



function P from subsets of Ω to the real numbers.

1°. $P(\Omega) = 1$.

2°. If $A \subset \Omega$, then $P(A) \geq 0$

3° if A_1 and A_2 disjoint, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

if $A_1, A_2, \cdots$, mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Properties: (A). $P(A^c) = 1 - P(A)$.

(B). $P(\phi) = 0$.

(C). If $A \subset B$, then $P(A) \leq P(B)$.

(D). Addition law: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example A. A fair coin is thrown twice. A="1st toss Heads", B="2nd toss heads".

sample space $\Omega = \{hh, ht, th, tt\}$

1st	2nd
h	h
t	t

assume each outcome is equally likely, with $\frac{1}{4}$.

$C = A \cup B$, $A = \{hh, ht\}$, $B = \{hh, th\} \Rightarrow C = \{hh, ht, th\}$.

We have $P(C) = \frac{3}{4}$, $P(A) = \frac{2}{4} = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) \neq P(A) + P(B)$.

notice $A \cap B \neq \phi$, $A \cap B = \{hh\}$, $P(A \cap B) = \frac{1}{4}$.

$$P(C) = P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{holds!}$$

$$\frac{3}{4} = \frac{2}{4} + \frac{2}{4} - \frac{1}{4}.$$

Example B. AIDS infection. (see the textbook, p. 6).

§1.4. Computing Probabilities: Counting Methods.

$\Omega = \{\omega_1, \omega_2, \cdots, \omega_N\}$, $P(\omega_i) = p_i$,

then. If $A = \{\omega_1, \omega_2\}$, then $P(A) = P(\omega_1) + P(\omega_2) = p_1 + p_2$.

$B = \{\omega_{i_1}, \omega_{i_2}, \cdots \omega_{i_k}\}$, $P(B) = P(\omega_{i_1}) + P(\omega_{i_2}) + \cdots + P(\omega_{i_k})$

Example A: A fair coin is thrown twice. $\Omega = \{hh, ht, th, tt\}$.

A = "at least one head is thrown".

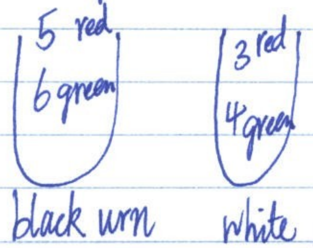
then $A = \{hh, ht, th\}$. $P(A) = \frac{3}{4} = 0.75$.

if $P(\omega_1) = P(\omega_2) = \cdots = P(\omega_N)$, equally likely, then $P(\omega_i) = \frac{1}{N}$, $i = 1, 2, \cdots, N$

then $B = \{\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_k}\}$, $P(B) = P(\omega_{i_1}) + \cdots + P(\omega_{i_k}) = \frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N} = \frac{k}{N}$.

i.e. $P(B) = \dfrac{\#\text{ of elements in } B}{\#\text{ of total elements in } \Omega} = \dfrac{\#\text{ of ways } B \text{ can occur}}{\text{total number of outcomes}}$.

Example B: Simpson's Paradox.



$P(R \mid \text{black}) = \frac{5}{11} = 0.455$ ✓

$P(R \mid \text{white}) = \frac{3}{7} = 0.429$.

6 red, 3 green (black) 9 red, 5 green (White)

$P(R|black) = \frac{6}{9} = 0.667$ ✓

$P(R|white) = \frac{9}{14} = 0.643$.

Now, put balls together, 11 red, 9 green (black) 12 red, 9 green (white)

$P(R|black) = \frac{11}{20} = 0.55$

$P(R|white) = \frac{12}{21} = 0.571$. ✓

The multiplication principle:

If one experiment has m outcomes and another experiment has n outcomes, then there are mn possible outcomes for the two experiments.

e.g. 1. toss 1 toss 2

h h

t t

$\Omega = \{hh, ht, th, tt\}$.

$4 = 2 \times 2$.

extended:

2. 3 intersection lights. $2 \times 2 \times 2 = 8$.

Example A. playing cards have 13 face values and 4 suits. There are $4 \times 13 = 52$ face value/suit combinations.

B. class has 12 boys, 18 girls. select one boy, one girl, representives. how many representives or different ways? $12 \times 18 = 216$.

~~different~~

C. An 8-bit binary word is a sequence of 8 digits, of which each may be either a 0 or a 1. How many different 8-bit words are there?

Answer: $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$.