Alex Beckwith HW 5

① Completed using Python in attached file.

②

balls	black	white	red	all
total	p	b	r	n
chosen	B	W	R	

ways to choose red $= {}_RC_r$

black $= {}_BC_p$

white $= {}_WC_w$

all $= {}_{(B+W+R)}C_n$

(2a) Joint distribution

$$P(b=B, w=W, r=R) = \frac{({}_WC_w)({}_BC_p)({}_RC_r)}{{}_{(W+B+R)}C_n}$$

(b)

$$P(b=B, w=W) = \frac{({}_WC_w)({}_BC_p)({}_RC_r)}{{}_{(W+B+R)}C_n}$$ (because $n-B-W=R$)

(c) $$P(a<W<b) = \frac{{}_{n-p-r}C_w}{{}_WC_w}$$

③ Please see code pdf.

⑥ Choosing point = uniform distribution w/ constant density "d"

Area of ellipse $= \pi ab$

def of ellipse $= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Total prob $= 1$
$= d \cdot \pi ab$

so $d = 1/\pi ab$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \cdot \frac{a^2}{a^2} = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

implies and

$$-\frac{b}{a}\sqrt{a^2-x^2} \leq y \leq \frac{b}{a}\sqrt{a^2-x^2} \qquad -\frac{a}{b}\sqrt{b^2-y^2} \leq x \leq \frac{a}{b}\sqrt{b^2-y^2}$$

Marginals $$P(Y=y) = \int_{-\frac{a}{b}\sqrt{b^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} \frac{1}{\pi ab}\, dx = 2\frac{a}{b} \cdot \frac{1}{\pi ab}\sqrt{b^2-y^2} = \frac{2}{\pi b^2}\sqrt{b^2-y^2}$$

$$\& \; P(X=x) = \frac{2}{\pi a^2}\sqrt{a^2-x^2}$$

⑦ $F(x,y) = (1-e^{-\alpha x})(1-e^{-\beta y})$ for $x \geq 0,\ y \geq 0,\ \alpha > 0,\ \beta > 0$

$$F_X = F_{xy}(x,\infty) = (1-e^{-\alpha x})$$

$$F_Y = F_{xy}(\infty,y) = (1-e^{-\beta y})$$

⑧ a $f(x,y) = \frac{6}{7}(x+y)^2$

(i) $$P(X>Y) = \frac{6}{7}\int_0^1\int_0^x (x+y)^2\,dy\,dx$$

$$= \frac{6}{7}\int_0^1 \left(x^2y + xy^2 + \frac{y^3}{3}\Big|_0^x\right)dx$$

$$= \frac{6}{7}\left(\frac{7}{3}\right)\cdot\frac{x^4}{4}\Big|_0^1 = \frac{2}{4} = \frac{1}{2}$$

(ii) $$P(X+Y\leq 1) = \frac{6}{7}\int_0^1\int_0^{1-x} x^2+2xy+y^2\,dy\,dx$$

$$= \frac{6}{7}\int_0^1 \left(xy^2+\frac{y^3}{3}\Big|_0^{1-x}\right)dx$$

$$= \frac{6}{7}\int_0^1 x(1-x)^2 + \frac{(1-x)^3}{3}\,dx$$

$$= \frac{6}{7}\left(x - x^2 + \frac{x^3}{3}\Big|_0^1\right) = \frac{6}{21} = \frac{2}{7}$$

(iii) $$P(X\leq\tfrac{1}{2}) = \int_0^{1/2}\int_0^1 \frac{6}{7}(x^2+2xy+y^2)\,dy\,dx$$

$$= \frac{6}{7}\int_0^{1/2}\left(x^2y + xy^2 + y^3/3\Big|_0^1\right)$$

$$= \frac{6}{7}\int_0^{1/2} x^2+x+\frac{1}{3}\,dx$$

$$= \frac{6}{7}\left(\frac{x^3}{3}+\frac{x^2}{2}+\frac{1}{3}x\Big|_0^{1/2}\right)$$

$$= \frac{6}{7}\left(\frac{1}{24}+\underbrace{\frac{1}{8}}_{3/24}+\underbrace{\frac{1}{6}}_{4/24}\right) = \frac{2}{7}$$

⑧b Marginal

$$F_x(x,y) = \frac{6}{7}\int_0^1 x^2+2xy+y^2\,dy$$

$$= \frac{6}{7}\left(x^2y+xy^2+\frac{y^3}{3}\Big|_0^1\right)$$

$$= \frac{6}{7}\left(x^2+x+\frac{1}{3}\right)$$

The distribution is symmetrical, so

$$F_y(x,y) = \frac{6}{7}\left(y^2+y+\frac{1}{3}\right)$$

13. Fair coin thrown once - if heads, throw twice

Results →

	throw1 H	throw1 T
Throw2 H	HH	TH
Throw2 T	HT	X

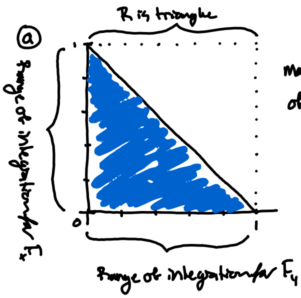
freq func for total heads

$P(X=k)$

heads	prob
0	1/4
1	1/2
2	1/4
otherwise	0

17. (X,Y) is point chosen randomly on $R = \{(x,y): |x| + |y| \le 1\}$

(a)



(b) Marginal density = area of triangle

$F_x = \frac{1}{2}$

$F_y = \frac{1}{2}$

19. 2 Components w/ exponential lifetimes.

$T_1 = \mathrm{Exp}(\alpha)$

$T_2 = \mathrm{Exp}(\beta)$

a) $$P(T_1 > T_2) = \int_0^\infty \int_0^{t_1} f_{T_1}(t_1) f_{T_2}(t_2)\, dt_2\, dt_1$$

(cdf of exp)

$$= \int_0^\infty f_{T_1}(t_1)\left(1 - e^{-\beta t_1}\right) dt_1$$

$$= \int_0^\infty f_{T_1}(t_1)\, dt_1 - \int_0^\infty \alpha e^{(-\alpha+\beta)t_1}\, dt_1$$

$$= 1 - \left(-\frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t_1}\Big|_0^\infty\right)$$

$$= 1 - + \frac{\alpha}{\alpha+\beta} = \frac{\alpha+\beta}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$$

b) $$P(T_1 > 2T_2) = \int_0^\infty \int_0^{t_1/2} f_{T_1}(t_1) f_{T_2}(t_2)\, dt_2\, dt_1$$

(cdf of exp)

$$= \int_0^\infty f_{T_1}(t_1)\left(1 - e^{-\frac{\beta t_1}{2}}\right) dt_1$$

$$= \int_0^\infty f_{T_1}(t_1)\, dt_1 - \int_0^\infty \alpha e^{(-\alpha+\frac{\beta}{2})t_1}\, dt_1$$

$$= 1 - \left(-\frac{\alpha}{\alpha+\beta/2} e^{-(\alpha+\frac{\beta}{2})t_1}\Big|_0^\infty\right)$$

$$= 1 - + \frac{\alpha}{\alpha+\beta/2} = \frac{\alpha+\beta/2}{\alpha+\beta/2} - \frac{\alpha}{\alpha+\beta/2} = \frac{\beta/2}{\alpha+\beta/2}$$