Chapter 3 part 2.

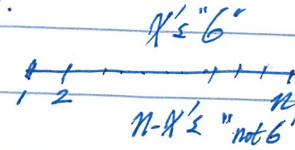
Multinomial distribution — a generalization of binomial distribution

Example: Toss a die n times, let X be the number of tosses one observes number 6.

then $X \sim B(n, \frac{1}{6})$.

$$P(X=x) = \binom{n}{x}\left(\frac{1}{6}\right)^{x}\left(1-\frac{1}{6}\right)^{n-x}, \quad x=0,1,2,\cdots,n.$$

$$= \frac{n!}{x!(n-x)!}\left(\frac{1}{6}\right)^{x}\left(1-\frac{1}{6}\right)^{n-x}.$$



Let Y = the number of tosses one observes outcome "not 6".

then $Y \sim B(n, \frac{5}{6})$. $\quad P(Y=y) = \binom{n}{y}\left(\frac{5}{6}\right)^{y}\left(1-\frac{5}{6}\right)^{n-y}, \quad y=0,1,\cdots,n.$

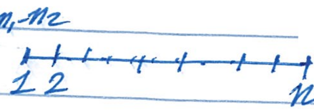
Note: $P(X=x) = P(X=x, Y=n-x) = \frac{n!}{x!(n-x)!}\underbrace{\left(\frac{1}{6}\right)}_{p_1}^{x}\underbrace{\left(1-\frac{1}{6}\right)}_{p_2}^{n-x} \hat{=} \binom{n}{x,\ n-x}p_1^{x}\cdot p_2^{n-x}$

Now, define more r.v's. $\quad N_1$ = number of tosses result number 6. $\quad p_1=\frac{1}{6}$

N_2 = number of tosses result numbers 1 or 2. $p_2=\frac{2}{6}$

What the joint distribution for (N_1, N_2)?

$$P(N_1=n_1, N_2=n_2) = \binom{n}{n_1,\ n_2,\ n-n_1-n_2}\left(\frac{1}{6}\right)^{n_1}\left(\frac{2}{6}\right)^{n_2}\left(1-\frac{1}{6}-\frac{2}{6}\right)^{n-n_1-n_2}$$



$$= \frac{n!}{n_1!\ n_2!\ (n-n_1-n_2)!}\left(\frac{1}{6}\right)^{n_1}\left(\frac{1}{3}\right)^{n_2}\cdot\left(\frac{1}{2}\right)^{n-n_1-n_2}$$

$n_1=0,1,\cdots,n$
$n_2=0,1,\cdots,n$
$n_1+n_2=0,1,\cdots,n$

Let N_3 = number of tosses result number 3, or 4 or 5.

then $P(N_1=n_1, N_2=n_2) = P(N_1=n_1, N_2=n_2, N_3=n_3)$

$$= \binom{n}{n_1,\ n_2,\ n_3}\left(\frac{1}{6}\right)^{n_1}\left(\frac{1}{3}\right)^{n_2}\left(\frac{1}{2}\right)^{n_3}.$$

where $n_1+n_2+n_3=n$, $\quad p_1=\frac{1}{6}$, $p_2=\frac{1}{3}$, $p_3=\frac{1}{2}$, $\quad p_1+p_2+p_3=1$.

From joint distribution of (N_1, N_2), find marginal distribution of N_1 and N_2.

$$p(n_1, n_2) = P(N_1 = n_1, N_2 = n_2) = \frac{n!}{n_1!\, n_2!\, (n-n_1-n_2)!} \left(\frac{1}{6}\right)^{n_1} \left(\frac{1}{3}\right)^{n_2} \left(\frac{1}{2}\right)^{n-n_1-n_2}.$$

$0 \le n_1 \le n$, $0 \le n_2 \le n$, $0 \le n_1 + n_2 \le n$.

$$P(N_1 = n_1) = \sum_{n_2} p(n_1, n_2) = \sum_{n_2=0}^{n-n_1} \frac{n!}{n_1!\, n_2!\, (n-n_1-n_2)!} \left(\frac{1}{6}\right)^{n_1} \left(\frac{1}{3}\right)^{n_2} \left(\frac{1}{2}\right)^{n-n_1-n_2}$$

$$= \frac{n!}{n_1!\, (n-n_1)!} \left(\frac{1}{6}\right)^{n_1} \sum_{n_2=0}^{n-n_1} \frac{(n-n_1)!}{n_2!\, (n-n_1-n_2)!} \left(\frac{1}{3}\right)^{n_2} \left(\frac{1}{2}\right)^{n-n_1-n_2}$$

using $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, with $a = \frac{1}{3}$, $b = \frac{1}{2}$, $n = n - n_1$

above $P(N_1 = n_1) = \frac{n!}{n_1!\, (n-n_1)!} \left(\frac{1}{6}\right) \cdot \left(\frac{1}{3} + \frac{1}{2}\right)^{n-n_1}$

$$= \binom{n}{n_1} \left(\frac{1}{6}\right)^{n_1} \left(1 - \frac{1}{6}\right)^{n-n_1}, \quad n_1 = 0, 1, \cdots, n.$$

i.e. $N_1 \sim B(n, \frac{1}{6})$.

Similarly, one could obtain $P(N_2 = n_2) = \binom{n}{n_2} \left(\frac{1}{3}\right)^{n_2} \left(1 - \frac{1}{3}\right)^{n-n_2}$, $n_2 = 0, 1, \cdots n$.

i.e. $N_2 \sim B(n, \frac{1}{3})$.

Definition for Multinomial distribution.

Suppose that each of n independent trials can result in one of r types of outcomes and that on each trial the probabilities of the r outcomes are $p_1, p_2, \cdots, p_r$. Let N_i be the total number of outcomes of type i in the n trials, $i = 1, 2, \cdots, r$. The joint frequency function for $(N_1, N_2, \cdots, N_r)$ is

$$P(N_1 = n_1, N_2 = n_2, \cdots, N_r = n_r) = \binom{n}{n_1, n_2, \cdots, n_r} p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}.$$

$n_1 + n_2 + \cdots + n_r = n$, $p_1 + p_2 + \cdots + p_r = 1$. $0 \le n_i \le n$, $i = 1, 2, \cdots r$

Example 1°. n voters are randomly selected. democrate, republican, independent $r=3$.

$$\begin{array}{ccc} \text{democrate} & \text{republican} & \text{independent} \\ p_1 & p_2 & p_3 \\ n_1 & n_2 & n_3 \end{array}$$

$$P(N_1=n_1, N_2=n_2, N_3=n_3) = \binom{n}{n_1, n_2, n_3} p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3}. \quad 0 \le n_i \le n. \quad n_1+n_2+n_3=n.$$

$$p_1+p_2+p_3=1.$$

2° on race. Black, White, Hispanic, Asian. $r=4$.

$$\begin{array}{cccc} \text{Black} & \text{White} & \text{Hispanic} & \text{Asian} \\ n_1 & n_2 & n_3 & n_4 \\ p_1 & p_2 & p_3 & p_4 \end{array}$$

$$P(N_1=n_1, N_2=n_2, N_3=n_3, N_4=n_4) = \binom{n}{n_1, n_2, n_3, n_4} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}.$$

$$0 \le n_i \le n, \quad n_1+n_2+n_3+n_4=n. \quad p_1+p_2+p_3+p_4=1.$$

What is the marginal distribution for $N_1, N_2, \cdots, N_r$?

For 1°

$$P(N_1=n_1) = \sum_{n_2}\sum_{n_3} P(N_1=n_1, N_2=n_2, N_3=n_3) = \binom{n}{n_1} p_1^{n_1} (1-p_1)^{n-n_1}$$

i.e. $N_1 \sim B(n, p_1)$. $\left\{ \begin{array}{l} \text{consider "democrate" as "success" event.} \\ \text{others (republican + independent) as} \\ \text{"failure" event} \end{array} \right\}$

For 2°.

$$P(N_1=n_1) = \sum_{n_2}\sum_{n_3}\sum_{n_4} P(N_1=n_1, N_2=n_2, N_3=n_3, N_4=n_4)$$

$$= \cdots = \binom{n}{n_1} p_1^{n_1} (1-p_1)^{n-n_1}, \qquad n_1=0,1,\cdots,n.$$

Again, $N_1 \sim B(n, p_1)$. Also $N_i \sim B(n, p_i)$, $i=2,3,4$.

Bivariate Normal distribution

$(X, Y) \sim N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$.

$$(X,Y) \sim f(x,y) = \frac{1}{2\pi \cdot \sigma_X \cdot \sigma_Y \cdot \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\cdot\rho\cdot(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right] \right\}$$

$-\infty < \mu_X < \infty$, $\sigma_X > 0$. $-\infty < \mu_Y < \infty$, $\sigma_Y > 0$. $-1 < \rho < 1$.

e.g. model for the joint distribution of heights of fathers and sons.

If $\rho = 0$. $(X,Y) \sim f(x,y) = \frac{1}{2\pi \cdot \sigma_X \cdot \sigma_Y} \exp\left\{ -\frac{1}{2} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right] \right\}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}.$$

p.d.f $N(\mu_X, \sigma_X^2)$. p.d.f. $N(\mu_Y, \sigma_Y^2)$.

Notation: $(X, Y) \sim N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$

What is $X \sim f_X(x)$? marginal density of r.v. X?

$X \sim f_X(x)$, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$, let $u = \frac{x-\mu_X}{\sigma_X}$, $v = \frac{y-\mu_Y}{\sigma_Y}$.

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi \cdot \sigma_X \cdot \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u^2 - 2\cdot\rho\cdot u \cdot v + v^2 \right] \right\} dv$$

$$u^2 + v^2 - 2\rho \cdot u \cdot v = v^2 - 2\rho \cdot u \cdot v + (\rho \cdot u)^2 - (\rho u)^2 + u^2$$
$$= (v - \rho u)^2 + (1-\rho^2) u^2$$

$$f_X(x) = \frac{1}{2\pi \cdot \sigma_X \cdot \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)} \left[(v-\rho u)^2 + (1-\rho^2) u^2 \right]} dv$$

$$= \frac{1}{2\pi\cdot\sigma_X\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{(v-\rho u)^2}{2(1-\rho^2)}}\cdot e^{-\frac{(1-\rho^2)u^2}{2(1-\rho^2)}}\,dv$$

$$= \frac{1}{2\pi\cdot\sigma_X\cdot\sqrt{1-\rho^2}} e^{-\frac{u^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{(v-\rho u)^2}{2(1-\rho^2)}}\,dv$$

let $t = \frac{v-\rho u}{\sqrt{1-\rho^2}}$

$$= \frac{1}{2\pi\sigma_X\cdot\sqrt{1-\rho^2}} e^{-\frac{u^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\cdot\sqrt{1-\rho^2}\cdot dt$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{u^2}{2}}\cdot\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{u^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \sim N(\mu_X, \sigma_X^2).$$

(recall $u = \frac{x-\mu_X}{\sigma_X}$)

We have $X \sim N(\mu_X, \sigma_X^2)$,
$Y \sim N(\mu_Y, \sigma_Y^2)$.

§3.4. Independent Random Variables

Definition: r.v. X and Y are called independent, if

$$F(x,y) = P(X\le x, Y\le y) = P(X\le x)P(Y\le y) = F_X(x)\cdot F_Y(y).$$

(recall A, B are indep, if $P(A\cap B) = P(A)\cdot P(B)$)

discrete r.v. case. (X,Y) with jpmf. $P(X=x_i, Y=y_j) = p_{ij}$,

marginal p.m.f. $P(X=x_i) = \sum_j p_{ij} = p_{i\cdot}$ $\quad i=1,2,\cdots,n$

$P(Y=y_j) = \sum_i p_{ij} = p_{\cdot j}$, $\quad j=1,2,\cdots,m$

then, X, Y are independent, if $\underline{P(X=x_i, Y=y_j) = P(X=x_i)P(Y=y_j)}$,

or $p_{ij} = p_{i\cdot}\cdot p_{\cdot j}$, for all $i = 1,2,\cdots n$; $j=1,2,\cdots,m$.

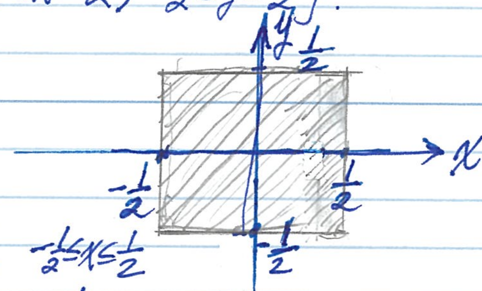
Continous case: $F(x,y)=F_X(x)\cdot F_Y(y)$, $\int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du = \int_{-\infty}^{x} f_X(u)du\cdot\int_{-\infty}^{y} f_Y(v)dv$.

take partial derivative both sides: $f(x,y)=f_X(x)\cdot f_Y(y)$. $\quad \frac{\partial^2}{\partial x\partial y}F(x,y)=\frac{\partial}{\partial x}F_X(x)\frac{\partial}{\partial y}F_Y(y)$

e.g.1. $(X,Y)\sim N(\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho)$. if $\rho=0$, $f(x,y)=f_X(x)\cdot f_Y(y)$.

thus, X and Y are independent, if $\rho=0$.

Expl A. $(X,Y)\sim$ Uniform on S. $S=\{(x,y)\mid -\frac{1}{2}\le x\le\frac{1}{2},\ -\frac{1}{2}\le y\le\frac{1}{2}\}$.



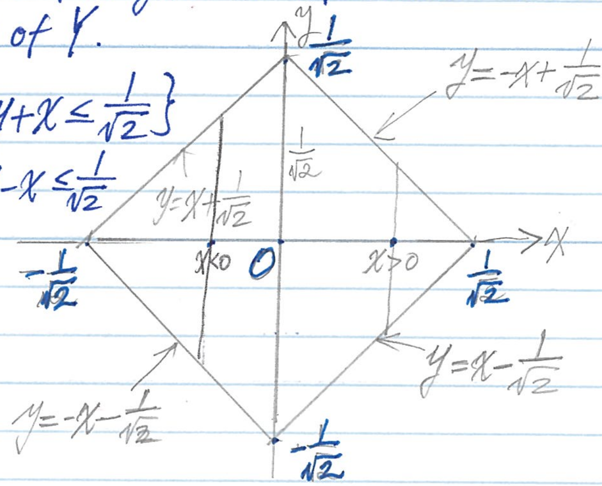
thus $f(x,y)=\begin{cases}1, & (x,y)\in S\\ 0, & (x,y)\notin S.\end{cases}$

$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{-\frac{1}{2}}^{\frac{1}{2}} 1\,dy=1$, $\left(-\frac{1}{2}\le x\le\frac{1}{2}\right)$ i.e. $f_X(x)=\begin{cases}1, & -\frac{1}{2}\le x\le\frac{1}{2}\\ 0, & \text{else}.\end{cases}$

similarly, $Y\sim f_Y(y)=\begin{cases}1, & -\frac{1}{2}\le y\le\frac{1}{2}\\ 0, & \text{else}.\end{cases}$

Thus, one has $f(x,y)=f_X(x)\cdot f_Y(y)$ for all $x,y\in(-\infty,\infty)$.

X and Y are independent! Knowing the value of X gives no information about the possible values of Y.



Expl. B. $(X,Y)\sim$ uniform on $A=\{(x,y)\mid -\frac{1}{\sqrt2}\le y+x\le\frac{1}{\sqrt2}$, $-\frac{1}{\sqrt2}\le y-x\le\frac{1}{\sqrt2}\}$

for $-\frac{1}{\sqrt2}\le x\le\frac{1}{\sqrt2}$,

$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy \underset{(x>0)}{=} \int_{x-\frac{1}{\sqrt2}}^{-x+\frac{1}{\sqrt2}} 1\,dy=\sqrt2-2x$.

$x<0$, $f_X(x)=\int_{-x-\frac{1}{\sqrt2}}^{x+\frac{1}{\sqrt2}} 1\,dy = 2x+\sqrt2$.

i.e. $X\sim f_X(x)=\begin{cases}\sqrt2-2x, & 0\le x\le\frac{1}{\sqrt2},\\ 2x+\sqrt2, & -\frac{1}{\sqrt2}\le x\le 0.\end{cases}$ One can verify $f_X(x)$ is p.d.f. by $\begin{cases} f_X(x)\ge 0.\\ \int_{-\infty}^{\infty} f_X(x)dx=1.\end{cases}$

in this case, $1\equiv f(x,y)\ne f_X(x)\cdot f_Y(y)$.

when $(x,y)\in A$. dependent!

problem 14. $(X, Y) \sim f(x,y) = \begin{cases} x e^{-x(y+1)}, & 0 \le x < \infty,\ 0 \le y < \infty \\ 0, & \text{otherwise}. \end{cases}$

a. find marginal p.d.f for X and Y. are X and Y independent?
b. find the conditional dencity of X and Y.

Ans. (a) $0 \le x < \infty$, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} x e^{-x(y+1)}\,dy = x \cdot e^{-x} \int_0^{\infty} e^{-xy}\,dy$

let $xy = u$, $= e^{-x} \int_0^{\infty} e^{-u}\,du = e^{-x}$. thus $X \sim f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{else}. \end{cases}$

$y \ge 0$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} x e^{-x(y+1)}\,dx$ let $x(y+1) = u$, $x = \frac{u}{y+1}$

$$= \int_0^{\infty} \frac{u}{y+1} \cdot e^{-u} \cdot \frac{du}{y+1} = \frac{1}{(1+y)^2} \int_0^{\infty} u e^{-u}\,du = \frac{1}{(1+y)^2} \Gamma(2) = \frac{1}{(1+y)^2}.$$

thus $Y \sim f_Y(y) = \begin{cases} \frac{1}{(1+y)^2}, & y \ge 0 \\ 0, & \text{else}. \end{cases}$ (verify $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$)

Since $f(x,y) \ne f_X(x) \cdot f_Y(y)$, X and Y are not independent!

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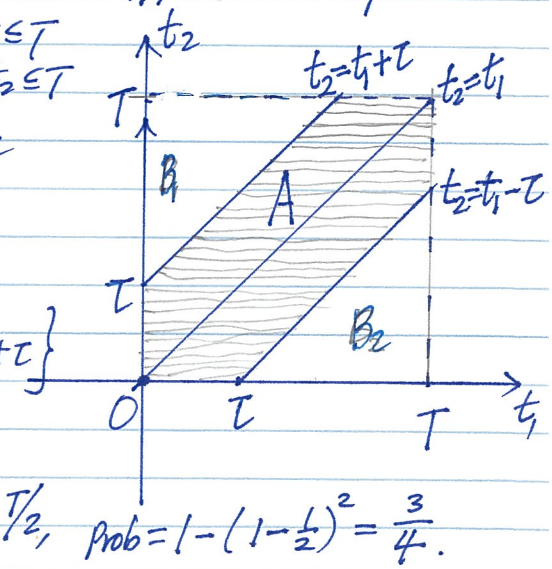
Expl. E. Suppose that a node in a communications network has the property that if two packets of information arrive within time τ of each other, they "collide" and then have to be retransmitted. If the times of arrival of the two packets are independent and uniform on $[0, T]$, what is the probability that they collide?

Ans: Let T_1, T_2 be the times of arrival of two packets, Thus T_1, T_2 are indep and uniform on $[0, T]$.

$$(T_1, T_2) \sim f(t_1, t_2) = \begin{cases} \frac{1}{T^2}, & 0 \le t_1 \le T,\ 0 \le t_2 \le T \\ 0, & \text{else} \end{cases}$$



$$P(|T_1 - T_2| \le \tau) = P(-\tau \le T_2 - T_1 \le \tau)$$

$$= P(T_1 - \tau \le T_2 \le T_1 + \tau) = P((T_1, T_2) \in A),$$

$$A = \{(t_1, t_2) \mid t_1 - \tau \le t_2 \le t_1 + \tau\}$$

$$= \iint_A f(t_1, t_2)\,dt_1\,dt_2 = \frac{1}{T^2} \cdot \iint_A dt_1\,dt_2 = \frac{1}{T^2} \cdot \text{area}(A)$$

$$= \frac{1}{T^2} \cdot [T^2 - \underbrace{(T-\tau)^2}_{\text{area of } (B_1+B_2)}] = 1 - \left(1 - \frac{\tau}{T}\right)^2.$$

if $\tau = T/2$, $\text{Prob} = 1 - \left(1 - \frac{1}{2}\right)^2 = \frac{3}{4}$.

§3.5 Conditional distribution

$(X,Y) \sim p(x_i, y_j) = P(X=x_i, Y=y_j)$, $i=1,2,\cdots,n$, $j=1,2,\cdots,m$.

$X \backslash Y$	y_1	y_2	$\cdots$	y_m	$P(X=x_i)$
x_1	$p(1,1)$	$p(1,2)$	$\cdots$	$p(1,m)$	$p_{1\cdot}$
x_2	$p(2,1)$	$p(2,2)$	$\cdots$	$p(2,m)$	$p_{2\cdot}$
$\vdots$	$\vdots$	$\cdot\cdot$			$\vdots$
x_n	$p(n,1)$,	$p(n,2)$	$\cdots$	$p(n,m)$	$p_{n\cdot}$
$p(Y=y_j)$	$p_{\cdot 1}$	$p_{\cdot 2}$		$p_{\cdot m}$	1

$p_{1\cdot} = p_{11} + p_{12} + \cdots + p_{1m}$

$p_{\cdot 1} = p_{11} + p_{21} + \cdots + p_{n1}$

$p_{ij} = p(i,j) = p(X=x_i, Y=y_j) = p(x_i, y_j)$

Conditional prob. of $X=x_i$ given $Y=y_j$ is

$$P(X=x_i | Y=y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)} = \frac{p(x_i, y_j)}{p_Y(y_j)} = \frac{p(i,j)}{p_{\cdot j}}$$

notation $p_{X|Y}(x_i|y_j) := P(X=x_i|Y=y_j)$.

Expl A. A coin is tossed 3 times.
$X = \#$ of heads on the 1st toss.
$Y = \#$ of heads in 3 tosses.

$X \backslash Y$	0	1	2	3	$P(X=x)$
0	1/8	2/8	1/8	0	1/2
1	0	1/8	2/8	1/8	1/2
$P(Y=y)$	1/8	3/8	3/8	1/8	1

find dist. of X given $Y=1$. i.e. $P(X=0|Y=1)$ and $P(X=1|Y=1)$.

Ans. $P(X=0|Y=1) = \dfrac{P(X=0, Y=1)}{P(Y=1)} = \dfrac{2/8}{3/8} = 2/3$. i.e. $p_{X|Y}(0|1) = 2/3$

$P(X=1|Y=1) = P(X=1, Y=1)/p(Y=1) = \dfrac{1/8}{3/8} = 1/3$. $p_{X|Y}(1|1) = 1/3$.

thus

$X/Y=1$	0	1
$p_{X\|Y}(x\|1)$	2/3	1/3

Expl B. Suppose a particle counter is imperfect and independently detects each incoming particle with prob. p. If the dist of the number of incoming particles in a unit of time is a Poisson distribution with parameter λ, what is the dist. of number of counted particles?

Ans. Let N denote the true number of particles and X the counted number.
conditional dist of X, given $N=n$ is $B(k,n,p)$ or $B(n,p)$.
i.e. $P(X=k|N=n) = \binom{n}{k} p^k (1-p)^{n-k}$. want $P(X=k) = ?$

$$P(X=k)=P(X=k;\Omega)=P\{X=k;\ N=0,1,2\cdots\}$$

$$=P\{\underbrace{X=k}_{A};\ \bigcup_{n=0}^{\infty}\underbrace{\{N=n\}}_{B_n}\}$$

$$=P(A\cap B_0)+P(A\cap B_1)+\cdots\cdots$$

$$=P(X=k,N=0)+P(X=k,N=1)+\cdots\cdots$$

$$+P(X=k,N=k-1)+P(X=k,N=k)\cdots$$

$$+P(X=k,N=k+1)+\cdots\cdot$$

$$=P(X=k,N=k)+P(X=k,N=k+1)+\cdots$$

$$=P(N=k)P(X=k|N=k)+P(N=k+1)P(X=k|N=k+1)$$

$$+\cdots$$

$$=\frac{\lambda^k}{k!}e^{-\lambda}\cdot\binom{k}{k}p^k(1-p)^{k-k}+\frac{\lambda^{k+1}}{(k+1)!}e^{-\lambda}\cdot\binom{k+1}{k}p^k(1-p)^{k+1-k}+\cdots\cdot$$

$$=\sum_{n=k}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda}\cdot\binom{n}{k}p^k(1-p)^{n-k}$$

$$=\sum_{n=k}^{\infty}\frac{\lambda^n}{\cancel{n!}}e^{-\lambda}\cdot\frac{\cancel{n!}}{k!(n-k)!}p^k(1-p)^{n-k}$$

$$=\frac{e^{-\lambda}}{k!}\sum_{n=k}^{\infty}\frac{\lambda^n\cdot p^k\cdot(1-p)^{n-k}}{(n-k)!} \rightarrow (\lambda p)^k(\lambda(1-p))^{n-k}$$

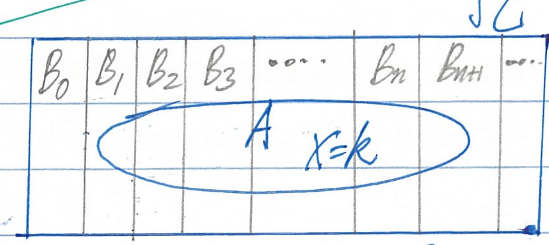
$$=\frac{e^{-\lambda}}{k!}(\lambda p)^k\left(\sum_{n=k}^{\infty}\frac{(\lambda-\lambda p)^{n-k}}{(n-k)!}\right) \qquad =\sum_{j=0}^{\infty}\frac{(\lambda-\lambda p)^j}{j!} \qquad j=n-k$$

$$=e^{\lambda-\lambda p} \quad \text{from} \quad e^x=\sum_{j=0}^{\infty}\frac{x^j}{j!}$$

$$=\frac{e^{-\lambda}\cdot(\lambda p)^k}{k!}e^{\lambda-\lambda p}=\frac{(\lambda p)^k}{k!}e^{-\lambda p}.$$

i.e. $P\{X=k\}=\dfrac{(\lambda p)^k}{k!}e^{-\lambda p},\ k=0,1,2\cdots$ i.e. $X\sim P(\lambda p)$.

Notes



$\Omega=B_0\cup B_1\cup B_2\cup\cdots$

$$P\{X=k\}=P(A)$$

$$=P(A\cap\Omega)$$

$$=P(A\cap B_0)+P(A\cap B_1)+P(A\cap B_2)$$

$$+\cdots$$

$$=\sum_{n=0}^{\infty}P(A\cap B_n)$$

Continuous case. $(X, Y) \sim f(x,y)$.

X = fish's length
Y = fish's age.

Conditional density of Y given $X = x$, $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$.

note: $P\{y \le Y \le y+dy \mid x \le X \le x+dx\} \approx \int_y^{y+dy} f_{Y|X}(y|x)\,dy \approx \int_y^{y+dy} f_{Y|X}(v|x)\,dv$

prefer use another variable v.

on the other hand, left side $= \dfrac{P\{x \le X \le x+dx,\ y \le Y \le y+dy\}}{P\{x \le X \le x+dx\}}$

$$= \frac{\int_y^{y+dy}\int_x^{x+dx} f(u,v)\,du\,dv}{\int_x^{x+dx} f_X(u)\,du} \approx \frac{\int_y^{y+dy} f(x,v)\,dx\ dv}{f_X(x)\cdot dx}$$

$$= \int_y^{y+dy} \frac{f(x,v)}{f_X(x)}\,dv$$

Compare two expressions, one has $f_{Y|X}(v|x) = \dfrac{f(x,v)}{f_X(x)}$

or $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$.

Summary. $(X, Y) \sim f(x,y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy\ ;\quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}\ ;\quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Example A. From previous example $(X,Y) \sim f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y \\ 0, & \text{else}. \end{cases}$

$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0. \end{cases}$ $\quad f_Y(y) = \begin{cases} \lambda^2 y e^{-\lambda y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$

find $f_{Y|X}(y|x)$, $f_{X|Y}(x|y)$

Ans: One only needs to consider $x \ge 0$.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x,y)}{\lambda e^{-\lambda x}} = \begin{cases} \dfrac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}}, & y \ge x \\ \dfrac{0}{\lambda e^{-\lambda x}}, & y < x \end{cases}$$

$$= \begin{cases} \lambda e^{-\lambda(y-x)}, & y \ge x \\ 0, & y < x. \end{cases}$$

e.g. given $X=5$, $\quad Y|X=5 \sim f_{Y|X}(y|5) = \begin{cases} \lambda e^{-\lambda(y-5)}, & y \ge 5 \\ 0, & y < 5. \end{cases}$

From above, $P(Y>10|X=5) = \int_{10}^{\infty} f_{Y|X}(y|5)\,dy = \int_{10}^{\infty} \lambda e^{-\lambda(y-5)}\,dy$

$$= \int_{10}^{\infty} \lambda e^{-\lambda y} \cdot e^{5\lambda}\,dy = e^{5\lambda} \cdot (-e^{-\lambda y})\Big|_{10}^{\infty} = e^{5\lambda} \cdot (0 + e^{-10\lambda}) = e^{-5\lambda}.$$

Similarly, $X|Y=y \sim f_{X|Y}(x|y)$.

for $y>0$ only. $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{f(x,y)}{\lambda^2 y e^{-\lambda y}} = \begin{cases} \dfrac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}}, & 0 \le x \le y \\ 0, & x > y \end{cases}$

$$= \begin{cases} \frac{1}{y}, & 0 \le x \le y \\ 0, & \text{else}. \end{cases}$$

i.e. $X|Y=y \sim f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 \le x \le y \\ 0, & \text{otherwise} \end{cases}$. $\quad X|Y=y \sim U[0,y]$.

e.g. given $Y=10$, $X|Y=10 \sim U[0,10]$.

$P(X>5|Y=10) = \int_5^{\infty} f_{X|Y}(x|10)\,dx = \int_5^{10} \frac{1}{10}\,dx = \frac{1}{2}$.

Example B. $(X, Y) \sim f(x,y) = \begin{cases} \frac{15}{2} \cdot x \cdot (2-x-y), & 0<x<1,\ 0<y<1. \\ 0, & \text{otherwise} \end{cases}$

find $f_{X|Y}(x|y)$, $0<y<1$.

Ans: for $0<y<1$, $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$,

for $0<y<1$, $f_Y(y) = \int f(x,y)\,dx = \int_0^1 \frac{15}{2} x \cdot (2-x-y)\,dx = \cdots = \frac{15}{12}(4-3y)$.

thus, $f_{X|Y}(x|y) = \frac{\frac{15}{2} x \cdot (2-x-y)}{\frac{15}{12}(4-3y)} = \frac{6x(2-x-y)}{4-3y}$ for $0<x<1$

Thus, $X|Y=y \sim f_{X|Y}(x|y) = \begin{cases} \frac{6x \cdot (2-x-y)}{4-3y}, & 0<x<1 \\ 0, & \text{otherwise}. \end{cases}$

Example C. $(X, Y) \sim f(x,y) = \begin{cases} \frac{e^{-\frac{x}{y}} \cdot e^{-y}}{y}, & 0<x<\infty,\ 0<y<\infty \\ 0, & \text{otherwise}. \end{cases}$

Find $P(X>1 \mid Y=y)$, $y>0$.

Ans

$P(X>1 \mid Y=y) = \int_1^\infty f_{X|Y}(x|y)\,dx$, $\quad f_Y(y) = \int f(x,y)\,dx = \int_0^\infty \frac{e^{-\frac{x}{y}} \cdot e^{-y}}{y}\,dx$

$= \frac{e^{-y}}{y} \int_0^\infty e^{-\frac{x}{y}}\,dx = e^{-y} \int_0^\infty \frac{1}{y} e^{-\frac{x}{y}}\,dx$

$= e^{-y} \cdot 1 = e^{-y}$, $y>0$.

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}} e^{-y}}{y} \frac{1}{e^{-y}}$

$= \frac{1}{y} e^{-\frac{x}{y}}$, $0<x<\infty$

thus $P(X>1 \mid Y=y) = \int_1^\infty \frac{1}{y} e^{-\frac{x}{y}}\,dx \overset{\frac{x}{y}=u}{=} \int_{\frac{1}{y}}^\infty e^{-u}\,du$

$= (-e^{-u})\Big|_{\frac{1}{y}}^\infty = 0 + e^{-\frac{1}{y}} = e^{-\frac{1}{y}}$.

i.e. $P(X>1 \mid Y=y) = e^{-\frac{1}{y}}$. e.g. $P(X>1 \mid Y=1) = e^{-1}$, $P(X>1 \mid Y=2) = e^{-\frac{1}{2}}$, etc.