Chapter 3. part 3

§3.6. Functions of jointly distributed r.v.'s

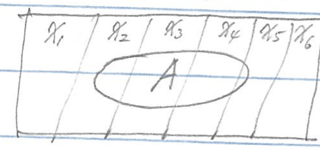
§3.6.1. Sums and Quotients

$(X,Y) \sim \; p(x_i, y_j) = P(X=x_i;\ Y=y_j)$
$\qquad\qquad f(x,y).$

$Z = X+Y$, $P(Z=z) = ?$ discrete case. known $p(x,y)=P(X=x, Y=y)$

Ans. $P(Z=z) = P(X+Y=z)$

$X=x_i$'s



$$= P(X+Y=z,\ X=x \text{ for all } x)$$
$$= \sum_{\text{for all } x} P(X+Y=z,\ X=x)$$
$$= \sum_{\text{all } x} P(X=x,\ Y=z-x)$$
$$= \sum_{x} p(x, z-x).$$

further, if X and Y are independent, then $p(x,y)=p_X(x)\,p_Y(y)$, one has

$$P(Z=z) = \sum_x p_X(x)\cdot p_Y(z-x).$$

Example. A fair coin is tossed 3 times. X = number of heads on 1st toss.
Y = number of total heads among 3 tosses.

Let $Z=X+Y$, find $p_Z(z)=P(Z=z)=?$

X \ Y	0	1	2	3
0	1/8	2/8	1/8	0
1	0	1/8	2/8	1/8

Ans. $Z=X+Y$, possible values of Z are $0,1,2,3,4$.

$P(Z=0) = P(X+Y=0) = P(X+Y=0,\ X=0 \text{ or } 1)$
$\qquad = P(X+Y=0,\ X=0) + P(X+Y=0,\ X=1)$
$\qquad = p(0,0)+p(1,-1) = 1/8 + 0 = 1/8.$

$P(Z=1) = P(X+Y=1) = P(X+Y=1,\ X=0) + P(X+Y=1,\ X=1)$
$\qquad = p(0,1)+p(1,0) = 2/8 + 0 = 2/8$

$P(Z=2) = P(X+Y=2) = p(0,2)+p(1,1) = 1/8 + 1/8 = 2/8$

$P(Z=3) = P(X+Y=3) = p(0,3) + p(1,2) = 0 + 2/8 = 2/8.$

$P(Z=4) = P(X+Y=4) = p(0,4) + p(1,3) = 0 + 1/8 = 1/8.$

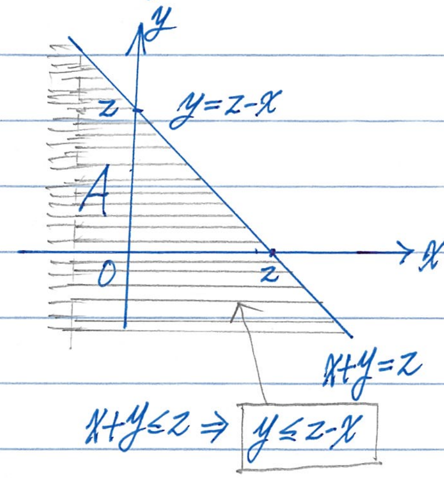
put it together.

Z	0	1	2	3	4
$p(z)$	1/8	2/8	2/8	2/8	1/8

Continuous case. $(X,Y) \sim f(x,y)$, $Z = X+Y$, find $Z \sim f_Z(z) = ?$

two steps method. 1° $Z \sim F_Z(z) = P(Z \le z)$

2°. $f_Z(z) = F_Z'(z)$



1° $F_Z(z) = P(Z \le z) = P(X+Y \le z)$

$$= \iint_A f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dy\,dx$$

2° $f_Z(z) = F_Z'(z) = \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dy\,dx \right\}_z'$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{z-x} f(x,y)\,dy \right\}_z' dx = \int_{-\infty}^{\infty} f(x, z-x)\cdot(z-x)_z'\,dx$$

$$= \int_{-\infty}^{\infty} f(x, z-x)\,dx$$

Note: $\left\{ \int_{-\infty}^{z} f(x,y)\,dy \right\}_z' = f(x,z)$

$\left\{ \int_{-\infty}^{z-x} f(x,y)\,dy \right\}_z' = f(x,z-x)\cdot(z-x)_z'$

i.e. $Z = X+Y \sim f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx.$

further, if $X \perp\!\!\!\perp Y$, independent, i.e. $f(x,y) = f_X(x) f_Y(y)$,

then, above $f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)\,dx.$

Example: Suppose the lifetime of a component $\sim \exp(\lambda)$. An identical and independent backup component is available. The system operates as long as one of components is functiona[l]. Therefore, the distribution of the lifetime of the system is that of the sum of two independent $\exp(\lambda)$. Let $T_1 \sim \exp(\lambda)$, $T_2 \sim \exp(\lambda)$, $S = T_1 + T_2$, $T_1 \perp\!\!\!\perp T_2$, find $f_S(s) =$

Ans.: In order to use above formula $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$.

we use $X \sim f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & x\le 0, \end{cases}$ $\quad Y \sim f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y>0 \\ 0, & y\le 0. \end{cases}$

$z>0$.

Then, $f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(z-x)\,dx$

Note: $f_Y(z-x) = \begin{cases} \lambda e^{-\lambda(z-x)}, & z-x>0 \quad x<z \\ 0, & z-x<0 \end{cases}$

$$= \int_0^{\infty} \lambda e^{-\lambda x} \cdot f_Y(z-x)\,dx$$

$$= \int_0^{z} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)}\,dx = \int_0^{z} \lambda \cancel{e^{-\lambda x}} \cdot \lambda e^{-\lambda z} \cdot \cancel{e^{\lambda x}}\,dx = \int_0^{z} \lambda^2 e^{-\lambda z}\,dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^{z} dx = \lambda^2 z e^{-\lambda z}.$$

Hence, $S \sim f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z>0 \\ 0, & z<0 \end{cases}$

Gamma$(2, \lambda)$.

$$\frac{\lambda^2}{\Gamma(2)} x^{2-1} e^{-\lambda x} = \lambda^2 x e^{-\lambda x}.$$

One has $\exp(\lambda) + \exp(\lambda) \sim \text{Gamma}(2, \lambda)$

$\uparrow$

$\text{Gamma}(1,\lambda) + \text{Gamma}(1,\lambda) \sim \text{Gamma}(2,\lambda)$.

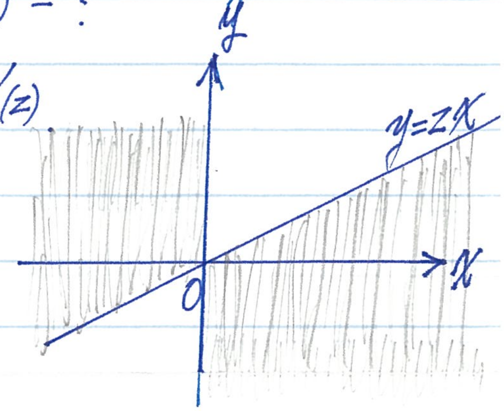
or $\Gamma(1,\lambda) + \Gamma(1,\lambda) \sim \Gamma(2,\lambda)$

more general $\Gamma(\alpha,\lambda) + \Gamma(\beta,\lambda) \sim \Gamma(\alpha+\beta,\lambda)$.

Quotient? $Z = X/Y$ or Y/X.

$(X, Y) \sim f(x,y)$, $Z = Y/X$, find $Z \sim f_Z(z) = ?$

two steps method: $F_Z(z) = P(Y/X \le z)$, $f_Z(z) = F_Z'(z)$



[note: if $x > 0$, $\frac{y}{x} \le z \Rightarrow y \le zx$

$x < 0$, $\frac{y}{x} \le z \Rightarrow y \ge zx$]

$$F_Z(z) = P\left(\frac{Y}{X} \le z\right) = P(Y \le zX, X > 0) + P(Y \ge zX, X < 0)$$

$$= \int_0^{\infty}\int_{-\infty}^{zx} f(x,y)\,dy\,dx + \int_{-\infty}^{0}\int_{zx}^{\infty} f(x,y)\,dy\,dx$$

$$f_Z(z) = F_Z'(z) = \int_0^{\infty}\left[\int_{-\infty}^{zx} f(x,y)\,dy\right]_z' dx + \int_{-\infty}^{0}\left[\int_{zx}^{\infty} f(x,y)\,dy\right]_z' dx$$

$$= \int_0^{\infty} f(x,zx)\cdot(zx)_z'\,dx + \int_{-\infty}^{0}\left[f(x,zx)\cdot(-zx)_z'\right]dx$$

$$= \int_0^{\infty} x\,f(x,zx)\,dx + \int_{-\infty}^{0}(-x)\cdot f(x,zx)\,dx = \int_{-\infty}^{\infty}|x|\cdot f(x,zx)\,dx.$$

if X and Y are indep. $f_Z(z) = \int_{-\infty}^{\infty}|x|\cdot f_X(x)\cdot f_Y(zx)\,dx$.

Expl. B. $X, Y \sim N(0,1)$. $X \perp\!\!\!\perp Y$. find $Z = Y/X$ p.d.f. ① two-steps method ② formula.

Ans: $f_Z(z) = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(zx)^2}{2}} dx = \frac{1}{2\pi} \cdot 2 \int_0^{\infty} x \cdot e^{-\frac{x^2}{2} - \frac{(zx)^2}{2}} dx$

$= \frac{1}{2\pi} \int_0^{\infty} e^{-(\frac{z^2+1}{2}) \cdot x^2} \cdot dx^2$, let $x^2 = u$.

$= \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{z^2+1}{2} u} du$ using $\int_0^{\infty} \lambda e^{-\lambda x} = 1$, with $\lambda = \frac{z^2+1}{2}$

$= \frac{1}{2\pi} \cdot \frac{1}{\frac{z^2+1}{2}} = \frac{1}{\pi} \frac{1}{z^2+1}$,

i.e. $Z = Y/X \sim f_Z(z) = \frac{1}{\pi \cdot (z^2+1)}$, $-\infty < z < \infty$.

$Z \sim$ Cauchy dist. [verify $\int_{-\infty}^{\infty} f_Z(z) dz = 1$?]

§ 3.6.2. joint distribution of functions of random variables

$(X_1, X_2) \sim f_{X_1,X_2}(x_1, x_2)$, $\begin{cases} Y_1 = g_1(X_1, X_2) \\ Y_2 = g_2(X_1, X_2) \end{cases}$, $(Y_1, Y_2) \sim f_{Y_1,Y_2}(y_1, y_2) = ?$

Assume 1° $\begin{cases} y_1 = g_1(x_1, x_2) \\ y_2 = g_2(x_1, x_2) \end{cases} \Rightarrow \begin{cases} x_1 = h_1(y_1, y_2) \\ x_2 = h_2(y_1, y_2) \end{cases}$. uniquly solve for x_1, x_2.

2°. $g_1(x_1, x_2)$, $g_2(x_1, x_2)$ have continuously partial derivatives at all (x_1, x_2).

$\frac{\partial g_1(x_1,x_2)}{\partial x_1}$, $\frac{\partial g_1(x_1,x_2)}{\partial x_2}$, $\frac{\partial g_2(x_1,x_2)}{\partial x_1}$, $\frac{\partial g_2(x_1,x_2)}{\partial x_2}$.

determinant

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1} \cdot \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \cdot \frac{\partial g_2}{\partial x_1} \neq 0$$

then $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2) \cdot |J(x_1,x_2)|^{-1}$.

Note. $X \sim f_X(x)$, $Y = g(X)$, $y = g(x) \longrightarrow x = g^{-1}(y) = h(y)$.

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

$$= f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X(x) \cdot \left|\frac{dy}{dx}\right|^{-1}.$$

Note $h'(y) = \dfrac{d[h(y)]}{dy} = \dfrac{dx}{dy}$

Example 1. $X_1 \sim \exp(\lambda_1)$, $X_2 \sim \exp(\lambda_2)$, $X_1 \perp\!\!\!\perp X_2$,

$\begin{cases} Y_1 = X_1 \\ Y_2 = X_1 + X_2, \end{cases}$ find $f_{Y_1,Y_2}(y_1,y_2)$.

Ans: $\begin{cases} y_1 = x_1 \\ y_2 = x_1 + x_2 \end{cases} \longrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_2 - y_1. \end{cases}$

$$J(x_1,x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1.$$

$f_{X_1}(x_1) = \begin{cases} \lambda_1 e^{-\lambda_1 x_1}, & x_1 > 0 \\ 0, & \text{else.} \end{cases}$ $\quad f_{X_2}(x_2) = \begin{cases} \lambda_2 e^{-\lambda_2 x_2}, & x_2 > 0 \\ 0, & \text{else} \end{cases}$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \begin{cases} \lambda_1 \lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2}, & x_1 > 0,\ x_2 > 0 \\ 0 & \text{else} \end{cases}$$

Now, $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2) \cdot |J(x_1,x_2)|^{-1} = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot 1$

$$= \lambda_1 \cdot \lambda_2 \cdot e^{-\lambda_1 x_1 - \lambda_2 x_2}, \quad x_1 > 0,\ x_2 > 0$$

$$= \lambda_1 \cdot \lambda_2\, e^{-\lambda_1 y_1 - \lambda_2 (y_2 - y_1)}, \quad y_1 > 0,\ y_2 - y_1 > 0$$

$$= \lambda_1 \lambda_2\, e^{-(\lambda_1 - \lambda_2) y_1 - \lambda_2 y_2}, \quad y_1 > 0,\ y_2 > y_1 \quad \{\text{or } 0 < y_1 < y_2\}$$

Further remark. from $f_{Y_1,Y_2}(y_1,y_2)$, find $f_{Y_1}(y_1)$, $f_{Y_2}(y_2)$.

$$f_{Y_1}(y_1)=\int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)dy_2 = \int_{y_1}^{\infty} \lambda_1\lambda_2 e^{-(\lambda_1-\lambda_2)y_1}\cdot e^{-\lambda_2 y_2} dy_2$$

$$= \lambda_1\cdot\lambda_2\cdot e^{-(\lambda_1-\lambda_2)y_1}\int_{y_1}^{\infty} e^{-\lambda_2 y_2} dy_2 = \lambda_1 e^{-(\lambda_1-\lambda_2)y_1}\left(-e^{-\lambda_2 y_2}\right)\Big|_{y_1}^{\infty}$$

$$= \lambda_1 e^{-(\lambda_1-\lambda_2)y_1} * \left[0+e^{-\lambda_2 y_1}\right] = \lambda_1 e^{-\lambda_1 y_1}.$$

i.e. $Y_1 \sim f_{Y_1}(y_1)=\begin{cases}\lambda_1 e^{-\lambda_1 y_1}, & y_1>0\\ 0, & \text{else}.\end{cases}$

$$f_{Y_2}(y_2)=\int_{-\infty}^{\infty} f(y_1,y_2)dy_1 = \int_0^{y_2}\lambda_1\lambda_2 e^{-(\lambda_1-\lambda_2)y_1-\lambda_2 y_2}dy_1$$

$$= \lambda_1\lambda_2 e^{-\lambda_2 y_2}\int_0^{y_2} e^{-(\lambda_1-\lambda_2)y_1}dy_1$$

$$= \lambda_1\lambda_2 e^{-\lambda_2 y_2}\cdot -\frac{e^{-(\lambda_1-\lambda_2)y_1}}{\lambda_1-\lambda_2}\Big|_0^{y_2} = \lambda_1\lambda_2 e^{-\lambda_2 y_2}\cdot\left[-\frac{e^{-(\lambda_1-\lambda_2)y_2}}{\lambda_1-\lambda_2}+\frac{1}{\lambda_1-\lambda_2}\right]$$

$$= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left[1-e^{-(\lambda_1-\lambda_2)y_2}\right]e^{-\lambda_2 y_2}. \quad y_2>0.$$

In particular, if $\lambda_1=\lambda_2$, then $(\lambda:=\lambda_1=\lambda_2)$

$$f_{Y_2}(y_2)=\int_0^{y_2}\lambda^2 e^{-(\lambda-\lambda)y_1-\lambda y_2}dy_1 = \int_0^{y_2}\lambda^2 e^{-\lambda y_2}dy_1 = \lambda^2 e^{-\lambda y_2}\int_0^{y_2}dy_1$$

$$= \lambda^2 e^{-\lambda y_2}\cdot y_2$$

i.e. $Y_2 \sim f_{Y_2}(y_2)=\begin{cases}\lambda^2\cdot y_2 e^{-\lambda y_2}, & y_2>0\\ 0, & \text{else}.\end{cases}$

$Y_2 \sim \text{Gamma}(2,\lambda)$. recall $X_1\sim\exp(\lambda)=\text{Gamma}(1,\lambda)$

$X_2\sim\exp(\lambda)=\text{Gamma}(1,\lambda)$

this verifies $Y_2=X_1+X_2\sim\text{Gamma}(2,\lambda)$.

Example 2. $X, Y \overset{iid}{\sim} N(0,1)$. rectangular coordinates X, Y.

polar coordinates R, Θ. $\quad R^2 = X^2 + Y^2$

$\tan\Theta = Y/X$,

$(R, \Theta) \sim f_{R,\Theta}(r, \theta) = ?$

Ans:
$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases} \Longrightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$J(x,y) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} & \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{r}$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

Apply formula

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(x,y) \cdot |J(x,y)|^{-1}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \cdot \left|\frac{1}{r}\right|^{-1}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \cdot r, \qquad -\infty < x, y < \infty \rightarrow \begin{matrix} 0 < r < \infty \\ 0 \le \theta \le 2\pi \end{matrix}$$

i.e. $$(R, \Theta) \sim f_{R,\Theta}(r,\theta) = \begin{cases} \frac{1}{2\pi} r e^{-\frac{r^2}{2}}, & 0 \le \theta \le 2\pi,\ 0 < r < \infty \\ 0, & \text{otherwise}. \end{cases}$$

Marginal dist. for R and Θ.

$$f_R(r)=\int f_{R,\Theta}(r,\theta)\,d\theta=\int_0^{2\pi}\frac{1}{2\pi}re^{-\frac{r^2}{2}}d\theta=\frac{1}{2\pi}re^{-\frac{r^2}{2}}\int_0^{2\pi}d\theta=re^{-\frac{r^2}{2}}.$$

i.e., $R\sim f_R(r)=\begin{cases} re^{-\frac{r^2}{2}}, & r>0 \\ 0, & r\le 0. \end{cases}$

$$f_\Theta(\theta)=\int f_{R,\Theta}(r,\theta)\,dr=\int_0^{\infty}\frac{1}{2\pi}re^{-\frac{r^2}{2}}dr=\frac{1}{2\pi}\int_0^{\infty}re^{-\frac{r^2}{2}}dr$$

(let $\frac{r^2}{2}=t$) $\quad =\frac{1}{2\pi}\int_0^{\infty}e^{-\frac{r^2}{2}}d\left(\frac{r^2}{2}\right)=\frac{1}{2\pi}\int_0^{\infty}e^{-t}dt=\frac{1}{2\pi}.$

i.e. $f_\Theta(\theta)=\begin{cases}\frac{1}{2\pi}, & \theta\in[0,2\pi] \\ 0, & \text{else}\end{cases}$

Notice $f_{R,\Theta}(r,\theta)=f_R(r)\cdot f_\Theta(\theta)$ for all r,θ.

Thus, R and Θ are independent r.v.s

Example 3. $X\sim \text{Gamma}(\alpha,\lambda)$, $Y\sim\text{Gamma}(\beta,\lambda)$, $X \perp\!\!\!\perp Y$.

$U=X+Y$, $V=\frac{X}{X+Y}$, find $f_{U,V}(u,v)=?$

Ans: $\begin{cases} u=x+y \\ v=\frac{x}{x+y}\end{cases} \longrightarrow \begin{cases} x=uv \\ y=u-uv.\end{cases}$

$$J(x,y)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2}\end{vmatrix}=-\frac{1}{x+y}.$$

$f_X(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$, $\quad f_Y(y)=\frac{\lambda^\beta}{\Gamma(\beta)}y^{\beta-1}e^{-\lambda y}$,

$$f_{U,V}(u,v)=f_{X,Y}(x,y)\cdot|J(x,y)|^{-1}$$

$$=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\cdot\frac{\lambda^\beta}{\Gamma(\beta)}y^{\beta-1}e^{-\lambda y}\cdot\left|-\frac{1}{x+y}\right|^{-1}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} (uv)^{\alpha-1} e^{-\lambda u} \cdot (u-uv)^{\beta-1} \cdot \left|-\frac{1}{u}\right|^{-1}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha+\beta-1} e^{-\lambda u} \cdot v^{\alpha-1} (1-v)^{\beta-1}$$

$$= \underbrace{\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} u^{\alpha+\beta-1} e^{-\lambda u}}_{U \sim Gamma(\alpha+\beta, \lambda)} \cdot \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1} (1-v)^{\beta-1}}_{V \sim Beta(\alpha, \beta)}$$

domain for (u, v). $x = uv > 0 \rightarrow u > 0, \ v > 0$

$y = u(1-v) > 0 \quad 1-v > 0, \ v < 1$

$\begin{cases} u = x+y > 0 \\ v = \frac{x}{x+y} > 0 \end{cases}$

i.e. $u > 0, \ 0 < v < 1$.

Thus, $f_{U,V}(u, v) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha+\beta-1} e^{-\lambda u} \cdot v^{\alpha-1} (1-v)^{\beta-1}, \quad u > 0, \ 0 < v < 1.$

From above $f_{U,V}(u, v)$, it is easy to find

$$f_U(u) = \int f_{U,V}(u, v) dv = \cdots = \begin{cases} \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha+\beta-1} \cdot e^{-\lambda u}, & u > 0 \\ 0, & u < 0. \end{cases}$$

$$f_V(v) = \int f_{U,V}(u, v) du = \cdots = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1} (1-v)^{\beta-1}, & 0 < v < 1 \\ 0, & \text{else} \end{cases}$$

Thus, $f_{U,V}(u, v) = f_U(u) \cdot f_V(v)$ for all u, v.

i.e. U and V are independent!

More examples

Example 1. Given $p(x_i, y_j) = p_{ij}$,
find $Z = 2X+Y$, $W = X-Y$.

$X \backslash Y$	0	1	2	3
0	1/8	2/8	1/8	0
1	0	1/8	2/8	1/8

Ans: from $x = 0, 1$, $y = 0, 1, \cdots 3$, one can see the values for Z are $0, 1, 3, \cdots, 5$.

thus, $p(Z=0) = p(2X+Y=0) = p(2X+Y=0;\ X = 0 \text{ or } 1)$

$= p(2X+Y=0, X=0) + p(2X+Y=0, X=1)$

$= p(Y=0, X=0) + p(Y=-2, X=1) = 1/8 + 0 = 1/8.$

$p(Z=1) = p(2X+Y=1) = p(2X+Y=1, X=0) + p(2X+Y=1, X=1)$

$= p(Y=1, X=0) + p(Y=-1, X=1) = \frac{2}{8} + 0 = \frac{2}{8}.$

$p(Z=2) = p(2X+Y=2) = p(X=0, 2X+Y=2) + p(X=1, 2X+Y=2)$

put X first for convenience!

$= p(X=0, Y=2) + p(X=1, Y=0)$

$= 1/8 + 0 = 1/8.$

$p(Z=3) = p(2X+Y=3) = p(X=0, Y=3) + p(X=1, Y=1) = 0 + 1/8 = 1/8.$

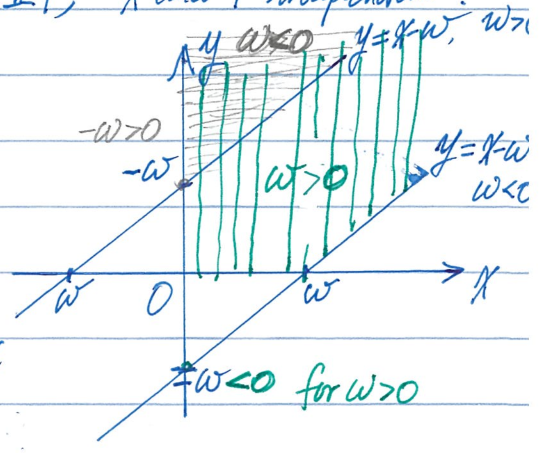
$p(Z=4) = p(2X+Y=4) = p(X=0, Y=4) + p(X=1, Y=2) = 0 + 2/8 = 2/8.$

$p(Z=5) = p(2X+Y=5) = p(X=0, Y=5) + p(X=1, Y=3) = 0 + 1/8 = 1/8.$

thus, one has

Z	0	1	2	3	4	5
$p(z)$	1/8	2/8	1/8	1/8	2/8	1/8

Example 2. $X \sim \exp(1)$, $Y \sim \exp(1)$, $W = X-Y$, $X \perp\!\!\!\perp Y$, X and Y independent.
find $W \sim f_W(w)$.



Ans: $F_W(w) = P(W \le w) = p(X-Y \le w)$

$= P(Y \ge X-w)$

if $w<0$, $F_W(w) = P(Y \ge X-w) = \int_0^\infty \int_{x-w}^\infty e^{-x} e^{-y}\, dy\, dx$

$= \int_0^\infty e^{-x} \cdot [\ e^{-(x-w)}]\, dx$

$$= \int_0^{\infty} e^{w} \cdot e^{-2x} \, dx = e^{w} \int_0^{\infty} e^{-2x} dx = \tfrac{1}{2} e^{w}.$$

if $w > 0$,

$$F_W(w) = P(Y \geq X - w) = \iint_A f(x,y)\,dx\,dy = \int_0^{\infty} \int_0^{y+w} e^{-x} e^{-y} \, dx \, dy$$

use type II region

$$= \int_0^{\infty} e^{-y} \int_0^{y+w} e^{-x} dx \, dy = \int_0^{\infty} e^{-y} \left[1 - e^{-(y+w)}\right] dy$$

$$= \int_0^{\infty} (e^{-y} - e^{-2y-w}) dy = 1 - \tfrac{1}{2} e^{-w},$$

put together, $F_W(w) = \begin{cases} \frac{1}{2} e^{w}, & w < 0 \\ 1 - \frac{1}{2} e^{-w}, & w > 0. \end{cases}$

Now, $f_W(w) = F_W'(w) = \begin{cases} \frac{1}{2} e^{w}, & w < 0 \\ 0 - \frac{1}{2} e^{-w}(-1) = \frac{1}{2} e^{-w}, & w > 0 \end{cases}$

$$f_W(w) = \begin{cases} \frac{1}{2} e^{w}, & w < 0 \\ \frac{1}{2} e^{-w}, & w \geq 0 \end{cases} = \frac{1}{2} e^{-|w|}, \quad w \in (-\infty, \infty)$$

double exponential distribution.

It can be verified that 1° $f(w) \geq 0$. 2°. $\int_{-\infty}^{\infty} f(w) dw = 1$.

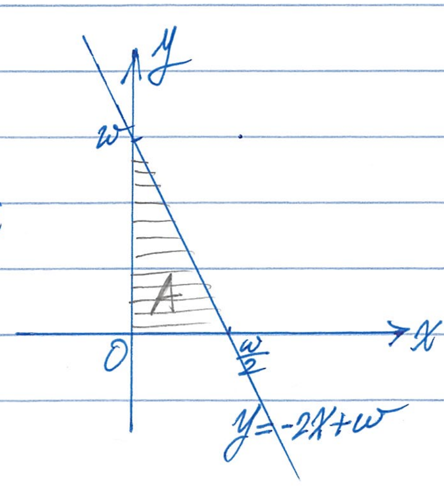
$$2^\circ. \int_{-\infty}^{\infty} f(w) dw = \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-|w|} dw = 2 \int_0^{\infty} \tfrac{1}{2} e^{-w} dw = \int_0^{\infty} e^{-w} dw = 1.$$

→ it is an even function.

Example 3. find $W = 2X + Y$'s pdf $f_W(w) = ?$

step 1. $F_W(w) = P(W \leq w) = P(2X + Y \leq w)$

$$= \iint_A f(x,y)\,dx\,dy = \int_0^{\frac{w}{2}} \int_0^{-2x+w} e^{-x} e^{-y} \, dy \, dx$$



step 2. $f_W(w) = F_W'(w) = \cdots$