Chapter 4 part 2

§4.2. Variance and Standard deviation.

indication of how dispersed the probability distribution is about its center, or how spread out on the average are the values of the r.v. about its expectation.

Definition: r.v. X, $E(X)$. $Var(X) = E\{[X-E(X)]^2\}$.

standard deviation of X is $\sqrt{Var(X)}$.

A. X is a discrete r.v.

X	x_1	x_2	$\cdots$	x_n	$\cdots$
$p(x)$	$p(x_1)$	$p(x_2)$	$\cdots$	$p(x_n)$	$\cdots$

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i \cdot p(x_i),$$

$$Var(X) = E[(X-\mu)^2] = (x_1-\mu)^2 \cdot p(x_1) + (x_2-\mu)^2 p(x_2) + \cdots + (x_n-\mu)^2 p(x_n) + \cdots$$

$$= \sum_{i=1}^{\infty} (x_i-\mu)^2 \cdot p(x_i).$$

B. X is a continuous r.v. $X \sim f(x)$. $\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$.

$$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx.$$

Theorem A. If $Var(X)$ exists, $Y = a + b\cdot X$, then $Var(Y) = b^2 \cdot Var(X)$.

proof: $E(Y) = a + b\cdot E(X)$. $Var(Y) = E\{[Y-E(Y)]^2\} = E\{[a+b\cdot X - a - b\cdot E(X)]^2\}$

$= E\{[b\cdot X - b\cdot \mu_X]^2\} = E\{b^2[X-\mu_X]^2\} = b^2 \cdot E\{[X-\mu_X]^2\}$

$= b^2 \cdot Var(X)$, where $\mu_X = E(X)$.

Expl. A. X is a Bernoulli r.v. find $Var(X)$.

$p = \frac{1}{2} = 0.5$

Plot hist on $p = 0.1$.

Ans.

X	1	0
$p(x)$	p	$1-p$

$E(X) = 1\cdot p + 0\cdot(1-p) = p$.

$Var(X) = E\{[X-p]^2\} = (1-p)^2 \cdot p + (0-p)^2 \cdot (1-p)$

$= p\cdot(1-p)^2 + p^2 \cdot (1-p) = p\cdot(1-p)[(1-p)+p]$

$= p\cdot(1-p)$.

$X \sim B(n,p)$, $n = 100$.

Expl B. $X \sim N(\mu, \sigma^2)$, $X \sim f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

find $Var(X)$

Ans: $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \mu$ (previous example)

$$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) dx = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

let $\frac{x-\mu}{\sigma} = z$, ($x = \mu + \sigma z$, $dx = dz \cdot \sigma$)

$$= \int_{-\infty}^{\infty} \sigma^2 z^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2}} \cdot \sigma \cdot dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 \cdot e^{-\frac{z^2}{2}} dz$$

let $\frac{z^2}{2} = u$, $z = \sqrt{2} \cdot \sqrt{u}$, $dz = \sqrt{2} \frac{1}{2\sqrt{u}} du$,

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2u e^{-u} \cdot \sqrt{2} \cdot \frac{1}{2\sqrt{u}} du = \frac{2\sqrt{2} \cdot \sigma^2}{\sqrt{2\pi}} \int_0^{\infty} u^{\frac{1}{2}} e^{-u} du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} u^{\frac{3}{2}-1} e^{-u} du = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \sigma^2.$$

$\left(\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right), \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right)$

i.e. $X \sim N(\mu, \sigma^2)$, $E(X) = \mu$, $Var(X) = \sigma^2$.

100 meter dash times

Theorem B. $Var(X) = E(X^2) - [E(X)]^2$.

Expl C. $X \sim U[0,1]$. find $\mu = E(X)$, $\sigma^2 = Var(X)$.

Ans. $\mu = E(X) = \int_0^1 x \cdot 1 dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$.

$Var(X) = E(X^2) - [E(X)]^2$, $E(X^2) = \int_0^1 x^2 \cdot 1 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Theorem C. Chebyshev's inequality

r.v. X, $\mu = E(X)$, $\sigma^2 = Var(X)$. for any $t > 0$, $P(|X-\mu| > t) \le \frac{\sigma^2}{t^2}$.

§4.3. Covariance and Correlation

a measure of their joint variability, or their degree of association.

Definition: X, Y are r.vs. $\mu_X = E(X)$, $\mu_Y = E(Y)$.

covariance of X and Y is

length, age

$$\mathrm{Cov}(X, Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

sealevel temperature

another formula:
$$\begin{aligned}\mathrm{Cov}(X,Y) &= E(XY - X\cdot\mu_Y - \mu_X\cdot Y + \mu_X\cdot\mu_Y)\\ &= E(XY) - \mu_Y\cdot E(X) - \mu_X\cdot E(Y) + E(\mu_X\cdot\mu_Y)\\ &= E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y\\ &= E(XY) - \mu_X\mu_Y\end{aligned}$$

i.e., $\mathrm{Cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$.

X\Y	1	2	
0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$
1	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{2}$

Expl. A. $(X,Y) \sim f(x,y) = \begin{cases} 2x+2y-4xy, & 0\le x\le 1,\ 0\le y\le 1\\ 0, & \text{else}\end{cases}$

find $\mathrm{Cov}(X,Y)$.

Ans: $\mathrm{Cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$.

$E(X) = \iint x f(x,y)\,dx\,dy = \int_0^1\int_0^1 x\cdot(2x+2y-4xy)\,dx\,dy = \cdots = \frac{1}{2}$.

$E(Y) = \int_0^1\int_0^1 y(2x+2y-4xy)\,dx\,dy = \cdots = \frac{1}{2}$.

$E(X\cdot Y) = \iint xy f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(2x+2y-4xy)\,dx\,dy = \frac{2}{9}$.

$\mathrm{Cov}(X,Y) = \frac{2}{9} - \frac{1}{2}\cdot\frac{1}{2} = \frac{2}{9} - \frac{1}{4} = \frac{8-9}{36} = -\frac{1}{36}$.

properties: $\mathrm{Cov}(a+X, Y) = \mathrm{Cov}(X,Y)$. $[\mathrm{Cov}(a,Y)=0]$

$\mathrm{Cov}(aX, bY) = a\cdot b\cdot \mathrm{Cov}(X,Y)$.

$\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z)$.

$$\mathrm{Cov}(aW+bX, cY+dZ) = ac\,\mathrm{Cov}(W,Y) + bc\,\mathrm{Cov}(X,Y) + ad\,\mathrm{Cov}(W,Z) + bd\cdot\mathrm{Cov}(X,Z).$$

Theorem A. $U = a + \sum_{i=1}^{n} b_i X_i$, $V = c + \sum_{j=1}^{m} d_j Y_j$,

then $Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i d_j Cov(X_i, Y_j)$.

Corollary A: $Var(a + \sum_{i=1}^{n} b_i X_i) = Cov(a + \sum_{i=1}^{n} b_i X_i,\ a + \sum_{j=1}^{n} b_j X_j)$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j Cov(X_i, X_j).$$

if X_i are independent, $Cov(X_i, X_j) = 0$, if $i \neq j$.

then $Var(a + \sum_{i=1}^{n} b_i X_i) = \sum_{i=1}^{n} b_i^2 Cov(X_i, X_i) = \sum_{i=1}^{n} b_i^2 Var(X_i)$.

or $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$.

\# $Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(X,Y) + Cov(Y,X) + Cov(Y,Y)$

$$= Var(X) + 2\,Cov(X,Y) + Var(Y).$$

Note. $E(X+Y) = E(X) + E(Y)$, always true.

Definition: correlation coefficient

$$\rho(X,Y) = \rho = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

Expl D. $(X,Y) \sim f(x,y) = \begin{cases} 2x + 2y - 4xy, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$

Find $\rho(X,Y)$.

Ans: $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$.

from Expl A, $Cov(X,Y) = -1/36$.

$f_X(x) = \int_0^1 (2x + 2y - 4xy)\,dy = 2x + y^2\Big|_0^1 - 2x \cdot y^2\Big|_0^1$

$$= 2x + 1 - 2x = 1.$$

$X \sim U[0,1]$. $E(X) = \frac{1}{2}$. $Var(X) = 1/12$. Same for $Y \sim U[0,1]$.

thus, $\rho(X,Y) = \dfrac{-1/36}{\sqrt{\frac{1}{12} \cdot \frac{1}{12}}} = \dfrac{-1/36}{1/12} = -1/3$.

Notations: $\mu_X = E(X)$. $\sigma_X^2 = Var(X)$. $\sigma_X = \sqrt{Var(X)}$, $\mu_Y = E(Y)$, $\sigma_Y^2 = Var(Y)$

$\sigma_{XY} = Cov(X,Y)$. $\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$.

Theorem B. $-1 \le \rho \le 1$. Furthermore, $\rho = \pm 1$ if and only if $P(Y = a + bX) = 1$ for some constants a and b.

Proof:

$$0 \le Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) = Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{Y}{\sigma_Y}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$

$$= \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{\sigma_Y^2} + \frac{2Cov(X,Y)}{\sigma_X \sigma_Y} = 1 + 1 + 2\cdot\rho$$

$$= 2 + 2\rho = 2(1+\rho)$$

i.e. $0 \le 2(1+\rho)$, $1+\rho \ge 0$, $\rho \ge -1$.

From $0 \le Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 2(1-\rho)$, obtain $2(1-\rho) \ge 0$, $\rho \le 1$.

thus, one has $-1 \le \rho \le 1$.

If $\rho = 1$, then $Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 0$. thus $P\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = c\right) = 1$

i.e. $P\left(Y = \frac{\sigma_Y}{\sigma_X} X - \sigma_Y \cdot c\right) = 1$, i.e. $P(Y = a + bX) = 1$.

$b = \frac{\sigma_Y}{\sigma_X} > 0$.

If $\rho = -1$, $P\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} = c\right) = 1$, $P\left(Y = -\frac{\sigma_Y}{\sigma_X} X + c\right) = 1$, $b = -\frac{\sigma_Y}{\sigma_X} < 0$

Expl F. Bivariate normal distribution. (P81, Expl F)

$(X,Y) \sim BN(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho)$.

if $(X,Y) \sim f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}\right.\right.$

$$\left.\left. + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - 2\rho \cdot \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y}\right]\right\}$$

Also. $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$.

find $Cov(X,Y)$, $\rho(X,Y)$.

Ans: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy$

let $u = \frac{x-\mu_X}{\sigma_X}$, $v = \frac{y-\mu_Y}{\sigma_Y}$, $dx = \sigma_X du$, $dy = \sigma_Y dv$

$$Cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_X u \cdot \sigma_Y v \cdot \frac{1}{2\pi \cdot \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[u^2+v^2-2\rho u v\right]\right\} \sigma_X \sigma_Y du\, dv$$

$$= \frac{\sigma_X \sigma_Y}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot v \cdot e^{-\frac{1}{2(1-\rho^2)}\left[u^2+v^2-2\rho \cdot u \cdot v\right]} du\, dv$$

consider inner integer $\int_{-\infty}^{\infty} u\, e^{-\frac{1}{2(1-\rho^2)}\left[u^2-2\rho \cdot u v+v^2\right]} du$

$$= \int_{-\infty}^{\infty} u\, e^{-\frac{1}{2(1-\rho^2)}\left[u^2-2\rho v \cdot u+\rho^2 v^2-\rho^2 v^2+v^2\right]} du = \int_{-\infty}^{\infty} u\, e^{-\frac{1}{2(1-\rho^2)}\left[(u-\rho v)^2+(1-\rho^2)v^2\right]} du$$

$$= \int_{-\infty}^{\infty} u\, e^{-\frac{(u-\rho v)^2}{2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}(1-\rho^2)v^2} du \quad = e^{-\frac{v^2}{2}} \underbrace{\int_{-\infty}^{\infty} u\, e^{-\frac{(u-\rho v)^2}{2(1-\rho^2)}} du}$$

Now, consider r.v $W \sim N(\rho v, 1-\rho^2)$, then $EW = \rho v$.

i.e. $\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{(x-\rho v)^2}{2\cdot(1-\rho^2)}} dx = \rho v$

obtain $\int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\rho v)^2}{2(1-\rho^2)}} dx = \sqrt{2\pi} \cdot \sqrt{1-\rho^2} \cdot \rho \cdot v$

thus $Cov(X,Y) = \frac{\sigma_X \sigma_Y}{2\pi\cancel{\sqrt{1-\rho^2}}} \int_{-\infty}^{\infty} v \cdot e^{-\frac{v^2}{2}} \cdot \sqrt{2\pi} \cdot \cancel{\sqrt{1-\rho^2}} \cdot \rho v\, dv$

$$= \frac{\sigma_X \sigma_Y \cdot \rho}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v^2 e^{-\frac{v^2}{2}} dv = \frac{\sigma_X \sigma_Y \rho}{\sqrt{2\pi}} \cdot 2 \int_0^{\infty} v^2 e^{-\frac{v^2}{2}} dv$$

let $\frac{v^2}{2} = t$, $v^2 = 2t$,
$v = \sqrt{2}\sqrt{t}$, $dv = \sqrt{2} \cdot \frac{dt}{2\sqrt{t}} = \frac{dt}{\sqrt{2t}}$

$$= \frac{2\sigma_X \sigma_Y \rho}{\sqrt{2\pi}} \int_0^{\infty} 2t\, e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma_X \sigma_Y \rho}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt = \frac{2 \cdot \sigma_X \sigma_Y \cdot \rho}{\sqrt{\pi}} \Gamma\left(\tfrac{3}{2}\right),$$

[recall $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$]

$\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$$= \frac{2 \cdot \sigma_X \sigma_Y \cdot \rho}{\sqrt{\pi}} \frac{1}{2} \cdot \sqrt{\pi} = \sigma_X \cdot \sigma_Y \cdot \rho.$$

i.e., $Cov(X,Y) = \sigma_X \cdot \sigma_Y \cdot \rho$.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{\sigma_X \cdot \sigma_Y \cdot \rho}{\sqrt{\sigma_X^2 \cdot \sigma_Y^2}} = \rho.$$

Some examples.

7-40 $(X,Y) \sim f(x,y) = \begin{cases} \frac{1}{y} e^{-(y+\frac{x}{y})}, & x>0,\ y>0 \\ 0, & \text{elsewhere.} \end{cases}$

find $E(X), E(Y), Cov(X,Y)$.

$$E(X) = \iint x f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty x \cdot \frac{1}{y} e^{-(y+\frac{x}{y})}\,dx\,dy$$

$$= \int_0^\infty e^{-y} \underbrace{\int_0^\infty \frac{x}{y} e^{-\frac{x}{y}}\,dx}_{= y}\,dy = \int_0^\infty e^{-y} \cdot y\,dy = 1.$$

[use $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$, $\Gamma(n) = (n-1)!$]

$$E[Y] = \int_0^\infty \int_0^\infty y \frac{1}{y} e^{-(y+\frac{x}{y})}\,dx\,dy$$

$$= \int_0^\infty e^{-y} \underbrace{\int_0^\infty e^{-\frac{x}{y}}\,dx}_{= y}\,dy = \int_0^\infty e^{-y} y\,dy = 1.$$

$$E(XY) = \int_0^\infty \int_0^\infty xy \cdot \frac{1}{y} e^{-(y+\frac{x}{y})}\,dx\,dy = \int_0^\infty y^2 e^{-y}\,dy = \Gamma(3) = 2.$$

therefore $Cov(X,Y) = E(XY) - E(X)E(Y) = 1$.

7-45 If X_1, X_2, X_3, X_4 are (pairwise) uncorrelated r.v.s, each having mean 0 and variance 1. Compute the correlations of

(a) $X_1 + X_2$ and $X_2 + X_3$; (b) $X_1 + X_2$ and $X_3 + X_4$.

7-45 (a)

$$\rho(X_1+X_2, X_2+X_3) = \frac{Cov(X_1+X_2, X_2+X_3)}{\sqrt{Var(X_1+X_2)\cdot Var(X_2+X_3)}}$$

$$Cov(X_1+X_2; X_2+X_3) = Cov(X_1,X_2) + Cov(X_1,X_3) + Cov(X_2,X_2) + Cov(X_2,X_3)$$
$$= 0+0+Var(X_2)+0 = 1.$$

$$Var(X_1+X_2) = Cov(X_1+X_2, X_1+X_2)$$
$$= Cov(X_1,X_1) + Cov(X_1,X_2) + Cov(X_2,X_1) + Cov(X_2,X_2)$$
$$= Var(X_1) + 0 + 0 + Var(X_2)$$
$$= 2.$$

thus $\rho(X_1+X_2, X_2+X_3) = \frac{1}{\sqrt{2\cdot 2}} = \frac{1}{2}$.

(b) $$\rho(X_1+X_2, X_3+X_4) = \frac{Cov(X_1+X_2, X_3+X_4)}{\sqrt{Var(X_1+X_2)\cdot Var(X_3+X_4)}}$$

$$Cov(X_1+X_2, X_3+X_4) = Cov(X_1,X_3) + Cov(X_1,X_4) + Cov(X_2,X_3) + Cov(X_2,X_4)$$
$$= 0+0+0+0$$
$$= 0.$$

thus $\rho(X_1+X_2, X_3+X_4) = 0$