Robust positively invariant sets for state-dependent and scaled disturbances

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***Abstract*— This paper introduces methods of deriving and computing maximal robust positively invariant sets for linear discrete time systems with additive model uncertainty. Two types of uncertainty are considered: state-dependent uncertainty, which handles multiplicative parametric model uncertainty as well as linearisation errors for nonlinear systems, and scaled sets of uncertainty. We provide a framework for analysing both types of uncertainty with illustrative examples.**

***Index Terms*—**

I. Introduction

Ever since the introduction of disturbance invariant sets in [1], the analytic properties of these sets have been fairly well studied, see [2]. Here by disturbance invariant sets we mean a subset of the state constraint set $\mathscr{X} \subseteq \mathcal{X}_0 \subseteq \mathbb{R}^n$ containing initial conditions for the perturbed linear system

$$x^+ = \Psi x + v, \tag{1}$$

where Ψ is Hurwitz, $x, x^+ \in \mathcal{X}_0$ the state and the successor state, and where $v \in \mathscr{V}$ denotes the perturbation, such that the successor state is contained in the disturbance invariant set $x^+ \in \mathscr{X}$ for all possible realisations of the perturbation $v \in \mathscr{V}$. That is summarised in the implicit definition

$$\mathscr{X} = \{x : \Psi x + v \in \mathscr{X} \; \forall v \in \mathscr{V}\}. \tag{2}$$

These sets are known as robust positive invariant (RPI) sets. In many applications we are interested in the largest such set $\mathcal{X}^\infty$, i.e. the set containing all other RPI sets $\mathscr{X} \subseteq \mathcal{X}^\infty$, this set is called the maximal robust positive invariant (MRPI) set. These sets are used e.g. as terminal regions in robust model predictive control. Even though analytical properties of these sets were derived early after their introduction, it took more than two decades until algorithms were introduced to numerically compute MRPI sets [3]–[5]. Earlier algorithms did not guarantee finite determinability of $\mathcal{X}^\infty$, see e.g. [6], or did not guarantee to produce the maximal RPI set, e.g. [7]. In the afore mentioned algorithms the perturbation was considered to belong to a fixed compact set, in this paper we consider sets that vary with parameters. First we consider the set of disturbances to be state dependent, e.g.

$$\mathcal{V}(x) = \{v : Gv \leq H(x)\}, \tag{3}$$

where $H(x)$ is a piecewise affine function. This means that in the first part of the paper we discuss the computation of $\mathcal{X}^\infty$ for the case where $\mathscr{V} = \mathcal{V}(x)$. The case where the disturbance depends on the state has been studied analytically before but not extensively, see [8]. These kind of disturbance constraints can be used to account for linearisation errors, as will be shown in section IV, but can also be used to treat multiplicative uncertainty.

The second case we consider is the case of a scaled set of disturbance, i.e.

$$\mathcal{V}(\theta) = \{v : Gv \leq (1+\theta)\mathbf{1}\}, \tag{4}$$

with $\theta > -1$. To distinguish the resulting sets from regular MRPI sets we use $\mathcal{Z}^\infty$ to emphasise that $\mathcal{Z}^\infty$ itself is not an MRPI set, however by fixing $\theta = \hat{\theta}$ the set $\mathcal{Z}^\infty|_{\hat{\theta}}$ is an MRPI set. To our best knowledge, this setup has not been presented in the literature. This type of disturbance constraints can be used to study the sensitivity of the MRPI set to changes in the disturbance *strength*.

The remainder of this paper is structured: In section II we introduce the notion of *parametric convexity* and of its general implications, this is sufficient for the convexity of the MRPI set. The algorithm to compute the MRPI set for state dependent disturbance constraints is presented in section III, we also prove its finite determinability. In section IV we demonstrate the algorithm using a numerical exemplary system, a levitating ball setup. The case of scaled disturbance constraints is discussed in section V where the algorithm for the parametrised MRPI set as well as the proof for its finite determinability are also presented. Section VI illustrates the use of parametrised MRPI sets using again the levitating ball example and illustrating some of possible robustness analyses of the parametrised MRPI sets with a numerical example. The paper is concluded in section VII.

In the remainder of this paper we use the notion of a polyhedra for sets that can be represented as the intersection of a finite number of halfspaces, polytopic sets for bounded polyhedral. By $\mathbf{1}$ we denote the column vector of ones in appropriate dimension, $\wedge$ denotes the logical and.

II. Parametrically convex set operations

In this section we discuss sets that depend on a state-like parameter (so called point-to-set maps, see [9]), and extend the existing set algebra [2] to accommodate such sets. We present the general case first, then derive the computation of the case with piece-wise affine sets.

Definition 2.1 (Parametric Convexity): Let $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$, let $\mathcal{P}(Y)$ denote the power set of Y and $T : X \to \mathcal{P}(Y)$, $X \ni s \mapsto T(s) \subset Y$ be a continuous point-to-set map. The map T is called *parametrically convex* if it satisfies

$$T(\lambda s_1 + (1-\lambda)s_2) \subseteq \lambda T(s_1) \oplus (1-\lambda)T(s_2) \tag{5}$$

for all $s_1, s_2 \in X$ and $\lambda \in [0,1]$.
In (5), $\oplus$ denotes the Minkowski set addition

$$\mathcal{A} \oplus \mathcal{B} = \{c : c = a + b \; \forall a \in \mathcal{A}, b \in \mathcal{B}\}. \tag{6}$$

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In the following we will need to be able to compute differences of sets analogous to the Pontryagin difference in [10] but using parametrised sets, this is defined as follows:

Definition 2.2 (Parametric Pontryagin Difference): Let $S \subseteq X$ and let $T : X \to \mathcal{P}(X)$ be a continuous point-to-set map, then the *parametric Pontryagin difference* $S \ominus T(S)$ is

$$S \ominus T(S) = \{x \in X : \{x\} \oplus T(x) \subseteq S\}\,, \tag{7}$$

where $T(S)$ denotes the image of S under the map T. Notice that for constant maps T definition (7) is equivalent to the well known Pontryagin difference. For parametric Pontryagin differences of convex sets and parametrically convex maps we have the following result.

Lemma 2.3: Let $S \subseteq X$ be a convex set and let $T : X \to \mathcal{P}(X)$ be a parametrically convex point-to-set map, then $S \ominus T(S)$ is convex.

Proof: Define $Z = S \ominus T(S)$ and let $z_1, z_2 \in Z$, then by definition of the parametric Pontryagin difference, we have

$$\{z_i\} \oplus T(z_i) \subseteq S,\ i = 1, 2. \tag{8}$$

To see that Z is convex we show that line segments between all possible z_1 and z_2 are subsets of Z, i.e. for all $\lambda \in [0, 1]$,

$$\begin{aligned}
&\{\lambda z_1 + (1-\lambda)z_2\} \oplus T\left(\lambda z_1 + (1-\lambda)z_2\right)\\
&\qquad\subseteq \{\lambda z_1 + (1-\lambda)z_2\} \oplus \lambda T(z_1) \oplus (1-\lambda)T(z_2)\\
&\qquad\subseteq \lambda \underbrace{\left(\{z_1\} \oplus T(z_1)\right)}_{\subseteq S} \oplus (1-\lambda) \underbrace{\left(\{z_2\} \oplus T(z_2)\right)}_{\subseteq S}\\
&\qquad\subseteq Z
\end{aligned}$$

where the last inclusion follows from the convexity of $\mathcal{S}$. ■

Consider a point-to-set map defined by (3) where, for now, H is elementwise convex in $x \in \mathscr{X} \subseteq \mathbb{R}^n$, so that $H_i(\lambda x_1 + (1-\lambda)x_2) \leq \lambda H_i(x_1) + (1-\lambda)H_i(x_2)$ for $\lambda \in [0, 1]$ and $x_1, x_2 \in X$. For such sets we have the following result:

Lemma 2.4: The point-to-set map $\mathcal{V}(x)$ defined in (3) is parametrically convex for all $x \in X$.

Proof: To show that $\mathcal{V}(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda\mathcal{V}(x_1) \oplus (1-\lambda)\mathcal{V}(x_2)$ for all $\lambda \in [0, 1]$ we note that

$$\begin{aligned}
&\mathcal{V}(\lambda x_1 + (1-\lambda)x_2)\\
=&\{v :\ Gv \leq H(\lambda x_1 + (1-\lambda)x_2)\}\\
\subseteq&\{v :\ Gv \leq \lambda H(x_1) + (1-\lambda)H(x_2)\}\\
=&\{v :\ Gv \leq \lambda H(x_1)\} \oplus \{v :\ Gv \leq (1-\lambda)H(x_2)\}\\
=&\lambda\mathcal{V}(x_1) \oplus (1-\lambda)\mathcal{V}(x_2).
\end{aligned}$$

■

Lemmas 2.3 and 2.4 imply that the parametric Pontryagin difference between a convex set $\mathcal{X}$ and the point-to-set map $\mathcal{V}(x)$, i.e. $\mathcal{X} \ominus \mathcal{V}(\mathcal{X})$, is a convex set. Furthermore, we will see that if $\mathcal{X}$ is polyhedral and $\mathcal{V}(x)$ is point wise polytopic, then $\mathcal{X} \ominus \mathcal{V}(\mathcal{X})$ is again a polytopic set. This will become more clear in the next section when we describe the computation of the MRPI set (2).

III. Maximal Robust Positive Invariant Sets for State Dependent Disturbance

In this section we describe an iterative algorithm to compute the MRPI set (2) for a linear system (1) subject to disturbance $v \in \mathcal{V}(x)$ where $\mathcal{V}(x)$ is defined as in (3) with a piecewise affine $H(x)$. Notice that the set $\mathcal{V}(x)$ is pointwise polytopic for finite $x \in \mathcal{X}_0$ and can hence be represented as the convex hull of its vertices $\mathcal{V}(x) = \text{conv}\{v_i(x)\}$. Since $\mathcal{V}(x)$ has a piece-wise affine dependence on x the vertices $v_i(x)$ are also piece-wise affine in x. The set $\mathcal{X}^\infty$ as defined in (2) requires to satisfy $\Psi x + v \in \mathcal{X}^\infty$ for all $x \in \mathcal{X}^\infty$ and $v \in \mathcal{V}(x)$. To compute the MRPI set we start from the given state constraint set $\mathcal{X}_0 = \{x :\ \Xi_{0,i}x \leq \xi_{0,i}\ \forall i \in \mathcal{I}_0\}$ and *cut off* all points that do not satisfy the invariance condition in k steps, $k \in \mathbb{N}$. So that all points remaining satisfy the invariance condition (2). I.e. we iteratively introduce constraints that separate all points for which the successor state can lie outside $\mathcal{X}_0$. So for the first iteration we need to enforce the constraint:

$$\begin{aligned}
&\Xi_{0,i}(\Psi x + v) \overset{!}{\leq} \xi_{0,i}\ \forall v \in \text{conv}\{v_i(x)\}\\
&\Xi_{0,i}\Psi x + \max_{v\in\mathcal{V}(x)} \Xi_{0,i}v \leq \xi_{0,i}\\
&\Xi_{0,i}\Psi x + \underbrace{\max_j \Xi_{0,i}v_j(x)}_{v^*_{0,i}(x)} \leq \xi_{0,i}.
\end{aligned} \tag{9}$$

to each inequality. Notice that $v^*_{0,i}(x)$ is not necessarily given by one unique maximiser, but is the solution of a multiparametric linear program and hence will be given by a vertex $v_i(x)$ of $\mathcal{V}(x)$ for each x on that facet. Since each vertex is a piece-wise affine function of x the maximum $v^*_{0,i}(x)$ will also be piece-wise affine and therefore the set $\mathcal{X}_1 = \mathcal{X}_0 \cap \{x : \Xi_{0,i}\Psi x + v^*_{0,i}(x) \leq \xi_{0,i} \forall i \in \mathcal{I}_0\}$ has the representation $\mathcal{X}_1 = \{x : \Xi_{1,i}x \leq \xi_{1,i}\ \forall i \in \mathcal{I}_1\}$. The next iterate is defined by

$$\begin{aligned}
&\mathcal{X}_2 = \mathcal{X}_1 \cap\\
&\ \{x : \Xi_{0,i}\Psi(\Psi x + v) + v^*_{0,i}(x) \leq \xi_{0,i}\ \forall i \in \mathcal{I}_0, v \in \mathcal{V}(x)\}\\
&= \mathcal{X}_1 \cap \{x : \Xi_{0,i}\Psi^2 x + v^*_{1,i}(x) + v^*_{0,i}(x) \leq \xi_{0,i}\ \forall i \in \mathcal{I}_0\}.
\end{aligned} \tag{10}$$

And we analogously define

$$\mathcal{X}_{k+1} = \mathcal{X}_k \cap \{x : \Xi_{0,i}\Psi^k x + \sum_{l=0}^{k-1} v^*_{l,i}(x) \leq \xi_{0,i}\ \forall i \in \mathcal{I}_0\}, \tag{11}$$

where we have

$$\begin{aligned}
&v^*_{l,i}(x) = \max_j \Xi_{0,i}\Psi^{l-1}v_j(x)\\
&= \begin{array}{cc}\max & \Xi_{0,i}\Psi^{l-1}v\\ \text{s.t.} & v \in \mathcal{V}(x)\end{array} = \begin{array}{cc}\max & \Xi_{0,i}\tilde{v}\\ \text{s.t.} & \tilde{v} \in \Psi^{l-1}\mathcal{V}(x)\end{array}
\end{aligned} \tag{12}$$

In a closed form the iterates can be expressed as

$$\begin{aligned}
\mathcal{X}_{k+1} =& \mathcal{X}_k \cap \left(\Psi^{-1}\mathcal{X}_k \ominus \Psi^{k-1}\mathcal{V}(\mathcal{X}_k)\right) = \mathcal{X}_k \cap D_k\\
=& \bigcap_{l\leq k+1} \left(\Psi^{-l}\mathcal{X}_0 \bigominus_{i\leq l-1} \Psi^i\mathcal{V}(\mathcal{X}_{l-1})\right).
\end{aligned} \tag{13}$$

We will use (13) to prove the finite determinability of $\mathcal{X}^\infty$, i.e. that there exists a finite number N such that $x \in \mathcal{X}_N$

implies $\Psi x + v \in \mathcal{X}_N$ for all $v \in \mathcal{V}(x)$ and the set therefore is robustly positive invariant.

Lemma 3.1: Let the system constraints be contained in a band $\mathcal{X}_0 \subseteq B = \{x : \Gamma x \leq \mathbf{1} \wedge -\Gamma x \leq \mathbf{1}\}$, let the pair (Ψ, Γ) be observable and let $\mathcal{V}(x)$ be defined by (3) with a piecewise affine H, then $\mathcal{X}_N \subseteq \mathcal{X}_{N+1}$ for a finite N. Hence the MRPI set $\mathcal{X}^\infty = \mathcal{X}_N$ is a polytope.

Proof: Notice that with (13) it is easy to see that in early iterations the parametric Pontryagin difference can yield the empty set. The empty set however is an admissible polytope, i.e. an admissible MRPI set. For the remainder of this proof we assume that the MRPI set is non-empty. For this notice that on any compact set $\mathcal{S} \subset \mathbb{R}^n$ we have $\bigcup_{s\in\mathcal{S}} \mathcal{V}(s)$ is compact.

The proof has two main steps: First we will prove that $\mathcal{X}_p$ is compact for $p \leq n$ where n is the state dimension. The second step is to prove that in (13) the set D_k grows exponentially, i.e. that for any given compact set $\mathcal{C}$ there exists a $\tilde{N}$ such that $\mathcal{C} \subseteq D_{\tilde{N}}$. The proof is concluded by setting $\mathcal{C} = \mathcal{X}_p$ and using $N = \tilde{N}$. For the first step, notice that the observability of (Ψ, Γ) is equivalent to the observability matrix Ω having full rank, i.e.

$$\Omega = \begin{pmatrix} \Gamma \\ \vdots \\ \Gamma\Psi^{n-1} \end{pmatrix}$$

having a trivial kernel. But this then implies that the set $\mathcal{P}_{n+1} = \{x : \Omega x \leq \mathbf{1} \wedge -\Omega x \leq \mathbf{1}\} = \bigcap_{l \leq n+1} \Psi^{-l} B$ is bounded. Notice that the containment $\mathcal{X}_k \subseteq \mathcal{P}_k$ holds for all k and hence the set $\mathcal{X}_{n+1}$ is also bounded. The case that $\tilde{N} < n$ occurs when Γ has more than rank one. To see that D_k grows exponentially we use that $\mathcal{V}(\mathcal{X}_{n+1})$ is bounded, let K_1 denote the smallest ball that contains $\mathcal{V}(\mathcal{X}_{n+1})$ and let $\bar{\lambda}$ denote the magnitude of the largest eigenvalue of Ψ. Furthermore, let K_1 be the biggest ball that is contained in $\mathcal{X}_0$. D_k has the representation

$$D_k = \underbrace{\Psi^{-k}\mathcal{X}_0}_{(i)} \ominus \underbrace{\left(\bigoplus_{i \leq k-1} \Psi^i \mathcal{V}(\mathcal{X}_k) \right)}_{(ii)}. \tag{14}$$

Since Ψ is asymptotically stable we know that $\bar{\lambda} < 1$ and so we have $\bar{\lambda}^{-l} K_1 \subseteq \Psi^{-l} K_1 \subseteq \Psi^{-l}\mathcal{X}_0 = (i)$ which implies exponential growth of D_k as long as (ii) is bounded. Recall that we assume that the MRPI set is non-empty. Hence, to bound (ii) we use $k \geq n+1$ where $(ii) \subseteq \oplus_{i\leq k-1} \bar{\lambda}^i K_2 \subseteq \frac{1}{1-\bar{\lambda}} K_2$ holds. This concludes the proof: We proved that D_k grows exponentially, that it contains an exponentially growing ball. We proved that the subtrahend is bounded inside a ball of finite radius, and hence we conclude that after a finite number of iterations N intersecting with D_{N+1} will no longer change $\mathcal{X}_N$. ■

We have seen that we can compute a the maximal robust positive invariant set $\mathcal{X}^\infty$ for linear systems with state dependent constraints. In the next section we will illustrate the concept with an example.

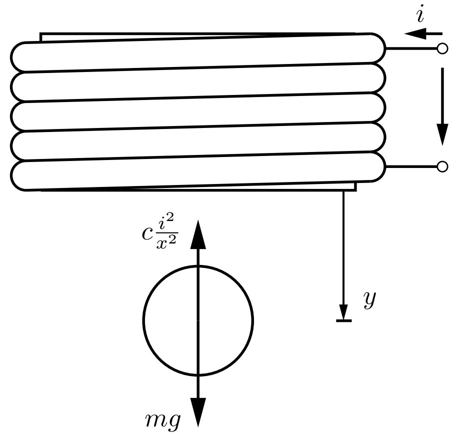

Fig. 1. Levitating ball system.

IV. Example

In this section we discuss the calculation of the MRPI set for a linearised simplified model of the levitating ball system depict in figure 1. The system dynamics for the ball are given by $m\ddot{y} = mg - c\frac{i^2}{y^2}$, where m, g, c, i and y denote the mass of the ball, the gravitational constant, a constant factor, the current and the distance between the coil and the centre of the ball respectively. For illustration purposes we neglect inductive dynamics and use the current $u = i$ as an input and the position y and its first derivative $\dot{y}$ as the states, i.e. $x = (y, \dot{y})^T$. We find that an equilibrium is present when $x_2 = 0$ and $u = \sqrt{\frac{gm}{c}} x_1$ for any positive position $x_1 > 0$. Linearising the nonlinear differential equation $\dot{x} = f(x, u)$ around $\hat{x}, \hat{u}$ we obtain

$$\Delta\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{2c\hat{u}^2}{m\hat{x}_1^3} & 0 \end{pmatrix}}_{\frac{\partial f}{\partial x}(\hat{x},\hat{u})} \Delta x + \underbrace{\begin{pmatrix} 0 \\ -\frac{2c\hat{u}}{m\hat{x}_1^2} \end{pmatrix}}_{\frac{\partial f}{\partial u}(\hat{x},\hat{u})} \Delta u. \tag{15}$$

We derive discrete time system dynamics using the explicit Euler formula $x^+ = x + T_s f(x, u) =: \tilde{f}(x, u)$, with the sampling rate T_s. We hence define the system matrix $A = (I + T_s \frac{\partial f}{\partial x}(\hat{x}, \hat{u}))$ and the input matrix $B = T_s \frac{\partial f}{\partial u}(\hat{x}, \hat{u})$. Notice that using a non-autonomous model does not affect our analysis since we analyse the closed loop system governed by a linear state feedback controller $u = Kx$. To cope with disturbance optimally we use K being the solution to robust Lyapunov condition $V(x) - V((A+BK)x + Dw) \leq \gamma^2 w^T w$ with $V(x) = x^T P x \geq 0$, i.e. $x^T P x - ((A + BK)x + Dw)^T P((A+BK)x+Dw) \geq x^T(Q + K^T RK)x - \gamma^2 w^T w$ for a minimised γ^2, see e.g. [11]. In order to obtain a representation of additive disturbances depending on the state and input like (3) we use the mean-value theorem for vector valued functions, see e.g. [12]:

Lemma 4.1 (Mean Value Theorem): Let $\mathcal{X} \subset \mathbb{R}^n$ be open, $g : \mathcal{X} \to \mathbb{R}^m$ continuously differentiable, and $x \in \mathcal{X}, h \in \mathbb{R}^n$ vectors such that the whole line segment $x + th$ remains in $\mathcal{X}$ for $0 \leq t \leq 1$. Then

$$g(x + h) = g(x) + \left(\int_0^1 \frac{\partial g}{\partial x}(x + th) dt \right) \cdot h. \tag{16}$$

Using the mean value theorem 4.1 and the linearisation for $\Delta x = \hat{x} + \tilde{x}$ and $\Delta u = \hat{u} + \tilde{u}$ we get the successor state:

$$\Delta x^+ = \underbrace{\tilde{f}(\hat{x},\hat{u})}_{\hat{x}} + \int_0^1 \frac{\partial \tilde{f}}{\partial x}(\hat{x}+t\tilde{x},\hat{u}+t\tilde{u})dt \cdot \tilde{x} +$$

$$\int_0^1 \frac{\partial \tilde{f}}{\partial u}(\hat{x}+t\tilde{x},\hat{u}+t\tilde{u})dt \cdot \tilde{u} + A\tilde{x} + B\tilde{u} - A\tilde{x} - B\tilde{u}$$

$$\Leftrightarrow \tilde{x}^+ = A\tilde{x} + B\tilde{x} + \left(\int_0^1 \frac{\partial \tilde{f}}{\partial x}(\hat{x}+t\tilde{x},\hat{u}+t\tilde{u})dt - A\right)\tilde{x} +$$

$$\left(\int_0^1 \frac{\partial \tilde{f}}{\partial u}(\hat{x}+t\tilde{x},\hat{u}+t\tilde{u})dt - B\right)\tilde{u} \tag{17}$$

This implies that for any state $\tilde{x}$ of linearised we obtain the expression $\tilde{x}^+ = A\tilde{x} + B\tilde{u} + H^x\tilde{x} + H^u\tilde{u}$, however the computation of H^x and H^u requires solving a nonlinear integral. Assume that for $x \in \mathcal{X}$ and $u \in \mathcal{U}$, where $\mathcal{X}$ and $\mathcal{U}$ are compact sets, we had extremal values of H^x and H^u, i.e. $H^x\tilde{x} + H^u\tilde{u} \in \text{conv}_k\{H^x_k\tilde{x} + H^u_k\tilde{u}\}$ for all $(\tilde{x},\tilde{u}) \in \mathcal{X} \times \mathcal{U}$. Clearly, we can then introduce the element wise bound disturbance

$$\mathcal{V}(x,u) = \Big\{ v : \min_k\{H^x_{k,i}x + H^u_{k,i}u\} \le v_i \wedge$$
$$v_i \le \max_k\{H^x_{k,i}x + H^u_{k,i}u\}\ i = 1,\ldots,n \Big\}. \tag{18}$$

With this set we can guarantee that $\tilde{x}^+ = A\tilde{x} + B\tilde{u} + v$ accounts for all nonlinearities within $\mathcal{X} \times \mathcal{U}$ if we constraint $v \in \mathcal{V}(\tilde{x},\tilde{u})$. For general nonlinear systems finding the extremal values of (H^x, H^u) can not be done easily. To obtain values for (H^x_k, H^u_k) we sample $\mathcal{X} \times \mathcal{U}$ and evaluate the integral expressions defining (H^x_k, H^u_k) pointwise. For this we use the numerical values for the example of the levitating ball: $Ts = 30ms, C = 1, m = 100g, \hat{x}_1 = 50mm$ and $\mathcal{X} = \{x : |\hat{x}_1 - x_1| \le 1mm \wedge |x_2| \le 105\frac{mm}{s}\}$, $\mathcal{U} = \{u : |\hat{u} - u| \le 10mA\}$. Using a total of 25 samples for the computation of (H^x, H^u) we obtain the invariant set shown in figure 2, which is less conservative than using fixed bounds on the nonlinearities as we will see in the next example. The algorithm terminates after 3 iterations.

V. Maximal Robust Positive Invariant Sets for Parametrised Disturbance

In this section we describe the computation of the MRPI set for (1) but using a disturbance set that is parametrised for scaling, i.e. $\mathcal{V}(\theta) = \{v : Gv \le (1+\theta)\mathbf{1}\} = (1+\theta)\mathcal{V}(0)$, for $\theta > -1$. We will be able to combine the uniform scaling of the disturbance set $\mathcal{V}(\theta)$ with non-uniform scaling of the input constraint set $\mathcal{U}(\alpha) = \{u : Fu \le (I + \text{diag}(\alpha_1,\ldots,\alpha_p))\,\mathbf{1}\}$. The necessity of uniform scaling of the disturbance constraints is due to a finesse we use to avoid solving multiparametric linear programs in every substep of the proposed iteration. A representation of the MRPI set of a system parametrised with respect to a scaling parameter allows us to study the system's

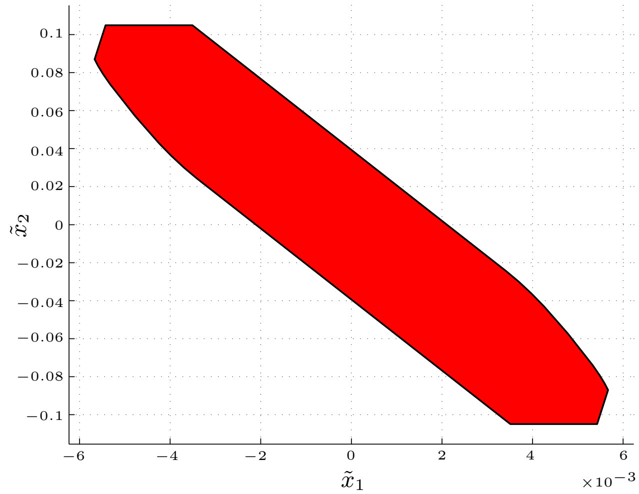

Fig. 2. The maximal robust positive invariant set for the levitating ball system.

sensitivity to *stronger/weaker* disturbance, the sensitivity to input constraints can be useful to choose particular actuators.

Remark 5.1: The set $\mathcal{V}(\theta)$ is nonempty and contains the origin for all $\theta > -1$ and hence the maximum

$$0 < \begin{array}{rl} \max & c^T v \\ \text{s.t.} & Gv \le (1+\theta)\mathbf{1} \\ & \theta > -1 \end{array}$$

is positive for any nonzero c.

In the following we describe an algorithm to compute the MRPI set $\mathcal{Z}^\infty$ contained in $\mathcal{Z} = \{(x,\theta) : \mathcal{F}_i x + \mathcal{G}_i\theta \le 1, \forall i \le m\}$. As in the state dependent case we iteratively introduce constraints *separating* points for which the successor state can lie outside the previous set, i.e. starting from $Z_0 = \mathcal{Z}$ we determine the first iterate by enforcing all individual constraints onto all possible successor states: $Z_1 = Z_0 \cap D_0$ where D_0 is defined by

$$\begin{aligned} D_0 &= \{\mathcal{F}_i(\Psi x + v) + \mathcal{G}_i\theta \le 1 \forall v \in \mathcal{V}(\theta), i \le m\} \\ &= \left\{ \mathcal{F}_i\Psi x + \begin{array}{rl} \max & F_i v \\ \text{s.t.} & Gv \le (1+\theta)\mathbf{1} \\ & \theta > -1 \end{array} + \mathcal{G}_i\theta \le 1 \forall i \le m \right\} \\ &= \left\{ \mathcal{F}_i\Psi x + (1+\theta)\underbrace{\begin{array}{rl} \max & F_i v \\ \text{s.t.} & Gv \le \mathbf{1} \end{array}}_{v^*_{0,i}} + \mathcal{G}_i\theta \le 1 \forall i \le m \right\} \\ &= \{\mathcal{F}_i\Psi x + (\mathcal{G}_i + v^*_{0,i})\theta \le 1 - v^*_{0,1} \forall i \le m\} \end{aligned} \tag{19}$$

Using the same principle we define $Z_{k+1} = Z_k \cap D_k$ with D_k given by

$$D_k = \{\mathcal{F}_i\Psi^{k+1}x + \left(\mathcal{G}_i + \sum_{0\le l\le k} v^*_{l,i}\right)\theta \le 1 - \sum_{0\le l\le k} v^*_{l,i}\, i \le m\} \tag{20}$$

where we use

$$v^*_{l,i} = \begin{array}{rl} \max & \mathcal{F}_i\Psi^l v \\ \text{s.t.} & Gv \le \mathbf{1} \end{array} \tag{21}$$

Notice that (21) can be represented in various ways:

$$\begin{array}{ll} \max & \mathcal{F}_i\Psi^l v \\ \text{s.t.} & v \in \bar{\mathcal{V}} \end{array} = \begin{array}{ll} \max & \mathcal{F}_i\tilde{v} \\ \text{s.t.} & \Psi^{-l}\tilde{v} \in \bar{\mathcal{V}} \end{array} = \begin{array}{ll} \max & \mathcal{F}_i v \\ \text{s.t.} & \tilde{v} \in \Psi^l\bar{\mathcal{V}} \end{array}$$

where $\bar{\mathcal{V}} = \mathcal{V}(0)$ for notational convenience. For any fixed $\hat{\theta} > -1$ we have the closed form description

$$\begin{aligned} Z_{k+1}|_{\hat{\theta}} &= Z_k|_{\hat{\theta}} \cap \left(\Psi^{-1}Z_k|_{\hat{\theta}} \ominus \Psi^{k-1}\mathcal{V}(\hat{\theta})\right) \\ &= \bigcap_{l\leq k}\left(\Psi^{-l}\mathcal{Z}|_{\hat{\theta}} \bigominus_{i\leq l-1} \Psi^i\mathcal{V}(\hat{\theta})\right). \end{aligned} \tag{22}$$

The iteration terminates when $Z_k \subseteq Z_{k+1}$. As in section III we will require $\mathcal{Z}|_{\hat{\theta}}$ to be contained in an observable band, i.e. that the set $\{x : \mathcal{F}x \leq \mathbf{1} - \mathcal{G}\hat{\theta}\} = \mathcal{Z}|_{\hat{\theta}} \subseteq \mathcal{B} = \{x : \Gamma x \leq \mathbf{1} \wedge -\Gamma x \leq \mathbf{1}\}$ for all $\hat{\theta} > -1$. We have the

Lemma 5.2: Let the system constraints be contained in a band $\mathcal{Z}|_{\hat{\theta}} \subseteq \mathcal{B} = \{x : \Gamma x \leq \mathbf{1} \wedge -\Gamma x \leq \mathbf{1}\}$ for any fixed $\hat{\theta}$, let the pair (Ψ, Γ) be observable and let $\mathcal{V}(0)$ be bounded, then $Z_N \subseteq Z_{N+1}$ for a finite N. Hence the MRPI set $\mathcal{Z}^\infty = Z_N$ is a finite polyhedron.

You wrote to highlight $\mathcal{V}(0)$ being bounded was an assumption. It is necessary so I don't know what to do with this.

Proof: The proof is, completely analogue to the proof of Lemma 3.1. First we argue that for each fixed $\hat{\theta}$ the set $Z_k|_{\hat{\theta}}$ becomes compact using the same observability argument as in Lemma 3.1. We then use the representation (22) to argue that D_k as in (20) grows exponentially and therefore contains any compact set after a finite number of iterations. The fact that θ was fixed to $\hat{\theta}$ does not change anything since our argument is constructed for a fixed matrix Γ where the rows had to be scaled by $\frac{1}{1-\mathcal{G}_i\hat{\theta}}$ and we could rescale them to accommodate any other choice of $\hat{\theta} > -1$. ■

We now have a way to scale the the disturbance set uniformly, as mentioned earlier we can extend the algorithm to accommodate non uniformly parametrised input constraints to accommodate more degrees of freedom for system analysis by simply using $\mathcal{Z} = \{(x, \theta, \alpha) : \mathcal{F}x + \mathcal{G}\theta + \mathcal{H}\alpha \leq \mathbf{1}\}$. This does not affect the algorithm since at each step of the iteration the elements of $\mathcal{H}$ remain unchanged. In the next section we will illustrate the algorithm.

VI. Example

In this section we compute the MRPI set for the levitating ball example system presented in section IV as well as for a purely numerical model to illustrate both the effectiveness of the afore presented algorithm for state dependent disturbances as well as the system analysis tool that is the parametrised MRPI set. First we present the parametrised MRPI set for the levitating ball, notice that in order to obtain comparable results we need to get fixed bounds on the effect of nonlinearities on each state, i.e. a fixed set containing the set $\mathcal{V}(x, u)$ in (18) for all $(x, u) \in \mathcal{X} \times \mathcal{U}$. Using the same constraint set $\mathcal{X} \times \mathcal{U}$ to determine the maximal and minimal values for nonlinear effects we obtain a constant set which is non symmetric around the origin due to nonlinearity. We

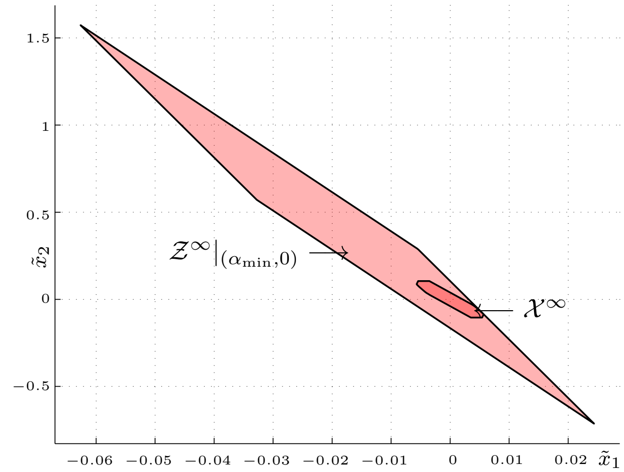

Fig. 3. Minimal scaling for MRPI set computed for constant disturbance which contains MRPI set for piecewise affine disturbances is $\alpha = 1.7555$. Note that since the nonlinearities of the system dynamics are not symmetrical around the equilibrium the MRPI set with fixed constraints on the disturbance is non-symmetric.

also introduce a scaling parameter α such that $\mathcal{U}(0) = \mathcal{U}$ for $\alpha = 0$. For this setup we obtain the MRPI set $\mathcal{Z}^\infty = \{(x, \theta, \alpha) : \Lambda_i^x x + \Lambda_i^\theta \theta + \Lambda_i^\alpha \alpha \leq \lambda_i \, \forall i \leq m_\infty\}$. Notice that since α was introduced as a scaling parameter for $\mathcal{U}(\alpha) = \{u : Fu \leq (1 + \alpha)\mathbf{1}\}$ all entries in $\mathcal{H}$ are non positive and remain unchanged throughout the computation of $\mathcal{Z}^\infty$ so that Λ^α has only non positive entries, i.e. increasing α will enlarge the parametrised MRPI set $\mathcal{Z}^\infty|_{\alpha_1} \subseteq \mathcal{Z}^\infty|_{\alpha_2}$ for $\alpha_1 \leq \alpha_2$.

First we want to compare the MRPI set $\mathcal{X}^\infty$ obtained with state dependent disturbance constraints with the parametrised one $\mathcal{Z}^\infty$. To do this we compute the scaling parameter $\alpha_{\min}$ such that $\mathcal{X}^\infty \subseteq \mathcal{Z}^\infty|_{(\theta=0,\alpha=\alpha_{\min})}$. We can do this by solving m_∞ linear programs

$$\gamma_i^* = \begin{cases} \max \Lambda_i^x x \\ \quad \text{s.t. } x \in \mathcal{X}^\infty \end{cases}$$

The minimal value for $\alpha_{\min}$ is then given by the maximal α satisfying $\gamma_i^* + \Lambda_i^\theta \cdot 0 + \Lambda_i^\alpha \alpha \leq \lambda_i$. Solving this we obtain the minimal value $\alpha_{\min} = 1.7555$ and the MRPI sets shown in Figure 3. The computation terminates after seven iterations and produces a polyhedron $\mathcal{Z}^\infty$ supported by $m_\infty = 10$ planes. Caution is advised when interpreting $\alpha > 0$, since the system is nonlinear extrapolations reveal very little insight, the only information we can safely extract is that the current set of inputs $\mathcal{U}(0) = \mathcal{U}$ is not large enough to cope with the perturbation set given by upper bounds of the nonlinear effects.

In the second example, we compute the parametrised MRPI set for the system

$$x^+ = \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{B} u + \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{D} w.$$

We use the constraint sets $\mathcal{U}(\alpha) = \{u : Fu \leq (I + \text{diag}(\alpha))\mathbf{1} \wedge -Fu \leq (I + \text{diag}(\alpha))\mathbf{1}\}$ and $\mathcal{V}(\theta) = D\mathcal{W}(\theta)$

where $\mathcal{W}(\theta) = \{w : |w_1| \leq 0.1(1+\theta) \wedge |w_2| \leq 0.15(1+\theta)\}$. The constraint matrix F is given by

$$F = \begin{pmatrix} 2 & 2 \\ 4 & 0 \\ 2 & -2 \end{pmatrix}.$$

All constraints were mainly chosen to be illustrative. We can now analyse the effect of changing individual constraints on the input by changing α as well as the effect of perturbations. To initialise the iteration we use $\mathcal{Z}(x, \theta, \alpha) = \{(x, \theta, \alpha) : Kx \in \mathcal{U}(\alpha) \wedge K_w x \in \mathcal{W}(\theta) \wedge \alpha, \theta > -\mathbf{1}\}$, with K and K_w being the solution to the aforementioned semidefinite program $x^T P x - ((A+BK)x+Dw)^T P((A+BK)x+Dw) \geq x^T(Q + K^T RK)x - \gamma^2 w^T w$, i.e. $K_w = (\gamma^2 - D^T PD)D^T P(A+BK)$. We use $Q = \text{diag}(2, 1)$ and $R = I$. Using these numerical values we determine the MRPI set $\mathcal{Z}^\infty$. After 22 iterations the algorithm terminates producing a total of 61 facets. Notice that we not only know the sign of the elements of $\mathcal{H}$, but we also know that each facet of $\mathcal{Z}(x, \theta, \alpha)$ depends at most on one α_i, the algorithm does not affect the elements corresponding on α and reduction methods can not reduce rows depending on two individual variables, therefore we know that Λ^α will have at most one nonzero entry per row. We can therefore compute which α_i will change the MRPI set the most. This can be done by element wise calculating

$$\max_i \frac{-\Lambda_i^\alpha}{\|\Lambda_i^x\|_2}.$$

In our example the greatest sensitivity corresponds to α_3 with a numerical value of 2.6458×10^9, i.e. the third input constraint. Analogous to the first example we might want to know how much disturbance the closed loop system can take such that a given set is contained in the MRPI set $\mathcal{C} \subseteq \mathcal{Z}^\infty|_{(\theta,0)}$. We used the non-positivity of $\mathcal{H}$ to argue the non-positivity of Λ^α, however we can not use a similar argument for Λ^θ, in fact it is easy to see that both signs are likely to be present in Λ^θ. The condition $\theta > -1$ implies that Λ^θ has at least one negative element, on the other hand the elements of Λ^θ are given by $\mathcal{G}_i + \sum_{l \leq k} v_{l,i}^*$. We know $\mathcal{G}$ and hence we know that most entries are zero, so that Λ^θ has positive entries. Let (Λ^1, λ^1) and (Λ^2, λ^2) denote all rows such that $\Lambda^{1,\theta} > 0$ and $\Lambda^{2,\theta} \leq 0$. This implies that for any fixed $\hat{\alpha}$ the set $\mathcal{Z}^\infty|_{\hat{\alpha}}$ is compact, since $\mathcal{Z}^\infty|_{(\hat{\alpha},\hat{\theta})}$ is compact and there exists a maximal $\theta_{\max}$ such that $\Lambda^{1,x}x + \Lambda^{1,\alpha}\hat{\alpha} + \Lambda^{1,\theta}\theta \leq \lambda^1$ can be satisfied analogue there is a minimal $\theta_{\min}$. We can solve the optimisation programs

$$\gamma_i = \left\{ \begin{array}{ll} \max & \Lambda^{1,x} x \\ \text{s.t.} & x \in \mathcal{C} \end{array} \right. \quad \delta_i = \left\{ \begin{array}{ll} \max & \Lambda^{2,x} x \\ \text{s.t.} & x \in \mathcal{C} \end{array} \right. .$$

The extremal values for which $\mathcal{C}$ is contained in the MRPI set are given by the smallest $\theta_{\max}$ satisfying $\gamma_i + \Lambda_i^{1,\alpha}\hat{\alpha} + \Lambda_i^{1,\theta}\theta_{\max} \leq \lambda_i^1$ and the largest $\theta_{\min}$ satisfying $\delta_i + \Lambda_i^{2,\alpha}\hat{\alpha} + \Lambda_i^{2,\theta}\theta_{\min} \leq \lambda_i^2$ The numerical values for the example are given by $[\theta_{\min}, \theta_{\max}] = [-0.9999, 6.2582]$. A two and three dimensional illustration of the parametrised MRPI set is given in Figure 4 and 5 respectively.

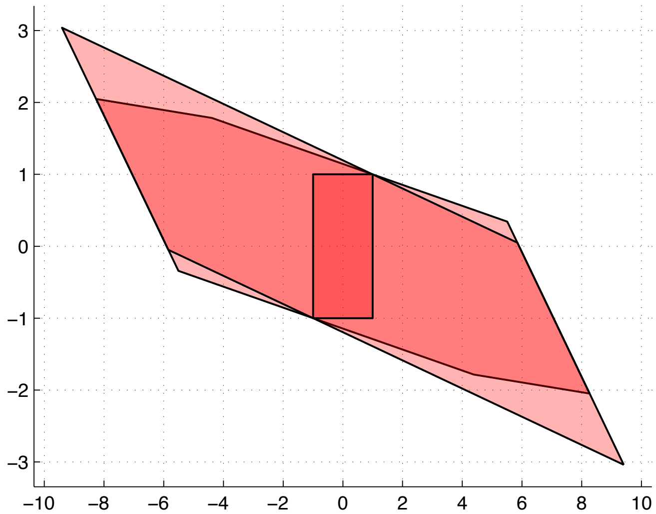

Fig. 4. The extremal MRPI sets $\mathcal{Z}^\infty|_{(\mathbf{0},\theta_{min})}$ and $\mathcal{Z}^\infty|_{(\mathbf{0},\theta_{\max})}$ containing the unit box.

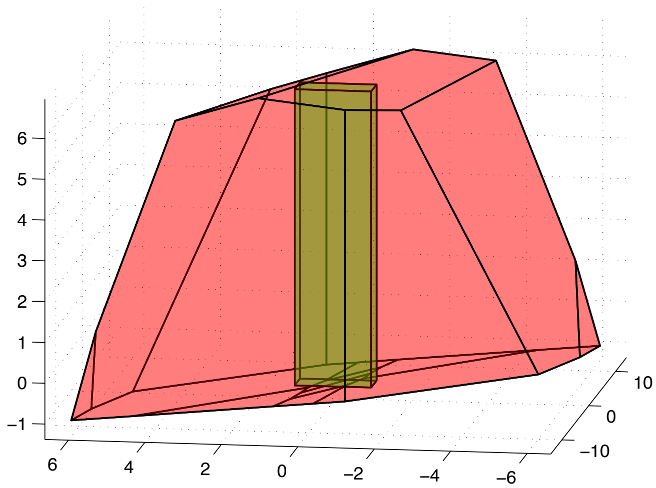

Fig. 5. The parametrised MRPI set for $\alpha = \mathbf{0}$. Containing the unit box in the interior between $\theta_{\min}$ and $\theta_{\max}$.

VII. Conclusions

In this paper we discussed extensions to existing computational methods to determine MRPI sets for linear systems subject to additive disturbance in two cases. The case of state dependent case which can be applied to determine approximations to MRPI sets for linearised nonlinear systems as was shown for the example of a levitating ball system. We also introduced the computation of MRPI sets using scaling parameters which allows various system analyses using uniform scaling for the disturbance sets and non-uniform scaling for the input constraints. The effectiveness of state dependent disturbance sets was demonstrated by comparing the two MRPI methods.

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