A Final Coalgebra Theorem

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Abstract

We prove that every set-based functor on the category of classes has a final coalgebra. This result strengthens the final coalgebra theorem announced in the book "Non-well-founded Sets", by the first author.

1 Introduction

The theorem of this note is an improvement of a result that was first stated, but not proved in its full generality, in [NWFS]. The original inspiration for the result came from [SCCS]. There a language of infinitary expressions called, *agents*, is made into a labelled transition system via an operational semantics. Different agents can exhibit the same operational behaviour. A notion of *bisimulation relation* between agents is used to define the maximal bisimulation equivalence relation which captures formally the intuitive behavioural equivalence relation. The quotient of the class of agents with respect to this equivalence relation can itself be made into a labelled transition system which can be viewed as a mathematical interpretation of the language of agents. If the labels for a labelled transition system are taken from a set *Act* then the system may be viewed as a coalgebra for the functor $pow(Act \times -)$. Here *pow* is the natural covariant power class functor which associates with each class X the class $powX$ of its subsets. It turns out that the quotient of the coalgebra of agents by the maximal bisimulation relation can be characterised up to isomorphism as a final coalgebra for the functor. So this basic construction of Milner's is providing a proof of the existence of a final coalgebra for the functor.

The final coalgebra theorem is a general result about the existence of final coalgebras for a very general collection of functors on the category of classes. The functor *pow* itself is essentially a special case of $pow(Act \times -)$ where *Act* is a singleton set. The final coalgebras for *pow* give the full models for $ZFC^- + AFA$, a version of axiomatic set theory in which the Foundation Axiom, FA, is replaced by a dual Anti-Foundation Axiom, AFA, which expresses the existence of many non-well-founded sets.

The notion of a final coalgebra for a functor can be used to characterise up to isomorphism many other interesting mathematical structures. In this respect it can play the same kind of role as the dual notion of an initial algebra for a functor. For example associated with each single sorted signature, Ω, is a functor, F_Ω, on the category of classes whose initial algebras are the algebras isomorphic to the algebra of terms of the signature. Terms can be viewed as well-founded trees and if the well-foundedness restriction is dropped from the trees then the resulting algebra of possibly non-well-founded trees is essentially a final coalgebra for F_Ω. In this note we are not concerned to explore such instances of the final coalgebra theorem but rather give a proof of the result itself.

The reader may wonder why the category of classes is being used here rather than the more familiar category of sets. The reader may even be concerned about the foundational correctness of the use of such a superlarge category. The point is that the powerset functor on the category of sets cannot have a final coalgebra for cardinality reasons. And it is functors involving *pow* that provide the most interesting instances of final coalgebras, such as in connection with SCCS and set theory. The possible foundational problems concerning the category of classes can be overcome in a reasonably standard way, but it seems inappropriate to focus on those problems in this note. Use of the category of classes could be avoided by introducing cardinality restrictions at suitable points. But such a presentation would surely obscure the ideas.

A key issue in generalising Milner's quotient construction was the problem of formulating a suitable generalisation, to a coalgebra for a functor, of the notions of bisimulation and bisimulation equivalence on a labelled transition system. The notions we use in this note are that of pre-congruence and congruence, as defined in section 4. At the time [NWFS] was written the author of that work had in mind the slightly different notions of bisimulation and bisimulation equivalence on a coalgebra as defined in section 6 of this note. Although every bisimulation is a pre-congruence, a suitable converse cannot be proved without making an additional assumption about the functor. In [NWFS] the additional assumption used was that the functor preserves weak pullbacks. The relationship between bisimulations and pre-congruences is worked out in section 6.

The Final Coalgebra Theorem is formulated in the next section along with the necessary definitions and some key lemmas, including the small subcoalgebra lemma and the Main Lemma. The small subcoalgebra lemma is proved in section 3, while section 4 is concerned with the general construction of a maximal congruence. The proof of the final Coalgebra Theorem is completed in section 5 with a proof of the Main Lemma.

The final section of this note is concerned with a generalisation of the final coalgebra theorem obtained by replacing the category of classes by a superlarge category satisfying some fairly weak assumptions true for the category of classes.

2 The Theorem

A *coalgebra* for an endo-functor Φ on a category consists of (A, α) where A is an object of the category and $\alpha : A \to \Phi A$. A *homomorphism* $\pi : (A, \alpha) \to (B, \beta)$ between two coalgebras (A, α) and (B, β) is a map $\pi : A \to B$ such that $\beta \circ \pi = (\Phi\pi) \circ \alpha$. The coalgebras and homomorphisms form a category. So we can formulate the notion of a *final coalgebra*. This is a coalgebra (A, α) such that for any coalgebra (B, β) there is a unique homomorphism $(B, \beta) \to (A, \alpha)$.

The superlarge category of classes is the category whose objects are classes and whose maps are all class functions between classes. An endo-functor Φ on this category is called *set-based* if for each class A and each $a \in \Phi A$ there is a set $A_0 \subseteq A$ and $a_0 \in \Phi A_0$ such that $a = \Phi\iota_{A_0,A}a_0$, where $\iota_{A_0,A}$ is the inclusion map $A_0 \hookrightarrow A$. Our aim is to give a proof of the following result.

Theorem 2.1 (The Final Coalgebra Theorem.) *Every set-based functor has a final coalgebra.*

This is an improvement on the corresponding result in [NWFS]. There the functor was required to be standard and preserve weak pullbacks. A functor Φ is *standard* if and only if it is set-based and preserves inclusion maps; i.e. if $A \subseteq B$ then $\Phi A \subseteq \Phi B$ and $\Phi\iota_{A,B} = \iota_{\Phi A,\Phi B}$.

So the present theorem no longer requires the assumptions that the functor preserve inclusion maps and weak pullbacks. The assumption that the functor is set-based is not significantly more general than the stronger assumption that the functor is standard. Nevertheless it has the advantage of being category theoretic. In fact, it is not hard to show that a functor is set-based if and only if it is almost naturally isomorphic (i.e., naturally isomorphic on non-empty classes) to a standard functor. The remaining assumption that the functor is set-based will only play an explicit role in the proof of the following key lemma.

A coalgebra (A_0, α_0) is a *subcoalgebra* of (A, α) if $A_0 \subseteq A$ and the inclusion map $\iota_{A_0,A}$ is a homomorphism $\iota_{A_0,A} : (A_0, \alpha) \hookrightarrow (A, \alpha)$. The subcoalgebra is *small* if A_0 is a set.

Lemma 2.2 (The Small Subcoalgebra Lemma.) *If (A, α) is a coalgebra and X is a subset of A then $X \subseteq A_0$ for some small subcoalgebra (A_0, α_0) of (A, α).*