Lemma : $o(1) = O(1)$

Proof : On the one hand,

$$o(1) \Leftrightarrow f \to 0,$$

on the other hand,

$$O(1) \Leftrightarrow |f| < A.$$

Now, if $f \to 0$, then $|f| < A$, whence

$$o(1) = O(1).$$

□

Lemma : $O(1) \neq o(1)$

Proof :

$$O(1) \neq o(1) \Leftrightarrow |f| < A \nRightarrow f \to 0$$

That's indeed the case if $f = B < A$. □

Lemma : $O(1) = o(x)$

Proof : That means

$$|f| < A \Rightarrow \frac{f}{x} \to 0$$

which is trivially true. □

Lemma :

$$f \sim \varphi \Leftrightarrow f = \varphi(1 + o(1))$$

Proof :

$$f \sim \varphi \Leftrightarrow \frac{f}{\varphi} \to 1 \Leftrightarrow \frac{f}{\varphi} - 1 \to 0 \Leftrightarrow \frac{f - \varphi}{\varphi} \to 0 \overset{\text{defn}}{\Leftrightarrow} f - \varphi = o(\varphi) \Leftrightarrow f = \varphi + o(\varphi) \Leftrightarrow f - \varphi = \varphi o(1)$$

because

$$g = o(\varphi) \Leftrightarrow \frac{g}{\varphi} \to 0 \Leftrightarrow \frac{g}{\varphi} = o(1) \Leftrightarrow g = \varphi o(1)$$

$\square$