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On Performance of Sparse Fast Fourier Transform Algorithms Using the Flat Window Filter

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ABSTRACT The problem of computing the Sparse Fast Fourier Transform(sFFT) of a K-sparse signal of size N has received significant attention for a long time. The first stage of sFFT is hashing the frequency coefficients of $\hat{x}$ into $B(\approx K)$ buckets named frequency bucketization. The process of frequency bucketization is achieved through the use of filters: Dirichlet kernel filter, aliasing filter, flat filter, etc. The frequency bucketization through these filters can decrease runtime and sampling complexity in low dimensions. It is a hot topic about the sFFT Algorithms using the flat filter because of its convenience and efficiency since its emerge and widely application. The next stage of sFFT is the spectrum reconstruction by identifying frequencies that are isolated in their buckets. Up to now, there are more than thirty different sFFT algorithms using the sFFT idea as mentioned above by their unique methods. An important question now is how to analyze and evaluate the performance of these algorithms in theory and practice. In this paper, it is mainly discussed about the sFFT algorithms using the flat filter. In the first part, the paper introduces the techniques in detail including two types of frameworks, five different methods to reconstruct spectrum and corresponding algorithms. We get the conclusion of the performance of these five algorithms including runtime complexity, sampling complexity and robustness in theory. In the second part, we make three categories of experiments for computing the signals of different SNR, different N, and different K by a standard testing platform and record the run time, percentage of signal sampled and L_0, L_1, L_2 error both in exactly sparse case and general sparse case. The result of experiments is consistent with the inferences obtained in theory. It can help us to optimize these algorithms and use them correctly in the right areas.

INDEX TERMS Sparse Fast Fourier Transform(sFFT), flat window filter, sub-linear algorithms

I. INTRODUCTION

The Discrete Fourier Transform(DFT) is one of the most important and widely used techniques in signal processing and mathematical computing. The most popular algorithm to compute the DFT is the fast Fourier transform(FFT) invented by Cooly and Tukey. The algorithm can compute the DFT of a signal of size N in $O(N\log N)$ time and use $O(N)$ samples. FFT greatly simplifies the operation process, however, with the emergence of big data problems, the FFT is no longer fast enough. Furthermore, sometimes it is hard to acquire a sufficient amount of data to compute the DFT. These two problems become the the major computational bottleneck in many applications. It motivates the need for the algorithms that can compute the Fourier transform in sublinear time and that use only a subset of the input data. People thought of many ideas in order to realize such an algorithm. Later, they focused on the study of the characteristics of the signal itself. The research found that a large number of signals are sparse in the frequency domain, only K frequencies are non-zeros or are significantly large. This feature is common and inherent in signals covers many fields (e.g. audio, video data, medical image, etc). In this case, when $K << N$,one can retrieve the information with high accuracy using only the coefficients of the K most significant frequencies. So the sFFT has been proposed and achieved good results. The research of sFFT has been a hot topic in signal processing research since its birth, it was named one of the 10 Breakthrough Technologies in MIT Technology Review in 2012.

The firsts stage of sFFT algorithm is bucketization such that the value of the bucket is the sum of the values of the frequency coefficients that hash into the bucket. The number of buckets is denoted by B and the size of one bucket is denoted by L. The process of bucketization is achieved through the use of filters. The effect of Dirichlet kernel filter is to make the signal convoluted a rectangular window in the time domain, it can be equivalent to the signal multiply a Dirichlet kernel window of size $L(L << N)$ in the frequency domain. The typical application using the Dirichlet kernel filter is AAFFT algorithm. The effect of aliasing filter is to make the signal multiply a comb window in the time domain, it can be equivalent to the signal convoluted a comb window of size $B(\approx K)$ in the frequency domain. The typical application using the aliasing filter is FFAST algorithm. The effect of flat filter is to make the signal multiply a mix window in the time domain, it can be equivalent to the signal convoluted a flat window of size $L(L << N)$ in the frequency domain. The typical application using the flat filter is sFFT1.0 algorithm. After bucketization, the algorithm then focuses on the non-empty buckets and computes the positions and values of the large frequency coefficients in those buckets in what we call the spectrum reconstruction or identifying frequencies. As we can see as follows, there are more than thirty algorithms using the sFFT idea and more than ten sFFT algorithms using the flat filter. A central question now is how to analyze and evaluate the performance of these algorithms for computing signals by the compare of themselves or other types of algorithms. It should be proved whether the runtime complexity, sampling complexity and robustness performance are consistent with the theory or not. Are there any better ways to improve these algorithms when using it in practice? The results of these performance analysis is the guide for us to optimize these algorithms and use them correctly in the different areas.

The first sFFT algorithm with sub-linear runtime and sub-sampling property was given in [1], which gave a randomized algorithm with runtime and sampling complexity $O(K^2 poly(\log N))$. This was later improved to $O(K poly(\log N))$ [2], [3] through the use of binary search technique for spectrum reconstruction and the use of unequally-spaced FFTs. The algorithm is so called Ann Arbor fast Fourier transform (AAFFT), the version of them are AAFFT0.5 and AAFFT0.9.

In [4], [5] an algorithm so called Fast Fourier Aliasing-based Sparse Transform(FFAST) which focuses on exactly K-sparse signals was given. Their approach is based on downsampling of the input signal using a constant number of co-prime downsampling factors guided by CRT. These aliasing patterns of different downsampled signals are formulated as parity-check constraints of good erasure-correcting sparse-graph codes. FFAST costs $O(K\log K)$ to compute the exactly signals and only use $O(K)$ samples. In [6], [7] the author adapt the FFAST framework to the case where the time-domain samples are corrupted by a white Gaussian noise. The author show that the extended noise robust algorithm R-FFAST computes DFT using $O(K\log^3 N)$ samples, in $O(K\log^4 N)$ runtime. These two algorithms perform well when N is a product of some smaller prime numbers.

In [8] an algorithm is proposed and so called sFFT by downsampling in the time domain(sFFT-DT). The idea behind sFFT-DT is to downsample the original input signal first and then all subsequent operations are conducted on the downsampled signals. To overcome aliasing problem, the author consider the locations and values of K non-zero entries as variables and the aliasing problem is found to be equivalent to MPP problem, which can be solved via orthogonal polynomials or syndrome decoding with a CS(compressive sensing) based solver.

In [9] a deterministic algorithm so called Gopher Fast Fourier Transform(GFFT) which based on CRT was given. The GFFT is a aliasing-based search algorithm. The approximation error bounds in [9] are further improved in [10]. Later, an algorithm so called Christlieb Lawlor Wang Sparse Fourier Transform(CLW-SFT), which used phase encoding method was given in [11]. The noiseless version of this algorithm is an adaptive algorithm [12] which has runtime $O(K\log K)$. In [11] the author developed this algorithm by using the multiscale error-correcting method to cope with high-level noise with runtime $O(K^2\log K)$. In [13] the author evaluate the performance of DMSFT (generated from GFFT) and CLW-DSFT (generated from CLW-SFT) and compare their runtime and robustness characteristics with other algorithms. These four algorithms all have a hypothesis that the algorithms can sample anywhere they want.

In [14] an algorithm is proposed and so called Deterministic Sparse FFT(DSFFT). In the algorithm K needs not to be known in advance but will be determined during the algorithm. The method is based on the divide-and-conquer approach and may require the solution of a Vandermonde system of size at most $K \times K$ at each iteration step j if $K^2 < 2^j$.

The sFFT algorithms using the flat window filter so called sFFT1.0-sFFT4.0 that can compute the exactly K-sparse signals in time $O(K\log N)$ and general K-sparse signals in time $O(K\log N\log(N/K))$ were given in [15], [16]. These algorithms leverage characteristic of flat window filter. The sFFT1.0 and sFFT2.0 algorithms can identify and estimate the K largest coefficients in one shot. The sFFT3.0 algorithm can estimate the position by using only two samples of the filtered signal inspired by the frequency offset estimation in the exactly sparse case. Later, a new robust algorithm so called Matrix Pencil FFT(MPFFT) was proposed in [17] on the basic of sFFT3.0 algorithm. The major new ingredient is a mode collision detector based on the matrix pencil method. This method enables the algorithm to use fewer samples of the input signal.

In [18] the paper proposes an overview of sFFT and summarize a three-step approach in the stage of spectrum reconstruction and provides a standard testing platform that can be used to evaluate different sFFT algorithms. There are also some researches try to conquer the sFFT problem from a lot of aspects: complexity [19], [20], performance [21], [22], software [23], [24], higher dimensions [25], [26], implementation [27], hardware [28] and special setting [29], [30] perspectives.

The identification of different sFFT algorithms can be know through a brief analysis as above. The Dirichlet kernel filter is not efficient because it only bins some frequency coefficients into one bucket one time. As to the aliasing filter it is difficult to solve the worst case because there may be many frequency coefficients in the same bucket accidentally if B only can be supposed as a power of two by the reason of the scaling operation is of no use. In comparison to them, using the flat filter is very convenience and efficiency.

This paper is structured as follows. Section II and Section III provides a brief overview of the sFFT technique. Section IV introduce and analyze two frameworks and five spectrum reconstruction methods of algorithms. In the one shot framework, sFFT1.0 and sFFT2.0 algorithm use voting method with the help of the stochastic characteristics. In the framework of iteration the sFFT3.0 and sFFT4.0 algorithm use phase encoding method with the help of the time shift characteristics. The MPSFT algorithm use matrix pencil method with the help of prony characteristics. In section V, we do three categories of comparison experiments. The first kind of experiment is to compare them with each other. The second is to compare them with other sFFT algorithms. The third is to compare them of optimization with them without optimization. The analyze of the experiment results satisfy theoretical inference.

II. NOTATION

In this section, we initially present some notation and basic definitions about sFFT. We use $\omega_N = e^{-2\pi \mathbf{i}/N}$ as the N-th root of unify. Let $\mathbf{F}_N \in \mathbb{C}^{N\times N}$ be the DFT matrix of size N defined as follows:

$$\mathbf{F}_N[j,k] = \frac{1}{N}\omega_N^{jk} \tag{1}$$

The DFT of a vector $x \in \mathbb{C}^N$ (consider a signal of size N, where N is a power of two) is a vector $\hat{x} \in \mathbb{C}^N$ defined as follows:

$$\hat{x} = \mathbf{F}_N x \tag{2}$$

$$\hat{x}_i = \frac{1}{N}\sum_{j=0}^{N-1} x_j \omega_N^{ij} \tag{3}$$

It is necessary to consider the inverse of the DFT matrix above. $\mathbf{F}_N^{-1} \in \mathbb{C}^{N\times N}$ defined as follows:

$$\mathbf{F}_N^{-1}[j,k] = \omega_N^{-jk} \tag{4}$$

The inverse DFT of is a vector x defined as follows:

$$x = \mathbf{F}_N^{-1}\hat{x} = \mathbf{F}_N^{-1}(\mathbf{F}_N x) \tag{5}$$

$$x_i = \sum_{j=0}^{N-1} \hat{x}_j \omega_N^{-ij} \tag{6}$$

For $x_{-i} = x_{N-i}$, we may define convolution as follows:

$$(x * y)_i = \sum_{j=0}^{N-1} x_j y_{i-j} \tag{7}$$

For coordinate-wise product $(xy)_i = x_i y_i$ and the DFT of xy is performed as described in Equation 8:

$$\widehat{xy} = \hat{x} * \hat{y} \tag{8}$$

For exactly signals, $\hat{x}$ is exactly K-sparse if it has exactly K non-zero frequency coefficients while the remaining $N-K$ coefficients are zero. For general signals, $\hat{x}$ is general K-sparse if the largest K frequency coefficients $>>$ remaining $N-K$ coefficients. The goal of the sFFT is to recover a K-sparse approximation $\hat{x}$ by finding frequency positions f and estimating values $\hat{x}_f$ of the K largest coefficients.

III. TECHNIQUES

In this section, we start with an overview of the techniques that we will use in the sFFT.

A. RANDOM SPECTRUM PERMUTATION

The random permutation include two operations, one is shift operation, another is scaling operation. Let $\tau \in \mathbb{R}$ be the offset parameter. Let matrix $\mathbf{S}_\tau \in \mathbb{R}^{N\times N}$ representing the shift operation, is defined as follows:

$$\mathbf{S}_\tau[j,k] = \begin{cases} 1, & j-\tau \equiv k (\mathrm{mod} N) \\ 0, & \text{o.w.} \end{cases} \tag{9}$$

Let $\sigma \in \mathbb{R}$ be the scaling parameter. Let matrix $\mathbf{P}_\sigma \in \mathbb{R}^{N\times N}$ representing the scaling operation, is defined as follows:

$$\mathbf{P}_\sigma[j,k] = \begin{cases} 1, & \sigma j \equiv k (\mathrm{mod} N) \\ 0, & \text{o.w.} \end{cases} \tag{10}$$

Suppose $\sigma^{-1} \in \mathbb{R}$ exists mod N, σ^{-1} satisfies $\sigma^{-1}\sigma \equiv 1(\mathrm{mod} N)$. If a vector $x' \in \mathbb{C}^N$, $x' = \mathbf{S}_\tau \mathbf{P}_\sigma x$, such that:

$$\begin{aligned} x'_i &= x_{\sigma(i-\tau)} \\ x'_{\sigma^{-1}i+\tau} &= x_i \end{aligned} \tag{11}$$

The random permutation isolates spectral components from each other, and it is performed as follows: if $x' = \mathbf{S}_\tau \mathbf{P}_\sigma x$, such that:

$$\begin{aligned} \hat{x'}_{\sigma i} &= \hat{x}_i \omega^{\sigma\tau i} \\ \hat{x'}_i &= \hat{x}_{\sigma^{-1} i} \omega^{\tau i} \end{aligned} \tag{12}$$

B. WINDOW FUNCTION

The window funciton is a mathematical tool and can be seen as a matrix multiply the original signal. We introduce three filters used in the sFFT algorithm mentioned in this paper.

The first filter is the frequency aliasing filter. Through the filter, the signal in the time domain is subsampled such that the corresponding signal in the frequency domain is aliased. Let $L \in \mathbb{Z}^+$ be the subsampling factor. Let $B \in \mathbb{Z}^+$ be the subsampling number. Let matrix $\mathbf{D}_L \in \mathbb{R}^{B\times N}$, representing the subsampling operator, is defined as follows:

$$\mathbf{D}_L[j,k] = \begin{cases} 1, & k = jL \\ 0, & \text{o.w.} \end{cases} \tag{13}$$

Let vector $y_{L,\tau}, \hat{y}_{L,\tau} \in \mathbb{C}^B$, be the filtered signal obtained by shift operation and aliasing filter. If $y_{L,\tau} = \mathbf{D}_L \mathbf{S}_\tau x,\ \hat{y}_{L,\tau} =$

$\mathbf{F}_B\mathbf{D}_L\mathbf{S}_\tau x$, we get formula (14) and if τ=0 we get formula (15).

$$\begin{aligned}\hat{y}_{L,\tau}[i] =& \hat{x}[i]\omega^{-\tau i} + \hat{x}[i+B]\omega^{-\tau(i+B)} + ... \\ & \hat{x}[i+(L-1)B]\omega^{-\tau(i+(L-1)B)}\end{aligned} \tag{14}$$

$$\hat{y}_{L,0}[i] = \hat{x}[i] + \hat{x}[i+B] + ... + \hat{x}[i+(L-1)B] \tag{15}$$

The second filter is the frequency flat filter. We use a filter vector G that is concentrated both in time and frequency domain, G is zero except at a small number of time coordinates with $\text{supp}(G) \subseteq [-w/2, w/2]$ and its Fourier transform $\hat{G}$ is negligible except at a small fraction L ($\approx \varepsilon N$) of the frequency coordinates (the pass region). The paper [15] claim there exists a standard window function $G(\varepsilon, \varepsilon', \delta, w)$ satisfies the formula (16). The filter can be obtained by convoluted a Gassian function with a box car window function and $\text{supp}(G) = w = O(1/\varepsilon \log(1/\delta)))$. One can potentially use a Dolph-Chebyshev window function with minimal big-Oh constant. In this paper, we use filter $G \in \mathbb{C}^N$ be a $(L/N, L/2N, \delta, w)$ flat window. The width of the filter in the time domain is denoted by w, the width of the passband region in the frequency domain is denoted by L, the number of buckets is denoted by B and $B = N/L$.

$$\begin{aligned}\hat{G}_i \in [1-\delta, 1+\delta] \; for \; i \in [-\epsilon' N, \epsilon' N] \\ \hat{G}_i \in [-\delta, \delta] \; for \; i \notin [-\epsilon N, \epsilon N] \\ \hat{G}_i \in [0,1] \text{ for all } i\end{aligned} \tag{16}$$

Let matrix $\mathbf{Q}_L \in \mathbb{C}^{N \times N}$ be a diagonal matrix whose diagonal entries represent filter coefficients in the time domain, is defined as follows:

$$\mathbf{Q}_L[j,k] = \begin{cases} G_j, & j = k \\ 0, & \text{o.w.} \end{cases} \tag{17}$$

The third filter is the frequency subsampled filter. Through the filter, the signal in the time domain is aliased such that the corresponding signal in the frequency domain is subsampled. Let matrix $\mathbf{U}_L \in \mathbb{R}^{B \times N}$ represents the aliasing operator as follows:

$$\mathbf{U}_L[j,k] = \begin{cases} 1, & j - k \equiv 0(\text{mod}B) \\ 0, & \text{o.w.} \end{cases} \tag{18}$$

Let vector y_L, $\hat{y}_L \in \mathbb{C}^B$, be the filtered signal obtained by subsampled filter. If $y_L = \mathbf{U}_L x$, $\hat{y}_L = \mathbf{F}_B\mathbf{U}_L x$ we get formula(19).

$$\hat{y}_L[i] = \hat{x}[iL] \tag{19}$$

C. FREQUENCY BUCKETIZATION

The process of bucketization in this paper is achieved through the use of flat filter, subsampled filter and shift operation, scaling operation. It can be equivalent to the signal multiply $\mathbf{F}_B\mathbf{U}_L\mathbf{Q}_L\mathbf{S}_\tau\mathbf{P}_\sigma$. The filtered signal is performed as follows:

If $y_{L,\tau,\sigma} = \mathbf{U}_L\mathbf{Q}_L\mathbf{S}_\tau\mathbf{P}_\sigma x$, $\hat{y}_{L,\tau,\sigma} = \mathbf{F}_B\mathbf{U}_L\mathbf{Q}_L\mathbf{S}_\tau\mathbf{P}_\sigma x$, such that:

$$\begin{aligned}\hat{y}_{L,\tau,\sigma}[0] \approx & \hat{G}_{\frac{L}{2}}\hat{x}_{\sigma^{-1}\left(-\frac{L}{2}\right)}\omega_N^{\tau\left(-\frac{L}{2}\right)} + \ldots \\ & \hat{G}_{-\frac{L}{2}+1}\hat{x}_{\sigma^{-1}(\frac{L}{2}-1)}\omega_N^{\tau(\frac{L}{2}-1)} \\ \hat{y}_{L,\tau,\sigma}[1] \approx & \hat{G}_{\frac{L}{2}}\hat{x}_{\sigma^{-1}\left(\frac{L}{2}\right)}\omega_N^{\tau\left(\frac{L}{2}\right)} + \ldots \\ & \hat{G}_{-\frac{L}{2}+1}\hat{x}_{\sigma^{-1}(\frac{3L}{2}-1)}\omega_N^{\tau(\frac{3L}{2}-1)} \\ \hat{y}_{L,\tau,\sigma}[i] \approx & \hat{G}_{\frac{L}{2}}\hat{x}_{\sigma^{-1}\left(\frac{(2i-1)L)}{2}\right)}\omega_N^{\tau\left(\frac{(2i-1)L}{2}\right)} + \ldots \\ & \hat{G}_{-\frac{L}{2}+1}\hat{x}_{\sigma^{-1}(\frac{(2i+1)L}{2}-1)}\omega_N^{\tau(\frac{(2i+1)L}{2}-1)}\end{aligned} \tag{20}$$

If the set I is a set of coordinates position, the position $f = (\sigma^{-1}u)\text{mod}N \in I$, suppose there is no hash collision in the bucket i, i = round(u/L), round() means to make decimals rounded. Through formula(20), we can get the formula(21)

$$\begin{aligned}\hat{y}_{L,\tau,\sigma}[i] \approx \hat{G}_{iL-u}\hat{x}_{\sigma^{-1}u}\omega_N^{\tau u} \text{ for } u \in [\frac{(2i-1)L}{2}, \frac{(2i+1)L}{2} - 1] \\ \hat{x}_f \approx \hat{y}_{L,\tau,\sigma}[i]\omega_N^{-\tau u}/\hat{G}_{iL-u} \text{ for } u = \sigma f \text{mod}N, i = \text{round}(u/L)\end{aligned} \tag{21}$$

As we see above, frequency bucketization includes 3 steps: random spectrum permutation($x' = \mathbf{S}_\tau\mathbf{P}_\sigma x$, it cost 0 runtime and unknow samples), flat window filter($x'' = \mathbf{Q}_L x'$, it cost w runtime and w samples), fourier transform of the aliasing signal($\hat{y}_{L,\tau,\sigma} = \mathbf{F}_B\mathbf{U}_L x''$, it cost $B\log B$ runtime and 0 samples). So totally frequency bucketization one round cost $w + B\log B$ runtime and w samples

IV. ALGORITHMS ANALYSIS

As mentioned above the goal of frequency bucketization is to decrease runtime and sampling complexity in advantage of low dimensions, after bucketizationafter the filtered signal $\hat{y}_{L,\tau,\sigma}$ can be obtained by original signal x. In this section, we introduce two frameworks, five methods and corresponding algorithms to recover the spectrum $\hat{x}$ of the filtered signal $\hat{y}_{L,\tau,\sigma}$ by their own way.

A. THE SFFT1.0 ALGORITHM BY ONE-SHOT FRAMEWORK

The first framework can directly reconstruct the spectrum by one-shot, does not need iteration. The process to reconstruct the spectrum of sFFT1.0 algorithm includes two kinds of rounds, the first are location rounds and another are estimation rounds. Every location round one time generate a list of candidate coordinates I_r. Candidate coordinates $i \in I_r$ have a certain probability of being indices of one of the K significant coefficients in spectrum. By running multiple rounds, this probability can be increased so it is certain to vote the candidate coordinates with a high probability after $R(\approx \log N)$ times' rounds. The next step is to do estimation rounds used to exactly determine the value of identified frequency $\hat{x}_f$ isolated in the bucket in the reason of the value of the bucket is approximate the frequency that identified in the bucket if there is no hash collision. The block diagram of the sFFT algorithms system of one-shot framework is shown in Figure 1. We explain the details as below.

Stage1 Bucketization: Run R times' round for set $\tau = \{\tau_1, \tau_2, ...\tau_R\}$ and set $\sigma = \{\sigma_1, \sigma_2, ...\sigma_R\}$, Calculate $\hat{y}_{L,\tau,\sigma} = \mathbf{F}_B \mathbf{U}_L \mathbf{Q}_L \mathbf{S}_\tau \mathbf{P}_\sigma x$ representing filtered spectrum.

Stage2-Step1 Location rounds: After R times' round, return R sets of coordinates $I_1, ..., I_R$(set I_r representing union of $2K$ sets J from B sets $J \in \{J_{r,0}, J_{r,1}, ...J_{r,B-1}\}$ in the No.r' round, set $J_{r,i} = \left\{\sigma_r^{-1}\frac{(2i-1)L}{2}, \sigma_r^{-1}(\frac{(2i-1)L}{2}+1), ..., \sigma_r^{-1}(\frac{(2i+1)L}{2}-1)\right\}$). Then do the vote, count the number s_i of occurrences of each found coordinate i, that is: $s_i = \| \{r | i \in I_r\} \|_0$ ($\| \|_0$ representing $\ell_0 -$ norm). Only keep the coordinates occurred in at least fifty percentage proportion($I = \{i \in I_1 \cup ... \cup I_R | s_i > R/2\}$).

Stage2-Step2 Estimation rounds: After location rounds the set I can be obtained then estimate R sets of frequency coefficients $\hat{x}^1, ..., \hat{x}^R$. The method is if position $f \in I$, we can get the value of position f through formula(21). For identified position f, R different $\hat{x}_f^r$ can be obtained in R times' round, finally use the median value of the sets as the final estimator.

Finally we analyze the performance of sFFT1.0 algorithm. In stage1 it cost $R(w + BlogB)$ runtime, in stage2-step1 it cost $2RK(N/B)$ runtime, in stage2-step 2 it cost $2RK$ runtime, totally it cost $O(R(w+BlogB+KN/B))$ runtime. And in fact the runtime satisfy Lemma 4.1.

Lemma 4.1: Suppose $R = O(\text{log}N)$, $w = B\text{log}(N/\delta)$, $\delta = 1/(N^c)$, it cost $O(\text{log}N\sqrt{NK\text{log}N})$ runtime in sFFT1.0 algorithm

Proof 4.1:

$$\begin{aligned}
&O(R(w + B\log B + KN/B)) \\
=&O\left(\log N \log(N/\delta)B + \log N K N B^{-1}\right) \\
\geq&O(\log N\sqrt{NK\log N})(\text{ for } B = \sqrt{NK\log^{-1}(N/\delta)})
\end{aligned}$$

In stage 1 it needs w samples one time. In the first round, the signal not chosen is in probability of $(N - w)/N$, suppose the probability does not change, in average the samples chosen after R times' round is in the number of $N\left(1 - (\frac{N-w}{N})^R\right) = N\left(1 - \left(\frac{N-\sqrt{KN\log(N/\delta)}}{N}\right)^{(\log N)}\right)$

B. THE SFFT2.0 ALGORITHM BY ONE-SHOT FRAMEWORK

As is shown in the Figure 1, sFFT2.0 is very similar to sFFT1.0. Additional another bucketization and location round are used with the frequency aliasing filter to restrict the locations of the large coefficient. Let M be the size of the aliasing filter and M divides N, it does a pre-processing stage as below. Firstly Obtain $\hat{y}_{M,0} = \mathbf{F}_M \mathbf{D}_M \mathbf{S}_0 x$, then union of $2K$ sets from M sets $\{0, M, ..., (L-1)M\}, ..., \{M-1, 2M-1, ..., N-1\}$ by selecting the $2K$ largest coefficients of magnitude of $\hat{y}_{M,0}[i]$ for $i \in [0, M-1]$, the union of $2K$ sets is set I' , assuming that all large coefficients j have j mod M in I'. That is, we restrict out sets I_r talked above to contain only coordinates i with i mod $M \in I'$,we expect that $|I_r| \approx 2K/M(2KN/B)$ rather than the previous $|I_r| \approx 2KN/B$.

Finally we analyze the performance of sFFT2.0 algorithm. In stage1 it is the same as the sFFT1.0, in stage2-step1 it cost $R(2K/M \cdot 2K(N/B) + M) + M\log M$ runtime, in stage2-step 2 it is the same, totally it cost $O(R(w+BlogB+K^2N/(BM) + M) + M\log M)$ runtime. And in fact the runtime satisfy Lemma 4.2.

Lemma 4.2: Suppose $R = O(\text{log}N)$, $w = B\text{log}(N/\delta)$, $\delta = 1/(N^c)$, it cost $O\left(\log N\left(K^2N\log(N)\right)^{1/3}\right)$ runtime in sFFT2.0 algorithm

Proof 4.2:

$$\begin{aligned}
&O(R(w + BlogB + K^2N/(BM) + M) + M\log M) \\
=&O\left(\log N\log(N/\delta)B + \log NK^2NM^{-1}B^{-1} + M\log M\right) \\
\geq&O(\log NK\sqrt{\log(N/\delta)NM^{-1}} + M\log M) \\
\approx&O(\log NK\sqrt{\log(N/\delta)NM^{-1}} + M\log N) \\
\geq&O\left(\log N\left(K^2N\log(N)\right)^{1/3}\right)
\end{aligned}$$

Compared to sFFT1.0, the runtime is factor $(N\log N)^{1/6}$ smaller. In average the sample of sFFT2.0 chosen is in the number of $N\left(1 - (\frac{N-w}{N})^R\right) + M$, compared to the sFFT1.0, B decreases so w decreases so that the samples decreases as well.

C. THE SFFT3.0 ALGORITHM BY ITERATIVE FRAMEWORK

Compared to the one shot framework the iterative framework has two improvements. The first advantage of iterative framework is that once a frequency coefficient of the signal was found and estimated, it can be subtracted from the signal. This fact can be used to reduce the amount of work to be done in subsequent steps. It is not necessary to update the whole input signal. Instead, it is sufficient to update the B-dimensional buckets. This way the removal of the effects of already found coefficients can be done in $O(B)$ time. The second important improvement in iterative framework is an improved method for finding the signal's significant frequency coordinates rather than voting method by R times' rounds. In one-shot framework, R times' rounds are run and their results combined in order to get correct locations at a high probability. In the iterative algorithms, two or $\log_2 L$ rounds is enough by their own ways.

In the No.m' iteration, let K_m be the expected sparsity, R_m be how many rounds in the No.m' iteration, B_m be the the number of buckets, L_m be the size of one bucket, w_m be the support of filter G, $\hat{y}_{L,\tau,\sigma}$ be filtered spectrum, $\hat{y}_{update}$ be update spectrum, $\hat{x}^{m-1}$ be last result, $\hat{y'}_{L,\tau,\sigma}$ be spectrum need to recover, $\hat{x'}^m$ be the recovered spectrum, $\hat{x}^m$ be the new result, set $\tau = \{\tau_1, \tau_2, ...\tau_R\}$ and set $\sigma =$

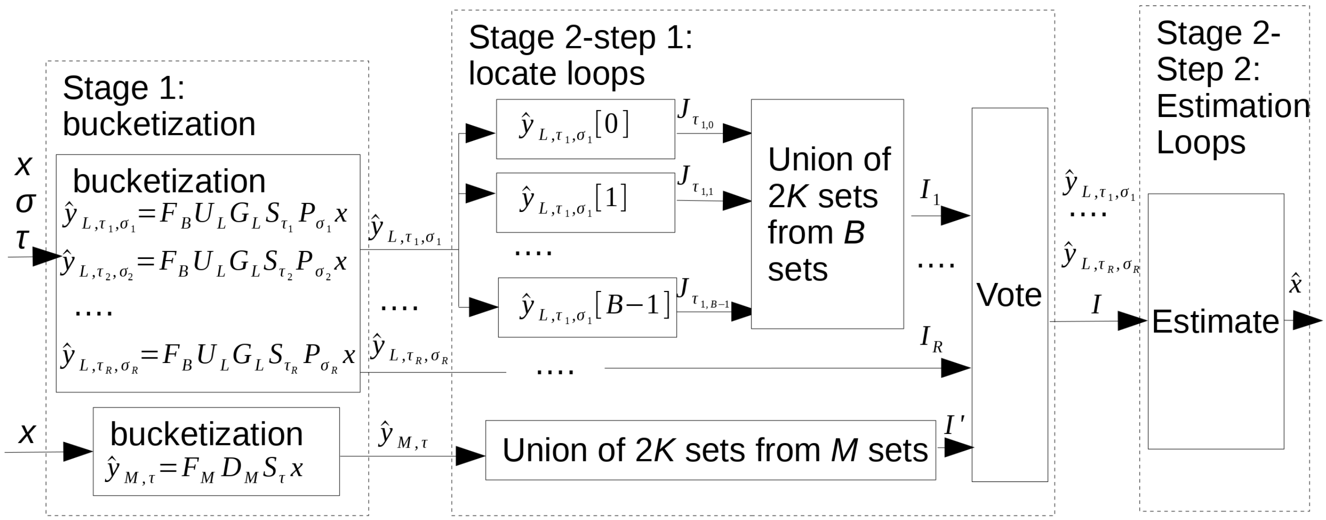


FIGURE 1. A system block diagram of one-shot framework.

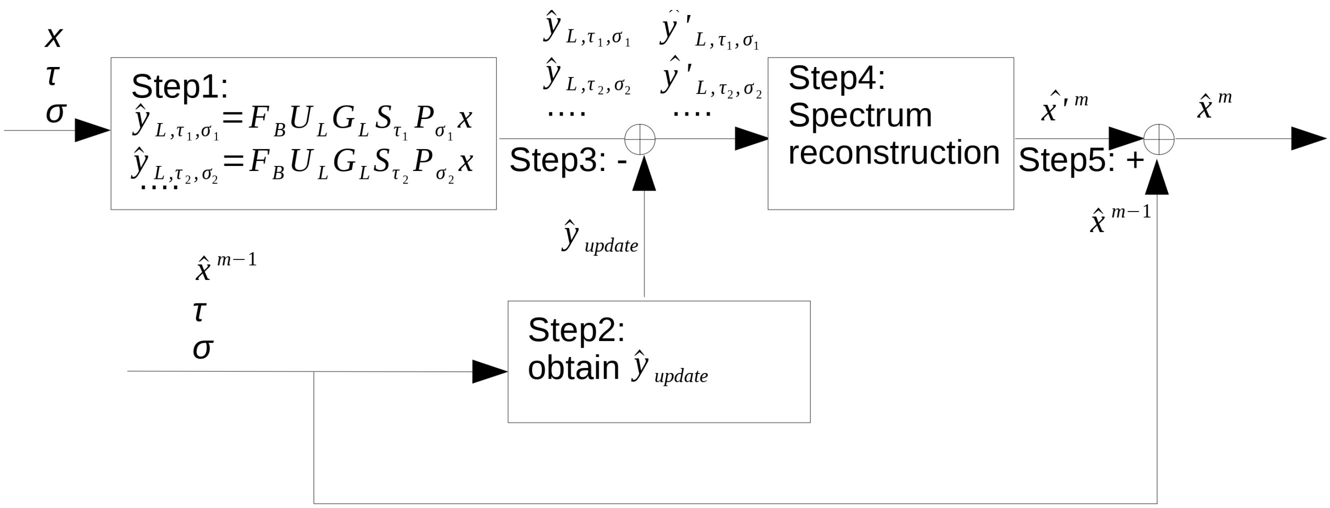


FIGURE 2. A system block diagram of iterative framework.

$\{\sigma_1, \sigma_2, ...\sigma_R\}$ be parameter. It can be seen that R_m = 2 in sFFT3.0 algorithm, $R_m = \log_l L_m$ in sFFT4.0 algorithm, $Rm = \log_2 L_m$ in MPSFT algorithm. The detail course in No.m' iteration is shown in Fig. 2 and explained as follows.

Step1: Run R_m bucketization rounds for K_m, B_m, L_m, set σ and set τ to calculate $\hat{y}_{L,\tau,\sigma} = \mathbf{F}_B\mathbf{U}_L\mathbf{Q}_L\mathbf{S}_\tau\mathbf{P}_\sigma x$ representing filtered spectrum.

Step2: Run R_m times' rounds for $\hat{x}^{m-1}$, set τ, set σ and formula(21) to obtain $\hat{y}_{update}$ representing spectrum have already gained.

Step3: $\hat{y'}_{L,\tau,\sigma}$ = $\hat{y}_{L,\tau,\sigma}$ - $\hat{y}_{update}$ representing spectrum need to recover.

Step4: recover the spectrum $\hat{x'}^m$ of $\hat{y'}_{L,\tau,\sigma}$ by different methods.

Step5: $\hat{x}^m$ = $\hat{x}^{m-1}$ + $\hat{x'}^m$ representing the result of this iteration.

Step6: If it is last iteration, the final result is $\hat{x}^m$, otherwise $\hat{x}^m$ will be the input to make $\hat{y}_{update}$ for the next iteration.

It is sufficient to locate the position only using $R = 2$ in stead of $R \approx \log N$ rounds by the phase encoding method in sFFT3.0 algorithm in the exactly sparse case. The process is in the first round we set $\tau_1 = 0$, and the second round we set $\tau_2 = 1$, then suppose in a bucket i, it contains only one large frequency, so we get $\hat{y}_{L,0,\sigma}[i] \approx \hat{G}_{iL-u}\hat{X}_{\sigma^{-1}(u)}\omega_N^{0\cdot(u)}$ and $\hat{y}_{L,1,\sigma}[i] \approx \hat{G}_{iL-u}\hat{X}_{\sigma^{-1}(u)}\omega_N^{1\cdot(u)}$, then $\omega_N^{1\cdot(u)} \approx \frac{\hat{y}_{L,0,\sigma}[i]}{\hat{y}_{L,1,\sigma}[i]}$ so we can locate the position $f = (\sigma^{-1}u)\mathrm{mod}N$. As to estimate the value identified, it is similar to the process of sFFT1.0 algorithm. Finally we analyze the performance of the sFFT3.0. And in fact it satisfied Lemma 4.3.

Lemma 4.3: In sFFT3.0 algorithm, it cost $O(K \log N)$ runtime and $O(K\log N)$ samples

Proof 4.3: In the first iteration, suppose $w_1 = B_1\log(N/\delta)$, $K_1 = K$, it cost $2(w_1 + B_1\log B_1 + K_1) = O(B_1\log N)$ runtime and find at least $K/2$ true frequency, In the second iteration, suppose $B_2 = B_1/2$, $w_2 = B_2\log(N/\delta)$, $K_2 = K/2$, it cost $2(w_2 + B_2\log B_2)$ runtime in the step1, it cost $2K_1$ runtime in the step2, it cost $2B_2$ runtime in the step3, it cost $2K_2$ runtime in the step4, it cost K_2 runtime in the step5, it total cost $O(w_2 + B_2\log B_2 + K_1) < O(B_1\log N)/2$ runtime in the second iteration, so the total runtime is $O(B_1\log N) + O(B_1\log N/2) + \cdots = O(B_1\log N) = O(K\log N)$. As to the sampling complexity, it is clearly that the samples chosen after two rounds in the first iteration is in the number of $2w_1 = 2B_1\log(N/\delta) = O(K\log N)$, as to the samples of the following iterations, the samples can be ignored because of the decrease of B.

D. THE SFFT4.0 ALGORITHM BY ITERATIVE FRAMEWORK

Compared with the sFFT3.0 algorithm, only step4 of sFFT4.0 algorithm is different. In sFFT3.0 algorithm, it is sufficient to locate the position only running two rounds in the noiseless case in advance of it satisfies Lemma 4.4. In sFFT4.0 algorithm, it is sufficient to locate the position only running $R(=\log_l L)$ times' round by the multiscale phase encoding method in advance of it satisfies Lemma 4.5.(let l be the multiscle parameter).

Lemma 4.4: In the bucket i suppose the located position is denoted by u', the real position is denoted by u, the noisy is denoted by $S_\tau[i]$, it satisfied formula(22), so $S_0[i]$ is the noisy in the bucket i for τ = 0, $S_1[i]$ is the noisy in the bucket i for τ = 1, function $\Phi(\theta)$ satisfies $e^{\Phi(\theta)\mathbf{i}} = \theta,\ \Phi(\theta) \in [0, 2\pi)$. In sFFT3.0 algorithm, the algorithm guarantee must be required as formula(23)

$$S_\tau[i] = \hat{y}_{L,\tau,\sigma}[i] - \hat{G}_{iL-u}\hat{x}_{\sigma^{-1}u}\omega_N^{\tau u} \tag{22}$$

$$\left|\Phi\left(\frac{\hat{y}_{L,0,\sigma}[i]}{\hat{y}_{L,1,\sigma}[i]}\right) - \Phi\left(\frac{\hat{y}_{L,0,\sigma}[i] - S_0[i]}{\hat{y}_{L,1,\sigma}[i] - S_1[i]}\right)\right| \le \frac{\pi}{N} \tag{23}$$

Proof 4.4:

$$\omega_N^{1\cdot(u')} = \frac{\hat{y}_{L,0,\sigma}[i]}{\hat{y}_{L,1,\sigma}[i]} \Rightarrow u' = \text{round}(\Phi(\frac{\hat{y}_0[i]}{\hat{y}_1[i]})\frac{N}{2\pi})$$

$$\omega_N^{1\cdot(u)} = \frac{\hat{y}_{L,0,\sigma}[i] - S_0[i]}{\hat{y}_{L,1,\sigma}[i] - S_1[i]} \Rightarrow u = \Phi(\frac{\hat{y}_0[i] - S_0[i]}{\hat{y}_1[i] - S_1[i]})\frac{N}{2\pi}$$

in order to $u' = u \Rightarrow$ absolute value ≤ 0.5

$$\Rightarrow \left|\Phi\left(\frac{\hat{y}_{L,0,\sigma}[i]}{\hat{y}_{L,1,\sigma}[i]}\right) - \Phi\left(\frac{\hat{y}_{L,0,\sigma}[i] - S_0[i]}{\hat{y}_{L,1,\sigma}[i] - S_1[i]}\right)\right| \le \frac{\pi}{N}$$

Lemma 4.5: In the bucket i suppoe L be the range in this location, l be the multiscle parameter, r be the size of one scale($r = L/l$), u_0 be the initial position, u'_l be the located value from located position $u'(u'_l = (u' - u_0)/r,\ u'_l \in [0, l])$, u_l be the real value from real position $u(u_l = (u - u_0)/r,\ u_l \in [0, l])$, $\tau_1 = 0,\ \tau_2 \approx N/L$, In sFFT4.0 algorithm, the algorithm guarantee must be required as formula(24),

$$\left|\Phi\left(\frac{\hat{y}_{L,\tau_1,\sigma}[i]}{\hat{y}_{L,\tau_2,\sigma}[i]}\right) - \Phi\left(\frac{\hat{y}_{L,\tau_1,\sigma}[i] - S_{\tau_1}[i]}{\hat{y}_{L,\tau_2,\sigma}[i] - S_{\tau_2}[i]}\right)\right| \le \frac{\pi}{l} \tag{24}$$

Proof 4.5:

$$\omega_N^{\tau_2 u'} = \frac{\hat{y}_{L,\tau_1,\sigma}[i]}{\hat{y}_{L,\tau_2,\sigma}[i]} \Rightarrow (u_0 + u'_l r)\tau_2 \text{mod} N\frac{2\pi}{N} = \Phi(\frac{\hat{y}_{\tau_1}[i]}{\hat{y}_{\tau_2}[i]})$$

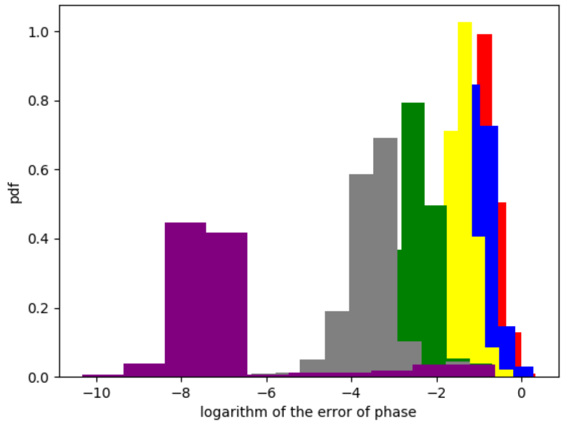
$$\omega_N^{\tau_2 u} = \frac{\hat{y}_{\tau_1}[i] - S_{\tau_1}[i]}{\hat{y}_{\tau_2}[i] - S_{\tau_2}[i]} \Rightarrow (u_0 + u_l r)\tau_2 \text{mod} N\frac{2\pi}{N} = \Phi(\frac{\hat{y}_{\tau_1}[i] - S_{\tau_1}[i]}{\hat{y}_{\tau_2}[i] - S_{\tau_2}[i]})$$

in order to $u'_l = u_l \Rightarrow$ absolute value ≤ 0.5

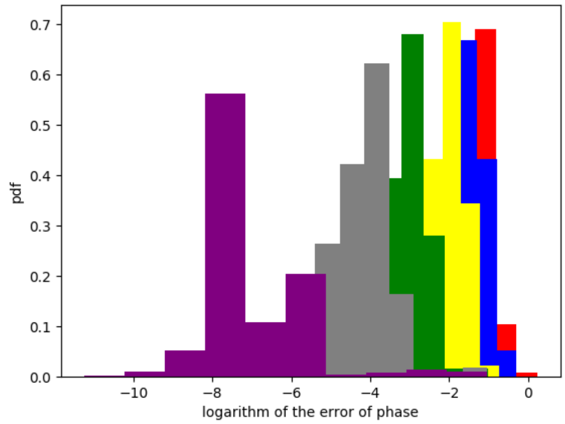
$$\Rightarrow \left|\Phi\left(\frac{\hat{y}_{L,\tau_1,\sigma}[i]}{\hat{y}_{L,\tau_2,\sigma}[i]}\right) - \Phi\left(\frac{\hat{y}_{L,\tau_1,\sigma}[i] - S_{\tau_1}[i]}{\hat{y}_{L,\tau_2,\sigma}[i] - S_{\tau_2}[i]}\right)\right| \le \frac{\pi}{l}$$

It is clear that the restrictive conditions of formula(23) are very harsh in sFFT3.0 algorithm when N is large so sFFT3.0 algorithm is not robustness. From the lemma4.5, it is most robustness when we use binary search ($l = 2$), because the confidence space($\pi/2$) is very big, but it is not effective. We want to know how to set the confidence space of multiscle parameter l, we do it by Monte Carlo experiment, $\left|\Phi\left(\frac{\hat{y}_{L,\tau_1,\sigma}[i]}{\hat{y}_{L,\tau_2,\sigma}[i]}\right) - \Phi\left(\frac{\hat{y}_{L,\tau_1,\sigma}[i] - S_{\tau_1}[i]}{\hat{y}_{L,\tau_2,\sigma}[i] - S_{\tau_2}[i]}\right)\right|$ is denoted by $\Delta\Phi(\theta)$, we do two categories experiments to calculate the logarithm of the error of phase $\log_{10}(\Delta\Phi(\theta))$ and computing probability distribution function(PDF) of the value $\log_{10}(\Delta\Phi(\theta))$ in all valuable buckets and all rounds by the input signals of different N under different SNR circumstances only if the bucket is not aliasing, then we get the Figure 3. From the Figure 3, we can see with the develop of SNR(from the red space to purple space), the probability of small error increases. Compare of the two cases: small N and big N under condition of the same K and same SNR, if N is big, it means in one bucket the noisy has less energy compared with effective signal, it can be concluded both in the theory and in experiments that the variance of error becomes smaller when N is big. From the Figure 3, if we want to keep the probability greater than 0.99 under condition of SNR = -20(red space), the threshold should be more than $10^{0.5} \approx 3.2$, it seems impossible. Under condition of SNR = -10(blue space), the threshold should be more than $10^0 \approx 1$, it can be solved by binary search method because the confidence space is $\pi/2 \approx 1.7$. Under condition of SNR=0(yellow space), the threshold should be more than $10^{-0.5} \approx 0.3$, and under condition of SNR>0, it is certain to keep the the high probability if the threshold is more than 0.3. Under condition of SNR=120(purple space), if the threshold is $\pi/N = \pi/8192 \approx 0.004 > 10^{-3}$, or the threshold is $\pi/N = \pi/1048576 \approx 0.000003 > 10^{-6}$, it can also keep the high probability to satisfy formula(23).(Remarks: It is easy to know the PDF of the error of phase will not change much with different τ and different σ)

In the real sFFT4.0 algorithm, we use l =8, the confidence space is $\pi/8 \approx 0.4$, it can keep the high probability to satisfy formula(24) under condition of SNR$\geq$0. As to the runtime complexity we can easy to get it should run $\log_l L(=\log_8(N/B))$ times' rounds instead of two times' rounds in every iteration, so the runtime and sampling complexity is $O(K\log N\log_8(N/K))$.



(a)



(b)

FIGURE 3. PDF of logarithm of the error of phase $\log_{10}(\Delta\Phi(\theta))$ vs SNR. (a)K=50 and N=8192, (b)K=50 and N=1048576

E. THE MPSFT ALGORITHM BY ITERATIVE FRAMEWORK

Compared with the sFFT4.0 algorithm, only the discriminant equation to location of MPSFT algorithm is different. The matrix pencil method, like the Prony method, is a standard technique in signal processing for mode frequency identification. In this section, we use the matrix pencil method into the MPSFT algorithm to achieve two effects. Firstly, it identifies modes much more accurately. Secondly, it helps detect errors in our mode identification step and greatly reduces the number of spurious modes being found. Rely on these, we can use only a little cost to solve the collision problem.

Suppose the number of significant frequencies in the bucket i is denoted by a. In most buckets a=0, in a part of buckets a=1, only in a small part of buckets a>=2. Then the formula(21) can be translated to the formula(25), the problem to reconstruct spectrum is translated to how to calculate $2a$ variables as below: a amplitudes($\hat{G}_{\text{poly}(f_0)}\hat{x}_{f_0}$,...$\hat{G}_{\text{poly}(f_{a-1})}\hat{x}_{f_{a-1}}$) and a positions($\omega^{f_0\sigma}$,...$\omega^{f_{a-1}\sigma}$). It needs $2a$ equations, where $\hat{y}_{L,\tau,\sigma}[i]$ is known and denoted by $m_\tau = \hat{y}_{L,\tau,\sigma}[i]$ using fixed L and σ, p_j representing unknown $\hat{G}_{iL-\sigma f_j}\hat{x}_{f_j}$, z_j representing unknown $\omega_N^{\sigma f_j}$. By taking the above into consideration, the problem can be formulated by BCH codes as formula(26) by using $\tau = 0, 1, \cdots, 2a-1$.

$$\hat{y}_{L,\tau,\sigma}[i] \approx \hat{G}_{iL-\sigma f_0}\hat{x}_{f_0}\omega_N^{\sigma\tau f_0} + ... + \hat{G}_{iL-\sigma f_{a-1}}\hat{x}_{f_{a-1}}\omega_N^{\sigma\tau f_{a-1}} \tag{25}$$

$$\begin{bmatrix} z_0^0 & z_1^0 & \cdots & z_{a-1}^0 \\ z_0^1 & z_1^1 & \cdots & z_{a-1}^1 \\ \cdots & \cdots & \cdots & \cdots \\ z_0^{2a-1} & z_1^{2a-1} & \cdots & z_{a-1}^{2a-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \cdots \\ p_{a-1} \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \\ \cdots \\ m_{2a-1} \end{bmatrix} \tag{26}$$

$$\mathbf{M}_{a_m} = \begin{bmatrix} m_0 & m_1 & \cdots & m_{a_m-1} \\ m_1 & m_2 & \cdots & m_{a_m} \\ \cdots & \cdots & \cdots & \cdots \\ m_{a_m-1} & m_{a_m} & \cdots & m_{2a_m-1} \end{bmatrix}_{a_m \times a_m} \tag{27}$$

Suppose there are at most a_m significant frequencies in the bucket. By singular value decomposition(SVD) of the matrix $\mathbf{M}_{a_m}$ defined as formula(27) in bucket i, we obtain a_m singular values for each frequency. For example we can set a_m equal to two, so we obtain two singular values by SVD. If two singular values are both small, it means there is no significant frequency. If there is only one big singular value, it means there is one significant frequency. If both of them are big, it means there are more than one significant frequencies. The way to solve the collision problem is as above, as to distinguish the position of the frequency, the method is very similar to the sFFT4.0 algorithm using binary search and the discriminant is inspired by the matrix pencil method. The detail can seen [17]. It is sufficient to locate the position and estimate the value of frequencies only using $R = 2\log_2 L$ times' rounds in MPSFT algorithm, the runtime and sampling complexity in algorithm is approximately equal to $O(K \log N \log_2(N/K))$.

After analyzing two types of frameworks, five different methods to reconstruct spectrum and corresponding algorithms, Table 1[1] can be concluded with the addition information of other sFFT algorithms and fftw.

From the Table 1 we can see sFFT3.0 algorithm is lowest runtime and sampling complexity, but it is non robustness. Other algorithms using flat window are good robustness but compare them with other sFFT algorithms it is no advantage in the sampling complexity except sFFT4.0 algorithm.

[1]The performance of algorithms using flat window are got as above. The performance of other algorithms are got from [2],[3],[5],[6],[8]. The analysis of robustness will be explained in next section.

TABLE 1. The performance of the sFFT algorithms in theory

algorithm	runtime complexity	sampling complexity	robustness
sFFT1.0	$O(K^{\frac{1}{2}}N^{\frac{1}{2}}\log^{\frac{3}{2}}N)$	$N\left(1-\left(\frac{N-w}{N}\right)^{\log N}\right)$	medium
sFFT2.0	$O(K^{\frac{2}{3}}N^{\frac{1}{3}}\log^{\frac{4}{3}}N)$	$N\left(1-\left(\frac{N-w}{N}\right)^{\log N}\right)$	medium
sFFT3.0	$O(K\log N)$	$O(K\log N)$	none
sFFT4.0	$O(K\log N\log_8(N/K))$	$O(K\log N\log_8(N/K))$	bad
MPFFT	$O(K\log N\log_2(N/K))$	$O(K\log N\log_2(N/K))$	good
sFFT-DT	$O(K\log K+N)$	$O(K\log K+N)$	medium
FFAST	$O(K\log K)$	$O(K\log K)$	none
R-FFAST	$O(K\log^4 N)$	$O(K\log K)$	medium
AAFFT	$O(K\text{poly}(\log N))$	$O(K\text{poly}(\log N))$	medium
FFTW	$O(N\log N)$	N	good

V. EXPERIMENTAL EVALUATION

In this section we evaluate the performance of five sFFT algorithms using flat window filter: sFFT1.0, sFFT2.0, sFFT3.0, sFFT4.0 and MPSFT. All of them were implemented in C or C++ language in order to empirically evaluate their runtime characteristics. We firstly compare these algorithms' runtime, percentage of signal sampled and robustness characteristics with each other. Then we compare these algorithms' characteristics with other algorithms: fftw, sFFT-DT, FFAST and AAFFT. Finally we compare these algorithms' runtime characteristics with themselves optimized. All experiments are run on a CentOS7.6 computer with 4 Intel(R) Core(TM) i5-4570 3.20GHz CPU, a cache size of 6144 KB and 8 GB of RAM.

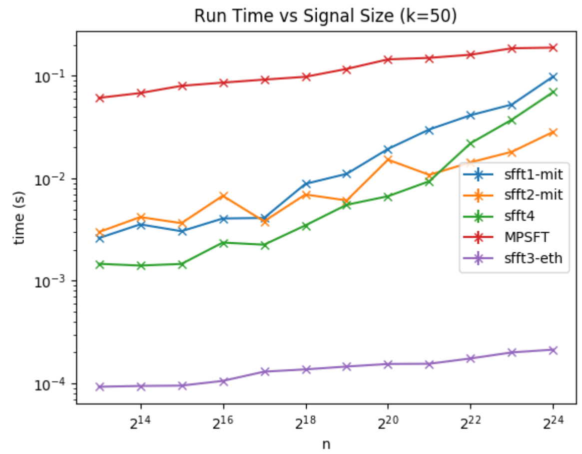
A. EXPERIMENTAL SETUP

In experiment, the test signals are gained in a manner that K frequencies are selected from N frequencies uniformly at random and assigned a magnitude of 1 and a uniformly random phase and the rest frequencies are set to zero in exactly case. When in general sparse case, the test signals are gained similarly but they are combined with additive white Gaussian noise, whose variance varies depending on the SNR required. Each point in the figure is the average result over 5 runs with 5 different instances as desired. The parameters of these algorithms are chosen so that can make a balance between time efficiency and robustness.

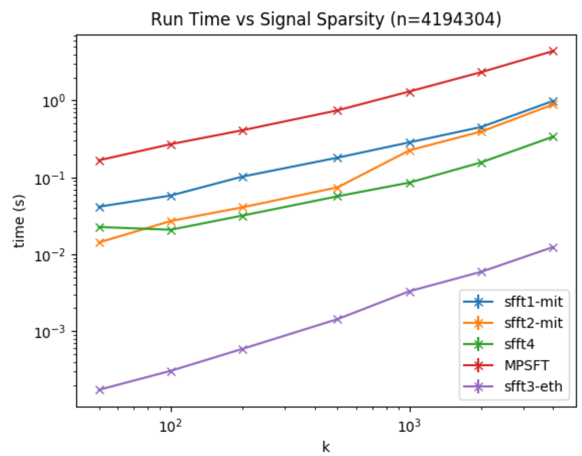
B. COMPARISON EXPERIMENT ABOUT DIFFERENT ALGORITHMS USING FLAT FILTER OF THEMSELVES

We plot Figure 4 representing runtime vs. signal size and vs. signal sparsity for sFFT1.0, sFFT2.0, sFFT3.0, sFFT4.0 and MPSFT algorithm in exactly sparse case.[2] As mentioned above, runtime is determined by two factors. One is how many rounds(it manly depend on R) and how much time cost in one round(it manly depend on w). So from the Figure 4 we can see 1)The runtime of these 5 algorithms are approximately linear in the log scale as a function of N and in the standard scale as a function of K. The reason is

[2]The general sparse case means SNR=20db is very similar to the exactly sparse case except sFFT3.0 algorithm. The detail of code, data, report can be provided in https://github.com/zkjiang/-/tree/master/docs/sfft project



(a)



(b)

FIGURE 4. (a)Runtime vs. signal size (b)Runtime vs. signal sparsity in the exactly case

R and w is with the growth of $\log N$ and K. 2)Results of ranking the runtime complexity of 5 algorithms is sFFT3.0 > sFFT4.0 > sFFT2.0 > sFFT1.0 > MPSFT. The reason is their individual's R is about 2, $2\log_8 L$, approxiate $\log N$, $\log N$ and $2\log_2 L$.

We plot Figure 5 representing percentage of signal sampled vs. signal size and vs. signal sparsity for sFFT1.0, sFFT2.0, sFFT3.0, sFFT4.0, and MPSFT algorithm in the exactly sparse case.[3] As mentioned above, percentage of signal sampled is also determined by two factors: how many rounds and how many samples sampled in one round. So from the Figure 5 we can see 1)The percentage of signal sampled of these 5 algorithms are approximately linear in the log scale as a function of N and in the standard scale as a function of K. 2)Results of ranking the sampling complexity of 5 algorithms is sFFT3.0 > sFFT4.0 > MPSFT > sFFT2.0 > sFFT1.0 because of the different R.

[3]The general sparse case is very similar except sFFT3.0 algorithm.

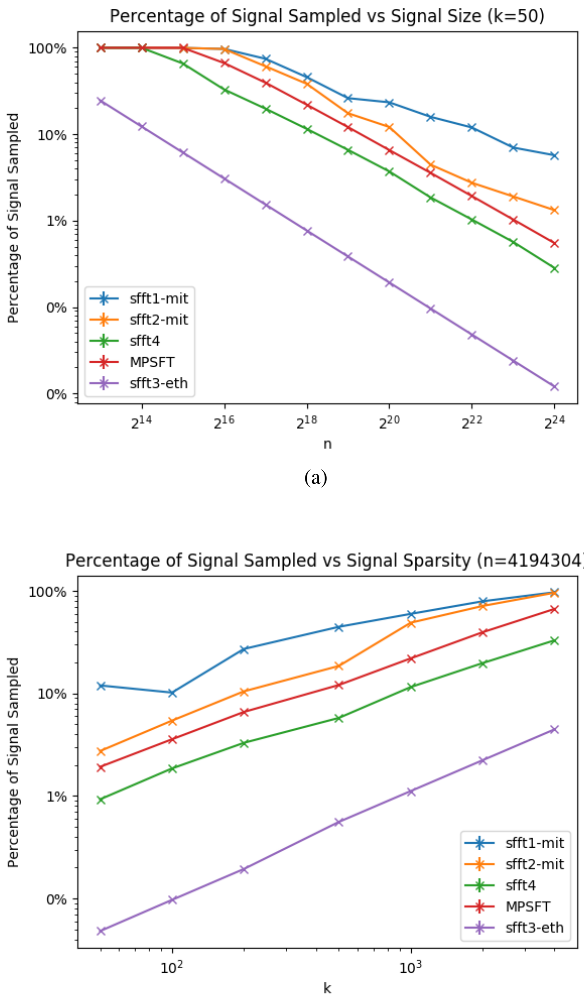


FIGURE 5. (a)Percentage of signal sampled vs. signal size (b)vs. signal sparsity

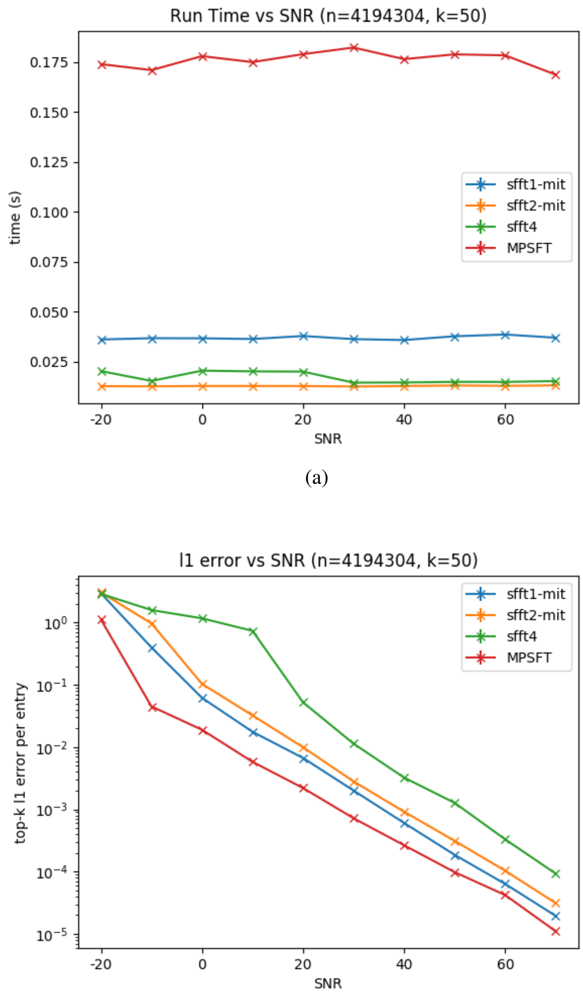


FIGURE 6. (a)Run time vs. SNR (b)$L1$-error vs. SNR

We plot Figure 6 representing runtime and $L1$-error vs SNR for sFFT1.0, sFFT2.0, sFFT3.0, sFFT4.0, and MPSFT algorithm.[4] From the Figure 6 we can see 1)The runtime is approximately equal vs SNR. 2) To a certain extent these 4 algorithms are all robustness, but when SNR is low, only MPSFT satisfies the ensure of robustness. When SNR is medium, sFFT1.0 and sFFT2.0 can also meet the ensure of robustness. And only when SNR is bigger than 20db, sFFT4.0 can deal with the noise interference. The reason is that the way of binary search is better than voting method under large noisy situation. And the way of mulitscale search is not good when it use in noisy situation according to the formula(26) and Figure 3.

[4]the $L0$-error, $L1$-error $L2$-error of all experiments can be provided in https://github.com/zkjiang/-/tree/master/docs/sfft project /experiment data

C. COMPARISON EXPERIMENT ABOUT ALGORITHMS USING THE FLAT FILTER AND OTHER ALGORITHMS

We plot Figure 7 representing run times vs. signal Size and vs. signal sparsity for sFFT1.0, sFFT4.0, AAFFT, R-FFAST, SFFT-DT and fftw algorithm in the general sparse case.[5] From the Figure 7, we can see 1)These algorithms are approximately linear in the log scale as a function of N except fftw algorithm. These algorithms are approximately linear in the standard scale as a function of K except fftw and SFFT-DT algorithm. 2) Results of ranking the runtime complexity of these 6 algorithms is sFFT4.0 > sFFT1.0 > AAFFT > SFFT-DT > fftw > R-FFAST when N is large. The reason is Least Absolute Shrinkage and Selection Operator(LASSO) method used in R-FFAST cost a lot of time. And SVD and CS method used in SFFT-DT also cost a lot of time. 2)Results of ranking the runtime complexity of these 6 algorithms is

[5]The exactly sparse case is very similar except sFFT3.0 algorithm and FFAST algorithm. FFAST and R-FFAST algorithm is not available when K is very large

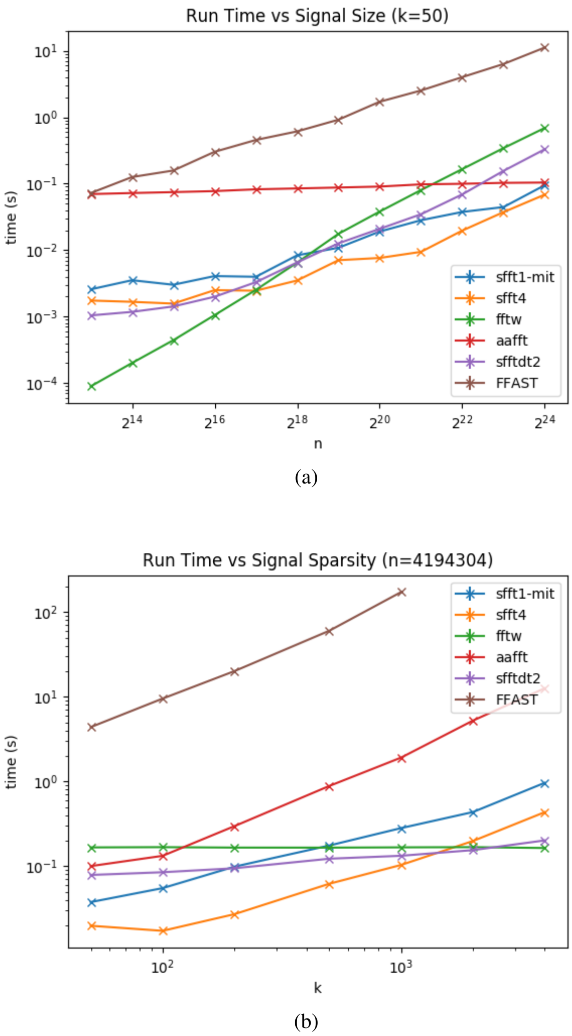


FIGURE 7. (a)Runtime vs. signal size (b)Runtime vs. signal sparsity

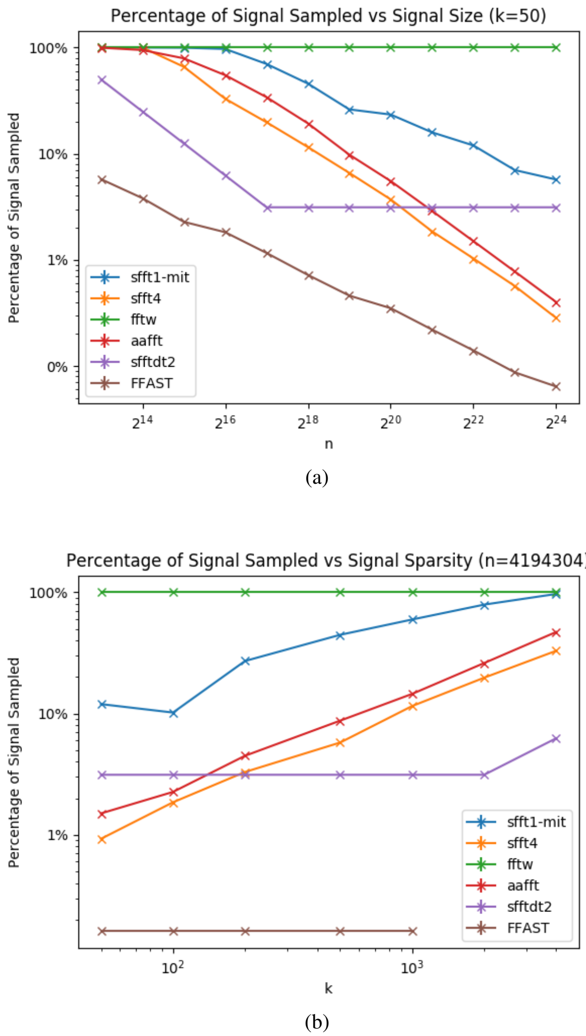


FIGURE 8. (a)Percentage of signal sampled vs. signal Size (b)vs. signal sparsity

fftw > SFFT-DT > sFFT4.0 > sFFT1.0 > AAFFT > R-FFAST when K is large. The reason is algorithms using aliasing filter saving the time by using a small number of buckets in the first stage compared to algorithms using flat filter when K is large.

We plot Figure 8 representing percentage of signal sampled vs. signal size and vs. signal sparsity for sFFT1.0, sFFT4.0, AAFFT, R-FFAST, SFFT-DT and fftw algorithm in the general sparse case.[6] From Figure 8, we can see 1)These algorithms are approximately linear in the log scale as a function of N except fftw and SFFT-DT algorithm. The reason is sampling in low-dimension in sFFT algorithms can decrease sampling complexity and it is a limit to the size of bucket in SFFT-DT by using CS and SVD method. These algorithms are approximately linear in the standard scale as a function of K except R-FFAST and SFFT-DT algorithm. The reason is algorithms using aliasing filter saving the time by using less number of buckets. 2)Results of ranking the sampling complexity of these 6 algorithms is R-FFAST > sFFT4.0 > AAFFT > SFFT-DT > sFFT4.0 > fftw when N is large. 2)Results of ranking the sampling complexity is SFFT-DT > sFFT4.0 > AAFFT > sFFT1.0 > fftw when K is large. The reason is that algorithms using aliasing filter need less buckets than other algorithms, the number of the buckets they use in R-FFAST algorithm is only connect to the prime numbers gained by N and the number of the buckets they use in SFFT-DT algorithm is only connect to limit to the size of bucket.

We plot Figure 9 representing runtime and $L1$-error vs SNR for sFFT1.0, sFFT4.0, AAFFT, SFFT-DT and fftw algorithm. From the Figure 9 we can see 1)the runtime is approximately equal vs. SNR. 2)To a certain extent these 5 algorithms are all robustness, but when SNR is low, only fftw algorithm satisfies the ensure of robustness. When SNR

[6]The exactly sparse case is very similar except sFFT3.0 algorithm. FFAST and R-FFAST is not available when K is very large. It is a limit to the size of bucket in SFFT-DT(L is not allowed more than 1024)

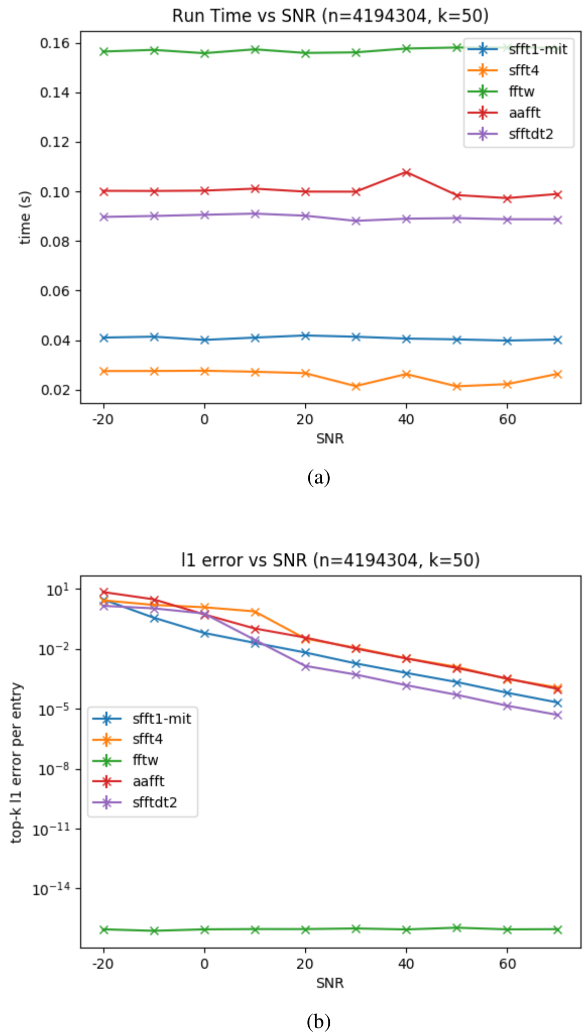


FIGURE 9. (a)Run time vs. SNR (b)$L1$-error vs SNR

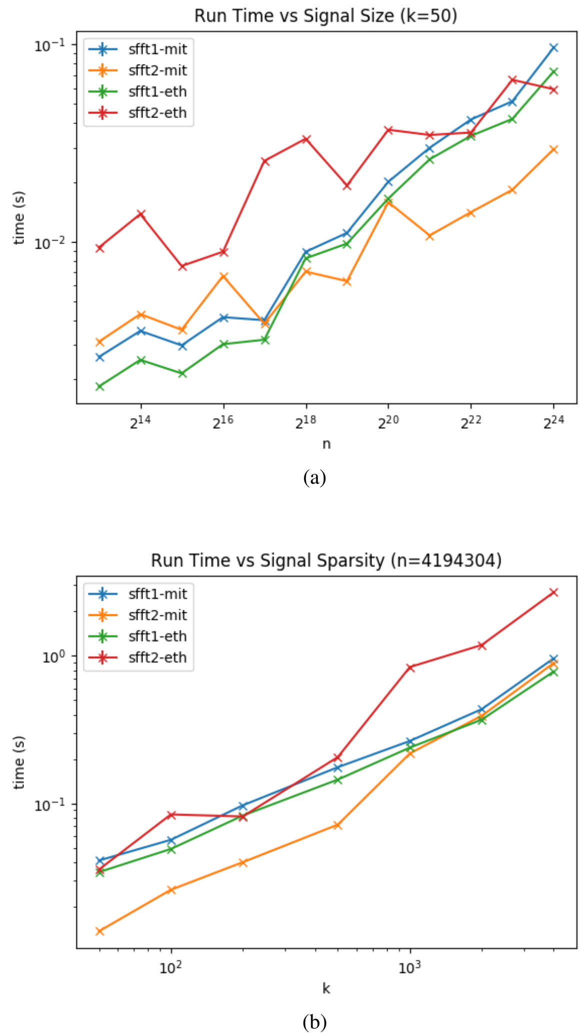


FIGURE 10. (a)Runtime vs. signal size (b)Runtime vs. signal sparsity

is medium, sFFT1.0, AAFFT and SFFT-DT algorithm can also meet the ensure of robustness. And only when SNR is bigger than 20db, sFFT4.0 algorithm can deal with the noise interference.

D. COMPARISON EXPERIMENT ABOUT SAME ALGORITHM OF OPTIMIZATION AND WITHOUT OPTIMIZATION

We plot Figure 10 representing runtime vs. signal size and vs. signal sparsity for sFFT1.0-mit, sFFT2.0-mit, sFFT1.0-eth and sFFT2.0-eth algorithm in the general sparse case. From the Figure 10, we can see the runtime of a same algorithm is accelerated a lot by the use of software optimization.

VI. CONCLUSION

In the first part, the paper introduces the techniques used in sFFT algorithm including random spectrum permutation, window function and frequency bucketization. In the second part, we analyze 5 typical algorithms using flat filter in detail including sFFT1.0 algorithm using voting method, sFFT2.0 algorithm using heuristic vote method by one shot framework and sFFT3.0 algorithm using phase encoding method, sFFT4.0 algorithm using multiscale phase encoding method, MPSFT algorithm using matrix method by iterative framework. We get the conclusion of the performance of these 5 algorithms including runtime complexity, sampling complexity and robustness in theory in Table 1. In the third part, we make three categories of experiments for computing the signals of different SNR, different N, and different K by a standard testing platform through 9 different sFFT algorithms and record the runtime, percentage of signal sampled and $L0, L1, L2$ error in every different situations both in exactly sparse case and general sparse case. The analyze of the experiment result satisfies theoretical inference.

The main contribution of this paper is 1)develop a standard testing platform which can test more than ten typical sFFT algorithms in different situations on the basic of an old platform. 2)get a conclusion of the character and performance of the 5 typical sFFT algorithms using the flat window filter: sFFT1.0 algorithm, sFFT2.0 algorithm, sFFT3.0 algorithm, sFFT4.0 algorithm, and MPSFT algorithm in theory and in practice.

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