MEASURE AND INTEGRATION
D-MATH, ETH Zürich

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Abstract

The aim of this course is to provide notions of abstract measure and integration which are more general and robust than the classical notion of Jordan measure and Riemann integral. For a presentation of Jordan measure and Riemann integral we suggest to look at the lecture notes of Analysis I & II by Michael Struwe
(https://people.math.ethz.ch/∼struwe/Skripten).
The concept of measure of a general set in $\mathbb{R}^n$ originated from the classical notion of volume of a cube in $\mathbb{R}^n$ as product of the length of its sides. Starting from this, by a covering process one can assign to any subset a nonnegative number which "quantifies its volume". Such an association leads to the introduction of a set function called "Lebesgue measure" which is defined over all subsets of $\mathbb{R}^n$. Although the Lebesgue measure was initially developed in Euclidean spaces, the theory is independent of the geometry of the background space and applies to abstract spaces as well.
Measure theory is now applied to most areas of mathematics such as functional analysis, geometry, probability, dynamical systems and other domains of mathematics.
All results presented here are classical and their presentation follows the one in Struwe's lecture notes. The proofs are mostly inspired by those in the books by Evans & Gariepy, and by Cannarsa & D'Aprile.

Some References

- Robert G. Bartle. *The elements of Integration and Lebesgue Measure*, Wiley Classics Library, 1995.
- Piermarco Cannarsa and Teresa D'Aprile. *Introduction to measure theory and functional analysis*, Vol. 89, Springer, 2015.
- Lawrence Craig Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*, revised edition, CRC Press, 2015.
- Gerald B. Folland. *Modern Techniques and Their Applications,* second edition, John Wiley & Sons, INC, 1999.
- Lecture Notes by Prof. Michael Struwe, FS 13, https://people.math.ethz.ch/~struwe/Skripten/AnalysisIII-FS2013-12-9-13.pdf

Contents

Chapter 1

Measure Spaces

In this chapter we develop measure theory from an abstract point of view in order to construct a large variety of measures on a generic space X.

In the particular case of $X = \mathbb{R}^n$, a special role is played by Radon measures which have important regularity properties.

1.1 Algebras and σ-Algebras of Sets

1.1.1 Notation and Preliminaries

We denote by X a nonempty set, by $\mathcal{P}(X)$ or 2^X the set of all subsets of X and $\emptyset$ the empty set. For any subset A of X we denote by A^c its complement, i.e.

$$A^c = \{x \in X \,|\, x \notin A\} \,. \tag{1.1.1}$$

For any $A, B \in \mathcal{P}(X)$ we set

$$A \setminus B = A \cap B^c. \tag{1.1.2}$$

Let $(A_n)_n$ be a sequence in $\mathcal{P}(X)$. The De Morgan's identity holds

$$\Big(\bigcup_{n=1}^{\infty} A_n \Big)^c = \bigcap_{n=1}^{\infty} A_n^c \,. \tag{1.1.3}$$

We define[1]

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$
$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \,. \tag{1.1.4}$$

If $L = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, then we set $L = \lim_{n\to\infty} A_n$ and we say that $(A_n)_n$ converges to L.

Remark 1.1.1. Note that:

1. $\liminf\limits_{n\to\infty} A_n \subseteq \limsup\limits_{n\to\infty} A_n$.

2. If $(A_n)_n$ is increasing ($A_n \subseteq A_{n+1}, n \in \mathbb{N}$), then

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n \,. \tag{1.1.5}$$

Proof. Since A_n is increasing we have

$$\bigcup_{m=1}^{\infty} A_m \subseteq \bigcup_{m=n}^{\infty} A_m, \quad \forall n \in \mathbb{N} \,.$$

Therefore

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \bigcup_{m=1}^{\infty} A_m.$$

On the other hand for every $n \in \mathbb{N}$ we have $\bigcap\limits_{m=n}^{\infty} A_m = A_n$ and therefore

$$\liminf_{n\to\infty} A_n = \bigcup_{m=1}^{\infty} A_m. \quad \square$$

[1] We can "think" about the limit supremum and infimum for sets in the following way:

$$\limsup_{n\to\infty} A_n = \{x \in A_n, \quad \text{for infinitely many } n\}$$

$$\liminf_{n\to\infty} A_n = \{x \in A_n, \quad \text{for all but finitely many } n\}$$

3. If $(A_n)_n$ is decreasing ($A_n \supseteq A_{n+1}, n \in \mathbb{N}$), then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n \,. \tag{1.1.6}$$

1.1.2 Algebras and σ-Algebras

Definition 1.1.2. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$, $\mathcal{A}$ is called an* ***algebra in X*** *if*

a) $X \in \mathcal{A}$.

b) $A, B \in \mathcal{A} \Longrightarrow A \cup B \in \mathcal{A}$.

c) $A \in \mathcal{A} \Longrightarrow A^c \in \mathcal{A}$.

Remark 1.1.3. If $\mathcal{A}$ is an **algebra** and $A,\ B \in \mathcal{A}$, then $A \cap B$ and $A \setminus B$ belong to $\mathcal{A}$. Therefore, the symmetric difference $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$ belongs to $\mathcal{A}$ as well. $\mathcal{A}$ is also stable under finite unions and intersections

$$A_1, \ldots, A_n \in \mathcal{A} \Longrightarrow \begin{cases} A_1 \cup \cdots \cup A_n \in \mathcal{A} \\ A_1 \cap \cdots \cap A_n \in \mathcal{A} \,. \end{cases}$$

Definition 1.1.4. *An algebra $\mathcal{E} \subseteq \mathcal{P}(X)$ is called a* ***$\boldsymbol{\sigma}$-algebra*** *if for any sequence $(A_n)_n$ of elements in $\mathcal{E}$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$.*

Remark 1.1.5. If $\mathcal{E}$ is a σ-algebra in X and $(A_n)_n \subseteq \mathcal{E}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{E}$ by De Morgan's identity (1.1.3). Moreover one can show (easy exercise!) that:

$$\liminf_{n\to\infty} A_n \in \mathcal{E}, \quad \limsup_{n\to\infty} A_n \in \mathcal{E}. \tag{1.1.7}$$

Exercise 1.1.6.

1. *$\mathcal{P}(X)$ and $\mathcal{A} = \{\emptyset, X\}$ are σ-algebras. $\mathcal{P}(X)$ is the largest σ-algebra in X and $\mathcal{A}$ the smallest.*

2. *In $X = [0,1)$, the class $\mathcal{A}_0$ consisting of $\emptyset$ and of all finite unions of half-closed intervals of the form*

$$A = \bigcup_{i=1}^{n} [a_i, b_i),\ 0 \le a_i < b_i \le a_{i+1} < b_{i+1} \ldots < b_n \le 1$$

is an algebra in $[0,1)$. Indeed,

$$A^c = [0, a_1) \cup [b_1, a_2) \cup \cdots \cup [b_n, 1) \in \mathcal{A}_0,$$

moreover, $\mathcal{A}_0$ is stable under finite unions. It suffices to observe that the union of two (not necessarily disjoint) intervals of the form $[a,b)$ and $[c,d)$ belongs to $\mathcal{A}_0$.

3. *In an infinite set X, consider the class $\mathcal{A}$.*

$$\mathcal{A} = \{A \in \mathcal{P}(X) \,|\, A \textit{ is finite, or } A^c \textit{ is finite}\}, \qquad (1.1.8)$$

$\mathcal{A}$ is an algebra. The only point that needs to be checked is that $\mathcal{A}$ is stable under a finite union. Let $A, B \in \mathcal{A}$. If A, B are both finite, then so is $A \cup B$. In all other cases, $(A \cup B)^c$ is finite. $\mathcal{A}$ but not a σ-algebra.

4. *In an uncountable set X, consider the class*

$$\mathcal{E} = \{A \in \mathcal{P}(X) \,|\, A \textit{ is countable or } A^c \textit{ is countable}\}. \qquad (1.1.9)$$

Then $\mathcal{E}$ is a σ-algebra which is strictly smaller than $\mathcal{P}(X)$ (here "countable" stands for "at most countable").

Definition 1.1.7. *Let $\mathcal{K} \subseteq \mathcal{P}(X)$. The intersection of all σ-algebras including K is called the* ***σ-algebra generated by*** *$\mathcal{K}$ and will be denoted by $\sigma(\mathcal{K})$.*

Exercise 1.1.8. *Let $\mathcal{K} \subseteq \mathcal{P}(X)$. Verify that the intersection of σ-algebras containing $\mathcal{K}$ is actually a σ-algebra and it is the smallest σ-algebra including $\mathcal{K}$.*

Example 1.1.9.

1. Let (X, d) be a metric space. The σ-algebra generated by all open sets of X is called the **Borel σ-algebra of X** and is denoted by $\mathcal{B}(X)$. The elements of $\mathcal{B}(X)$ are called **Borel sets**.

2. Let $X = \mathbb{R}$ and $\mathcal{I}$ be the set of all half-closed intervals $[a, b)$ with $a \leq b$. We show that $\sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R})$.

"$\sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathbb{R})$" : this follows from the fact that

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right).$$

" $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{I})$": Let V be an open set in $\mathbb{R}$, then it is a countable union of some family of open intervals. *Indeed, let (q_k) be the sequence including all rational numbers of V and denote by I_k the largest open interval contained in V and containing q_k. We clearly have $\cup_{k=0}^{\infty} I_k \subset V$, but also the opposite inclusion holds: it suffices to consider, for any $x \in V$, $r > 0$ such that $(x - r, x + r) \subset V$, and k such that $q_k \in (x - r, x + r)$ to obtain that $(x - r, x + r) \subset I_k$, by the maximality of I_k and then $x \in I_k$.*

Any open interval (a, b) can be represented as

$$(a, b) = \bigcup_{n=n_0}^{\infty} \left[a + \frac{1}{n}, b\right) \left(\frac{1}{n_0} < b - a\right).$$

Therefore $V \in \sigma(\mathcal{I})$. An analogous argument proves that $\mathcal{B}(\mathbb{R})$ is generated by half-closed intervals $(a, b]$ with $a \leq b$ by open intervals, by closed intervals and even by open and closed half-lines.

1.2 Measures

1.2.1 Additive and σ-Additive Functions

Definition 1.2.1. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and $\mu \colon \mathcal{A} \to [0, +\infty]$ with $\mu(\emptyset) = 0$.*

i) *We say that* ***μ is additive*** *if for any finite family $A_1, \ldots, A_n \in \mathcal{A}$ of mutually disjoint sets, we have*

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu(A_k). \tag{1.2.1}$$

ii) *We say that* ***μ is σ-additive*** *if, for any sequence $(A_n)_n \subseteq \mathcal{A}$ of mutually disjoint sets such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$, we have*

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k). \tag{1.2.2}$$

Remark 1.2.2.

1. If μ is σ-additive function on $\mathcal{A}$ then it is also additive.
2. If μ is additive on $\mathcal{A}$, it is monotone with respect to the inclusion: indeed, if $A, B \in \mathcal{A}$ and $B \subseteq A$, then $\mu(A) = \mu(B) + \mu(A \setminus B)$. Therefore, $\mu(A) \geq \mu(B)$.
3. Let μ be additive on $\mathcal{A}$ and let $(A_n)_n \subseteq \mathcal{A}$ be mutually disjoint sets such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$. Then
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \geq \mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu(A_k) \quad \forall n \in \mathbb{N}, \tag{1.2.3}$$
therefore
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \geq \sum_{k=1}^{\infty} \mu(A_k). \tag{1.2.4}$$
4. Any σ-additive function μ on $\mathcal{A}$ is **σ-subadditive**, that is, for any sequence $(A_n)_n$ in $\mathcal{A}$ such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \leq \sum_{k=1}^{\infty} \mu(A_k).$$
Indeed, let $(A_n)_n \subseteq \mathcal{A}$ be such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ and define $B_1 = A_1$ and $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k \in \mathcal{A}$. Then B_n are mutually disjoint, $\mu(B_n) \leq \mu(A_n)$ for all $n \in \mathbb{N}$ and
$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$
Therefore
$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i). \tag{1.2.5}$$
5. **In view of points 3 and 4 an additive function is σ-additive if and only if it is σ-subadditive.**

Definition 1.2.3. *A σ-additive function μ on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be*

- **finite**, *if $\mu(X) < +\infty$;*
- ***σ-finite***, *if there exists a sequence $(A_n)_n \subseteq \mathcal{A}$ such that*

$$\bigcup_{n=1}^{\infty} A_n = X \text{ and } \mu(A_n) < +\infty \quad \forall n \in \mathbb{N}.$$

Exercise 1.2.4. *In $X = \mathbb{N}$ consider the algebra*

$$\mathcal{A} = \big\{A \in \mathcal{P}(X) \,|\, A \text{ is finite or } A^c \text{ is finite}\big\}. \tag{1.2.6}$$

- *The function $\mu\colon \mathcal{A} \to [0, +\infty]$ defined as*

$$\mu(A) = \begin{cases} \# A & \text{if } A \text{ is finite} \\ \infty & \text{if } A^c \text{ is finite} \end{cases} \tag{1.2.7}$$

  is σ-additive ($\# A$ = number of elements of A).

- *The function $\nu\colon A \to [0, +\infty]$ defined as $\nu(\emptyset) = 0$ and*

$$\nu(A) = \begin{cases} \sum_{n \in A} \frac{1}{2^n} & \text{if } A \text{ is finite} \\ \infty & \text{if } A^c \text{ is finite} \end{cases} \tag{1.2.8}$$

  is additive but not σ-additive.

1.2.2 Measures and Measure Spaces

Let X be a set.

Definition 1.2.5. *A mapping μ: $\mathcal{P}(X) \to [0, +\infty]$ is called a **measure** on X if*

i) $\mu(\emptyset) = 0$ *and*

ii) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A \subseteq \bigcup_{k=1}^{\infty} A_k$.

Warning:
Most texts call such a mapping μ an outer measure reserving the name measure for μ restricted to the collection of μ-measurable subsets of X (see below).

Remark 1.2.6. The condition ii) in Definition 1.2.5 implies the subadditivity and the monotonicity.

Definition 1.2.7. *Let μ be a measure on X and $A \subseteq X$, then μ restricted to A written*

$$\mu \llcorner A$$

is the measure defined by

$$(\mu \llcorner A)(B) = \mu(A \cap B) \quad \forall B \subseteq X. \tag{1.2.9}$$

Definition 1.2.8. [Carathéodory criterion of measurability] *$A \subseteq X$ is called **μ-measurable** if for all $B \subseteq X$*

$$\boldsymbol{\mu(B) = \mu(B \cap A) + \mu(B \setminus A).} \tag{1.2.10}$$

Remark 1.2.9.

1) If $\mu(A) = 0$, then A is μ-measurable.
Indeed, let $B \subseteq X$. By the subadditivity, we have

$$\mu(B) \leq \mu(B \cap A) + \mu(B \setminus A). \tag{1.2.11}$$

On the other hand, we have

$$\mu(B \cap A) + \mu(B \setminus A) \leq \mu(A) + \mu(B) = \mu(B), \tag{1.2.12}$$

which yields the claim.

2) By the subadditivity the condition (1.2.10) is equivalent to

$$\mu(B) \geq \mu(B \cap A) + \mu\left(B \setminus A\right).$$

Theorem 1.2.10. *Let $\mu\colon \mathcal{P}(X) \to [0, +\infty]$ be a measure. Then the set*

$$\Sigma = \{A \subseteq X\colon\, A \textit{ is } \mu\textit{-measurable}\}$$

is a σ-algebra.

Proof. We proceed by steps.

i) $X \in \Sigma$: $\forall B \subseteq X$, it holds

$$\mu(B \cap X) + \mu(B \setminus X) = \mu(B) + \mu(\emptyset) = \mu(B). \tag{1.2.13}$$

ii) $A \in \Sigma \Longrightarrow A^c \in \Sigma$: $\forall B \in X$ it holds

$$B \cap A^c = B \setminus A, \quad B \setminus A^c = B \cap A. \tag{1.2.14}$$

Therefore

$$\mu(B \cap A^c) + \mu(B \setminus A^c) = \mu(B \setminus A) + \mu(B \cap A) = \mu(B). \tag{1.2.15}$$

iii) Let $A_1, \ldots, A_m \in \Sigma$. By induction on m we show that $\bigcup_{k=1}^m A_k \in \Sigma$.

$m = 1$ is trivial.

(induction step) $m - 1 \Longrightarrow m$: Let $A = \bigcup\limits_{k=1}^{m-1} A_k$.

By inductive hypothesis, we assume A is μ-measurable. Thus, for $\forall B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A). \tag{1.2.16}$$

Since A_m is also μ-measurable, we have

$$\begin{aligned} \mu(B \setminus A) &= \mu\big((B \setminus A) \cap A_m\big) + \mu\big((B \setminus A) \setminus A_m\big) \\ &= \mu\big((B \setminus A) \cap A_m\big) + \mu\big(B \setminus (A \cup A_m)\big). \end{aligned} \tag{1.2.17}$$

Now we observe that

$$B \cap (A \cup A_m) = (B \cap A) \cup [(B \setminus A) \cap A_m].$$

Thus by (1.2.17)

$$\begin{aligned} \mu(B) &= \mu(B \cap A) + \mu(B \setminus A) \\ &= \mu(B \cap A) + \mu\big((B \setminus A) \cap A_m\big) + \mu\big((B \setminus A) \setminus A_m\big) \\ &\geq \mu\big(B \cap (A \cup A_m)\big) + \mu\big(B \setminus (A \cup A_m)\big) \end{aligned} \tag{1.2.18}$$

From Remark 1.2.9 it follows that $\quad A \cup A_m = \bigcup_{k=1}^{m} A_k \in \Sigma$.

iv) Let $(A_k)_{k\in\mathbb{N}} \subset \Sigma$. We show that $A = \bigcup_{k=1}^{\infty} A_k$ is μ-measurable. We may suppose that $A_k \cap A_\ell = \emptyset$, $k \neq \ell$, otherwise we consider

$$\begin{aligned} &\widetilde{A}_1 = A_1,\ \widetilde{A}_2 = A_2 \setminus A_1,\ \widetilde{A}_3 = A_3 \setminus (A_1 \cup A_2) \\ &\widetilde{A}_n = A_n \setminus \Big(\bigcup_{k=1}^{n-1} A_k\Big). \end{aligned} \tag{1.2.19}$$

We can note that

$$\bigcup_{k=1}^{\infty} \widetilde{A}_k = \bigcup_{k=1}^{\infty} A_k.$$

Since each A_m is μ-measurable, we have

$$
\begin{aligned}
\mu\Big(B \cap \bigcup_{k=1}^{m} A_k\Big) &= \mu\Big(\Big(B \cap \bigcup_{k=1}^{m} A_k\Big) \cap A_m\Big) + \mu\Big(\Big(B \cap \bigcup_{k=1}^{m} A_k\Big) \setminus A_m\Big) \\
&= \mu(B \cap A_m) + \mu\Big(B \cap \bigcup_{k=1}^{m-1} A_k\Big) \\
&= \mu(B \cap A_m) + \mu\Big(\Big(B \cap \bigcup_{k=1}^{m-1} A_k\Big) \cap A_{m-1}\Big) + \mu\Big(\Big(B \cap \bigcup_{k=1}^{m-1} A_k\Big) \setminus A_{m-1}\Big) \\
&= \mu(B \cap A_m) + \mu(B \cap A_{m-1}) + \mu\Big(\Big(B \cap \bigcup_{k=1}^{m-2} A_k\Big)\Big) \\
&\quad \dots \\
&= \sum_{k=1}^{m} \mu(B \cap A_k).
\end{aligned} \tag{1.2.20}
$$

By using the monotonicity of μ, we get

$$
\begin{aligned}
\mu(B) &= \mu\Big(B \cap \bigcup_{k=1}^{m} A_k\Big) + \mu\Big(B \setminus \bigcup_{k=1}^{m} A_k\Big) \\
&\geq \sum_{k=1}^{m} \mu(B \cap A_k) + \mu(B \setminus A)
\end{aligned} \tag{1.2.21}
$$

We let $m \to +\infty$:

$$
\begin{aligned}
\mu(B) &\geq \sum_{k=1}^{\infty} \mu(B \cap A_k) + \mu(B \setminus A) \\
&\geq \mu(B \cap A) + \mu(B \setminus A)
\end{aligned}
$$

by the σ-subadditivity. Therefore $A \in \Sigma$, by the Remark 1.2.9 $\square$

Definition 1.2.11. *If μ: $\mathcal{P}(X) \to [0, +\infty]$ is a measure on X, Σ is the σ-algebra of μ-measurable sets, then (X, Σ, μ) is called a* ***Measure Space****.*

Exercise 1.2.12.

1. Let $x \in X$. Define, for every $A \in \mathcal{P}(X)$
$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A\,. \end{cases} \tag{1.2.22}$$
Then δ_x is a measure on $\mathcal{P}(X)$, called the **Dirac measure at $\boldsymbol{x}$**.

2. For every $A \in \mathcal{P}(X)$ we define
$$\mu(A) = \begin{cases} \#A & \text{if } A \text{ is finite} \\ +\infty & \text{otherwise}\,; \end{cases} \tag{1.2.23}$$
μ is a measure on $\mathcal{P}(X)$ and it is called the **counting measure**. Every A is μ-measurable.

3. $\mu\colon\ \mathcal{P}(X) \to [0, +\infty]$, $\mu(\emptyset) = 0$, $\mu(A) = 1$, $A \neq \emptyset$. In this case, $A \subseteq X$ is μ-measurable $\iff A = \emptyset$ or $A = X$.

Theorem 1.2.13. *Let (X, Σ, μ) be a* ***Measure Space*** *and let $A_k \in \Sigma$, $k \in \mathbb{N}$. Then the following conditions hold:*

i) ***σ-additivity****: $A_j \cap A_\ell = \emptyset$ $(j \neq \ell) \implies \mu\Big(\bigcup\limits_{k=1}^{\infty} A_k\Big) = \sum\limits_{k=1}^{\infty} \mu(A_k)$.*

ii) ***Continuity from below****: $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subset A_{k+1} \subseteq \ldots \implies \mu\Big(\bigcup\limits_{k=1}^{\infty} A_k\Big) = \lim\limits_{k\to+\infty} \mu(A_k)$.*

iii) ***Continuity from above****: $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k \supseteq A_{k+1} \supseteq \ldots$, $\mu(A_1) < +\infty$. Then*

$$\mu\Big(\bigcap_{k=1}^{\infty} A_k\Big) = \lim_{k\to+\infty} \mu(A_k).$$

Proof.

i) In the proof of Theorem 1.2.10 we showed that $\forall B \subseteq X$:

$$\mu\Big(B \cap \bigcup_{k=1}^{m} A_k\Big) = \sum_{k=1}^{m} \mu(B \cap A_k) \quad \forall m\,. \tag{1.2.24}$$

By taking $B = X$, we get in particular

$$\mu\Big(\bigcup_{k=1}^{m} A_k\Big) = \sum_{k=1}^{m} \mu(A_k). \tag{1.2.25}$$

By using the monotonicity and the σ-subadditivity, we get

$$\begin{aligned} \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) &\geq \lim_{m\to+\infty} \mu\Big(\bigcup_{k=1}^{m} A_k\Big) \\ &= \lim_{m\to+\infty} \sum_{k=1}^{m} \mu(A_k) \\ &= \sum_{k=1}^{\infty} \mu(A_k) \geq \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \end{aligned} \tag{1.2.26}$$

ii) Consider the pairwise disjoint family of sets $\widetilde{A}_k = A_k \setminus A_{k-1}$, $\widetilde{A}_1 = A_1$:

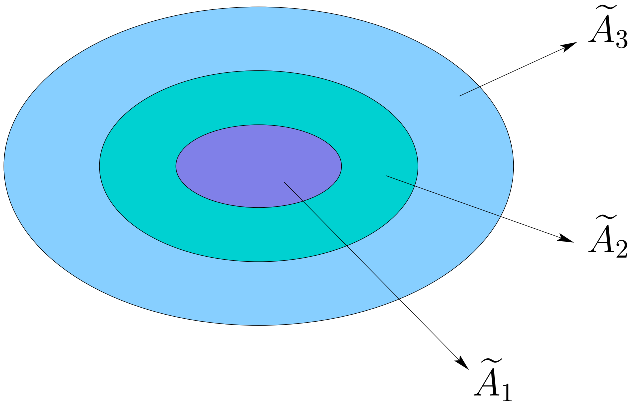


Fig. 1.1

By using i), we get

$$\begin{aligned}
\mu(\cup_{k=1}^{\infty} A_k) &= \mu(\cup_{k=1}^{\infty} \widetilde{A}_k) = \sum_{k=1}^{\infty} \mu(\widetilde{A}_k) = \lim_{m\to+\infty} \sum_{k=1}^{m} \mu(\widetilde{A}_k) \\
&= \lim_{m\to+\infty} \sum_{k=2}^{m} \mu(A_k) - \mu(A_{k-1}) + \mu(A_1) \\
&= \lim_{m\to+\infty} \mu(A_m).
\end{aligned} \tag{1.2.27}$$

iii) One considers $\widetilde{A}_k = A_1 \setminus A_k, k \in \mathbb{N}$. We have

$$\emptyset = \widetilde{A}_1 \subseteq \widetilde{A}_2 \subseteq \dots . \tag{1.2.28}$$

We observe that $\forall k \in \mathbb{N}$

$$\mu(A_1) = \mu(A_k) + \mu(\widetilde{A}_k).$$

Thus

$$\begin{aligned}
\mu(A_1) - \lim_{k\to+\infty} \mu(A_k) &= \lim_{k\to+\infty} \mu(\widetilde{A}_k) \\
&\underset{\text{ii)}}{=} \mu\Big(\bigcup_{k=1}^{\infty} \widetilde{A}_k\Big) = \mu\Big(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\Big) \\
&= \mu(A_1) - \mu\Big(\bigcap_{k=1}^{\infty} A_k\Big).
\end{aligned} \tag{1.2.29}$$

We can conclude. $\square$

Remark 1.2.14. If $\mu(A_1) = +\infty$, then iii) in Theorem 1.2.13 may be false. Consider

$$\mu(A) = \begin{cases} \#A & \text{if } A \text{ is finite} \\ +\infty & \text{otherwise}. \end{cases}$$

Take $A_n := \{m \in \mathbb{N} : m \geq n\}$. We have $\mu(A_1) = +\infty$ and

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = 0 \neq \lim_{k \to +\infty} \mu(A_k).$$

$\square$

1.2.3 Construction of Measures

Let $X \neq \emptyset$.

Definition 1.2.15. *Let $\mathcal{K} \subseteq \mathcal{P}(X)$. $\mathcal{K}$ is called a* **covering** *of X if*

i) $\emptyset \in \mathcal{K}$.

ii) $\exists (K_j)_{j \in \mathbb{N}} \subseteq \mathcal{K} : \quad X = \bigcup_{j=1}^{\infty} K_j.$

Example 1.2.16. The open intervals $I = \prod_{k=1}^{n} (a_k, b_k)$, $a_k \leq b_k \in \mathbb{R}$ are a covering of $X = \mathbb{R}^n$. Each algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ is a covering for X, since $\emptyset, X \in \mathcal{A}$.

Theorem 1.2.17. *Let $\mathcal{K}$ be a covering for X, $\lambda\colon \mathcal{K} \to [0,+\infty]$ with $\lambda(\emptyset) = 0$. Then*

$$\mu(A) = \inf\left\{ \sum_{j=1}^{\infty} \lambda(K_j) : K_j \in \mathcal{K},\ A \subseteq \bigcup_{j=1}^{\infty} K_j \right\} \tag{1.2.30}$$

is a measure on X.

Proof of Theorem 1.2.17. We observe that $\mu \geq 0$, $\mu(\emptyset) = 0$. We show that it is σ-subadditive. Let $A \subseteq \bigcup_{k=1}^{\infty} A_k$. The inequality $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ is trivial if the right-hand side is infinite. Therefore assume that $\sum_{k=1}^{\infty} \mu(A_k)$ is finite. For any $k \in \mathbb{N}$ and any $\varepsilon > 0$ there exists $(K_{j,k})_j \subseteq \mathcal{K}$ such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} K_{j,k} \text{ and } \sum_{j=1}^{\infty} \lambda(K_{j,k}) \leq \mu(A_k) + \frac{\varepsilon}{2^k}.$$

Since $A \subset \bigcup_{j,k} K_{j,k}$. By definition of μ, we have

$$\mu(A) \leq \sum_{j,k} \lambda(K_{j,k}) \leq \sum_{k=1}^{\infty} \mu(A_k) + \varepsilon. \tag{1.2.31}$$

We let $\varepsilon \to 0$ and the σ-subadditivity follows. $\square$

Exercise 1.2.18. $X \neq \emptyset$, $\mathcal{K} = \{\emptyset, X\}$, $\lambda\colon \mathcal{K} \to [0,+\infty]$, $\lambda(\emptyset) = 0$, $\lambda(X) = 1$. The measure on X induced by λ is exactly the measure μ defined by $\mu(\emptyset) = 0$ and $\mu(A) = 1$ if $A \neq \emptyset$.

If $\mathcal{K} = \mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\lambda\colon \mathcal{A} \to [0,+\infty]$ is additive, a natural question is: is the measure μ defined in Theorem 1.2.17 a σ-additive extension of λ? We will give a sufficient condition in order for this to hold.

Definition 1.2.19. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. A mapping λ: $\mathcal{A} \to [0, +\infty]$ is called a* **pre-measure** *if*

i) $\lambda(\emptyset) = 0$.

ii) $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$ *for every $A \in \mathcal{A}$ such that $A = \bigcup_{k=1}^{\infty} A_k$, $A_k \in \mathcal{A}$ mutually disjoint.*

λ is called **σ-finite** *if there exists a covering $X = \bigcup_{k=1}^{\infty} S_k$, $S_k \in \mathcal{A}$ and $\lambda(S_k) < +\infty$, $\forall k$ (we may assume without loss of generality the S_k's are mutually disjoint).*

Given a pre-measure λ, we obtain as above a measure μ defined by

$$\mu(A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) : A \subseteq \bigcup_{k=1}^{\infty} A_k,\ A_k \in \mathcal{A} \right\}. \tag{1.2.32}$$

Theorem 1.2.20. *(Carathéodory-Hahn extension) Let λ: $\mathcal{A} \to [0, +\infty]$ be a pre-measure on X, μ be defined as in (1.2.32). Then it holds:*

i) *μ: $\mathcal{P}(X) \to [0, +\infty]$ is a measure.*

ii) *$\mu(A) = \lambda(A) \quad \forall A \in \mathcal{A}$.*

iii) *$\forall A \in \mathcal{A}$ is μ-measurable.*

Proof of Theorem 1.2.20.

i) It follows from Theorem 1.2.17.

ii) Let $A \in \mathcal{A}$. Clearly it holds $\mu(A) \leq \lambda(A)$.

For the opposite inequality, let $A \subseteq \bigcup_{k=1}^{\infty} A_k$, A_k mutually disjoint, $A_k \in \mathcal{A}$. We set $\widetilde{A}_k = A_k \cap A \in \mathcal{A}$. Clearly, $\widetilde{A}_k$ are mutually disjoint and $A = \bigcup_{k=1}^{\infty} \widetilde{A}_k$. Since λ is a pre-measure, it follows

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(\widetilde{A}_k) \leq \sum_{k=1}^{\infty} \lambda(A_k). \tag{1.2.33}$$

Thus

$$\begin{aligned}\lambda(A) &\leq \inf\left\{\sum_{k=1}^{\infty}\lambda(A_k)\ A \subseteq \bigcup_{k=1}^{\infty} A_k,\ A_k \in \mathcal{A}\right\} \\ &= \mu(A).\end{aligned} \tag{1.2.34}$$

iii) Let $A \in \mathcal{A}$ and $B \subseteq X$ be arbitrary. For every $\varepsilon > 0$ choose $B_k \in \mathcal{A}$, $k \in \mathbb{N}$ with $B \subseteq \bigcup_{k=1}^{\infty} B_k$ and

$$\boxed{\sum_{k=1}^{\infty}\lambda(B_k) \leq \mu(B) + \varepsilon\,.}$$

Since $A, B_k \in \mathcal{A}$, we have

$$\lambda(B_k) = \lambda(B_k \cap A) + \lambda(B_k \setminus A). \tag{1.2.35}$$

Moreover, we have

$$\begin{aligned} B \cap A &\subseteq \bigcup_{k=1}^{\infty} B_k \cap A \\ B \setminus A &\subseteq \bigcup_{k=1}^{\infty}(B_k \setminus A).\end{aligned} \tag{1.2.36}$$

Thus, by definition of μ, we have

$$\begin{aligned}\mu(B \cap A) + \mu(B \setminus A) &\leq \sum_{k=1}^{\infty}\lambda(B_k \cap A) + \sum_{k=1}^{\infty}\lambda(B_k \setminus A) \\ &= \sum_{k=1}^{\infty}\lambda(B_k) \leq \mu(B) + \varepsilon\end{aligned} \tag{1.2.37}$$

We let $\varepsilon \to 0$ and we conclude the proof. □

Theorem 1.2.21. (Uniqueness of Charathéodory-Hahn extension)

Let $\lambda\colon \mathcal{A} \to [0, +\infty]$ be as in Theorem 1.2.20 and σ-finite, μ be the measure induced by λ, Σ be the σ-algebra of the μ-measurable sets and let $\widetilde{\mu}\colon \mathcal{P}(X) \to [0, +\infty]$ be another measure with $\widetilde{\mu}_{|\mathcal{A}} = \lambda$. Then $\widetilde{\mu}_{|\Sigma} = \mu$, namely the Charathéodory-Hahn extension is unique.

Proof. i). Let $A \subseteq \cup_{k=1}^{\infty} A_k$ with $A_k \in \mathcal{A}$, $k \in \mathbb{N}$. From the σ-sub-additivity of $\tilde{\mu}$ it follows

$$\tilde{\mu}(A) \leq \Sigma_{k=1}^{\infty} \tilde{\mu}(A_k) = \Sigma_{k=1}^{\infty} \lambda(A_k).$$

By taking the infimum over all $(A_k)_k$ we obtain

$$\tilde{\mu}(A) \leq \mu(A). \tag{1.2.38}$$

ii) We show that if $A \in \Sigma$ then also the inverse inequality holds. To this purpose we consider first of all the case there is $S \in \mathcal{A}$ such that $A \subseteq S$ and $\lambda(S) < +\infty$.

By i) we get

$$\tilde{\mu}(S \setminus A) \leq \mu(S \setminus A) \leq \mu(S) = \lambda(S) < \infty. \tag{1.2.39}$$

Since $A \in \Sigma$, $S \in \mathcal{A} \subset \Sigma$ it follows by the sub-additivity of $\tilde{\mu}$ that

$$\begin{aligned} \tilde{\mu}(A) + \tilde{\mu}(S \setminus A) &\leq \mu(A) + \mu(S \setminus A) = \mu(S) \\ &= \lambda(S) = \tilde{\mu}(S) \leq \tilde{\mu}(A) + \tilde{\mu}(S \setminus A). \end{aligned} \tag{1.2.40}$$

From (1.2.39) and (1.2.40) it follows that $\mu(A) = \tilde{\mu}(A)$.

Suppose now that $X = \cup_{k=1}^{\infty} S_k$ where $(S_k)_{k\in\mathbb{N}}$ are mutually disjoint, $S_k \in \mathcal{A}$, $\lambda(S_k) < \infty$, for all $k \in \mathbb{N}$. We consider the disjoint union $A = \cup_{k=1}^{\infty} A_k$, $A_k = A \cap S_k$, $k \in \mathbb{N}$. For every m it holds

$$\tilde{\mu}(\cup_{k=1}^{m} A_k) = \mu(\cup_{k=1}^{m} A_k).$$

By taking the limit $m \to +\infty$ we get

$$\tilde{\mu}(A) \geq \lim_{m\to\infty} \tilde{\mu}(\cup_{k=1}^{m} A_k) = \lim_{m\to\infty} \mu(\cup_{k=1}^{m} A_k) = \mu(A).$$

We can conclude the proof. $\square$

Remark 1.2.22. *i) From Theorem 1.2.21 it is not clear if every $A \in \Sigma$ is also $\tilde{\mu}$-measurable.*
ii) In general the statement of Theorem 1.2.21 cannot be improved as the following example shows.

Example 1.2.23. *Let* $X = [0,1]$*,* $\mathcal{A} = \{\emptyset, X\}$ *and* $\lambda\colon \mathcal{A} \to [0,+\infty]$*,* μ *as in the exercise 1.2.18 . We have* $\Sigma = \{\emptyset, X\}$*. We will see in the next section that Lebesgue measure* $\mathcal{L}^1$ *is also an extension of* λ *but* $\mathcal{L}^1([0,1/2]) = 1/2 \neq \mu([0,1/2]) = 1$.

1.3 Lebesgue Measure

Lebesgue measure is the extension of "volume" of the so-called elementary sets.

Definition 1.3.1.

i) *For* $a = (a_1, \ldots, a_n)$*,* $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ *we set*

$$(a,b) = \begin{cases} \prod_{i=1}^{n} (a_i, b_i) & \text{if } a_i < b_i,\ 1 \le i \le n \\ \emptyset & \text{otherwise.} \end{cases} \tag{1.3.1}$$

In an analogous way we define the intervals $[a,b]$*,* $(a,b]$*,* $[a,b)$*. In case of open (half-open) intervals* (a_i,b_i) *we allow* $a_i = -\infty$ *or* $b_i = +\infty$*.*

ii) *The volume of an interval is defined to be*

$$\begin{aligned} \mathrm{vol}\big((a,b)\big) &= \mathrm{vol}\big((a,b]\big) = \mathrm{vol}\big([a,b)\big) = \mathrm{vol}\big([a,b]\big) \\ &= \begin{cases} \prod_{i=1}^{n} (b_i - a_i) \le +\infty & \text{if } a_i < b_i,\ 1 \le i \le n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

iii) *An* **elementary set** *is the finite disjoint union of intervals with volume*

$$\mathrm{vol}\Big(\bigcup_{k=1}^{m} I_k\Big) = \sum_{k=1}^{m} \mathrm{vol}(I_k).$$

Remark 1.3.2.

i) Let $I = \bigcup_{k=1}^{m} I_k = \bigcup_{j=1}^{n} J_j$ where I_k, J_j are intervals with $I_k \cap I_\ell = \emptyset$ if $k \neq \ell$ and $J_h \cap J_i = \emptyset$, if $h \neq i$, then

$$\sum_{k=1}^{m} \mathrm{vol}(I_k) = \sum_{j=1}^{n} \mathrm{vol}(J_j).$$

ii) The class of all elementary sets is an algebra, the vol function is a pre-measure. In particular, we have the following property: if I is an elementary set such that $I = \bigcup_{k=1}^{\infty} I_k$, I_k elementary mutually disjoint sets, then one can see (easy exercise) that

$$\mathrm{vol}(I) = \sum_{k=1}^{\infty} \mathrm{vol}(I_k).$$

In what follows, we shall use the notion of dyadic cubes to obtain a basic decomposition of open sets in $\mathbb{R}^n$. For every $k \in \mathbb{N}$ let $\mathcal{D}_k$ be the collection of half open cubes

$$\mathcal{D}_k = \left\{ \prod_{i=1}^{n} \Big[\frac{a_i}{2^k}, \ \frac{a_i+1}{2^k}\Big), \ a_i \in \mathbb{Z} \right\}.$$

In other words, $\mathcal{D}_0$ is the collection of cubes with edge length 1 and vertices at points with integer coordinates. Bisecting each edge of a cube in $\mathcal{D}_0$, we obtain from it 2^n subcubes of edge length $\frac{1}{2}$. If we continue bisecting, we obtain finer and finer collections $\mathcal{D}_k$ of cubes such that each cube in $\mathcal{D}_k$ has edge length 2^{-k} and is the union of 2^n disjoint cubes in $\mathcal{D}_{k+1}$.

The cubes of the collection

$$\big\{Q : \ Q \in \mathcal{D}_k, \ k = 0, 1, 2, \dots\big\}$$

are called **dyadic cubes**.

Remark 1.3.3. Dyadic cubes have the following properties:

a) For every $k \geq 0$, $\mathbb{R}^n = \bigcup_{Q \in \mathcal{D}_k} Q$ with disjoint union.

b) If $Q \in \mathcal{D}_k$ and $P \in \mathcal{D}_h$ with $h \leq k$, then $Q \subset P$ or $P \cap Q = \emptyset$.

c) If $Q \in \mathcal{D}_k$, then $\mathrm{vol}(Q) = 2^{-kn}$.

Lemma 1.3.4. *Every open set in $\mathbb{R}^n$ can be written as a countable union of disjoint dyadic cubes.*

Proof. Let $\emptyset \neq V$ be an open set in $\mathbb{R}^n$. Let $\mathcal{S}_0$ be the collection of all cubes in $\mathcal{D}_0$ which lie entirely in V. Let $\mathcal{S}_1$ be the set of cubes in $\mathcal{D}_1$ which lie in V but which are not subcubes of any cube in $\mathcal{S}_0$. More generally, for $k \geq 1$, let $\mathcal{S}_k$ be the cubes in $\mathcal{D}_k$ which lie in V but which are not subcubes of any cube in $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{k-1}$ (see Fig. 1.2).

If $\mathcal{S} = \cup_k \mathcal{S}_k$, then $\mathcal{S}$ is countable since each $\mathcal{D}_k$ is countable and the cubes in $\mathcal{S}$ are non-overlapping by construction. Moreover, since V is open and the cubes in $\mathcal{D}_k$ become arbitrarily small as $k \to +\infty$, then by Remark 1.3.3 a) each point of V will eventually lie in a cube of some $\mathcal{S}_k$. Hence $V = \bigcup_{Q \in \mathcal{S}} Q$ and the proof is complete. $\square$

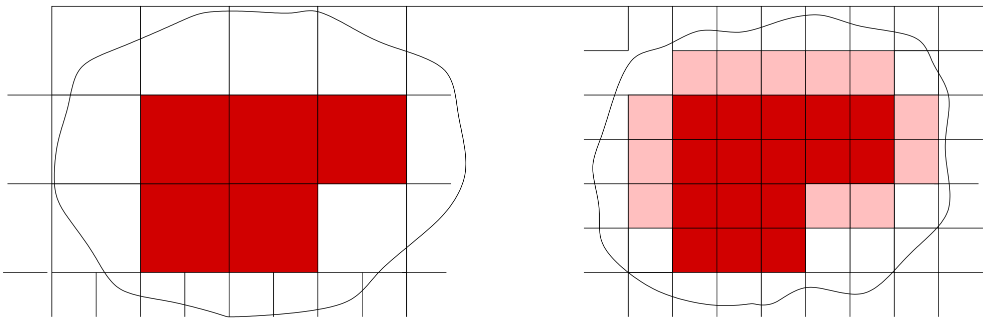


Fig: 1.2

Definition 1.3.5. *The Lebesgue measure $\mathcal{L}^n$ is the Carathéodory-Hahn extension of the volume defined on the algebra of elementary sets.*

Definition 1.3.6. *A measure μ on $\mathbb{R}^n$ is called* **Borel**, *if every Borel set is μ-measurable.*

As a consequence of Lemma 1.3.4 we get that $\mathcal{L}^n$ is a Borel measure and the σ-algebra of the $\mathcal{L}^n$-measurable sets contains the σ-algebra of the Borel sets.

In the sequel we are going to list some properties of Lebesgue measure.

Theorem 1.3.7. *For every $A \subseteq \mathbb{R}^n$ it holds*

$$\mathcal{L}^n(A) = \inf_{A \subseteq G} \mathcal{L}^n(G), \quad G \text{ open.} \tag{1.3.2}$$

Proof. "$\leq$": This inequality follows from the monotonicity: if $A \subseteq G \Longrightarrow \mathcal{L}^n(A) \leq \mathcal{L}^n(G)$.

"$\geq$": We may suppose without loss of generality that $\mathcal{L}^n(A) < +\infty$ (otherwise the inequality is trivial). For $\varepsilon > 0$, choose intervals $(I_\ell)_{\ell \in \mathbb{N}}$ with $A \subseteq \bigcup_{\ell=1}^\infty I_\ell$ and

$$\sum_{\ell=1}^\infty \operatorname{vol}(I_\ell) \leq \mathcal{L}^n(A) + \varepsilon.$$

Since we are assuming that $\mathcal{L}^n(A) < +\infty$ we have $\operatorname{vol}(I_\ell) < +\infty$ for every $\ell > 0$.

For every ℓ we choose an open bounded interval $\widetilde{I}_\ell \supset I_\ell$ such that $\operatorname{vol}(\widetilde{I}_\ell) \leq \operatorname{vol}(I_\ell) + \frac{\varepsilon}{2^\ell}$, $\ell \in \mathbb{N}$. We set $G = \bigcup_{\ell=1}^\infty \widetilde{I}_\ell$, G is an open set containing A. Moreover,

$$\begin{aligned} \mathcal{L}^n(G) &\leq \sum_{\ell=1}^\infty \operatorname{vol}(\widetilde{I}_\ell) \leq \sum_{\ell=1}^\infty \operatorname{vol}(I_\ell) + \sum_{\ell=1}^\infty \frac{\varepsilon}{2^\ell} \\ &\leq \mathcal{L}^n(A) + 2\varepsilon. \end{aligned} \tag{1.3.3}$$

We let $\varepsilon \to 0$ and get the result. $\square$

Theorem 1.3.8. *Let $A \subseteq \mathbb{R}^n$. The following conditions are equivalent:*

i) *A is $\mathcal{L}^n$-measurable.*

ii) $\forall \varepsilon > 0 \;\; \exists G \supseteq A$ *open:* $\mathcal{L}^n(G \setminus A) < \varepsilon$.

Proof of Theorem 1.3.8.

i) $\Longrightarrow$ ii):

• Assume first that $\mathcal{L}^n(A) < \infty$. For every $\varepsilon > 0$, choose $G \supseteq A$, G open with

$$\mathcal{L}^n(G) \leq \mathcal{L}^n(A) + \varepsilon.$$

Since A is $\mathcal{L}^n$-measurable, by choosing $B = G$ in the criterion of measurability, we get

$$\begin{aligned} \mathcal{L}^n(G) &= \mathcal{L}^n(G \cap A) + \mathcal{L}^n(G \setminus A) \\ &= \mathcal{L}^n(A) + \mathcal{L}^n(G \setminus A). \end{aligned} \tag{1.3.4}$$

Therefore,

$$\mathcal{L}^n(G \setminus A) = \mathcal{L}^n(G) - \mathcal{L}^n(A) \leq \varepsilon. \tag{1.3.5}$$

• If $\mathcal{L}^n(A) = \infty$, we set $A_k = A \cap [-k, k]^n$, $k \in \mathbb{N}$, with $A = \bigcup_{\ell=1}^{\infty} A_\ell$. For every $\varepsilon > 0$ choose $G_k \supseteq A_k$, G_k open with

$$\mathcal{L}^n(G_k \setminus A_k) < \frac{\varepsilon}{2^k}, \quad k \in \mathbb{N}.$$

Then $G = \bigcup_{\ell=1}^{\infty} G_k$ is open, $A \subseteq G$ and

$$\mathcal{L}^n(G \setminus A) \leq \sum_{k=1}^{\infty} \mathcal{L}^n(G_k \setminus A_k) < \varepsilon, \quad \text{since}$$

$$G \setminus A = \bigcup_{k=1}^{\infty} (G_k \setminus A) \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus A_k).$$

ii) $\Longrightarrow$ i): For $\varepsilon > 0$, choose $A \subseteq G$, G open with

$$\mathcal{L}^n(G \setminus A) \leq \varepsilon.$$

Since G is $\mathcal{L}^n$-measurable, we have for every $B \subseteq \mathbb{R}^n$:

$$\begin{aligned}\mathcal{L}^n(B) &= \mathcal{L}^n(B \cap G) + \mathcal{L}^n(B \setminus G) \\ &\geq \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A) - \mathcal{L}^n(G \setminus A) \\ &\geq \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A) - \varepsilon.\end{aligned} \tag{1.3.6}$$

In the second inequality, we have used the relation $B \setminus A \subseteq (B \setminus G) \cup (G \setminus A)$. We let $\varepsilon \to 0$ and we get the result. □

Corollary 1.3.9. *Let $A \subseteq \mathbb{R}^n$. Then A is $\mathcal{L}^n$-measurable if and only if it can be "approximated" from inside and outside: $\forall \varepsilon > 0 \quad \exists F \subseteq A \subseteq G$, F closed, G open such that*

$$\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \leq \varepsilon. \tag{1.3.7}$$

Proof. If (1.3.7) holds, then the condition ii) of Theorem 1.3.8 holds and therefore A is $\mathcal{L}^n$-measurable. Conversely, for $\varepsilon > 0$ choose $G \supseteq A^c$ open such that

$$\mathcal{L}^n(G \setminus A^c) < \varepsilon \quad \text{(Theorem 1.3.8)} .$$

Set $F = G^c \subseteq A$. Then F is closed with

$$\mathcal{L}^n(A \setminus F) = \mathcal{L}^n(A \cap G) = \mathcal{L}^n(G \setminus A^c) < \varepsilon$$

and (1.3.7) follows. □

Corollary 1.3.10. *$A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable iff it holds*

$$\forall \varepsilon > 0 \ \exists F \subseteq A \subseteq G,\ F \text{ closed, } G \text{ open: } \mathcal{L}^n(G \setminus F) < \varepsilon. \tag{1.3.8}$$

Proof. We observe that if $F \subseteq A \subseteq G$ then

$$G \setminus F = (G \setminus A) \cup (A \setminus F). \tag{1.3.9}$$

(1.3.7) $\implies$ (1.3.8): From (1.3.9) it follows that

$$\mathcal{L}^n(G \setminus F) \leq \mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F).$$

(1.3.8) $\implies$ (1.3.7): We observe that

$$G \setminus A \subseteq G \setminus F, \quad A \setminus F \subseteq G \setminus F.$$

Therefore

$$\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \leq 2\mathcal{L}^n(G \setminus F).$$

□

1.4 Comparison between Lebesgue and Jordan Measure

To conclude, we compare Lebesgue measure with Jordan measure.

We recall that $A \subseteq \mathbb{R}^n$ bounded is **Jordan-measurable** if $\underline{\mu}(A) = \overline{\mu}(A)$, where

$$\underline{\mu}(A) = \underline{\int}_{\mathbb{R}^n} \chi_A \, d\mu := \sup \left\{ \mathrm{vol}(E) \colon \; E \subseteq A, \; E \text{ elementary set} \right\}$$

$$\overline{\mu}(A) = \overline{\int}_{\mathbb{R}} \chi_A \, d\mu := \inf \left\{ \mathrm{vol}(E) : \quad A \subseteq E, \; E \text{ elementary set} \right\};$$

In this case we denote the common value by $\mu(A)$.

χ_A denotes the characteristic function of A:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \,. \end{cases}$$

Theorem 1.4.1. *Let $A \subseteq \mathbb{R}^n$ be bounded, then*

i) $\underline{\mu}(A) \leq \mathcal{L}^n(A) \leq \overline{\mu}(A)$.

ii) *If A is Jordan-measurable, then A is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(A) = \mu(A)$.*

Proof.

i) It holds

$$\begin{aligned}\mathcal{L}^n(A) &= \inf\left\{\sum_{k=1}^{\infty} \operatorname{vol}(I_k), \quad A \subseteq \bigcup_{k=1}^{\infty} I_k,\ I_k \text{ intervals}\right\}\\ &\leq \inf\left\{\sum_{k=1}^{m} \operatorname{vol}(I_k) \quad A \subseteq E = \bigcup_{k=1}^{m} I_k,\ I_k \text{ intervals}\right\}\\ &= \overline{\mu}(A).\end{aligned} \tag{1.4.10}$$

Moreover, for every $E = \bigcup_{k=1}^{m} I_k \subseteq A$, I_k mutually disjoint, it holds

$$\operatorname{vol}(E) = \mathcal{L}^n(E) \leq \mathcal{L}^n(A) \tag{1.4.11}$$

and therefore

$$\underline{\mu}(A) \leq \mathcal{L}^n(A).$$

ii) If A is Jordan-measurable, then $\underline{\mu}(A) = \overline{\mu}(A)$ and from i), we have

$$\mu(A) = \underline{\mu}(A) = \overline{\mu}(A) = \mathcal{L}^n(A). \tag{1.4.12}$$

Now we show that A is $\mathcal{L}^n$-measurable. Since A is bounded we have $\mathcal{L}^n(A) < +\infty$. Since A is Jordan-measurable, $\forall \varepsilon > 0\ \exists I_\varepsilon$, I^ε elementary sets such that $I^\varepsilon \supseteq A \supseteq I_\varepsilon$ and

$$\operatorname{vol}(I^\varepsilon) - \varepsilon < \mu(A) < \operatorname{vol}(I_\varepsilon) + \varepsilon.$$

We may assume that $G = I^\varepsilon$ is open. It follows that

$$\begin{aligned}\mathcal{L}^n(G \setminus A) &\leq \mathcal{L}^n(I^\varepsilon \setminus I_\varepsilon) = \operatorname{vol}(I^\varepsilon \setminus I_\varepsilon)\\ &= \operatorname{vol}(I^\varepsilon) - \operatorname{vol}(I_\varepsilon) < 2\varepsilon.\end{aligned} \tag{1.4.13}$$

Therefore A is $\mathcal{L}^n$-measurable according to Theorem 1.3.8. □

Theorem 1.4.2. *Lebesgue measure is invariant under* **isometries** *namely under* $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x_0 + Rx$, *where* R *is a rotation* $(R^t R = Id)$.

Proof. Exercise! □

We present a further property of Lebesgue measure.

Definition 1.4.3. *A Borel measure μ on $\mathbb{R}^n$ is called* ***Borel regular*** *if for every $A \subseteq \mathbb{R}^n$ there exists $B \supseteq A$ Borel set such that $\mu(A) = \mu(B)$.*

Corollary 1.4.4. *$\mathcal{L}^n$ is Borel regular.*

Proof. Without loss of generality let $\mathcal{L}^n(A) < \infty$ (otherwise $B = \mathbb{R}^n$). Choose G_k open such that $A \subseteq G_k$ and

$$\mathcal{L}^n(G_k) < \mathcal{L}^n(A) + \frac{1}{k}\,,\ k \in \mathbb{N}.$$

In particular $\mathcal{L}^n(G_1) < \mathcal{L}^n(A) + 1 < +\infty$. Without loss of generality we may assume $G_{k+1} \subseteq G_k, k \in \mathbb{N}$. Set $B = \bigcap_{k=1}^{\infty} G_k$. Then B is Borel and

$$\mathcal{L}^n(B) = \lim_{k\to+\infty} \mathcal{L}^n(G_k) = \mathcal{L}^n(A).$$

□

1.5 An example of a non-measurable set

In this section we will construct a subset of $\mathbb{R}$ which is not $\mathcal{L}^1$ measurable. We will need the axiom of the choice in the following form:

Zermelo's Axiom:
For any family of arbitrary nonempty disjoint sets indexed by a set A, $\{E_\alpha,\ \alpha \in A\}$, there exists a set consisting of exactly one element from each E_α, $\alpha \in A$.

For $x, y \in [0, 1)$ we define

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y < 1 \\ x + y - 1, & \text{if } x + y \geq 1. \end{cases} \tag{1.5.1}$$

We observe that if $E \subseteq [0,1)$ is a $\mathcal{L}^1$-measurable set, then $E \oplus x \subseteq [0,1)$ is also $\mathcal{L}^1$-measurable and $\mathcal{L}^1(E \oplus x) = \mathcal{L}^1(E) \ \ \forall x \in [0,1)$. Indeed, we have

$$E \oplus x = E_1 \cup E_2 \tag{1.5.2}$$

where

$$\begin{aligned} E_1 &:= E \cap [0, 1-x) \oplus x = E \cap [0, 1-x) + x \\ E_2 &:= E \cap [1-x, 1) \oplus x = E \cap [1-x, 1) + (x-1). \end{aligned} \tag{1.5.3}$$

E_1, E_2 are $\mathcal{L}^1$-measurable with

$$\begin{aligned} \mathcal{L}^1(E_1) &= \mathcal{L}^1\big(E \cap [0, 1-x)\big) \\ \mathcal{L}^1(E_2) &= \mathcal{L}^1\big(E \cap [1-x, 1)\big) \end{aligned}$$

and $E_1 \cap E_2 = \emptyset$. Therefore, $E \oplus x$ is measurable and

$$\begin{aligned} \mathcal{L}^1(E \oplus x) &= \mathcal{L}^1(E_1) + \mathcal{L}^1(E_2) \\ &= \mathcal{L}^1\big(E \cap [0, 1-x)\big) + \mathcal{L}^1\big(E \cap [1-x, 1)\big) \\ &= \mathcal{L}^1(E). \end{aligned}$$

In $[0,1)$ we define the following equivalence relation: $x, y \in [0,1)$, $x \sim y$ if $x - y \in \mathbb{Q}$. By the Axiom of Choice, there exists a set $P \subseteq [0,1)$ such that P consists of exactly one representative point from each equivalence class. Let $\mathbb{Q} \cap [0,1) = \{r_j\}_{j \in \mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1)$ with $r_0 = 0$, and for $j \in \mathbb{N}$ we define

$$P_j = P \oplus r_j. \tag{1.5.4}$$

Observe the following.

1. (P_j) is a disjoint family.

   Let $x \in P_n \cap P_m$, then $x = p_n \oplus r_n = p_m \oplus r_m$, r_n, $r_m \in \mathbb{Q}$. Therefore $p_n - p_m \in \mathbb{Q}$, namely p_n, p_m are in the same equivalence class. By construction $p_n = p_m$ and $r_n = r_m \Longrightarrow P_n \equiv P_m$.

2. $[0,1) = \bigcup\limits_{j=0}^{+\infty} P_j$.

Let $x \in [0,1)$ and $[x]$ be its equivalence class. There exists a unique $p \in P$ such that $x \sim p$. If $x = p$, then $x \in P_0 = P$; if $x > p$, then $x = p + r_i = p \oplus r_i$ for some $r_i \Longrightarrow x \in P_i$, if $x < p$ then $x = p + r_j - 1 = p \oplus r_j$ for some r_j, therefore $x \in P_j$. It follows that

$$[0,1) = \bigcup_{j=0}^{+\infty} P_j. \tag{1.5.5}$$

3. If P is Lebesgue measurable, then P_j would be Lebesgue measurable as well and

$$1 = \mathcal{L}^1\big([0,1)\big) = \sum_{i=1}^{\infty} \mathcal{L}^1(P_j) = \sum_{i=1}^{\infty} \mathcal{L}^1(P).$$

This is impossible since the right-hand side is $= +\infty$ or $= 0$.

Since P is not $\mathcal{L}^1$-measurable, there exists $B \subseteq \mathbb{R}$ such that

$$\mathcal{L}^1(B) < \mathcal{L}^1(B \cap P) + \mathcal{L}^1(B \setminus P).$$

$B \cap P$ and $B \setminus P$ are two disjoint sets for which the additivity of the measure does not hold. We observe that $\mathcal{L}^1(P) > 0$ (if $\mathcal{L}^1(P) = 0 \Longrightarrow P$ is $\mathcal{L}^1$-measurable). If $E \subseteq P$ is $\mathcal{L}^1$-measurable, then $\mathcal{L}^1(E) = 0$. Indeed, set $E_i = E \oplus r_i$, E_i is $\mathcal{L}^1$-measurable and $\mathcal{L}^1(E_i) = \mathcal{L}^1(E)$. Moreover, $\bigcup_{i=0}^{\infty} E_i = F \subset [0,1)$, F, $\mathcal{L}^1$-measurable with

$$\begin{aligned} 1 = \mathcal{L}^1([0,1]) \geq \mathcal{L}^1(F) &= \sum_{i=0}^{\infty} \mathcal{L}^1(E_i) \\ &= \sum_{i=0}^{\infty} \mathcal{L}^1(E) \\ \Longrightarrow \mathcal{L}^1(E) = 0 & \end{aligned} \tag{1.5.6}$$

Exercise 1.5.1. For every $A \subseteq \mathbb{R}$ with $\mathcal{L}^1(A) > 0$, there exists $B \subset A$ such that B is not $\mathcal{L}^1$-measurable.

Proof. Suppose for instance that $A \subseteq (0,1)$. Set for $i \geq 0$, $B_i = A \cap P_i$. If B_i were $\mathcal{L}^1$-measurable, then $\mathcal{L}^1(B_i) = 0$ (since $B_i \subseteq P_i$), therefore $\sum_{i=0}^{\infty} \mathcal{L}^1(B_i) = 0$. Since $A = \bigcup_{i=0}^{\infty} B_i$, we would get a contradiction. $\square$

Exercise 1.5.2. Show that every countable subset of $\mathbb{R}$ has Lebesgue measure zero.

Proof. Let $\alpha \in \mathbb{R}$. Then $\{\alpha\} \subseteq [\alpha - \varepsilon, \alpha + \varepsilon]\ \forall \varepsilon > 0$. Therefore

$$\mathcal{L}^1\{\alpha\} \leq \mathrm{vol}[\alpha - \varepsilon, \alpha + \varepsilon] = 2\varepsilon \quad \forall \varepsilon > 0 \tag{1.5.7}$$

and this gives $\mathcal{L}^1\{\alpha\} = 0$.

If $A = \{\alpha_1, \alpha_2, \ldots\} = \bigcup_{n=1}^{\infty} \{\alpha_n\}$ is a countable set, then

$$\mathcal{L}^1(A) \leq \sum_n \mathcal{L}^1\{\alpha_n\} = 0. \tag{1.5.8}$$

$\square$

1.6 An uncountable $\mathcal{L}^n$-null set: the Cantor Triadic Set

Let $X = [0, 1]$. For any fixed integer $b \geq 2$ and any $x \in [0, 1]$ one can expand x in base b: there exists a sequence of "digits" $d_i(x) \in \{0, \ldots, b - 1\}$ such that

$$x = \sum_{i=1}^{\infty} d_i(x)\, b^{-i}$$

which is also denoted $x = 0, d_1, d_2 \ldots$. This expansion is not entirely unique (for instance, in base 10, we have $0.1 = 0.099 = 0.0\overline{9}$). However, the set of those $x \in X$ for which there exists more than one expansion is countable.

The Cantor set is defined as follows: take $b = 3$, so the digits are in $\{0, 1, 2\}$ and let

$$C = \big\{x \in [0, 1] \,|\, d_i(x) \in \{0, 2\} \quad \forall i\big\}$$

be the set of those x which have no digit 1 in their 3-expansion.

Claim 1.6.1. $\mathcal{L}^1(C) = 0$, *but C is not countable.*

Proof of Claim 1.6.1.

- We observe that $C = \bigcap_{n=1}^{\infty} C_n$, where

$$C_n = \{x \in [0,1] \,|\, d_i(x) \neq 1 \quad \forall i \leq n\}.$$

Each C_n is a Borel subset of $[0,1]$; indeed, we have for instance

$$\begin{aligned} C_1 &= \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right] \\ C_2 &= \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right]. \end{aligned} \tag{1.6.1}$$

More generally, C_n is a Borel subset of $[0,1]$ which is a disjoint union of 2^n intervals of length 3^{-n}. Thus by additivity

$$\mathcal{L}^1(C_n) = 2^n \cdot 3^{-n} = \left(\tfrac{2}{3}\right)^n$$

and since $C_{n+1} \subseteq C_n$, $\mathcal{L}^1(C_1) < +\infty$, we have

$$\mathcal{L}^1(C) = \mathcal{L}^1\Big(\bigcap_n C_n\Big) = \lim_{n \to +\infty} \mathcal{L}^1(C_n) = 0. \tag{1.6.2}$$

- C is uncountable.

Indeed, the function

$$f\Big(\sum_{i=1}^{\infty} \frac{d_i}{3^i}\Big) = \sum_{i=1}^{\infty} \frac{d_i}{2} \; \frac{1}{2^i} \tag{1.6.3}$$

maps C onto $[0,1]$, (see Exercise 1.6.3 below).

Remark 1.6.2. *Fix some* $0 < \beta < 1$ *and let* $I_1 = [0, 1]$. *For each* $n \in \mathbb{N}$ *let* $I_{n+1} \subseteq I_n$ *be the collection of intervals obtained by excluding from every interval in* I_n, *say with length* ℓ, *its centered subinterval of length* $\beta\ell$. *We define by* $C_\beta = \cap_{n=1}^{\infty} I_n$ *the* fat Cantor set *corresponding to* β. *If the parameter* $\beta < \frac{1}{3}$ *then* C_β *is not Jordan measurable. Actually the inner Jordan measure* $\underline{\mu}(C_\beta) = 0$ *since* C_β *has empty interior, whereas the outer Jordan measure* $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta} > 0$. *The Lebesgue measure of* C_β *is* $\mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$. *Actually for all* $n \in \mathbb{N}$, $I_n \setminus I_{n+1}$ *has length* $2^{n-1}\beta^n$ *and thus* $I_1 \setminus C_\beta$ *has length*

$$\sum_{n=1}^{\infty} 2^{n-1}\beta^n = \frac{\beta}{1-2\beta}.$$

Exercise 1.6.3. Show that f in (1.6.3) is surjective but not injective, ($f(x)$ coincides at the opposing ends of one of the middle third removed. For instance:

$$\frac{7}{9} = 0.20\overline{2}, \quad \frac{8}{9} = 0.2200, \text{ so}$$

$$f\left(\frac{7}{9}\right) = 0.10\overline{1} = f\left(\frac{8}{9}\right) = 0.11. \text{)}$$

1.7 Lebesgue-Stieltjes Measure

Let $F \colon \mathbb{R} \to \mathbb{R}$ be nondecreasing and continuous from the left:

$$F(x_0) = \lim_{x \to x_0^-} F(x) \quad \forall x_0 \in \mathbb{R}.$$

For $a, b \in \mathbb{R}$ we set

$$\lambda_F[a, b) = \begin{cases} F(b) - F(a) & \text{if } a < b \\ 0 & \text{otherwise.} \end{cases} \tag{1.7.1}$$

Let Λ_F be the induced measure of λ_F:

$$\Lambda_F(A) = \inf\left\{ \sum_{k=1}^{\infty} \lambda_F[a_k, b_k),\ A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k) \right\}.$$

Λ_F is called the Lebesgue-Stieltjes Measure generated by F. Is Λ_F Borel or even Borel regular?

The following criterion is useful.

Definition 1.7.1. *A measure μ on $\mathbb{R}^n$ is called* ***metric*** *if $\forall A, B \subseteq \mathbb{R}^n$ with*

$$\operatorname{dist}(A, B) = \inf\{|a - b|,\ a \in A,\ b \in B\} > 0,$$

we have

$$\mu(A \cup B) = \mu(A) + \mu(B). \tag{1.7.2}$$

Theorem 1.7.2. *(Carathéodory's criterion for Borel-measures)*

A metric measure μ on $\mathbb{R}^n$ is Borel.

Proof of Theorem 1.7.2. It is enough to show that every closed set F of $\mathbb{R}^n$ is μ-measurable, namely $\forall B \subseteq \mathbb{R}^n$ we have

$$\mu(B) \geq \mu(B \cap F) + \mu(B \setminus F). \tag{1.7.3}$$

If $\mu(B) = \infty$ then (1.7.3) is obvious.

Assume instead $\mu(B) < \infty$. For $k \geq 0$, define

$$F_k = \Big\{x \in \mathbb{R}^n :\ d(x, F) \leq \frac{1}{k}\Big\}. \tag{1.7.4}$$

Since μ is metric and

$$\operatorname{dist}(B \setminus F_k,\ B \cap F) \geq \frac{1}{k} > 0,$$

we have by the monotonicity of μ

$$\begin{aligned} \mu(B \cap F) + \mu(B \setminus F_k) &= \mu\big((B \cap F) \cup (B \setminus F_k)\big) \\ &\leq \mu(B). \end{aligned} \tag{1.7.5}$$

Claim 1.7.3. *$\mu(B \setminus F_k) \to \mu(B \setminus F)$ as $k \to +\infty$.*

Proof of Claim 1.7.3. For $k \in \mathbb{N}$ we set

$$R_k = (F_k \setminus F_{k+1}) \cap B = \left\{x \in B : \frac{1}{k+1} < d(x,F) \leq \frac{1}{k}\right\}. \tag{1.7.6}$$

R_k are disjoint with

$$\begin{aligned} B \setminus F = B \setminus \bigcap_{\ell=1}^{\infty} F_\ell &= \bigcup_{\ell=1}^{\infty} (B \setminus F_\ell) \\ &= (B \setminus F_1) \cup \bigcup_{\ell=1}^{\infty} (B \setminus F_{\ell+1}) \setminus (B \setminus F_\ell) \\ &= (B \setminus F_k) \cup \bigcup_{\ell=k}^{\infty} (B \setminus F_{\ell+1}) \setminus (B \setminus F_\ell) \\ &= (B \setminus F_k) \cup \bigcup_{\ell=k}^{\infty} (F_\ell \setminus F_{\ell+1}) \cap B \\ &= (B \setminus F_k) \cup \bigcup_{\ell=k}^{\infty} R_\ell. \end{aligned}$$

Thus

$$\mu(B \setminus F_k) \leq \mu(B \setminus F) \leq \mu(B \setminus F_k) + \sum_{\ell=k}^{\infty} \mu(R_\ell). \tag{1.7.7}$$

We show that

$$\lim_{k \to +\infty} \sum_{\ell=k}^{\infty} \mu(R_\ell) = 0. \tag{1.7.8}$$

To this purpose we prove that

$$\sum_{\ell=1}^{\infty} \mu(R_\ell) < +\infty. \tag{1.7.9}$$

We observe that $\operatorname{dist}(R_i, R_j) > 0$, if $|i-j| \geq 2$. Since μ is metric, it follows by induction that

$$\begin{aligned}\sum_{k=1}^{m} \mu(R_{2k}) &= \mu\Big(\bigcup_{k=1}^{m} R_{2k}\Big) \le \mu(B) \\ \sum_{k=1}^{m} \mu(R_{2k+1}) &= \mu\Big(\bigcup_{k=1}^{m} R_{2k+1}\Big) \le \mu(B).\end{aligned} \tag{1.7.10}$$

By adding (1.7.9) and (1.7.10) and letting $m \to +\infty$, we prove that $\sum_{k=1}^{\infty} \mu(R_k) < +\infty$. $\square$

Theorem 1.7.4. *The Lebesgue-Stieltjes measure is Borel regular.*

Proof.

Step 1. We first show that Λ_F is Borel.

According to Theorem 1.7.2 it is enough to show that Λ_F is metric. Let $A, B \subseteq \mathbb{R}$ with $\delta = \operatorname{dist}(A, B) > 0$. For $\varepsilon > 0$, choose $a_k, b_k \in \mathbb{R}$ with

$$A \cup B \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_F[a_k, b_k) < \Lambda_F(A \cup B) + \varepsilon.$$

Up to a further subdivision of the intervals $[a_k, b_k)$, we can suppose that $\forall k \ge 0$ one has $|b_k - a_k| < \delta$.

Therefore, for every $k \in \mathbb{N}$, we have either $A \cap [a_k, b_k) = \emptyset$ or $B \cap [a_k, b_k) = \emptyset$. The covering $([a_k, b_k))_k$ is decomposed in a covering $\mathcal{A}$ of A and a covering $\mathcal{B}$ of B. It follows

$$\begin{aligned}\Lambda_F(A \cup B) &\le \Lambda_F(A) + \Lambda_F(B) \\ &\le \sum_{[a_k,b_k)\in\mathcal{A}} \Lambda_F\big([a_k, b_k)\big) + \sum_{[a_k,b_k)\in\mathcal{B}} \Lambda_F\big([a_k, b_k)\big) \\ &\le \sum_{k} \lambda_F\big([a_k, b_k)\big) \le \Lambda_F(A \cup B) + \varepsilon.\end{aligned} \tag{1.7.11}$$

We let $\varepsilon \to 0$ and we get the claim.

Step 2. We show that Λ_F is Borel regular. Let $A \subseteq \mathbb{R}$ with $\Lambda_F(A) < +\infty$ (otherwise we choose $B = \mathbb{R}$). For $j \in \mathbb{N}$ we choose the sequences $(a_k^j)_k$, $(b_k^j)_k$ with

$$A \subseteq \bigcup_{k=1}^{\infty} [a_k^j, b_k^j) =: B_j$$

and

$$\sum_{k=1}^{\infty} \lambda_F\big([a_k^j, b_k^j)\big) \leq \Lambda_F(A) + \frac{1}{j}\,.$$

We set $B = \bigcap_{j=1}^{\infty} B_j$, B is Borel with $A \subseteq B$ and $B \subseteq B_j$. It follows that

$$\begin{aligned}\Lambda_F(A) &\leq \Lambda_F(B) \leq \Lambda_F(B_j)\\ &\leq \sum_{k=1}^{\infty} \lambda_F\big([a_k^j, b_k^j)\big) \leq \Lambda_F(A) + \frac{1}{j}.\end{aligned} \tag{1.7.12}$$

We let $j \to +\infty$ and get $\Lambda_F(A) = \Lambda_F(B) \implies \Lambda_F$ is Borel regular. $\square$

The set of half-open intervals is not an algebra. Nevertheless, it holds the following result:

Theorem 1.7.5. *For $a < b$ there holds*

$$\Lambda_F\big([a,b)\big) = \lambda_F\big([a,b)\big) = F(b) - F(a) \tag{1.7.13}$$

Proof. Let's consider $a, b \in \mathbb{R}$ with $a < b$.

1. By definition we have

$$\Lambda_F\big([a,b)\big) \leq \lambda_F\big([a,b)\big)$$

2. Now we show that

$$\Lambda_F\big([a,b)\big) \geq \lambda_F\big([a,b)\big)$$

Let $([a_k, b_k)_k$ be a covering of $[a, b)$, namely $[a, b) \subseteq \cup_k [a_k, b_k)$. Since F is left-continuous for every $\varepsilon > 0$ there are $\delta > 0, \delta_k > 0$ such that

$$F(b) - F(b - \delta) \leq \varepsilon \tag{1.7.14}$$

$$F(a_k) - F(a_k - \delta_k) \leq 2^{-k}\varepsilon, \quad \forall k \geq 0 \tag{1.7.15}$$

We observe that

$$[a, b - \delta] \subseteq \bigcup_{k=0}^{\infty} (a_k - \delta_k, b_k).$$

Since $[a, b - \delta]$ is compact we can extract a finite sub-covering of the open intervals $((a_k - \delta_k, b_k))_k$, i.e.

$$[a, b - \delta] \subseteq \bigcup_{k=0}^{m} (a_k - \delta_k, b_k).$$

By discarding any unnecessary $(a_k - \delta_k, b_k)$ and reindexing the rest, we may assume without restriction that $a_k - \delta_k < b_{k-1}$ for all $k = 1, \dots m$. We have

$$\lambda_F([a, b - \delta)) = F(b - \delta) - F(a) \leq \sum_{k=0}^{m} F(b_k) - F(a_k - \delta_k).$$

Therefore

$$\begin{aligned}
\lambda_F([a, b)) &= F(b) - F(a) = F(b) - F(b - \delta) + F(b - \delta) - F(a) \\
&\leq \varepsilon + \sum_{k=0}^{m} F(b_k) - F(a_k - \delta_k) \\
&= \varepsilon + \sum_{k=0}^{m} F(b_k) - F(a_k) + \sum_{k=0}^{m} F(a_k) - F(a_k - \delta_k) \\
&\leq \sum_{k=0}^{\infty} F(b_k) - F(a_k) + \varepsilon + \sum_{k=0}^{\infty} 2^{-k}\varepsilon \\
&= \sum_{k=0}^{\infty} F(b_k) - F(a_k) + 3\varepsilon\,.
\end{aligned} \tag{1.7.16}$$

By taking in (1.7.16) the infimum over all covering of $[a, b)$ and then letting $\varepsilon \to 0$ we get

$$\lambda_F([a, b)) \leq \Lambda_F([a, b)),$$

and we conclude the proof of Theorem 1.7.5.

$\square$

Exercise 1.7.6.

i) For $F(x) = x$, $\Lambda_F = \mathcal{L}^1$.

ii) For $F(x) = 1$, $x > 0$ and $F(x) = 0$, $x \leq 0$, it follows that $\Lambda_F = \delta_0$.

1.8 Hausdorff Measures

Hausdorff measures are among the most important measures. They generalize Lebesgue measure and permit to measure the subsets of $\mathbb{R}^n$ of dimension less than n, such as the submanifolds (surfaces and curves in $\mathbb{R}^3$) and the so-called fractal sets (a fractal is a never ending pattern that repeats itself at different scales). They allow us to define the dimension of every set of $\mathbb{R}^n$ even with complicated geometry.

Definition 1.8.1. *For $s \geq 0$, $\delta > 0$ and $\emptyset \neq A \subseteq \mathbb{R}^n$ we set*

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{k \in I} r_k^s, \quad A \subseteq \bigcup_{k \in I} B(x_k, r_k),\ 0 < r_k < \delta \right\}. \qquad (1.8.1)$$

We also set $\mathcal{H}^0_\delta(\emptyset) = 0$. **The set of indices I is at most countable**.

In the definition of $\mathcal{H}^s_\delta(A)$ we require that $r_k > 0$ since we want to avoid the case 0^0. In the case $s > 0$ we get the same value $\mathcal{H}^s_\delta(A)$ with or without such a condition.

$\mathcal{H}^s_\delta$ is a measure on $\mathbb{R}^n$. It is defined in terms of the radius of the balls $B(x_k, r_k) := \{y \in \mathbb{R}^n\colon\ |y - x_k| < r_k\}$. We observe that $\delta \mapsto \mathcal{H}^s_\delta(A)$ is a non-increasing function:

$$\mathcal{H}^s_{\delta_1}(A) \leq \mathcal{H}^s_{\delta_2}(A) \text{ if } \delta_2 \leq \delta_1, \qquad (1.8.2)$$

Thus there exists the limit

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A). \qquad (1.8.3)$$

Definition 1.8.2. *$\mathcal{H}^s$ is called the* ***s-dimensional Hausdorff measure*** *on $\mathbb{R}^n$.*

Theorem 1.8.3. *For $s \geq 0$, $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$.*

Proof of Theorem 1.8.3.

i) $\boldsymbol{\mathcal{H}^s}$ **is a measure.**

Let $s \geq 0$. Clearly, $\mathcal{H}^s(\emptyset) = 0$.

Let $A \subseteq \bigcup_k^\infty A_k \subseteq \mathbb{R}^n$. Since for every $\delta > 0$ $\mathcal{H}^s_\delta$ is σ-subadditive, we have the following inequalities

$$\mathcal{H}^s_\delta(A) \leq \sum_k \mathcal{H}^s_\delta(A_k) \leq \sum_k \mathcal{H}^s(A_k). \tag{1.8.4}$$

We let $\delta \to 0$ and get the σ-sub-additivity of $\mathcal{H}^s$.

ii) $\boldsymbol{\mathcal{H}^s}$ **is metric and therefore Borel**.

Let $A, B \subseteq \mathbb{R}^n$ with $\delta_0 = \mathrm{dist}(A, B)$. We select $0 < \delta < \frac{\delta_0}{4}$.

Suppose $A \cup B \subseteq \bigcup_k B(x_k, r_k)$, $r_k < \delta$. Set

$$\begin{aligned} \mathcal{A} &= \left\{ B(x_k, r_k),\ B(x_k, r_k) \cap A \neq \emptyset \right\} \\ \mathcal{B} &= \left\{ B(x_k, r_k),\ B(x_k, r_k) \cap B \neq \emptyset \right\}. \end{aligned} \tag{1.8.5}$$

Then

$$\begin{aligned} A &\subseteq \bigcup_{B(x_k, r_k) \in \mathcal{A}} B(x_k, r_k) \\ B &\subseteq \bigcup_{B(x_k, r_k) \in \mathcal{B}} B(x_k, r_k)\,. \end{aligned} \tag{1.8.6}$$

We observe that if $B(x_k, r_k) \in \mathcal{A}$, then $B(x_k, r_k) \cap B = \emptyset$. Indeed, suppose by contradiction that there exist $x \in A$ and $y \in B$ such that $|x - x_k| < r_k$, $|y - x_k| < r_k$. Then

$$\begin{aligned} \delta_0 = d(A, B) \leq |x - y| &\leq |x - x_k| + |y - x_k| \\ &\leq 2r_k \leq 2\delta < \frac{\delta_0}{2}, \end{aligned} \tag{1.8.7}$$

which is a contradiction. Thus

$$\begin{aligned} \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) &\leq \sum_{B(x_k, r_k) \in \mathcal{A}} r_k^s + \sum_{B(x_k, r_k) \in \mathcal{B}} r_k^s \\ &= \sum_k r_k^s. \end{aligned} \tag{1.8.8}$$

Taking the infimum over $\bigcup_k B(x_k, r_k)$ covering $A \cup B$, we get

$$\mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \tag{1.8.9}$$

provided $\delta < \frac{\delta_0}{4}$. Let $\delta \to 0$ and get

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B). \tag{1.8.10}$$

It follows that $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$ and we conclude.

iii) **$\mathcal{H}^s$ is Borel regular.**

Let $A \subseteq \mathbb{R}^n$, suppose $\mathcal{H}^s(A) < +\infty \Longrightarrow \mathcal{H}^s_\delta(A) < +\infty \; \forall \delta > 0$.

For $\delta = \frac{1}{\ell}$, choose $\bigcup_k B(x_{k,\ell}, r_{k,\ell}) \supseteq A$ with $r_{k,\ell} < \frac{1}{\ell}$ and

$$\sum_k r^s_{k,\ell} \leq \mathcal{H}^s_{\frac{1}{\ell}}(A) + \frac{1}{\ell}.$$

Set $A_\ell = \bigcup_k B(x_{k,\ell}, r_{k,\ell})$ and $B = \bigcap_\ell A_\ell$. B is a Borel set with $A \subseteq B$.

For each ℓ, we have

$$\begin{aligned} \mathcal{H}^s_{\frac{1}{\ell}}(A) &\leq \mathcal{H}^s_{\frac{1}{\ell}}(B) \leq \mathcal{H}^s_{\frac{1}{\ell}}(A_\ell) \leq \sum_k r^s_{k,\ell} \\ &\leq \mathcal{H}^s_{\frac{1}{\ell}}(A) + \frac{1}{\ell}. \end{aligned} \tag{1.8.11}$$

We let $\ell \to +\infty$ and get

$$\mathcal{H}^s(B) = \mathcal{H}^s(A). \quad \square \tag{1.8.12}$$

Remark 1.8.4.

1) We observe that $\mathcal{H}^0$ is the so-called **counting measure**. Actually let k be a positive integer and E be a set with k elements and let $\delta_0 > 0$ be the minimal distance between the elements of E. Then every covering of balls of radius $0 < \delta < \delta_0$ consists of at least k sets. On the other hand if $E = \{x_1, \ldots, x_k\}$ then $\cup_{i=1}^{k} B(x_i, \delta)$ is a covering of E. It follows that $\mathcal{H}^0_\delta(E) = k$ for every $0 < \delta < \delta_0$. It follows that $\mathcal{H}^0(E) = k$ as well. If E is infinite than by monotonicity we have $\mathcal{H}^0(E) \geq k$ for all k and therefore $\mathcal{H}^0(E) = +\infty$. Finally $\mathcal{H}^0(\emptyset) = 0$ by definition.

2) In general H^s_δ are not Borel measures and therefore they are not metric.

In order to define the dimension of an arbitrary set, we need the following

Lemma 1.8.5. *Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < +\infty$. It holds*

i) $\mathcal{H}^s(A) < +\infty \Longrightarrow \mathcal{H}^t(A) = 0.$

ii) $\mathcal{H}^t(A) > 0 \Longrightarrow \mathcal{H}^s(A) = +\infty.$

Proof.

i) Let $\mathcal{H}^s(A) < +\infty$. For $\delta > 0$, $A \subseteq \bigcup_k B(x_k, r_k)$ with $r_k < \delta$, it holds

$$\mathcal{H}^t_\delta(A) \leq \sum_k r_k^t = \sum_k r_k^{t-s} r_k^s \;\leq\; \delta^{t-s} \sum_k r_k^s. \tag{1.8.13}$$

By considering the infimum with respect to $\bigcup_k B(x_k, r_k)$, we get

$$\mathcal{H}^t_\delta(A) \leq \delta^{t-s}\, \mathcal{H}^s_\delta(A). \tag{1.8.14}$$

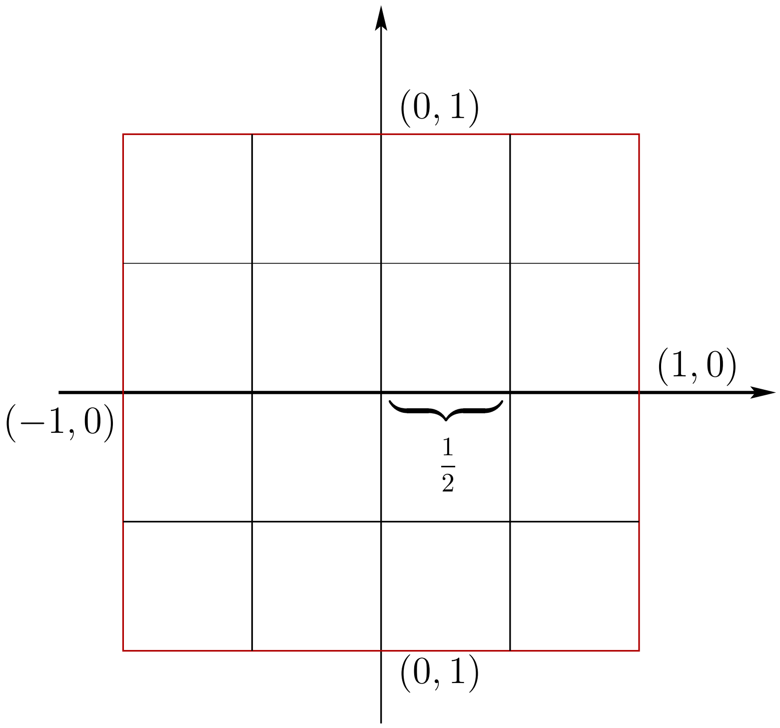
Let $\delta \to 0$ and we get $\mathcal{H}^t(A) = 0$.

ii) It follows from i). $\square$

Example 1.8.6. Let $Q = [-1,1]^n \subseteq \mathbb{R}^n$. Then it holds

$$2^{-n}\mathcal{L}^n(Q) \leq \mathcal{H}^n(Q) \leq 2^{-n} n^{\frac{n}{2}} \mathcal{L}^n(Q). \tag{1.8.15}$$

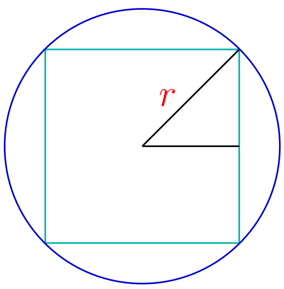
Proof. Let $\delta > 0$ and $k > 0$ be such that $r = 2^{-k-1}\sqrt{n} < \delta$. We decompose Q in subcubes Q_ℓ of edge length 2^{-k}, $1 \leq \ell \leq 2^{(k+1)n}$



$k = 1$

$n = 2$

$\ell = 16$



$$r = \sqrt{\sum_{i=1}^{n} \left(\frac{2^{-k}}{2}\right)^2} = n^{\frac{1}{2}}\, 2^{-(k+1)}$$

Fig. 1.3

Each Q_ℓ is included in a ball with the same center and radius r

Therefore $Q \subseteq \bigcup_{\ell=1}^{2^{(k+1)n}} B(x_\ell, r)$. Since $r = n^{\frac{1}{2}}\, 2^{-(k+1)} < \delta$, we have

$$\begin{aligned} \mathcal{H}^n_\delta(Q) \leq \sum_{1\leq \ell \leq 2^{(k+1)n}} r^n &= r^n\, 2^{(k+1)n} \\ &= n^{\frac{n}{2}}\, 2^{-(k+1)n}\, 2^{(k+1)n} \\ &= n^{\frac{n}{2}} = 2^n\, 2^{-n}\, n^{\frac{n}{2}} \\ &= \mathcal{L}^n(Q)\, n^{\frac{n}{2}}\, 2^{-n}. \end{aligned} \tag{1.8.16}$$

Conversely, if $(B(x_k, r_k))_k$ with $r_k < \delta$ is a covering of the ball $B(0,1)$, then the following estimate holds

$$\begin{aligned} w_n = \mathcal{L}^n\big(B(0,1)\big) &\leq \sum_k \mathcal{L}^n\big((B(x_k, r_k)\big) \\ &= w_n \sum_k r_k^n \end{aligned}$$

where

$$w_n = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}, \qquad \Gamma(t) = \int_0^{+\infty} x^{t-1}\, e^{-x} dx \;\; t > 0,$$

Γ is the Euler function.

Therefore $\sum_k r_k^n \geq 1$ and

$$\begin{aligned} \mathcal{H}^n_\delta(Q) &\geq \mathcal{H}^n_\delta\big(B(0,1)\big) \\ &= \inf\Big\{\sum_k r_k^n : \; B(0,1) \subseteq \bigcup_k B(x_k, r_k),\, r_k < \delta\Big\} \\ &\geq 1 = 2^{-n}\, \mathcal{L}^n(Q) \end{aligned} \tag{1.8.17}$$

Let $\delta \to 0$ in (1.8.16) and in (1.8.17) and we get the result. $\square$

Remark 1.8.7.

1. Actually, $\mathcal{L}^n(A) = w_n\, \mathcal{H}^n(A)$ for all $\mathcal{L}^n$-measurable sets $A \subseteq \mathbb{R}^n$.
2. For $A \subseteq \mathbb{R}^n$, if $s > n$, then it holds $\mathcal{H}^s(A) = 0$. Actually we can write $\mathbb{R}^n = \cup_\ell Q_\ell$ where Q_ℓ are dydic cubes. For Example 1.8.6 and Lemma 1.8.5 it follows that $\mathcal{H}^s(Q_\ell) = 0$ for every $\ell \in \mathbb{N}$. Therefore $\mathcal{H}^s(\mathbb{R}^n) = 0$.

Definition 1.8.8. *The* **Hausdorff dimension** *of a set* $A \subseteq \mathbb{R}^n$ *is defined to be*

$$\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 : \quad \mathcal{H}^s(A) = 0\}.$$

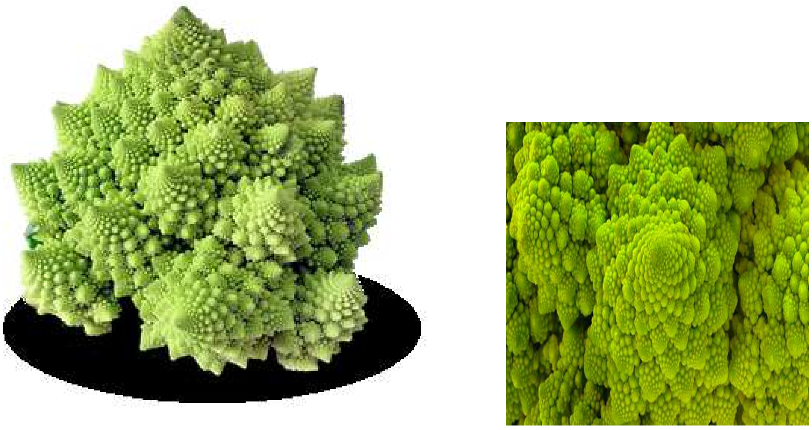


Figure 1.1: Example of fractal in everyday life: the cauliflower

Remark 1.8.9.

1. From Lemma 1.8.5 it follows that

$$\dim_{\mathcal{H}}(A) = \sup\{s \geq 0 : \quad \mathcal{H}^s(A) = +\infty\}.$$

2. Observe that $\dim_{\mathcal{H}}(A) \leq n$. Let $s = \dim_{\mathcal{H}}(A)$. Then $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = +\infty$ for all $t < s$; $\mathcal{H}^s(A)$ may be any number between 0 and ∞, inclusive. Furthermore, $\dim_{\mathcal{H}}(A)$ need not be an integer. Even if $\dim_{\mathcal{H}}(A) = k$ is an integer and $0 < \mathcal{H}^k(A) < \infty$, A need not be a "$k$-dimensional surface" in any sense.

3. If $A \subseteq \mathbb{R}^n$ is open, then $\dim_{\mathcal{H}}(A) = n$. Indeed, each open set contains a ball B such that $\mathcal{L}^n(B) > 0 \Longrightarrow \mathcal{H}^n(A) > 0 \Longrightarrow \dim_{\mathcal{H}}(A) \geq n \Longrightarrow \dim_{\mathcal{H}}(A) = n$.

Sets with non-integer Hausdorff dimension are important and most sets in nature are fractals with non-integer dimension.

A **fractal set** is a rough or fragmented geometric set that can be splitted into parts, each of which is a reduced-size copy of the whole.

1.8.1 Examples of non-integer Hausdorff-dimension sets

1. **Triadic Cantor Set**

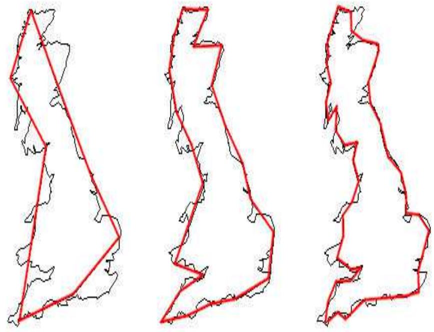


Figure 1.2: Approximation of the coastline of Great Britain by polygons

$C = \lim_{k \to +\infty} C_k = \bigcap_{k=0}^{\infty} C_k$, C_k closed sets with $C_0 = [0,1]$, C_k is obtained from C_{k-1} by removing the open middle third from each connected component of C_{k-1}. In this case:

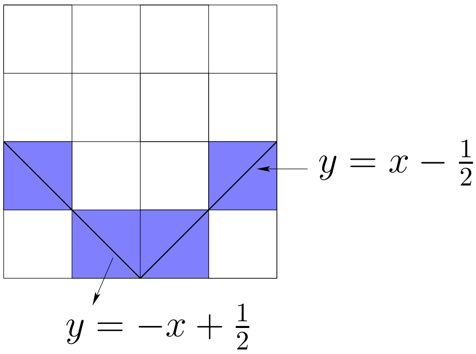
$$\dim_{\mathcal{H}}(C) = \frac{\ell n(2)}{\ell n(3)} =: d \quad 2^{-(d+1)} < \mathcal{H}^d(C) < 2^{-d},$$

where $\ell n(x) = \log_e(x)$.

Heuristics:

- $\forall s > 0$, $\lambda > 0$, $\forall A \subseteq \mathbb{R}^n$
$$\mathcal{H}^s(\lambda A) = \lambda^s \, \mathcal{H}^s(A).$$
- If there exists d such that $\mathcal{H}^d(C) \in (0, +\infty)$ then, since C is the union of two copies of $\frac{C}{3}$, we have
$$\mathcal{H}^d(C) = 2\mathcal{H}^d\Big(\frac{C}{3}\Big) = \frac{2}{3^d}\,\mathcal{H}^d(C) \Longrightarrow \frac{2}{3^d} = 1 \Longrightarrow$$
$$\boxed{d = \log_3 2 = \frac{\ell n\, 2}{\ell n\, 3}}.$$

2. **Cantor Dust**



$A_0 = [0,1]^2$

$A_1 = 1 \cup 2 \cup 3 \cup 4$

= union of squares through which the two lines $y = x - \frac{1}{2}$, $y = -x + \frac{1}{2}$ pass

Fig. 1.4

$A_2 \subseteq A_1$ is the union of 4^2 squares of length 4^{-2} obtained by repeating the same construction in every sub-squares of A_1.

Finally one considers $A = \bigcap_{k=1}^{\infty} A_k$. One has

$$\dim_{\mathcal{H}}(A) = 1 \tag{1.8.18}$$

Proof. We show that $\frac{1}{2} \le \mathcal{H}^1(A) \le \frac{\sqrt{2}}{2}$.

Each set A_k consists of 4^k squares of length 4^{-k}. The 4^k balls of radius $4^{-k}\frac{\sqrt{2}}{2}$ cover A_k and thus A. Thus

$$\mathcal{H}^1_\delta(A) \le \mathcal{H}^1_\delta(A_k) \le \frac{\sqrt{2}}{2} \quad \forall k \ge k_0 : \; r_{k_0} = 4^{-k_0}\frac{\sqrt{2}}{2} < \delta. \tag{1.8.19}$$

Let $\Pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection $\Pi(x,y) = x$. We have $\Pi(A) = [0,1]$. Let $A \subseteq \bigcup_{k=1}^{\infty} B(z_k, r_k)$ with $z_k = (x_k, y_k)$ and $r_k < \delta$. We get

$$\begin{aligned} [0,1] = \Pi(A) &\subseteq \bigcup_{k=1}^{\infty} \Pi\big(B(x_k, r_k)\big) \\ &= \bigcup_{k=1}^{\infty} (x_k - r_k, x_k + r_k). \end{aligned} \tag{1.8.20}$$

Thus

$$1 = \mathcal{L}^1[0,1] \leq \sum_{k=1}^{\infty} \mathcal{L}^1(I_k) = 2\sum_{k=1}^{\infty} r_k \Longrightarrow \sum_{k=1}^{\infty} r_k \geq \tfrac{1}{2}. \qquad (1.8.21)$$

where $I_k = (x_k - r_k, x_k + r_k)$. $\square$

3. **The Koch Curve**

Description:

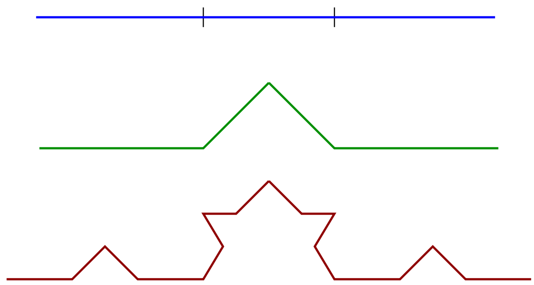


Fig. 1.5

Begin with a straight line. Divide it into 3 equal segments and replace the middle segment by the two sides of an equilateral triangle of the same length. Now repeat, taking each of the four resulting segments dividing them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle. Continue with the construction.

The **Koch Curve** is the limiting curve obtained by applying this construction on infinite number of times.

The length of the intermediate curve at the n^{th}-iteration of the construction is $(\frac{4}{3})^n$. Therefore the length of the Koch curve is infinite. Moreover, the length of the Koch curve between any two points of the curve is also infinite since there is a copy of the Koch curve. In this case, the Hausdorff dimension is $d = \frac{\ell n\, 4}{\ell n\, 3} > 1$ (the curve is made of 4 copies of itself rescaled by the factor $\frac{1}{3}$).

1.9 Radon Measure

Definition 1.9.1. *A measure μ on $\mathbb{R}^n$ is called a Radon measure if μ is Borel regular and $\mu(K) < \infty$, for every compact $K \subseteq \mathbb{R}^n$.*

Example 1.9.2.

i) $\mathcal{L}^n$ is a Radon measure on $\mathbb{R}^n$.

ii) $\mathcal{H}^s$ for $s < n$ is not a Radon measure.

iii) If μ is a Borel regular measure and $A \subseteq \mathbb{R}^n$ is μ-measurable with $\mu(A) < +\infty$, then the measure $\mu\lfloor A$ defined by

$$(\mu\lfloor A)(B) := \mu(A \cap B), \; B \subseteq \mathbb{R}^n \tag{1.9.1}$$

is a Radon measure. **See Theorem 3 in Evans & Gariepy's book**

For general Radon measures an analogous of Theorem 1.3.7 holds.

Theorem 1.9.3. *(Approximation by open and compact sets) Let μ be a Radon measure on $\mathbb{R}^n$.*

i) *For every $A \subseteq \mathbb{R}^n$ it holds*

$$\mu(A) = \inf\left\{\mu(G) : \quad A \subseteq G,\; G \text{ open}\right\}. \tag{1.9.2}$$

ii) *For every $A \subseteq \mathbb{R}^n$ μ-measurable it holds*

$$\mu(A) = \sup\left\{\mu(F) : \quad F \subseteq A,\; F \text{ compact}\right\}. \tag{1.9.3}$$

To prove Theorem 1.9.3 we need the following lemma:

Lemma 1.9.4. *For every Borel set $B \subseteq \mathbb{R}^n$ it holds the following: $\forall \varepsilon > 0 \;\; \exists G \supseteq B$, G open such that $\mu(G \setminus B) < \varepsilon$.* **(No proof)**

Proof of Theorem 1.9.3.

i) If $\mu(A) = +\infty$ then (1.9.2) is obvious. Let us suppose $\mu(A) < +\infty$.

Assume first that ***A* is a Borel set** and fix $\varepsilon > 0$. Then by Lemma 1.9.4 there exists G open such that $G \supseteq A$ and $\mu(G \setminus A) < \varepsilon$. Since $\mu(G) = \mu(G \cap A) + \mu(G \setminus A)$, we get $\mu(G) < \mu(A) + \varepsilon$ and i) follows.

Now let A be an arbitrary set. Since μ is Borel regular, there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$. Then

$$\begin{aligned} \mu(A) = \mu(B) &= \inf\{\mu(U),\ B \subseteq U \text{ open}\} \\ &\geq \inf\{\mu(U),\ A \subseteq U \text{ open}\}. \end{aligned} \tag{1.9.4}$$

ii) Let A be a μ-measurable.

- Case 1: $\mu(A) < +\infty$.

  Set $\nu = \mu \lfloor A$. ν is a Radon measure. We apply i): $\forall \varepsilon > 0$ there exists G open such that $G \supseteq \mathbb{R}^n \setminus A$ and

  $$\nu(G) \leq \nu(\mathbb{R}^n \setminus A) + \varepsilon = \varepsilon.$$

  Let $C = \mathbb{R}^n \setminus G$, C is closed and $C \subseteq A$. Moreover,

  $$\begin{aligned} \mu(A \setminus C) = \mu\big(A \cap (\mathbb{R}^n \setminus C)\big) &= \nu(\mathbb{R}^n \setminus C) \\ &= \nu(G) < \varepsilon. \end{aligned} \tag{1.9.5}$$

  Thus

  $$\mu(A) \leq \mu(C) + \varepsilon\,. \tag{1.9.6}$$

  ($\mu(A) = \mu(A \cap C) + \mu(A \setminus C)$, since C is measurable). Therefore

  $$\mu(A) = \sup\{\mu(C) \colon C \subseteq A,\ C \text{ closed}\}. \tag{1.9.7}$$

- Case 2: $\mu(A) = +\infty$.

  Define $D_k = \{x : k-1 \leq |x| < k\}$. Then $A = \bigcup_{k=1}^{\infty}(D_k \cap A)$.

  Since μ is a Radon measure, $\mu(D_k \cap A) < +\infty$. From case 1 it follows that $\forall k$ there exists a closed set $C_k \subseteq D_k \cap A$, C_k such that $\mu(C_k) \geq \mu(D_k \cap A) - \frac{1}{2^k}$. Now $\bigcup_{k=1}^{\infty} C_k \subseteq A$ and

$$\begin{aligned}\lim_{m\to+\infty} \mu\Big(\bigcup_{k=1}^{m} C_k\Big) = \mu\Big(\bigcup_{k=1}^{\infty} C_k\Big) &= \sum_{k=1}^{\infty} \mu(C_k) \\ &\geq \sum_{k=1}^{\infty} \Big[\mu(D_k \cap A) - \frac{1}{2^k}\Big] \\ &= \mu(A) - 1 = +\infty.\end{aligned}$$

We observe that $\bigcup_{k=1}^{m} C_k$ is closed $\forall m$, thus, also in the case $\mu(A) = +\infty$, (1.9.7) holds. Finally, we set

$$\overline{B}_m = \overline{B}(0, m) = \{x \in \mathbb{R}^n : |x| \leq m\}. \tag{1.9.8}$$

If C is closed, then $C \cap \overline{B}_m$ is compact and

$$\mu(C) = \lim_{m\to+\infty} \mu(C \cap \overline{B}_m).$$

Hence, for each μ-measurable set A:

$$\sup\{\mu(K) : \ K \subseteq A \text{ compact}\} = \sup\{\mu(C) : \ C \subseteq A \text{ closed}\}.$$

We conclude the proof. $\square$

Theorem 1.9.5. *Let μ be a Radon measure on $\mathbb{R}^n$ and let $A \subseteq \mathbb{R}^n$. The following two conditions are equivalent:*

i) *A is μ-measurable,*

ii) *$\forall \varepsilon > 0\ \exists G \supset A$, G open such that $\mu(G \setminus A) < \varepsilon$.*

The proof is the same of that of Theorem 1.3.8 by using Theorem 1.9.3 instead of Theorem 1.3.7.

Chapter 2

Measurable Functions

2.1 Inverse Image of a Function

Let X, Y be nonempty sets.

For any map $\varphi \colon X \to Y$ and $A \in \mathcal{P}(Y)$, we set

$$\varphi^{-1}(A) = \{x \in X :\ \varphi(x) \in A\} = \{\varphi \in A\}, \tag{2.1.1}$$

$\varphi^{-1}(A)$ is called the **inverse image** or **preimage** of A.

Some Properties

i) $\varphi^{-1}(A^c) = \big(\varphi^{-1}(A)\big)^c \quad \forall A \in \mathcal{P}(Y)$.

ii) If $A, B \in \mathcal{P}(Y)$:

$$\varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B).$$

iii) If $\{A_k\} \subseteq \mathcal{P}(Y)$:

$$\varphi^{-1}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \bigcup_{k=1}^{\infty} \varphi^{-1}(A_k).$$

Consequently, if $(Y, \mathcal{F}, \mu)$ is a **measure space**, then $\varphi^{-1}(\mathcal{F}) = \{\varphi^{-1}(A) \colon A \in \mathcal{F}\}$ is a σ-algebra in X.

2.2 Definition and Basic Properties

Let μ be a measure on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ μ-measurable.

Definition 2.2.1. *$f\colon \Omega \to [-\infty, \infty]$ is called* **μ-measurable** *if*

i) *$f^{-1}\{+\infty\}$, $f^{-1}\{-\infty\}$ are μ-measurable.*

ii) *$f^{-1}(U)$ is μ-measurable for every $U \subseteq \mathbb{R}$ open.*

Remark 2.2.2. The following two conditions are equivalent to ii) in Definition 2.2.1.
iii) $f^{-1}(B)$ is μ-measurable for each Borel set $B \subseteq \mathbb{R}$.
iv) $f^{-1}((-\infty, a))$ is μ-measurable, $\forall a \in \mathbb{R}$.

Remark 2.2.3. We consider $\overline{\mathbb{R}} = [-\infty, +\infty]$ with the topology generated by the open sets of $\mathbb{R}$ and the neighborhoods $[-\infty, a)$, $(a, +\infty]$, $a \in \mathbb{R}$ of $\pm\infty$.
Then $f\colon \Omega \to \overline{\mathbb{R}}$ is μ-measurable if and only if
v) $f^{-1}(U)$ is μ-measurable, $\forall\, U \subseteq \overline{\mathbb{R}}$ open
or if and only if
vi) $f^{-1}([-\infty, a))$ is μ-measurable, $\forall a \in \mathbb{R}$.

Proof of vi) $\implies$ i) and vi) $\implies$ iv):

<u>vi) $\implies$ i)</u>: We use the representation

$$f^{-1}(\{-\infty\}) = \bigcap_{k \in \mathbb{Z}} f^{-1}\big([-\infty, k)\big)$$

$$f^{-1}(\{+\infty\}) = \Omega \setminus \bigcup_{k \in \mathbb{Z}} f^{-1}\big([-\infty, k)\big).$$

<u>vi) $\implies$ iv)</u>: $f^{-1}((-\infty, a)) = f^{-1}([-\infty, a)) \setminus f^{-1}(\{-\infty\})$.

Exercise 2.2.4. *Let $f\colon \Omega \to \mathbb{R}$ be μ-measurable and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous. Then $g \circ f$ is μ-measurable.*

Theorem 2.2.5. *(Properties of Measurable Functions)*

i) *Let* $f, g\colon \Omega \to \mathbb{R}$ *be* μ*-measurable functions. Then:*

$f+g$, $f \cdot g$, $|f|$, $\operatorname{sign}(f)$, $\max(f,g)$, $\min(f,g)$ *and (if* $g(x) \neq 0$*)* $\frac{f}{g}$ *are* μ*-measurable, where*

$$(\operatorname{sign} f)(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

ii) *If* $f_k\colon \Omega \to \overline{\mathbb{R}}$ *are* μ*-measurable. Then:*

$\inf_k f_k$, $\sup_k f_k$ $\liminf_{k\to\infty} f_k$, $\limsup_{k\to\infty} f_k$ *are* μ*-measurable.*

Proof.

i) In view of Remark 2.2.2, $f\colon \Omega \to \mathbb{R}$ is μ-measurable iff $f^{-1}(-\infty, a)$ is μ-measurable $\forall a \in \mathbb{R}$.

Let f, g: $\Omega \to \mathbb{R}$ be μ-measurable, then

$$(f+g)^{-1}(-\infty, a) = \bigcup_{\substack{r+s<a \\ r,s\in\mathbb{Q}}} f^{-1}\big((-\infty, r)\big) \cap g^{-1}\big((-\infty, s)\big). \quad (2.2.1)$$

Since $(f^2)^{-1}(-\infty, a) = f^{-1}(-\infty, \sqrt{a}) \setminus f^{-1}(-\infty, -\sqrt{a}]$, if $a \geq 0$, and $(f^2)^{-1}(-\infty, a) = \emptyset$ is $a < 0$, then

$$f \cdot g = \tfrac{1}{2}\big[(f+g)^2 - f^2 - g^2\big] \quad (2.2.2)$$

is μ-measurable as well.

Moreover, it holds

$$\left(\frac{1}{g}\right)^{-1}((-\infty, a)) = \begin{cases} g^{-1}\left(\frac{1}{a}, 0\right) & a < 0 \\ g^{-1}\big((-\infty, 0)\big) & a = 0 \\ g^{-1}\big((-\infty, 0)\big) \cup g^{-1}\left(\left(\frac{1}{a}, +\infty\right)\right) & a > 0 \end{cases} \quad (2.2.3)$$

Therefore $\dfrac{1}{g}$ and $\dfrac{f}{g}$ are μ-measurable.

Finally set $s^+ = \max\{s, 0\}$, $s^- = \max\{-s, 0\}$. The maps $s \to s^+$, $s \to s^-$ are continuous. Therefore, by Exercise 2.2.4 we have

$$\begin{aligned} f^+, f^-, |f| &= f^+ + f^- \\ \sup(f, g) &= f + (g - f)^+ \\ \inf(f, g) &= f - (g - f)^- \end{aligned} \tag{2.2.4}$$

are μ-measurable.

The function $x \mapsto \text{sign}(x)$ is continuous except at the origin. If $U \subseteq \mathbb{R}$ is open, $\text{sign}^{-1}(U)$ is either open or of the form $V \cup \{0\}$ where V is open, so $\text{sign}(x)$ is Borel measurable function (in the sense that the pre-image of a Borel subset of $\mathbb{R}$ is a Borel set of $\mathbb{R}$). Therefore $\text{sign} f$ is μ-measurable.

ii) Let $f_k\colon \Omega \to [-\infty, +\infty]$ be μ-measurable. Then

$$\begin{aligned} \left(\inf_{k\geq 1} f_k\right)^{-1}[-\infty, a) &= \bigcup_{k=1}^{\infty} f_k^{-1}[-\infty, a) \\ \left(\sup_{k\geq 1} f_k\right)^{-1}[-\infty, a) &= \bigcup_{\ell=1}^{\infty} \bigcap_{k=1}^{\infty} f_k^{-1}\Big[-\infty, a - \frac{1}{\ell}\Big) \end{aligned} \tag{2.2.5}$$

$\implies \inf_k f_k$, $\sup_k f_k$ are μ-measurable.

We complete the proof by noting

$$\liminf_{k\to\infty} f_k = \sup_{m\geq 1} \inf_{k\geq m} f_k$$

$$\limsup_{k\to\infty} f_k = \inf_{m\geq 1} \sup_{k\geq m} f_k.$$

□

We now discuss the functions that are building blocks for the theory of integration.

Given $A \subseteq \mathbb{R}^n$, the function $\chi_A\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{2.2.6}$$

is called the **characteristic function** of the set A. It is easily checked that the characteristic function χ_A of A is μ- measurable if and only if A is measurable.

A **simple function** is a function of the form

$$f(x) = \sum_{i=1}^{+\infty} d_i\, \chi_{A_i}(x),\ d_i \in \mathbb{R},\ A_i \subseteq \mathbb{R}^n, A_i \text{ mutually disjoint.}$$

If A_i are μ-measurable, then f is called a **μ-measurable simple function**. Equivalently, $f\colon \mathbb{R}^n \to \mathbb{R}$ is a μ- measurable simple function if and only if f is μ- measurable and the range of f is an at most countable subset of $\mathbb{R}$.

The following theorem provides a useful way to decompose of a nonnegative μ-measurable function.

Theorem 2.2.6. *Let $f\colon \Omega \to [0, +\infty]$ be μ-measurable. Then there are μ-measurable sets $A_k \subseteq \Omega$ such that*

$$f = \sum_{k=1}^{\infty} \frac{1}{k}\, \chi_{A_K}\,.$$

Proof of Theorem 2.2.6: We set

$$A_1 = \{x \in \Omega : f(x) \geq 1\} = f^{-1}[1, +\infty], \quad A_1 \text{ is } \mu\text{-measurable.}$$

We define by induction the following sets:

$$A_k = \left\{x \in \Omega : f(x) \geq \frac{1}{k} + \sum_{j=1}^{k-1} \frac{1}{j}\, \chi_{A_j}\right\} k = 2, 3, \dots . \tag{2.2.7}$$

Claim 2.2.7. $f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\, \chi_{A_k}(x)$.

Proof of Claim 2.2.7.

"$f(x) \geq \sum\limits_{k=1}^{\infty} \dfrac{1}{k}\, \chi_{A_k}(x)$":

This inequality follows directly from the definition of A_k if $\sup\{k : x \in A_k\} = +\infty$. Otherwise, consider $k_0 = \max\{k\colon\ x \in A_k\}$, and we use the fact that $x \in A_{k_0}$. We get

$$f(x) \geq \sum_{k=1}^{k_0} \frac{1}{k} \chi_{A_k}(x)$$

and since $x \notin A_k$ for $k \geq k_0 + 1$ we can also write

$$f(x) \geq \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x).$$

"$f(x) \leq \sum\limits_{k=1}^{\infty} \dfrac{1}{k}\, \chi_{A_k}(x)$":

We consider different cases.

i) $f(x) = +\infty \Longrightarrow x \in A_k,\ \forall k$ and $\sum\limits_{k=1}^{\infty} \dfrac{1}{k}\, \chi_{A_k}(x) = \sum\limits_{k=1}^{\infty} \dfrac{1}{k} = +\infty = f(x)$.

ii) $f(x) = 0 \Longrightarrow x \notin A_k,\ \forall k \Longrightarrow \sum\limits_{k=1}^{\infty} \dfrac{1}{k}\, \chi_{A_k}(x) = 0$.

iii) $0 < f(x) < +\infty \Longrightarrow \forall k_0 > 0 :\ x \notin \bigcap_{k \geq k_0} A_k$, namely $\forall k_0 > 0 :\ \exists \overline{k} = \overline{k}(k_0) \geq k_0$ such that $x \notin A_{\overline{k}}$. (Indeed, if $\exists k_0$ and $x \in \bigcap_{k \geq k_0} A_k$ then $\chi_{A_k}(x) = 1\ \forall k \geq k_0$ and

$$+\infty = \sum_{k=k_0}^{\infty} \frac{1}{k}\, \chi_{A_k}(x) \leq f(x) < +\infty.$$

Hence for infinitely many k, we have $x \notin A_k$ and thus

$$0 \leq f(x) - \sum_{n=1}^{k} \frac{1}{n}\, \chi_{A_n}(x) < \frac{1}{k}. \tag{2.2.8}$$

By letting $k \to +\infty$, we get

$$f(x) \leq \sum_{n=1}^{\infty} \frac{1}{n}\, \chi_{A_n}(x). \quad \square \tag{2.2.9}$$

Remark 2.2.8.

1. Set $f_k = \sum_{j=1}^{k} \frac{1}{j}\chi_{A_j}(x)$. Then f_k is an increasing sequence that converges to f.

   If f is bounded, then the convergence is uniform:

$$\sup_{x\in\mathbb{R}} |f(x) - f_k(x)| \to 0 \text{ as } k \to +\infty.$$

2. Through the approximation of f^+ and f^- we can approximate every μ-measurable function $f\colon \Omega \to [-\infty, +\infty]$ by step-functions.

Proposition 2.2.9. *Let $f\colon \Omega \to \mathbb{R}$ be continuous, μ be a Borel measure. Then f is μ-measurable.*

Proof. Let $U \subseteq \mathbb{R}$ be an open set. Then $f^{-1}(U) = O \cap \Omega$ where $O \subset \mathbb{R}^n$ open. Since μ is a Borel measure (any open set in $\mathbb{R}^n$ is measurable), $f^{-1}(U)$ is μ-measurable. $\square$

Notation:

The expression "μ-a.e." means "**almost everywhere with respect of the measure $\boldsymbol{\mu}$**", that is except possibly on a set A with $\mu(A) = 0$.

2.3 Lusin's and Egoroff's Theorems

Let μ be a <u>Radon measure</u> on $\mathbb{R}^n$.

Theorem 2.3.1. *(Egoroff)*

Let $\Omega \subseteq \mathbb{R}^n$ be μ-measurable with $\mu(\Omega) < +\infty$. Let f_k: $\Omega \to \overline{\mathbb{R}}$ be μ-measurable, $\forall k \in \mathbb{N}$, $f\colon \Omega \to \overline{\mathbb{R}}$ be μ-measurable and $f(x)$ finite μ-a.e.. Moreover $f_k(x) \to f(x)$ as $k \to +\infty$ for μ-a.e. $x \in \Omega$. Then it holds: $\forall \delta > 0\ \exists A \subseteq \Omega$, A μ-measurable with $\mu(\Omega \setminus A) < \delta$ and

$$\sup_{x\in A} |f_k(x) - f(x)| \to 0 \text{ as } k \to +\infty, \tag{2.3.1}$$

namely $(f_k)_k$ converges uniformly to f on A. Since μ is a Radon measure and $\mu(\Omega) < +\infty$ it also holds: $\forall \delta > 0\ \exists F \subseteq \Omega$, F compact with $\mu(\Omega \setminus F) < \delta$ and

$$\sup_{x\in F} |f_k(x) - f(x)| \to 0 \text{ as } k \to +\infty, \tag{2.3.2}$$

namely $(f_k)_k$ converges uniformly to f on F.

Exercise 2.3.2. Show that the conclusion of Egoroff's theorem can fail if we drop the assumption that the domain has finite measure.

Solution:
Let $\mu = \mathcal{L}^1$ be the Lebesgue measure and $f_k(x) = \chi_{[-k,k]}(x)$, $f_k(x) \to 1$ pointwise as $k \to +\infty$. For every $\delta > 0$ there does not exists a μ-measurable set A such that $\mu(\mathbb{R} \setminus A) \leq \delta$ and $f_k(x)$ uniformly on A. Suppose there is $\delta > 0$ and a μ-measurable set $A \subseteq \mathbb{R}$, such that $\sup_A |f_k(x) - 1| \to 0$, and $\mu(\mathbb{R} \setminus A) \leq \delta$. Then $\exists N > 0$ such that $[-N, N]^c \subset A^c$ which is a contradiction, since $\mu(\mathbb{R} \setminus A) \leq \delta$ and $\mu([-N, N]^c) = +\infty$.

Proof of Theorem 2.3.1: Let $\delta > 0$. Define

$$C_{ij} = \bigcup_{k=j}^{\infty} \left\{ x \in \Omega : \ |f_k(x) - f(x)| > \frac{1}{2^i} \right\} \quad (i, j = 1, 2, \dots). \tag{2.3.3}$$

Then $C_{i,j+1} \subseteq C_{i,j}$ $\forall i, j$. Since $f_k(x) \to f(x)$ for μ-a.e x and $\mu(\Omega) < \infty$, we have

$$\lim_{j\to+\infty} \mu(C_{ij}) = \mu\Big(\bigcap_{j=1}^{\infty} C_{ij}\Big) = 0. \tag{2.3.4}$$

Hence for every i, $\exists N(i) > 0$ such that

$$\mu(C_{i,N(i)}) < \frac{\delta}{2^{i+1}}\,,$$

we set

$$A = \Omega\setminus\bigcup_{i=1}^{\infty} C_{i,N(i)}, \quad \mu(\Omega\setminus A) = \mu\Big(\bigcup_{i=1}^{\infty} C_{i,N(i)}\Big) < \sum_{i=1}^{\infty} \frac{\delta}{2^{i+1}} = \frac{\delta}{2}. \tag{2.3.5}$$

Moreover, $\forall x \in A$, $\forall i$, $\forall k \geq N(i)$

$$|f_k(x) - f(x)| \leq \frac{1}{2^i}\,. \tag{2.3.6}$$

Thus $f_k \to f$ is uniformly on A.

Now let $F \subseteq A$ be compact such that $\mu(A \setminus F) < \frac{\delta}{2}$. From (2.3.5) it follows,

$$\mu(\Omega \setminus F) \leq \mu(\Omega \setminus A) + \mu(A \setminus F) \leq \delta/2 + \delta/2. \tag{2.3.7}$$

We can conclude the proof. $\square$

The type of convergence involved in the conclusion of Egoroff's theorem is sometimes called **almost uniform convergence**.

Any continuous function is μ-measurable. The next result gives a continuity property that characterizes measurable functions.

Theorem 2.3.3. *(Lusin's Theorem)*

We assume that $\Omega \subseteq \mathbb{R}^n$ is μ-measurable and $\mu(\Omega) < +\infty$. Let $f\colon \Omega \to \overline{\mathbb{R}}$ be μ-measurable, finite μ-a.e.. Then $\forall \varepsilon > 0$ $\exists K \subseteq \Omega$ compact such that $\mu(\Omega \setminus K) < \varepsilon$ and $f|_K$ is continuous.

Remark 2.3.4. 1) By $f|_K$ we mean the restriction of f to the set K. The conclusion of the theorem states that if f is viewed as a function defined only on K, then f is continuous. However, the theorem does not make the stronger assertion that the function f defined on Ω is continuous at the points of K.

2) If we drop the condition $\mu(\Omega) < +\infty$, in general we cannot find a compact set F satisfying the conditions of Theorem 2.3.3. Nevertheless for every $\forall \varepsilon > 0$ $\exists C \subseteq \Omega$ closed such that $\mu(\Omega \setminus C) < \varepsilon$ and $f|_C$ is continuous.

Example 2.3.5.

$\Omega = [0, 1] \subseteq \mathbb{R}$

$f = \chi_{[0,1]\setminus\mathbb{Q}}$

$f\colon [0, 1] \to \mathbb{R}$ is not continuous but $f|_{[0,1]\setminus\mathbb{Q}}\colon [0, 1] \setminus \mathbb{Q} \to \mathbb{R}$ is continuous.

Example 2.3.5 shows also that we cannot take $\varepsilon = 0$ in Lusin's theorem.

Proof of Theorem 2.3.3: For each positive integer i, let $\{B_{ij}\} \subseteq \mathbb{R}$ be disjoint Borel sets such that $\mathbb{R} = \bigcup_{j=1}^{\infty} B_{ij}$ and $\operatorname{diam} B_{ij} = \sup\{|x-y|, x, y \in B_{ij}\} < \dfrac{1}{i}$.

Define $A_{ij} = f^{-1}(B_{ij})$, A_{ij} is μ-measurable and let

$$\widetilde{\Omega} = \bigcup_{j=1}^{\infty} A_{ij} \quad (\Omega = \widetilde{\Omega} \cup f^{-1}\{\pm\infty\}).$$

Since μ is a Radon measure, there exists $K_{ij} \subseteq A_{ij}$ compact such that

$$\mu(A_{ij} \setminus K_{ij}) < \frac{\varepsilon}{2^{i+j}}.$$

Then

$$\begin{aligned}
\mu\Big(\widetilde{\Omega}\setminus\bigcup_{j=1}^{\infty}K_{ij}\Big) &= \mu\Big(\bigcup_{j=1}^{\infty}A_{ij}\setminus\bigcup_{j=1}^{\infty}K_{ij}\Big)\\
&\le \mu\Big(\bigcup_{j=1}^{\infty}(A_{ij}\setminus K_{ij})\Big)\\
&\le \Big(\sum_{j=1}^{\infty}\mu(A_{ij}\setminus K_{ij})\\
&< \sum_{j=1}^{\infty}\frac{\varepsilon}{2^{i+j}} = \frac{\varepsilon}{2^{i}}\,.
\end{aligned} \tag{2.3.8}$$

Since

$$\lim_{n\to+\infty}\mu\Big(\widetilde{\Omega}\setminus\bigcup_{j=1}^{n}K_{ij}\Big) = \mu\Big(\widetilde{\Omega}\setminus\bigcup_{j=1}^{\infty}K_{ij}\Big) < \frac{\varepsilon}{2^{i}}\,,$$

$\exists N(i) > 0$ such that

$$\mu\Big(\widetilde{\Omega}\setminus\bigcup_{j=1}^{N(i)}K_{ij}\Big) < \frac{\varepsilon}{2^{i}}\,.$$

Set $D_i = \bigcup_{j=1}^{N(i)} K_{ij}$, D_i is compact. For each i and j we fix $b_{ij} \in B_{ij}$ and define g_i: $D_i \to \mathbb{R}$, $g_i(x) = b_{ij}$ if $x \in K_{ij}$ for all $j \le N(i)$. We observe that g_i is continuous since it is locally constant (the sets K_{ij} for $j \le N(i)$ are compact, disjoint sets and so they are at positive distance apart).

Furthermore, $|f(x) - g_i(x)| < \frac{1}{i}$ $\forall x \in D_i$ $(\operatorname{diam} B_{ij} < \frac{1}{i})$. Set $K = \bigcap_{i=1}^{\infty} D_i$, K is compact and

$$\mu(\widetilde{\Omega}\setminus K) = \mu\Big(\bigcup_{i=1}^{\infty}(\widetilde{\Omega}\setminus D_i)\Big) \le \sum_{i=1}^{\infty}\mu(\widetilde{\Omega}\setminus D_i) < \varepsilon.$$

Note that

$$\begin{aligned}
\Omega\setminus K &= \widetilde{\Omega}\setminus K \cup f^{-1}\{\pm\infty\}\setminus K\\
\mu(\Omega\setminus K) &\le \mu(\widetilde{\Omega}\setminus K) + \mu(f^{-1}(\pm\infty\}\setminus K)\\
&\le \varepsilon + 0\,.
\end{aligned}$$

Since $|f(x)-g_i(x)| < \frac{1}{i}$ $\forall x \in D_i$, we see that $g_i(x) \to f(x)$ is uniformly on K as $i \to +\infty$. Thus $f|_K$ is continuous. $\square$

2.4 Convergence in Measure

Let μ be an arbitrary measure on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ μ-measurable and let $f, f_k\colon \Omega \to \overline{\mathbb{R}}$ be μ-measurable and $|f(x)| < +\infty$ μ-a.e..

Definition 2.4.1. *The sequence $(f_k)_k$ converges in measure μ to f, in short: $f_k \xrightarrow{\mu} f$ as $k \to \infty$ if $\forall \varepsilon > 0$*

$$\lim_{k\to+\infty} \mu(\{x \in \Omega : |f(x) - f_k(x)| > \varepsilon\}) = 0.$$

Question:

Which is the relation between convergence in measure, pointwise convergence and uniform convergence?

Theorem 2.4.2. *Let $\mu(\Omega) < +\infty$. If $f_k \to f$ μ-a.e. as $k \to +\infty$ then $f_k \xrightarrow{\mu} f$ as $k \to +\infty$.*

Proof. From Theorem 2.3.1 it follows that for every $\delta > 0$ $\exists F_\delta \subseteq \Omega$ μ-measurable with $\mu(\Omega \setminus F_\delta) < \delta$ (if μ is a Radon measure we can actually assume that F_δ is compact) and $\forall \varepsilon > 0$ $\exists n_\varepsilon > 0$ such that

$$\sup_{x\in F_\delta} |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_\varepsilon.$$

Therefore, for $n \geq n_\varepsilon$

$$\{x \in \Omega \, |f_n(x) - f(x)| > \varepsilon\} \subseteq \Omega \setminus F_\delta. \tag{2.4.1}$$

Hence

$$\mu\{x \in \Omega : |f_n(x) - f(x)| > \varepsilon\} \leq \mu(\Omega \setminus F_\delta) < \delta$$

Since $\delta > 0$ is arbitrary, we can conclude. $\square$

Remark 2.4.3.

1) Theorem 2.4.2 does not hold in general if $\mu(\Omega) = +\infty$. Take $f_k(x) = \chi_{(-k,k)}(x)$, $\Omega = \mathbb{R}$, $\mu = \mathcal{L}^1$. Then $f_k(x) \to f \equiv 1$ in $\mathbb{R}$ but $f_k \overset{\mu}{\not\to} 1$: $\forall \varepsilon > 0$

$$\mathcal{L}^1(\{x \in \mathbb{R} : |f_k(x) - 1| \geq \varepsilon\}) = \mathcal{L}^1\big((-k,k)^c\big) = \infty\,. \qquad (2.4.2)$$

2) The converse of Theorem 2.4.2 does not hold. Take $\Omega = [0,1) \subseteq \mathbb{R}$, $\mu = \mathcal{L}^1$

$$f_k(x) = \chi_{[\frac{k-2^n}{2^n},\ \frac{k+1-2^n}{2^n})}(x),\ k \in \mathbb{N}.$$

For every $k \geq 1$, the index n is chosen so that $2^n \leq k < 2^{n+1}$.

We observe that $\mu(\{f_k(x) > 0\}) = \dfrac{1}{2^n} < \dfrac{2}{k} \to 0$ as $k \to +\infty$. Thus $f_k \xrightarrow{\mu} f \equiv 0$ as $k \to +\infty$. But for every $x \in \Omega$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, we have

$$f_k(x) = \begin{cases} 1 & \text{if } k = [2^n x] + 2^n \\ 0 & \text{for the remaining } k \in \{2^n + 1, \ldots, 2^{n+1} - 1\}, \end{cases}$$

where $[s] = \sup\{n \in \mathbb{N},\, n \leq s\}$. Thus f_k cannot converge to $f \equiv 0$! $\square$

However, it holds the following

Theorem 2.4.4. *Let $f_k \xrightarrow{\mu} f$. Then there exists a subsequence $\{f_{k_n}\}$ which converges to f μ-a.e. as $n \to +\infty$.*

Proof. Since $f_k \xrightarrow{\mu} f$, there exists $k_n \geq 1$ such that

$$\mu(\{x \in \Omega : |f_k(x) - f(x)| > 2^{-n}\}) < 2^{-n} \quad \forall k \geq k_n.$$

We define

$$A_n = \{x \in \Omega : |f_{k_n}(x) - f(x)| > 2^{-n}\} \qquad (2.4.3)$$

and for $h \geq 1$ fixed

$$E_h = \bigcup_{n \geq h} A_n.$$

By the σ-subadditivity of the measure, we have

$$\mu(E_h) \leq \sum_{n=h}^{\infty} \mu(A_n) \leq \sum_{n=h}^{\infty} \frac{1}{2^n} = 2^{-h+1}. \tag{2.4.4}$$

Let $x \in \Omega \setminus E_h$, then $x \notin A_n$ $\forall n \geq h$. Thus $\forall h \in \mathbb{N}$, we have $f_{k_n}(x)$ converges to $f(x)$ on $\Omega \setminus E_h$.

Set $E = \bigcap_{h=1}^{\infty} E_h$, since $\mu(E_1) < +\infty$ and E_h is decreasing, we have

$$\mu(E) = \lim_{h \to \infty} \mu(E_h) = 0. \tag{2.4.5}$$

Since $f_{k_n}(x)$ converges to $f(x)$ $\forall x \in \Omega \setminus E_h, \forall h$, then $f_{k_n}(x)$ converges to $f(x)$, $\forall x \in \Omega \setminus E$, and we conclude. $\square$

Chapter 3

Integration

In this chapter we assume that μ is a Radon measure on $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ is μ-measurable.

3.1 Definitions and Basic Properties

Definition 3.1.1. *A function $g\colon \Omega \to \overline{\mathbb{R}}$ is called a* ***simple function*** *if the image of g is at most countable.*

Notation:

$$f^+ = \max(f, 0), \;\; f^- = \max(-f, 0)$$

We recall that $\; f = f^+ - f^-, \;\; |f| = f^+ + f^-$.

Definition 3.1.2. *(Integral of simple functions)*

If $g\colon \Omega \to [0, +\infty]$ is a ***nonnegative, simple, μ-measurable*** *function, we define*

$$\int_\Omega g \, d\mu \equiv \sum_{0 \le y \le \infty} y \, \mu(g^{-1}\{y\}). \tag{3.1.1}$$

We will use the convention that $0 \cdot \infty = 0$.

Definition 3.1.3. *If* $g\colon \Omega \to [-\infty, +\infty]$ *is a simple,* μ-*measurable function, and either* $\int_\Omega g^+ d\mu < \infty$ *or* $\int_\Omega g^- d\mu < \infty$, *we call* g *a* ***μ-integrable simple function*** *and define*

$$\int g\,d\mu \equiv \int g^+\,d\mu - \int g^-\,d\mu. \tag{3.1.2}$$

Therefore if g *is* μ-*integrable simple function:*

$$\int g\,d\mu = \sum_{-\infty\le y\le\infty} y\mu(g^{-1}\{y\}). \tag{3.1.3}$$

Definition 3.1.4. *Let* $f\colon \Omega \to [-\infty, \infty]$. *We define the* ***upper integral***

$$\begin{aligned}\overline{\int_\Omega} f\,d\mu \;&=\; \inf\left\{\int_\Omega g\,d\mu : \; g \text{ a } \mu\text{-integrable simple function}, \right.\\ &\qquad\left. g \ge f \;\mu\text{-a.e.} \right\} \in \bar{\mathbb{R}}\end{aligned} \tag{3.1.4}$$

and the ***lower integral***

$$\begin{aligned}\underline{\int_\Omega} f\,d\mu \;&=\; \sup\left\{\int_\Omega g\,d\mu : \; g \text{ a } \mu\text{-integrable simple function}, \right.\\ &\qquad\left. g \le f \;\mu\text{-a.e.} \right\} \in \bar{\mathbb{R}}.\end{aligned} \tag{3.1.5}$$

Definition 3.1.5. *A* μ-*measurable function* $f\colon \Omega \to \overline{\mathbb{R}}$ *is called* **μ-integrable** *if* $\overline{\int_\Omega} f\,d\mu = \underline{\int_\Omega} f\,d\mu$, *in which case we write*

$$\int_\Omega f\,d\mu \equiv \overline{\int_\Omega} f\,d\mu = \underline{\int_\Omega} f\,d\mu. \tag{3.1.6}$$

Exercise 3.1.6. *Show that*

$$\underline{\int_{\Omega}} f\, d\mu \leq \overline{\int_{\Omega}} f\, d\mu. \tag{3.1.7}$$

Warning:
For us a function is "**integrable**" provided it has an integral, even if this integral equals $+\infty$ or $-\infty$ (see Evans-Gariepy).

Proposition 3.1.7. *Let $f\colon\ \Omega \to [0, +\infty]$ μ-measurable. Then f is μ-integrable.*

Proof. We may suppose that $\underline{\int_{\Omega}} f\, d\mu < +\infty$.

(Indeed, if $\underline{\int_{\Omega}} f\, d\mu = +\infty$, then $\overline{\int_{\Omega}} f\, d\mu = +\infty$ as well).

In particular, we have $f(x) < +\infty \quad \mu$-a.e.

- Case 1: Assume $\mu(\Omega) < +\infty$. For $\varepsilon > 0$ given, we set

$$A_k = \{x \in \Omega :\ k\varepsilon \leq f(x) < (k+1)\,\varepsilon\}, \quad k \geq 0. \tag{3.1.8}$$

We set $\Omega' := \cup_{k\geq 0} A_k = \cup_{k\geq 0} f^{-1}([k\varepsilon, (k+1)\varepsilon))$. We have $\mu(\Omega \setminus \Omega') = 0$.

We define the simple functions

$$\begin{aligned} e(x) &= \varepsilon \sum_{k=0}^{\infty} k\, \chi_{A_k}(x) \ \text{ and} \\ g(x) &= \varepsilon \sum_{k=0}^{\infty} (k+1)\, \chi_{A_k}(x). \end{aligned} \tag{3.1.9}$$

Clearly, we have $e(x) \leq f(x) < g(x)$ μ-a.e and

$$\begin{aligned}\int_\Omega e\,d\mu \le \underline{\int_\Omega} f\,d\mu \le \overline{\int_\Omega} f\,d\mu \\ \le \int_\Omega g\,d\mu \le \int_\Omega e\,d\mu + \varepsilon\mu(\Omega).\end{aligned} \tag{3.1.10}$$

We let $\varepsilon \to 0$ and get the result.

- Case 2: Let $\Omega \subseteq \mathbb{R}^n$ be a general μ-measurable set. We consider $\mathbb{R}^n = \bigcup_{\ell=1}^\infty Q_\ell$, Q_ℓ disjoint dyadic cubes.

  For $\varepsilon > 0$ there are simple functions $e_\ell, g_\ell\colon \Omega \cap Q_\ell \to \overline{\mathbb{R}}$ with
  $$e_\ell \le f \le g_\ell \text{ in } \Omega_\ell = \Omega \cap Q_\ell$$
  and
  $$\int_{\Omega_\ell} e_\ell\,d\mu \le \int_{\Omega_\ell} g_\ell\,d\mu \le \int_{\Omega_\ell} e_\ell\,d\mu + \frac{\varepsilon}{2^\ell}. \tag{3.1.11}$$
  Observe that $\sum_{\ell=1}^\infty e_\ell\,\chi_{\Omega_\ell}$ and $\sum_{\ell=1}^\infty g_\ell\,\chi_{\Omega_\ell}$ are simple functions with
  $$\sum_{\ell=1}^\infty e_\ell\,\chi_{\Omega_\ell} \le f(x) \le \sum_{\ell=1}^\infty g_\ell\,\chi_{\Omega_\ell}\,.$$
  We sum up in (3.1.11) over ℓ and we get
  $$\sum_{\ell=1}^\infty \int_{\Omega\cap Q_\ell} e_\ell\,d\mu \le \underline{\int_\Omega} f\,d\mu \le \overline{\int_\Omega} f\,d\mu \le \sum_{\ell=1}^\infty \int_{\Omega\cap Q_\ell} e_\ell\,d\mu + \varepsilon.$$

We let $\varepsilon \to 0$ and conclude the proof. □

Proposition 3.1.8. *(Monotonicity)*
Let $f_1, f_2\colon \Omega \to \overline{\mathbb{R}}$ be μ-integrable with $f_1 \ge f_2$ μ-a.e.. Then
$$\int_\Omega f_1\,d\mu \ge \int_\Omega f_2\,d\mu. \tag{3.1.12}$$

Proof. If $g \geq f_1$ and $f_1 \geq f_2$ μ-a.e., then $g \geq f_2$ μ-a.e.. Therefore

$$\begin{aligned}\int_\Omega f_1 \, d\mu = \overline{\int_\Omega} f_1 = \inf_g \int_\Omega g \, d\mu \geq \overline{\int_\Omega} f_2 \, d\mu \\ = \int_\Omega f_2 \, d\mu,\end{aligned} \tag{3.1.13}$$

where the infimum is taken over all simple μ-integrable function $g \geq f_1$ μ-a.e.. □

Definition 3.1.9.

i) *A function $f \colon \Omega \to \overline{\mathbb{R}}$ is* ***μ-summable*** *if f is μ-measurable and*
$$\int_\Omega |f| \, d\mu < +\infty.$$

ii) *A function $f \colon \Omega \to \overline{\mathbb{R}}$ is* ***locally μ-summable*** *in Ω, if $f|_K$ is μ-summable for each compact set $K \subseteq \Omega$.*

Proposition 3.1.10.

i) *If f is μ-summable, then it is μ-integrable.*

ii) *If $f = 0$ μ-a.e., then f is μ-integrable and $\int_\Omega f \, d\mu = 0$.*

Proof of Proposition 3.1.10:

i) We have $f = f_+ - f_-$. Since $0 \leq f_\pm \leq |f|$ the functions $f_\pm$ are μ-integrable and $\int_\Omega f_\pm \, d\mu < +\infty$. For $\varepsilon > 0$ $\exists$ simple functions $0 \leq e_\pm \leq f_\pm \leq g_\pm$ μ-a.e. with
$$\begin{aligned}0 \leq \int_\Omega e_\pm \, d\mu \leq \int_\Omega f_\pm \, d\mu \leq \int_\Omega g_\pm \, d\mu \\ \leq \int_\Omega e_\pm \, d\mu + \varepsilon < +\infty\end{aligned}$$

(see the proof of Proposition 3.1.7).

Thus $e := e_+ - g_-$, $g := g_+ - e_-$ are simple functions satisfying

$$e \le f \le g \ \mu\text{-a.e.}$$

We may assume without restriction that $e_\pm = g_\pm = 0$ on $\{x \in \Omega : \ f^\pm(x) = 0\}$. It holds

$$\begin{aligned}\int_\Omega e \, d\mu \le \underline{\int_\Omega} f \, d\mu \le \overline{\int_\Omega} f \, d\mu \le \int_\Omega g \, d\mu \\ \le \int_\Omega e \, d\mu + 2\varepsilon.\end{aligned}$$

We let $\varepsilon \to 0$ and get the result.

ii) It is trivial (choose $e = g = 0$). In this case we get $\underline{\int_\Omega} f \, d\mu \ge 0$ and $\overline{\int_\Omega} f \, d\mu \le 0$ and therefore

$$\underline{\int_\Omega} f \, d\mu = \overline{\int_\Omega} f \, d\mu = 0.$$

We conclude the proof. $\square$

Proposition 3.1.11. *Let $f\colon \Omega \to [0, +\infty]$ be μ-measurable.*

i) *If $\int_\Omega f \, d\mu = 0 \Longrightarrow f(x) = 0$ μ-a.e.*

ii) *If $\int_\Omega f \, d\mu < +\infty \Longrightarrow f(x) < +\infty$ μ-a.e.*

Proof of Proposition 3.1.11:

i) Assume f is **not zero** μ-a.e.. Consider the measurable sets

$$A_k = \left\{ x : f(x) \ge \frac{1}{k} \right\}, \quad k \ge 1.$$

Note that $A_k \subseteq A_{k+1}$ and

$$\bigcup_k A_k = \{x \in \Omega : f(x) > 0\}.$$

We have

$$0 < \mu\{x \in \Omega : f(x) > 0\} = \lim_{k \to +\infty} \mu(A_k). \tag{3.1.14}$$

Therefore there exists $k \geq 1$ such that $\mu(A_k) > 0$. The function $s(x) = \frac{1}{k}\chi_{A_k}(x)$ is a simple function such that $s(x) \leq f$ and $\int_\Omega s(x)\, d\mu = \frac{1}{k}\mu(A_k) > 0 \Longrightarrow \int_\Omega f\, d\mu > 0$, which is a contradiction.

ii) Suppose that $f(x) = +\infty$ $\forall x \in A$ with $\mu(A) > 0$. Then the simple function $s(x) = +\infty\, \chi_A(x)$ satisfy $0 \leq s \leq f$, with $\int_\Omega s(x)\, d\mu = +\infty$. Hence $\int_\Omega f\, d\mu = +\infty$ which is a contradiction.

□

Corollary 3.1.12. *Let $f_1, f_2 \colon \Omega \to \overline{\mathbb{R}}$ integrable with $f_1 = f_2$ μ-a.e. Then*

$$\int_\Omega f_1\, d\mu = \int_\Omega f_2\, d\mu. \tag{3.1.15}$$

Proof. It follows directly from Proposition 3.1.8. □

Theorem 3.1.13. *(Tchebychev Inequality)*

Let $f \colon \Omega \to \overline{\mathbb{R}}$ be μ-summable. Then for every $a > 0$ it holds

$$\mu(\{x \in \Omega : |f(x)| > a\}) \leq \frac{1}{a} \int_\Omega |f|\, d\mu. \tag{3.1.16}$$

Proof. Choose in Proposition 3.1.8 $f_1 = |f|$ and $f_2 = a\, \chi_{\{x \in \Omega : |f(x)| > a\}}$.
□

Corollary 3.1.14. *Let $f, f_k \colon \Omega \to \overline{\mathbb{R}}$ be μ-integrable with*

$$\lim_{k \to +\infty} \int_\Omega |f_k - f| \, d\mu = 0.$$

Then $f_k \xrightarrow{\mu} f$ as $k \to +\infty$ and $f_{k_n} \to f$ μ-a.e. for a subsequence f_{k_n}.

Proof. From Theorem 3.1.13 it follows that for $\forall \varepsilon > 0$

$$\mu\{x \in \Omega : |f_k - f| > \varepsilon\} < \frac{1}{\varepsilon} \int_\Omega |f_k - f| \to 0 \text{ as } k \to +\infty. \tag{3.1.17}$$

Therefore, $f_k \xrightarrow{\mu} f$ as $k \to +\infty$ and the second part is a consequence of Theorem 2.4.4. □

Theorem 3.1.15. *Let $f, g \colon \Omega \to \overline{\mathbb{R}}$ be **μ-summable**, $\lambda \in \mathbb{R}$. Then $f + g$, λf are μ-summable and*

$$\begin{aligned} \int_\Omega (f+g) \, d\mu &= \int_\Omega f \, d\mu + \int_\Omega g \, d\mu \\ \int_\Omega \lambda f \, d\mu &= \lambda \int_\Omega f \, d\mu. \end{aligned} \tag{3.1.18}$$

Proof.

i) The two relations hold for μ-integrable simple functions of the form

$$f(x) = \sum_{k=1}^{\infty} a_k \, \chi_{A_k} \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} b_k \, \chi_{B_k} \tag{3.1.19}$$

if both satisfy either $\int_\Omega f^+ d\mu < +\infty$ and $\int_\Omega g^+ d\mu < +\infty$ or $\int_\Omega f^- d\mu < +\infty$ and $\int_\Omega g^- d\mu < +\infty$. Such conditions are satisfied for instance if we assume that both f and g are μ-summable.

We observe that in the representation of f and g in (3.1.19) can always suppose that $A_k = B_k\ \forall k \in \mathbb{N}$. Otherwise we can consider $(A_k \cap B_\ell)_{k,\ell\in\mathbb{N}}$ (we leave all the statements in this part of the proof as an exercise).

ii) We know that $f+g$ is μ-measurable. For $\varepsilon > 0$ we choose simple μ-integrable functions $f_\varepsilon, f^\varepsilon, g_\varepsilon, g^\varepsilon$ such that

$$f_\varepsilon \leq f \leq f^\varepsilon \qquad \text{and} \qquad g_\varepsilon \leq g \leq g^\varepsilon$$

$$\int_\Omega f^\varepsilon d\mu - \int_\Omega f\,d\mu < \varepsilon \qquad \text{and} \qquad \int_\Omega f\,d\mu - \int_\Omega f_\varepsilon < \varepsilon$$

$$\int_\Omega g^\varepsilon d\mu - \int_\Omega g\,d\mu < \varepsilon \qquad \text{and} \qquad \int_\Omega g\,d\mu - \int_\Omega g_\varepsilon < \varepsilon$$

Thus $f_\varepsilon + g_\varepsilon$, $f^\varepsilon + g^\varepsilon$ are μ-measurable simple functions such that

$$f_\varepsilon + g_\varepsilon \leq f + g \leq f^\varepsilon + g^\varepsilon \tag{3.1.20}$$

Moreover since we are assuming that f, g are both μ-summable we have in particular $\int_\Omega (f^\varepsilon)^- d\mu < +\infty$ and $\int_\Omega (g^\varepsilon)^- d\mu < +\infty$, $\int_\Omega (f_\varepsilon)^+ d\mu < +\infty$ $\int_\Omega (f^\varepsilon)^+ d\mu < +\infty$. It follows that $f_\varepsilon + g_\varepsilon$, $f^\varepsilon + g^\varepsilon$ are μ-integrable since $\int_\Omega (f^\varepsilon + g^\varepsilon)^-\,d\mu < +\infty$ and $\int_\Omega (f_\varepsilon + g_\varepsilon)^+\,d\mu < +\infty$ and therefore

$$\int_\Omega (f^\varepsilon + g^\varepsilon)\,d\mu = \int_\Omega f^\varepsilon d\mu + \int_\Omega g^\varepsilon d\mu \tag{3.1.21}$$

$$\int_\Omega (f_\varepsilon + g_\varepsilon)\,d\mu = \int_\Omega f_\varepsilon d\mu + \int_\Omega g_\varepsilon d\mu. \tag{3.1.22}$$

By combining (3.1.20) and (3.1.21) and (3.1.22) we get

$$
\begin{aligned}
\overline{\int_\Omega} f+g\,d\mu &\leq \int_\Omega (f^\varepsilon+g^\varepsilon)\,d\mu = \int_\Omega f^\varepsilon d\mu + \int_\Omega g^\varepsilon d\mu \\
&\leq \int_\Omega f\,d\mu + \int_\Omega g\,d\mu + 2\varepsilon \\
\underline{\int_\Omega}(f+g)\,d\mu &\geq \int_\Omega f_\varepsilon\,d\mu + \int_\Omega g_\varepsilon\,d\mu \\
&\geq \int_\Omega f\,d\mu + \int_\Omega g\,d\mu - 2\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, it follows that $f+g$ is μ-integrable with

$$\int_\Omega (f+g)\,d\mu = \int_\Omega f\,d\mu + \int_\Omega g\,d\mu.$$

iii) We apply the result to $|f|$ and $|g|$ and we get

$$
\begin{aligned}
\int_\Omega |f+g| d\mu &\leq \int_\Omega (|f|+|g|)\,d\mu \\
&= \int_\Omega |f|\,d\mu + \int_\Omega |g|\,f\mu < +\infty.
\end{aligned}
\tag{3.1.23}
$$

Thus $f+g$ is μ-summable and a) is proved.

iv) Let $\varepsilon > 0$, choose simple functions $f_\varepsilon \leq f \leq f^\varepsilon$ such that

$$
\begin{aligned}
\int_\Omega f^\varepsilon d\mu - \int_\Omega f\,d\mu < \varepsilon \\
\int_\Omega f\,d\mu - \int_\Omega f_\varepsilon\,d\mu < \varepsilon.
\end{aligned}
$$

Thus $\lambda\, f_\varepsilon$, $\lambda\, f^\varepsilon$ are simple functions with

$$\lambda\, f_\varepsilon \leq \lambda f \leq \lambda f^\varepsilon \quad \text{if } \lambda \geq 0$$

and

$$\lambda\, f_\varepsilon \geq \lambda f \geq \lambda f^\varepsilon \quad \text{if } \lambda < 0.$$

We consider $\boldsymbol{\lambda > 0}$ (the case $\lambda = 0$ is trivial and $\lambda < 0$ is similar to the case $\lambda > 0$). It follows

$$\begin{aligned} \overline{\int_\Omega} \lambda f \, d\mu &\leq \int_\Omega \lambda f^\varepsilon d\mu = \lambda \int_\Omega f^\varepsilon d\mu \\ &\leq \lambda \int_\Omega f \, d\mu + \lambda\varepsilon \end{aligned} \tag{3.1.24}$$

and

$$\begin{aligned} \underline{\int_\Omega} (\lambda f) \, d\mu &\geq \lambda \int_\Omega f_\varepsilon \, d\mu \\ &\geq \lambda \int_\Omega f \, d\mu - \lambda\varepsilon. \end{aligned} \tag{3.1.25}$$

Thus λf is μ-integrable with

$$\int_\Omega (\lambda f) \, d\mu = \lambda \int_\Omega f \, d\mu. \tag{3.1.26}$$

Moreover,

$$\int_\Omega |\lambda f| \, d\mu = \int_\Omega |\lambda| \, |f| = |\lambda| \int_\Omega |f| \, d\mu < +\infty. \tag{3.1.27}$$

Thus λf is μ-summable as well and b) is proved. □

Corollary 3.1.16. *Let $f \colon \Omega \to \overline{\mathbb{R}}$ be μ-summable. Then*

$$\left| \int_\Omega f \, d\mu \right| \leq \int_\Omega |f| \, d\mu. \tag{3.1.28}$$

Proof. We have $-|f| \leq f \leq |f|$. Thus

$$-\int_\Omega |f| \, d\mu \leq \int_\Omega f \, d\mu \leq \int_\Omega |f| \, d\mu \tag{3.1.29}$$

and we can conclude. □

Lemma 3.1.17. *Let $f\colon \Omega \to \overline{\mathbb{R}}$ be μ-summable and $\Omega_1 \subseteq \Omega$ be μ-measurable. Then $f_1 = f|_{\Omega_1}$ and $f\chi_{\Omega_1}$ are μ-summable on Ω_1 and Ω respectively and*

$$\int_{\Omega_1} f_1 \, d\mu = \int_{\Omega} f \, \chi_{\Omega_1} \, d\mu.$$

Proof. We first note that f_1 and $f\chi_{\Omega_1}$ are μ-measurable functions.

Step 1: We show the result for simple μ-summable functions. Let $g(x) = \sum_{i=1}^{\infty} a_i \chi_{A_i}$ with $A_{i_1} \cap A_{i_2} = \emptyset$ if $i_1 \neq i_2$ and $\cup_i A_i = \Omega$. In this case we have:

$$\begin{aligned}
g_1(x) = g|_{\Omega_1}(x) &= \sum_{i=1}^{\infty} a_i \chi_{A_i \cap \Omega_1} \\
\int_{\Omega_1} g_1 d\mu &= \sum_{i=1}^{\infty} a_i \mu(A_i \cap \Omega_1) \\
g\chi_{\Omega_1}(x) &= \sum_{i=1}^{\infty} a_i \chi_{A_i} \chi_{\Omega_1} = \sum_{i=1}^{\infty} a_i \chi_{A_i \cap \Omega_1} \\
\int_{\Omega} g\chi_{\Omega_1} d\mu &= \sum_{i=1}^{\infty} a_i \mu(A_i \cap \Omega_1).
\end{aligned}$$

We also observe that both g_1 and $g\chi_{\Omega_1}$ are μ-summable because $|g\chi_{\Omega_1}| \leq |g|$.

Step 2: Let $\varepsilon > 0$ and choose simple μ-integrable functions g, h such that

$$g \leq f \leq h \qquad \mu \text{ a.e. } x \in \Omega$$

and

$$\begin{aligned}
\int_{\Omega} f d\mu &\leq \int_{\Omega} g d\mu + \varepsilon \\
\int_{\Omega} h d\mu &\leq \int_{\Omega} f d\mu + \varepsilon.
\end{aligned}$$

Claim 1: f_1 is μ-integrable.

Proof of Claim 1. We have

$$\begin{aligned}
0 &\leq \overline{\int_{\Omega}} f\, d\mu - \underline{\int_{\Omega}} f\, d\mu \\
&\leq \int_{\Omega_1} h_1 d\mu - \int_{\Omega_1} g_1 d\mu = \int_{\Omega} h\chi_{\Omega_1} d\mu - \int_{\Omega} g\chi_{\Omega_1} d\mu \\
&= \int_{\Omega} (h-g)\chi_{\Omega_1} d\mu \leq \int_{\Omega} (h-g) d\mu = \int_{\Omega} h d\mu - \int_{\Omega} g d\mu \leq 2\varepsilon.
\end{aligned}$$

We let $\varepsilon \to 0$ and get f_1 is μ-integrable.

Claim 2: $f\chi_{\Omega_1}$ is μ-integrable.

Proof of Claim 2.

$$\begin{aligned}
0 &\leq \overline{\int_{\Omega}} f\chi_{\Omega_1} d\mu - \underline{\int_{\Omega}} f\chi_{\Omega_1} d\mu \\
&\leq \int_{\Omega} h\chi_{\Omega_1} d\mu - \int_{\Omega} g\chi_{\Omega_1} d\mu \\
&= \int_{\Omega} (h-g)\chi_{\Omega_1} d\mu \leq \int_{\Omega} (h-g) d\mu = \int_{\Omega} h d\mu - \int_{\Omega} g d\mu \leq 2\varepsilon.
\end{aligned}$$

We let $\varepsilon \to 0$ and get $f\chi_{\Omega_1}$ is μ-integrable.

Claim 3: $\int_{\Omega_1} f d\mu = \int_{\Omega} f\chi_{\Omega_1}$.

Proof of Claim 3.

$$\begin{aligned}
\int_{\Omega_1} f_1 d\mu - \int_{\Omega} f\chi_{\Omega_1} d\mu &\leq \int_{\Omega_1} h_1 d\mu - \int_{\Omega} g\chi_{\Omega_1} d\mu \\
&= \int_{\Omega} h\chi_{\Omega_1} d\mu - \int_{\Omega} g\chi_{\Omega_1} d\mu \\
&= \int_{\Omega} (h-g)\chi_{\Omega_1} d\mu \leq 2\varepsilon.
\end{aligned}$$

On another hand:

$$\begin{aligned}
\int_{\Omega_1} f_1 d\mu - \int_{\Omega} f\chi_{\Omega_1} d\mu &\geq \int_{\Omega_1} g_1 d\mu - \int_{\Omega} h\chi_{\Omega_1} d\mu \\
&= \int_{\Omega} g\chi_{\Omega_1} d\mu - \int_{\Omega} h\chi_{\Omega_1} d\mu \\
&= \int_{\Omega} (g-h)\chi_{\Omega_1} d\mu \geq -2\varepsilon.
\end{aligned}$$

Therefore

$$\left| \int_{\Omega_1} f_1 d\mu - \int_{\Omega} f\chi_{\Omega_1} d\mu \right| \leq 2\varepsilon.$$

By letting $\varepsilon \to 0$ we get the result and we conclude the proof. □

Corollary 3.1.18. *Let $f\colon \Omega \to \overline{\mathbb{R}}$ be a μ-summable function and $\Omega_1 \subseteq \Omega$ with $\mu(\Omega_1) = 0$. Then*

$$\int_{\Omega_1} f\, d\mu = 0. \tag{3.1.30}$$

Proof. We observe that $f\,\chi_{\Omega_1} = 0$ μ-a.e.. Therefore $\int_{\Omega} f\,\chi_{\Omega_1}\, d\mu = 0$ and the result follows from Lemma 3.1.17. □

Proposition 3.1.19. *Let $f\colon \Omega \to \overline{\mathbb{R}}$ be μ-summable, $\Omega = \Omega_1 \cup \Omega_2$, Ω_1, Ω_2 μ-measurable sets, $\Omega_1 \cap \Omega_2 = \emptyset$. Then*

$$\int_{\Omega} f\, d\mu = \int_{\Omega_1} f\, d\mu + \int_{\Omega_2} f\, d\mu. \tag{3.1.31}$$

Proof. We have $f = f\,\chi_{\Omega_1} + f\,\chi_{\Omega_2}$. The result follows from Lemma 3.1.17 and Corollary 3.1.18. □

3.2 Comparison between Lebesgue and Riemann-Integral

3.2.1 Review of Riemann Integral

Let $I = [a, b]$ be an interval of $\mathbb{R}$ and let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of I.

If $f\colon I \to \mathbb{R}$ is a bounded function and P is a partition of I, we define the *upper* and the *lower Riemann sums* respectively by

$$\begin{aligned} S(P,f) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in (x_{i-1}, x_i]} f(x) \\ &= \sum_{i=1}^{n} (x_i - x_{i-1})\, M_i \\ s(P,f) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in (x_{i-1}, x_i]} f(x) \\ &= \sum_{i=1}^{n} (x_i - x_{i-1})\, m_i. \end{aligned} \tag{3.2.1}$$

Let $\mathcal{P}$ be the set of all partitions of I. We define

$$\begin{aligned} \mathcal{R}\overline{\int_a^b} f(x)\,dx &= \inf\{S(P,f), P \in \mathcal{P}\} \\ \mathcal{R}\underline{\int_a^b} f(x)\,dx &= \sup\{s(P,f), P \in \mathcal{P}\}. \end{aligned} \tag{3.2.2}$$

We say that a bounded function f is Riemann integrable if

$$\mathcal{R}\underline{\int_a^b} f(x)\,dx = \mathcal{R}\overline{\int_a^b} f(x)\,dx =: \int_a^b f(x)\,dx.$$

If

$$\varphi(x) = \sum_{i=1}^{n} c_i\, \chi_{(x_{i-1}, x_i]} \tag{3.2.3}$$

where $\{x_i\}_{i=0,\ldots,n}$ partition of $I = [a, b]$ and $c_1, \ldots, c_n \in \mathbb{R}$, its Lebesgue integral is defined by

$$\int_{[a,b]} \varphi(x)\,dx = \sum_{i=1}^{n} c_i (x_i - x_{i-1}). \tag{3.2.4}$$

Thus we can write

$$\begin{aligned} S(P,f) &= \int_{[a,b]} \overline{\varphi}(x)\,dx \\ s(P,f) &= \int_{[a,b]} \underline{\varphi}(x)\,dx, \end{aligned} \tag{3.2.5}$$

where

$$\overline{\varphi}(x) = \sum_{i=1}^{n} M_i\,\chi_{(x_{i-1},x_i]} \tag{3.2.6}$$

$$\underline{\varphi}(x) = \sum_{i=1}^{n} m_i\,\chi_{(x_{i-1},x_i]} \tag{3.2.7}$$

It is clear that $\underline{\varphi}(x) \le f(x) \le \overline{\varphi}(x)$. In view of the above considerations we can write

$$\begin{aligned} \mathcal{R}\overline{\int_a^b} f(x)\,dx &= \inf\Big\{ \int_{[a,b]} \overline{\varphi}(x)\,dx,\ \overline{\varphi} \text{ as in (3.2.3)},\ \overline{\varphi} \ge f \text{ in } I \Big\} \\ \mathcal{R}\underline{\int_a^b} f(x)\,dx &= \sup\Big\{ \int_{[a,b]} \underline{\varphi}\,dx,\ \underline{\varphi} \text{ as in (3.2.3)},\ \underline{\varphi} \le f \text{ in } I \Big\}. \end{aligned} \tag{3.2.8}$$

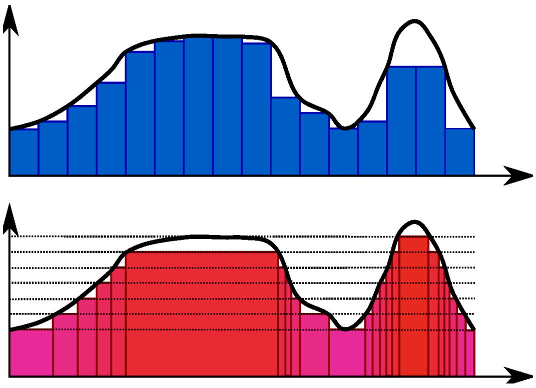


Figure 3.1: The Riemann integral considers the area under a curve as made out of vertical rectangles, the Lebesgue integral considers horizontal slabs that are not necessarily just rectangles.

Example 3.2.1. We know that the Dirichlet function $\chi_{\mathbb{Q}\cap[0,1]} =: f$ is not Riemann integrable. Indeed, for every partition P of $[0,1]$, we have $S(P,f) = 1$ and $s(P,f) = 0$. Therefore

$$\mathcal{R}\overline{\int_0^1} f(x)\,dx = 1,\ \mathcal{R}\underline{\int_0^1} f(x)\,dx = 0. \tag{3.2.9}$$

Moreover, if we consider the sequence

$$f_n(x) = \chi_{\{r_1,\dots,r_n\}}(x),$$

where $\{r_n\}$ is an enumeration of $\mathbb{Q}\cap[0,1]$, we have $f_n(x)$ is $\mathcal{R}$-integrable and $\mathcal{R}\int_0^1 f_n(x)\,dx = 0$ and $f_n \uparrow \chi_{[0,1]\cap\mathbb{Q}}$ as $n \to +\infty$.
It makes no sense to wonder if

$$\lim_{n\to+\infty} \mathcal{R}\int_0^1 f_n(x)\,dx = \mathcal{R}\int_0^1 \chi_{\mathbb{Q}\cap[0,1]}(x)\,dx.$$

Unlike the Riemann integral the Lebesgue integral is such that the characteristic functions of $\mathcal{L}^1$-measurable sets are $\mathcal{L}^1$-integrable and it behaves well under "passage to the limit" as we will see in the sequel.

Proposition 3.2.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded R-integrable function. Then it is $\mathcal{L}^1$-integrable and*

$$\mathcal{R}\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\mathcal{L}^1.$$

Proof. We observe that every simple function of the type $\varphi(x) = \sum_{i=1}^n c_i\,\chi_{(x_{i-1},x_i]}$ is a $\mathcal{L}^1$-integrable simple function since $(x_{i-1},x_i]$ are $\mathcal{L}^1$-measurable. Moreover the following holds:

$$\begin{aligned}
&\inf\left\{\int_{[a,b]} \overline{\varphi}(x)\,dx,\ \overline{\varphi} \text{ as in (3.2.3)},\ \overline{\varphi} \geq f \text{ in } I\right\}\\
&\geq \inf\left\{\int_{[a,b]} g\,d\mathcal{L}^1 :\ g \text{ a } \mathcal{L}^1\text{-integrable simple function, } g \geq f\ \mu\text{-a.e. in I}\right\}
\end{aligned}$$

$$\begin{aligned}
&\sup\left\{\int_{[a,b]} g\,d\mathcal{L}^1 :\ g \text{ a } \mathcal{L}^1\text{-integrable simple function, } g \leq f\ \mu\text{-a.e. in I}\right\}\\
&\geq \sup\left\{\int_{[a,b]} \underline{\varphi}\,dx,\ \underline{\varphi} \text{ as in (3.2.3)},\ \underline{\varphi} \leq f \text{ in } I\right\}
\end{aligned}$$

It follows that

$$\mathcal{R}\underline{\int_a^b} f(x)\,dx \leq \underline{\int_{[a,b]}} f\,d\mathcal{L}^1 \leq \overline{\int_{[a,b]}} f\,d\mathcal{L}^1 \leq \mathcal{R}\overline{\int_a^b} f\,dx.$$

□

3.3 Convergence Results

Let μ be a Radon measure on $\mathbb{R}^n$ and Ω be μ-measurable.

Theorem 3.3.1. *(Fatou's Lemma)*

Let $f_k\colon \Omega \to [0, +\infty]$ be μ-measurable, for all $k \in \mathbb{N}$. Then it holds

$$\int_\Omega \liminf_{k\to+\infty} f_k \, d\mu \le \liminf_{k\to+\infty} \int_\Omega f_k \, d\mu. \tag{3.3.1}$$

Proof. Note that all f_k's are μ-integrable and hence $f := \liminf_{k\to\infty} f_k$ as well. It is sufficient to show that

$$\int_\Omega g \, d\mu \le \liminf_{k\to\infty} \int_\Omega f_k \, d\mu$$

for every simple function $g = \sum_{j=0}^{\infty} a_j \, \chi_{A_j}$ with $g \le f$.

Without loss of generality we can assume that $g \ge 0$, $a_0 = 0$, $a_j > 0 \; \forall j \ge 1$, $A_i \cap A_j = \emptyset$ (for $i \neq j$).

We fix $0 < t < 1$ and for $j \in \mathbb{N}$ it holds $A_j = \bigcup\limits_{k=1}^{\infty} B_{j,k}$, where

$$B_{j,k} = \{x \in A_j : f_\ell(x) > t \, a_j \quad \text{for } \ell \ge k\}.$$

Clearly: $B_{j,k} \subseteq B_{j,k+1}$ and

$$\lim_{k\to\infty} \mu(B_{j,k}) = \mu(A_j). \tag{3.3.2}$$

Fix $J, k \in \mathbb{N}$:

$$\begin{aligned} \int_\Omega f_k \, d\mu &\ge \sum_{j=1}^{J} \int_{A_j} f_k \, d\mu \ge \sum_{j=1}^{J} \int_{B_{j,k}} f_k \, d\mu \\ &\ge t \cdot \sum_{j=1}^{J} a_j \, \mu(B_{j,k}). \end{aligned}$$

- We let first $k \to +\infty$:

$$\liminf_{k\to\infty} \int_\Omega f_k d\mu \overset{(3.3.2)}{\ge} t \cdot \sum_{j=1}^{J} a_j \, \mu(A_j).$$

- Then we let $J \to +\infty$:

$$\liminf_{k\to\infty} \int_\Omega f_k d\mu \geq t \sum_{j=1}^{\infty} a_j\, \mu(A_j) = t \cdot \int_\Omega g\, d\mu.$$

- Finally, we let $t \to 1$ and we conclude the proof.

$\square$

Example 3.3.2. The condition $f_k \geq 0$ in Theorem 3.3.1 is necessary. Take $\mu = \mathcal{L}^n$, $\Omega = \mathbb{R}^n$, $f_k = -\frac{1}{k^n}\, \chi_{B_k(0)}$. Then:

$$\liminf_{k\to\infty} \int_\Omega f_k, d\mu = -w_1,$$

$$\text{as} \quad \int_{\mathbb{R}^n} f_k(x)\, dx = -\mathcal{L}^n\big(B(0,1)\big) = -w_1,$$

$$\text{but} \quad \liminf_{k\to+\infty} f_k = f \equiv 0.$$

Theorem 3.3.3. *(Monotone Convergence Theorem / Beppo Levi's Theorem)*

Let $f_k\colon \Omega \to [0, +\infty]$ be μ-measurable for all $k \geq 1$ and be such that $f_1 \leq \dots \leq f_k \leq f_{k+1} \leq \dots$. Then it holds

$$\int_\Omega \lim_{k\to\infty} f_k\, d\mu = \lim_{k\to\infty} \int_\Omega f_k\, d\mu. \tag{3.3.3}$$

Proof. Since $(f_k)_k$ is a non decreasing sequence, for every j it holds $\int_\Omega f_j\, d\mu \leq \int_\Omega \lim_{k\to\infty} f_k\, d\mu$ $(j = 1, \dots)$ and hence:

$$\lim_{k\to\infty} \int_\Omega f_k\, d\mu \leq \int_\Omega \lim_{k\to\infty} f_k\, d\mu.$$

The opposite inequality follows from Fatou's Lemma:

$$\int_\Omega \lim_{k\to\infty} f_k\,d\mu = \int_\Omega \liminf_{k\to\infty} f_k\,d\mu$$
$$\overset{\text{Fatou's Lemma}}{\leq} \quad \liminf_{k\to\infty} \int_\Omega f_k\,d\mu = \lim_{k\to\infty} \int_\Omega f_k\,d\mu.$$

$\square$

Remark 3.3.4. We can also start with Beppo Levi's Theorem and deduce Fatou's Lemma by setting for every $k \geq 1$: $g_k := \inf_{\ell\geq k} f_\ell$ We have $0 \leq g_1 \leq g_{k-1} \leq g_k \leq f_\ell$ $(\ell \geq k \geq 1)$ and

$$\lim_{k\to+\infty} g_k = \liminf_{k\to+\infty} f_k, \int_\Omega g_k \leq \inf_{\ell\geq k} \int_\Omega f_\ell\,d\mu.$$

From Beppo Levi's Theorem it holds

$$\int_\Omega \liminf_{k\to\infty} f_k\,d\mu = \int_\Omega \lim_{k\to+\infty} g_k\,d\mu = \lim_{k\to+\infty} \int_\Omega g_k\,d\mu$$
$$\leq \liminf_{k\to+\infty} \int f_k\,d\mu.$$

$\square$

Theorem 3.3.5. *(Dominated Convergence / Lebesgue Theorem)*

Let $g\colon \Omega \to [0,\infty]$ *be* μ*-summable and* $f\colon \Omega \to \bar{\mathbb{R}}$, $\{f_k\}_k\colon \Omega \to \bar{\mathbb{R}}$ *be* μ*-measurable. Suppose* $|f_k| \leq g$ *and* $f_k \to f$ μ*-a.e. as* $k \to +\infty$*. Then:*

$$\lim_{k\to+\infty} \int_\Omega |f_k - f|\,d\mu = 0. \tag{3.3.4}$$

Moreover:

$$\lim_{k\to+\infty} \int_\Omega f_k\,d\mu = \int_\Omega f\,d\mu. \tag{3.3.5}$$

Proof. We observe that

$$|f| = \lim_{k\to\infty} |f_k| \leq g \quad \mu\text{-a.e.}.$$

Thus:

$$|f_k - f| \leq |f_k| + |f| \leq 2g \tag{3.3.6}$$

and hence $|f_k - f|$ is μ-summable. Then:

$$\begin{aligned}
\int_\Omega 2g\, d\mu &= \int_\Omega \liminf_{k\to\infty}(2g - |f_k - f|)\, d\mu \\
&\overset{\mathrm{Fatou'sLemma}}{\leq} \liminf_{k\to\infty} \int_\Omega (2g - |f_k - f|)\, d\mu \\
&= \int_\Omega 2g\, d\mu - \limsup_{k\to\infty} \int_\Omega |f_k - f|\, d\mu \\
\implies \quad & \lim_{k\to\infty} \int_\Omega |f_k - f|\, d\mu = 0.
\end{aligned} \tag{3.3.7}$$

□

The equation (3.3.5) follows from the fact that

$$\left| \int_\Omega f_k\, d\mu - \int_\Omega f\, d\mu \right| \leq \int_\Omega |f_k - f|\, d\mu.$$

3.4 Application: Differentiation under integral sign

Theorem 3.4.1. *Let $f\colon \mathbb{R}\times[0,1]\to\mathbb{R}$ be a given function satisfying the following conditions:*

i) $\forall x$: $y\mapsto f(x,y)$ *is* $\mathcal{L}^1$*-summable on* $[0,1]$.

ii) $\frac{\partial f}{\partial x}$ *exists and it is bounded in* $\mathbb{R}\times[0,1]$.

Then the map $y\mapsto \frac{\partial f}{\partial x}(x,y)$ *is, for every fixed* $x\in\mathbb{R}$, $\mathcal{L}^1$*-summable on* $[0,1]$ *and*

$$\frac{d}{dx}\left(\int_0^1 f(x,y)\,dy\right)=\int_0^1\frac{\partial f}{\partial x}(x,y)\,dy.$$

Proof. For $x\in\mathbb{R}$, $h\neq 0$, we have

$$\frac{1}{h}\left(\int_0^1 f(x+h,y)\,dy-\int_0^1 f(x,y)\,dy\right)=\left(\int_0^1\frac{1}{h}\big(f(x+h,y)-f(x,y)\big)dy\right).\tag{3.4.1}$$

Given $h_k\to 0$ as $k\to+\infty$, it holds

$$g_k(y)=\frac{1}{h_k}\big(f(x+h_k,y)-f(x,y)\big)\to\frac{\partial f}{\partial x}(x,y),\text{ as }k\to+\infty.\tag{3.4.2}$$

Moreover by the mean value theorem and the assumption ii) we have

$$|g_k(y)|\leq\sup_{\mathbb{R}\times[0,1]}\left|\frac{\partial f}{\partial x}(x,y)\right|\leq C.\tag{3.4.3}$$

$\frac{\partial f}{\partial x}(x,\cdot)$ is $\mathcal{L}^1$-measurable with respect to $y\in[0,1]$ and $\mathcal{L}^1$-summable since it is limit of $\mathcal{L}^1$-measurable function and bounded.

By Dominating Convergence Theorem we have:

$$\int_0^1 g_k(y)\,dy\to\int_0^1\frac{\partial f}{\partial x}(x,y)\,dy,\ \text{ as }k\to+\infty.$$

This holds $\forall$ sequence $h_k \to 0$. Hence:

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \lim_{h\to 0}\frac{1}{h}\left(\int_0^1 f(x+h,y)\,dy - \int_0^1 f(x,y)\,dy\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\int_0^1 \big(f(x+h,y)-f(x,y)\big)\,dy = \int_0^1 \frac{\partial f}{dx}(x,y)\,dy.$$

$\square$

3.5 Absolute Continuity of Integrals

Let $f\colon \Omega \to \overline{\mathbb{R}}$ be μ-summable. For $A \subseteq \Omega$ μ-measurable, we set

$$\nu(A) = \int_A f\,d\mu.$$

According to Corollary 3.1.18 we have

$$\mu(A) = 0 \Longrightarrow \nu(A) = 0. \tag{3.5.1}$$

Exercise 3.5.1. ν is σ-additive, ν is a Radon measure in the case $f \geq 0$ μ-a.e. .

Notation:

$$\boxed{\nu = \mu \llcorner f.}$$

In particular we have

$$\nu = \mu \llcorner \chi_A = \mu \llcorner A.$$

Let Σ_μ and Σ_ν be the σ-algebra respectively of the μ and ν-measurable sets $A \subseteq \mathbb{R}^n$.

Definition 3.5.2. *A measure ν such that $\Sigma_\mu \subseteq \Sigma_\nu$ and with the property (3.5.1) is called "absolutely continuous" with respect to μ and we write $\nu \ll \mu$.*

Theorem 3.5.3. *Let $f\colon \Omega \to \overline{\mathbb{R}}$ be μ-summable. Then $\forall \varepsilon > 0\ \exists \delta > 0$ such that*

$$\forall A \subseteq \Omega \ \mu\text{-measurable with } \mu(A) < \delta \implies \int_A |f|\, d\mu < \varepsilon.$$

Proof. We assume by contradiction that there exist $\varepsilon > 0$, and a sequence of μ-measurable subsets $A_k \subseteq \Omega$ with $\mu(A_k) < 2^{-k}$ and

$$\int_{A_k} |f|\, d\mu \geq \varepsilon \quad \forall k \in \mathbb{N}.$$

For $\ell \geq 1$ we define

$$B_\ell := \bigcup_{k=\ell}^{\infty} A_k.$$

The following properties hold

- $B_{\ell+1} \subseteq B_\ell$.
- $\forall \ell$: $\mu(B_\ell) \leq \sum_{k=\ell}^{\infty} \mu(A_k) < 2^{1-\ell}$.
- $\mu(B_1) < +\infty$.
- $\int_{B_\ell} |f|\, d\mu \geq \int_{A_k} |f|\, d\mu \geq \varepsilon \quad \forall k \geq \ell$.
- For $A = \bigcap_{\ell=1}^{\infty} B_\ell$: $\mu(A) = \lim_{\ell \to +\infty} \mu(B_\ell) = 0$.

Now observe that $g_\ell := |f| \cdot \chi_{B_\ell} \to |f| \cdot \chi_A$ as $\ell \to +\infty$ and $|g_\ell| \leq |f|$ μ-summable, $\forall \ell$.

By Dominated Convergence Theorem it follows

$$\begin{aligned} \varepsilon \ &\leq \ \lim_{\ell \to +\infty} \int_{B_\ell} |f|\, d\mu = \lim_{\ell \to +\infty} \int_\Omega g_\ell\, d\mu \\ &= \int_\Omega \lim_{\ell \to \infty} g_\ell\, d\mu = \int_A |f|\, d\mu = 0. \end{aligned}$$

We get a contradiction and we can conclude. $\square$

3.6 Vitali's Theorem

We formulate in this section a necessary and sufficient condition to pass to the limit under integral sign.

Let f, $f_k\colon \Omega \to \overline{\mathbb{R}}$ be μ-summable, $k \in \mathbb{N}$.

Definition 3.6.1. *The family $\{f_k\}$ is called **uniformly μ-summable** if $\forall \varepsilon > 0$ $\exists \delta > 0$: $\forall k \in \mathbb{N}$, $\forall A \subseteq \Omega$ μ-measurable with $\mu(A) < \delta$ we have $\displaystyle\int_A |f_k|\, d\mu < \varepsilon$.*

Theorem 3.6.2. *(Vitali's Theorem)*

If $\mu(\Omega) < +\infty$, the following conditions are equivalent:

i) $f_k \xrightarrow{\mu} f$ *(as $k \to +\infty$) and $\{f_k\}$ is uniformly μ-summable.*

ii) $\displaystyle\lim_{k\to+\infty} \int_\Omega |f_k - f|\, d\mu = 0.$

Remark 3.6.3. The condition $\mu(\Omega) < +\infty$ is in general necessary: $\Omega = \mathbb{R}^n$, $\mu = \mathcal{L}^n$, $f_k = \frac{1}{k^n}\chi_{B(0,k)}$. We have:

$$\{f_k\} \text{ is uniformly } \mu\text{-summable, } f_k \xrightarrow{\mu} 0 \text{ as } k \to +\infty. \tag{3.6.1}$$

But $\int_{\mathbb{R}^n} |f_k|\, d\mu = w_n = \mathcal{L}^n(B(0,1)) > 0$ and $\int_{\mathbb{R}^n} f\, d\mu = 0$.

Proof of Vitali's Theorem:

ii) $\Longrightarrow$ i):

If $\int_\Omega |f_k - f|\, d\mu \to 0$ as $k \to +\infty$, then $f_k \xrightarrow{\mu} f$ by Corollary 3.1.14. Now we show that $\{f_k\}$ is uniformly μ-summable.

For $\varepsilon > 0$, let $k_0 = k_0(\varepsilon) > 0$ be such that

$$\int_\Omega |f_k - f|\, d\mu < \varepsilon, \text{ for all } k \geq k_0.$$

Let $\delta > 0$ be such that $\forall A \subseteq \Omega$ μ-measurable with $\mu(A) < \delta$, we have

$$\int_A |f| \, d\mu < \varepsilon \tag{3.6.2}$$

and

$$\max_{1 \le k \le k_0} \int_A |f_k| \, d\mu < \varepsilon \tag{3.6.3}$$

(see Theorem 3.5.3).

Thus for $k \ge k_0$ and if $\mu(A) < \delta$, we have

$$\begin{aligned} \int_A |f_k| \, d\mu &\le \int_A |f| \, d\mu + \int_A |f_k - f| \, d\mu \\ &\le 2\varepsilon \, . \end{aligned} \tag{3.6.4}$$

Thus, we have the desired property.

i) $\Longrightarrow$ ii):

Let $f_k \xrightarrow{\mu} f$ as $k \to +\infty$ and be uniformly μ-summable. Since $f_k \xrightarrow{\mu} f$, there exists a sub-sequence $f_{k_n} \to f$ μ-a.e. as $n \to +\infty$ (see Theorem 2.4.4).

Given $\varepsilon > 0$, we choose $\delta > 0$ such that for $A \subseteq \Omega$, $\mu(A) < \delta$, we have

$$\int_A |f| \, d\mu < \varepsilon \text{ and } \int_A |f_k| \, d\mu < \varepsilon \quad \forall k \ge 1. \tag{3.6.5}$$

Correspondent to such a $\delta > 0$ we can find by **Egoroff-Theorem** a closed set $F \subseteq \Omega$ such that $\mu(\Omega \setminus F) < \delta$ and

$$\sup_{x \in F} |f_{k_n}(x) - f(x)| \to 0 \text{ as } n \to +\infty.$$

Let $\overline{n}$ be such that

$$\sup_{x \in F} |f_{k_n}(x) - f(x)| < \frac{\varepsilon}{\mu(\Omega)}, \quad \forall n \ge \overline{n}.$$

Thus for $n \geq \overline{n}$:

$$\begin{aligned}\int_\Omega |f_{k_n}(x) - f(x)|\, d\mu &= \int_{\Omega\setminus F} |f_{k_n} - f|\, d\mu + \int_F |f_{k_n} - f|\, d\mu \\ &\leq \int_{\Omega\setminus F} |f_{k_n}| + |f|\, d\mu + \int_F \sup_F |f_{k_n} - f|\, d\mu \\ &< 2\varepsilon + \frac{\varepsilon}{\mu(\Omega)}\, \mu(F) < 3\varepsilon.\end{aligned} \tag{3.6.6}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{n\to+\infty} \int_\Omega |f_{k_n}(x) - f(x)|\, d\mu = 0. \tag{3.6.7}$$

Claim 3.6.4. *The whole sequence f_k satisfies (3.6.7).*

Proof of Claim 3.6.4: Suppose that

$$\limsup_{k\to+\infty} \int_\Omega |f_k(x) - f(x)|\, d\mu > 0.$$

Then there exists a subsequence f_{k_n} such that

$$\lim_{n\to+\infty} \int_\Omega |f_{k_n}(x) - f(x)|\, d\mu = \limsup_{k\to+\infty} \int_\Omega |f_k(x) - f(x)|\, d\mu > 0.$$

At this point we can repeat the same arguments as above to f_{k_n} and get that $\exists\{f_{k_n'}\}$ subsequence of f_{k_n} such that

$$\lim_{n\to+\infty} \int_\Omega |f_{k_n'} - f|\, d\mu = 0,$$

which is a contradiction. $\square$

Vitali's Theorem permits to improve Lebesgue's Theorem.

Theorem 3.6.5. *Let $\mu(\Omega) < +\infty$ and $\{f_n\}$ be a sequence of μ-measurable functions such that $f_n \xrightarrow{\mu} f$ as $n \to \infty$. Suppose that $\forall n \in \mathbb{N}$ $|f_n(x)| \leq |g_n(x)|$ μ-a.e., where $\{g_n\}$ is a sequence of μ-summable functions such that*

$$\lim_{n\to+\infty} \int_\Omega |g_n(x) - g(x)|\, d\mu = 0$$

for a μ-summable functions g. Then

$$\lim_{n\to+\infty} \int_\Omega |f_n(x) - f(x)|\, d\mu = 0. \tag{3.6.8}$$

Proof. Since $\lim_{n\to+\infty} \int_\Omega |g_n(x) - g(x)|\, d\mu = 0$, by Vitali's Theorem, the sequence $\{g_n\}$ is uniformly μ-summable, namely $\forall \varepsilon > 0$ $\exists \delta > 0$ such that if $\mu(A) < \delta$, then

$$\sup_{n\in\mathbb{N}} \int_A |g_n(x)|\, d\mu \leq \varepsilon. \tag{3.6.9}$$

Hence, we have

$$\int_A |f_n(x)|\, d\mu \leq \int_A |g_n(x)|\, d\mu < \varepsilon \quad \forall n \in \mathbb{N}, \text{ if } \mu(A) < \delta. \tag{3.6.10}$$

Thus the sequence $(f_n)_n$ is also uniformly μ-summable and from Vitali's Theorem, we get the result and we conclude. $\square$

Theorem 3.6.6. *Let $\mu(\Omega) < +\infty$ and $u_k, u\colon \Omega \to \bar{\mathbb{R}}$ be μ-measurable functions such that*

$$\int_\Omega |u|^p d\mu < +\infty, \quad \int_\Omega |u_k|^p d\mu < +\infty$$
$$\int_\Omega |u_k - u|^p d\mu \to 0 \quad \text{as } k \to +\infty,$$

where $1 \leq p < +\infty$. Then

$$\int_\Omega u_k \, d\mu \to \int_\Omega u \, d\mu \quad \text{as } k \to +\infty. \tag{3.6.11}$$

Proof. The case $p = 1$ is trivial. Suppose $p > 1$.

1) Tchebychev Inequality yields

$$\begin{aligned}\mu\{x : |u_k(x) - u(x)| > \varepsilon\} = \mu\{x : |u_k(x) - u(x)|^p > \varepsilon^p\}\\ \leq \varepsilon^{-p} \int_\Omega |u_k(x) - u(x)|^p d\mu \to 0\\ \text{as } k \to +\infty.\end{aligned} \tag{3.6.12}$$

Hence $u_k \xrightarrow{\mu} u$ as $k \to +\infty$.

ii) $\{u_k\}_k$ are uniformly μ-summable.

Indeed for $\varepsilon > 0$ there exists $\delta > 0$, $\overline{k} > 0$ such that

$$\mu(A) < \delta \Longrightarrow \int_A |u|^p d\mu < \varepsilon \text{ and } \int_\Omega |u_k - u|^p d\mu < \varepsilon \quad \forall k \geq \overline{k}.$$

We have

$$|u_k|^p \leq (|u| + |u_k - u|)^p \leq 2^{p-1}(|u|^p + |u_k - u|^p). \tag{3.6.13}$$

Thus, for every $A \subseteq \Omega$ μ-measurable we have

$$\begin{aligned} \int_A |u_k|\, d\mu &\le \int_A (1 + |u_k|^p)\, d\mu \\ &\le \mu(A) + 2^{p-1} \int_A |u|^p + |u_k - u|^p d\mu \\ &\le \delta + 2^p \varepsilon \le (1 + 2^p)\, \varepsilon \end{aligned} \tag{3.6.14}$$

if $\mu(A) < \delta < \varepsilon$ and $k \ge \overline{k}$. Thus $\int_\Omega |u_k - u|\, d\mu \to 0$ as $k \to +\infty$ by Vitali's Theorem and thus

$$\int_\Omega u_k\, d\mu \to \int_\Omega u\, d\mu \;\text{ as } k \to +\infty. \tag{3.6.15}$$

□

Exercise 3.6.7.

1. By applying Lebesgue's Theorem to the counting measure on $\mathbb{N}$, verify that
$$\lim_{n \to +\infty} n \sum_{i=1}^{\infty} \sin\left(\frac{2^{-i}}{n}\right) = 1.$$

2. Let f: $[a, +\infty) \to \overline{\mathbb{R}}$ be a locally bounded function and locally Riemann integrable. Then f is $\mathcal{L}^1$-summable iff f is absolutely Riemann integrable in the generalized sense (namely $\int_a^{+\infty} |f(x)|\, d\mu = \lim_{j \to +\infty} \int_a^j |f(x)|\, d\mu$ exists and it is finite) and in this case
$$\int_{[a,+\infty)} f(x)\, d\mathcal{L}^1 = \lim_{j \to +\infty} \int_a^j f(x)\, d\mu.$$

Proof. **1.** Let $g_j := |f|\, \chi_{[a,j]}$. g_j are $\mathcal{L}^1$-summable since they are Riemann-integrable. Moreover, g_j is an increasing sequence which converges to $|f|$ pointwise.

Beppo-Levi's Theorem says that that $|f|$ is $\mathcal{L}^1$-summable iff

$$\lim_{j \to +\infty} \int_{\mathbb{R}} g_j\, d\mathcal{L}^1 < +\infty.$$

Since

$$\int_{\mathbb{R}} g_j \, d\mathcal{L}^1 = \int_a^j |f(x)| \, dx \quad \forall j \geq 0$$

this is equivalent to say that $|f|$ is Riemann integrable in a generalized sense.

2. Now set $f_j = f\chi_{[a,j]}$. f_j are $\mathcal{R}$-integrable $\Longrightarrow$ f_j are $\mathcal{L}^1$-integrable and $f_j \to f$ in $\mathbb{R}$ as $j \to +\infty$, $|f_j| \leq |f|$. We have just said that f is $\mathcal{L}^1$-summable iff it is absolutely $\mathcal{R}$-integrable in the generalized sense.

By Lebesgue Theorem, we have

$$\int_{[a,+\infty)} f \, d\mathcal{L}^1 = \lim_{j\to\infty} \int_{\mathbb{R}} f_j \, d\mathcal{L}^1 = \lim_{j\to+\infty} \int_a^j f(x) \, d\mathcal{L}^1. \tag{3.6.16}$$

It follows that if f is not absolutely $\mathcal{R}$-integrable in the generalized sense, then it is not $\mathcal{L}^1$-summable. For instance, $\frac{\sin x}{x}$ is not $\mathcal{L}^1$-summable but there exists $\lim_{j\to+\infty} \int_a^j \frac{\sin x}{x} dx < +\infty$.

In the same way we can prove that if f is locally $\mathcal{R}$-integrable on $X = [a, b)$ (or $X = (a, b]$, $X = (a, b)$), then f is $\mathcal{L}^1$-summable on X iff it is $\mathcal{R}$-absolutely integrable in the generalized sense (namely there exists finite $\lim_{\varepsilon\to0} \int_a^{b-\varepsilon} |f(x)| \, d\mu$) and in this case:

$$\int_{[a,b)} f \, d\mathcal{L}^1 = \lim_{\varepsilon\to0} \int_a^{b-\varepsilon} f(x) \, dx.$$

□

3.7 The $L^p(\Omega, \mu)$ spaces, $1 \leq p \leq \infty$

We suppose as usual that μ is a Radon-measure and Ω is a μ-measurable set.

Definition 3.7.1. *For* $f\colon \Omega \to \overline{\mathbb{R}}$, $1 \le p < \infty$, *let*

$$\|f\|_{L^p(\Omega,\mu)} := \Big(\int_\Omega |f|^p d\mu\Big)^{\frac{1}{p}} \le +\infty$$

and

$$\begin{aligned}\|f\|_{L^\infty(\Omega,\mu)} &:= \mu\text{−}\operatorname*{ess\,sup}_{x\in\Omega} |f(x)| \\ &= \inf\{C \in [0,+\infty] : \ |f| \le C \ \mu\text{-}a.e.\} \le +\infty.\end{aligned}$$

For $1 \le p \le \infty$ *we set*

$$\mathcal{L}^p(\Omega,\mu) = \{f : \Omega \to \overline{\mathbb{R}} : \ f \ \mu\text{-}measurable,\ \|f\|_{L^p(\Omega,\mu)} < +\infty\}.$$

Remark 3.7.2. For $f \in \mathcal{L}^\infty(\Omega,\mu)$, we have $|f(x)| \le \|f\|_{L^\infty}$ for μ-a.e. x.

Proof. Consider a sequence $\{c_k\}$ with $c_k \downarrow \|f\|_{L^\infty}$ as $k \to \infty$.

Set $A_k = \{x \in \Omega\colon |f(x)| > c_k\}$, $k \in \mathbb{N}$. By definition $\mu(A_k) = 0\ \forall k$ and thus $A = \bigcup_k A_k$ has measure zero as well. Therefore $|f(x)| \le \|f\|_{L^\infty}\ \forall x \in \Omega \setminus A$. □

Remark 3.7.3. Consider the Lebesgue measure $\mathfrak{L}^1$ on $[0,1]$ and the space $\mathcal{L}^p((0,1],\mathfrak{L}^1)$. Let us set for every $\alpha \in \mathbb{R}$

$$\varphi_\alpha(x) = x^\alpha \quad \forall x \in (0,1]. \tag{3.7.1}$$

Then $\varphi_\alpha \in \mathcal{L}^p((0,1])$ iff $\alpha\, p + 1 > 0$. □

Solution: $\varphi_\alpha \in \mathcal{L}^p((0,1])$ iff there exists finite the following limit

$$\begin{aligned}\lim_{\delta\to 0}\int_\delta^1 x^{p\alpha}\,dx &= \lim_{\delta\to 0} \frac{x^{p\alpha+1}}{1+p\alpha}\Big|_\delta^1 \\ &= \lim_{\delta\to 0}\Big[\frac{1-\delta^{1+p\alpha}}{p\alpha+1}\Big].\end{aligned} \tag{3.7.2}$$

The last limit is finite iff $\alpha p + 1 > 0$. In this case, we have

$$\int_{[0,1]} x^{p\alpha}\,d\mathfrak{L}^1 = \lim_{\delta\to 0}\int_\delta^1 x^{p\alpha}\,d\mathfrak{L}^1 = \frac{1}{1+\alpha p}. \tag{3.7.3}$$

Exercise 3.7.4. Show that $\mathcal{L}^p((0,1],\mathfrak{L}^1)$ is not an algebra. $\square$

We observe that $\|\cdot\|_p$ in general is not a norm since $\|\varphi\|_p = 0$ iff $\varphi(x) = 0$ for μ-a.e.

Recall: Let Y be a vector space. A norm on Y is a mapping $\|\cdot\|\colon Y \to [0,+\infty)$, $y \mapsto \|y\|$ such that

i) $\|y\| = 0 \iff y = 0$.

ii) $\|\alpha y\| = |\alpha|\,\|y\| \quad \forall \alpha \in \mathbb{R}, \forall y \in Y$.

iii) $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\| \quad \forall y_1, y_2 \in Y$.

The space Y endowed with the norm $\|\cdot\|$ is called a **normed space**. It is also a metric space with the distance given by $d(y_1,y_2) = \|y_1 - y_2\|$ $y_1, y_2 \in Y$.

In order to construct a vector space on which $\|\cdot\|_p$ is a norm, let us consider the following equivalence relation on $\mathcal{L}^p(\Omega,\mu)$:

$$\varphi \sim \psi \iff \varphi = \psi \ \mu\text{-a.e.}$$

Let us denote by $L^p(\Omega,\mu)$ the quotient space $\mathcal{L}^p(\Omega,\mu)/_\sim = \{[\varphi]\colon \varphi \in \mathcal{L}^p(\Omega,\mu)\}$. One can verify that $L^p(\Omega,\mu)$ is a vector space.

The precise definition of addition of two elements $[\varphi_1]$, $[\varphi_2] \in L^p(\Omega,\mu)$ is the following: let f_1, f_2 be two "representative" of $[\varphi_1]$ and $[\varphi_2]$, respectively such that f_1, f_2 are finite everywhere (such representatives exist: exercise). Then $[\varphi_1] + [\varphi_2] = [f_1 + f_2]$.

We set $\|[\varphi]\|_p = \|\varphi\|_p \ \forall [\varphi] \in L^p(\Omega,\mu)$.

This definition is independent on the particular element $\varphi \in [\varphi]$. Then, since the zero element of $L^p(\Omega,\mu)$ is the class consisting of all functions vanishing μ-a.e., it is clear that $\|[\varphi]\|_p = 0 \iff [\varphi] = 0$. To simplify the notation we will hereafter identify $[\varphi]$ with φ and we will talk about "Functions in $L^p(\Omega,\mu)$" when there is no danger of confusion with the understanding that we regard equivalent functions (i.e. functions differing only on a set of measure zero) as identical elements of the space $L^p(\Omega,\mu)$.

Theorem 3.7.5. *The space $L^p(\Omega,\mu)$ is a complete normed space for $1 \le p \le \infty$.*

Proof. **Step 1.** We first show that $L^p(\Omega,\mu)$ is a normed space and in particular that $\|\cdot\|_{L^p}$ is a norm.

i) The **positive homogeneity** is clear:

$$\|\lambda f\|_{L^\infty} = |\lambda|\|f\|_{L^\infty}$$

$$\|\lambda f\|_{L^p} = \Big(\int_\Omega \|\lambda f\|^p d\mu\Big)^{\frac{1}{p}} = |\lambda|\Big(\int_\Omega |f|^p d\mu\Big)^{\frac{1}{p}}.$$

□

ii) **Triangular Inequality**

We need some preliminary results.

Lemma 3.7.6. *(Young Inequality) Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then it holds*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \forall a, b \ge 0. \tag{3.7.4}$$

Proof of Lemma 3.7.6: We suppose $b > 0$. The function $a \mapsto ab - \frac{a^p}{p}$ with $a \ge 0$ has the maximum at $a^* = b^{\frac{1}{p-1}}$. Thus for all $a \ge 0$

$$\begin{aligned} ab - \frac{a^p}{p} &\le b^{\frac{p}{p-1}}\, b - \frac{b^{\frac{p}{p-1}}}{p} \\ &= b^{\frac{1}{p-1}}\Big(1 - \frac{1}{p}\Big) = \frac{b^q}{q}. \end{aligned} \tag{3.7.5}$$

□

The numbers p, q such that $\frac{1}{p} + \frac{1}{q} = 1$ are called **conjugate**.

Corollary 3.7.7. *(Hölder Inequality)*

Let $1 \leq p, q \leq \infty$ *be conjugate,* $f \in L^p(\Omega, \mu)$*,* $g \in L^q(\Omega, \mu)$*. Then it holds* $f \cdot g \in L^1(\Omega, \mu)$ *and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}. \tag{3.7.6}$$

Proof of Corollary 3.7.7: We suppose $p \leq q$ without loss of generality

- Case $p = 1$, $q = \infty$

  $$|fg| \leq |f| \, \|g\|_{L^\infty} \quad \mu\text{-a.e.}$$

  Thus by monotonicity of the integral, we have

  $$\begin{aligned} \int_\Omega |fg| \, d\mu &\leq \int |f| \, \|g\|_{L^\infty} \, d\mu \\ &= \|g\|_{L^\infty} \int_\Omega |f| \, d\mu. \end{aligned} \tag{3.7.7}$$

- Case $1 < p \leq q < +\infty$

  The conclusion is trivial if $\|f\|_{L^p} = 0$ or $\|g\|_{L^q} = 0$. Thus we assume that $\|f\|_{L^p} > 0$ and $\|g\|_{L^q} > 0$.

We set $\widetilde{f}(x) = \frac{|f(x)|}{\|f\|_{L^p}}$ and $\widetilde{g}(x) = \frac{|g(x)|}{\|g\|_{L^q}}$. We apply Lemma 3.7.6 to $\widetilde{f}, \widetilde{g}$:

$$\widetilde{f}\,\widetilde{g} \leq \frac{\widetilde{f}^p}{p} + \frac{\widetilde{g}^q}{q}. \tag{3.7.8}$$

Integrating over Ω, we get

$$\int_\Omega |\widetilde{f}\,\widetilde{g}| \, d\mu \leq \frac{1}{p} \int_\Omega \widetilde{f}^p \, d\mu + \frac{1}{q} \int_\Omega \widetilde{g}^q \, d\mu.$$

Thus

$$\frac{\displaystyle\int_\Omega |f(x)\, g(x)| \, d\mu}{\|f\|_{L^p} \, \|g\|_{L^q}} \leq 1. \tag{3.7.9}$$

and we conclude. $\square$

Corollary 3.7.8.

i) *Let* $f \in L^p$, $g \in L^q$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$. *Then* $fg \in L^r$ *and* $\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$.

ii) *If* $\mu(\Omega) < \infty$, *then*

$$L^s(\Omega) \subseteq L^r(\Omega), \ 1 \leq r < s \leq +\infty. \tag{3.7.10}$$

Proof.

i) The property is clear if $p = q = r = +\infty$. If $r < \infty$, the property follows from Corollary 3.7.7: $|f|^r \in L^{\frac{p}{r}}$, $|g|^r \in L^{\frac{q}{r}}$ and $\frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$.

ii) If $\mu(\Omega) < +\infty$. Set $g \equiv 1 \in L^{\frac{rs}{s-r}}$. Observe that $\frac{1}{r} = \frac{1}{s} + \frac{s-r}{rs}$ and apply i).

$\square$

Exercise 3.7.9. *Show that In general the inclusion in ii) is strict in the sense that there are functions in L^r that does not belong to L^s with $r < s$.*

Hint: *Take*

$$\Omega = \Big(0, \tfrac{1}{2}\Big), \; f(x) = \frac{1}{(-\log x)^2 \, x^{\frac{1}{r}}}, \quad f \in L^r,$$

but $f \notin L^s$, with $s > r$. Another example is $\Omega = (0,1)$, $f(x) = \log(\frac{1}{x}) \in L^p \; \forall p \geq 1$, but $f \notin L^\infty$.

Corollary 3.7.10. *(Minkowski Inequality)*

Let $1 \leq p \leq \infty$ and $f, g \in L^p(\Omega, \mu)$. Then $f + g \in L^p$ and

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof.

- $\underline{p = 1}$: This case follows from the pointwise estimate

$$|f + g| \leq |f| + |g|.$$

- $\underline{p = \infty}$: This case follows from the estimate

$$|(f + g)(x)| \leq \|f\|_{L^\infty} + \|g\|_{L^\infty} \; \mu\text{-a.e. } x \in \Omega.$$

- $\underline{1 < p < \infty}$.

From the estimate

$$|(f + g)(x)|^p \leq 2^{p-1}\big(|f(x)|^p + |g(x)|^p\big) \quad \forall x \in \Omega$$

it follows that $f + g \in L^p$ and $|f + g|^{p-1} \in L^{\frac{p}{p-1}}$. By applying the

Hölder inequality, we get

$$
\begin{aligned}
\|f+g\|_{L^p}^p &= \int_\Omega |f+g|\,|f+g|^{p-1}\,d\mu \\
&\leq \int_\Omega |f|\cdot|f+g|^{p-1}\,d\mu + \int_\Omega |g|\,|f+g|^{p-1}\,d\mu \\
&\leq \Big(\int_\Omega |f|^p\Big)^{\frac{1}{p}} \Big(\int_\Omega |f+g|^{p-1\cdot\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \qquad (3.7.11)\\
&\quad + \Big(\int_\Omega |g|^p\Big)^{\frac{1}{p}} \Big(\int_\Omega |f+g|^{p-1\cdot\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \\
&= (\|f\|_p + \|g\|_p)\,\big(\|f+g\|_p^{p-1}\big).
\end{aligned}
$$

By dividing both sides by $\|f+g\|_p^{p-1}$, we get

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}. \tag{3.7.12}$$

$\square$

Exercise 3.7.11. (Interpolation Inequality)
Let $1 \leq p < r < q < \infty$ and $f \in L^p \cap L^q$. Then $f \in L^r$ and

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta}\,\|f\|_{L^q}^{1-\theta},$$

where $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$, $\theta \in [0,1]$.

Exercise 3.7.12. Let $\mu(\Omega) < \infty$ and $1 \leq p < \infty$. Show that if $f\colon \Omega \to \overline{\mathbb{R}}$ is such that $f \cdot g \in L^1(\Omega,\mu)$, $\forall g \in L^p(\Omega,\mu)$, then $f \in L^q(\Omega,\mu)$ $\forall q \in [1,p')$, where p' is the conjugate of p.

Hint: Observe that $f \in L^1(\Omega,\mu)$. So by taking $g = |f|^{1/p}$ we deduce that $|f|^{1+1/p} \in L^1(\Omega,\mu)$. Iterate the argument.

We continue with the **proof of Theorem 3.7.5.**

Lemma 3.7.13. *($L^p(\Omega,\mu)$ is complete)*

Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\Omega,\mu)$. Then there exists $f \in L^p$ such that $\|f_k - f\|_{L^p} \to 0$ as $k \to +\infty$.

Proof.

$\underline{p=\infty}$: $\forall \varepsilon > 0\ \exists k_\varepsilon \in \mathbb{N}$ such that

$$\|f_\ell - f_m\|_{L^\infty} < \varepsilon \quad \forall \ell, m \geq k_\varepsilon.$$

For $\ell, m \in \mathbb{N}$ we set

$$E_{\ell,m} = \{x \in \Omega :\ |f_\ell(x) - f_m(x)| > \|f_\ell - f_m\|_{L^\infty}\}. \tag{3.7.13}$$

By definition of ess-sup, $E_{\ell,m}$ has measure zero . We define

$$E_0 = \bigcup_{\ell,m=1} E_{\ell,m}. \tag{3.7.14}$$

We have $\mu(E_0) = 0$. Now let $x \in \Omega \setminus E_0$, then $x \notin E_{\ell,m}\ \forall \ell, m$. Thus if $\ell, m \geq k_\varepsilon$, we have

$$|f_\ell(x) - f_x(x)| \leq \|f_\ell - f_m\|_\infty < \varepsilon. \tag{3.7.15}$$

This implies that $(f_k(x))_k$ is a Cauchy sequence in $\mathbb{R}$ ($\forall x \in \Omega \setminus E_0$) and since $\mathbb{R}$ is complete, it converges to a real number $f(x)$. Thus if $(f_k(x))_k$ is a Cauchy sequence in $L^\infty(\Omega)$, then it converges μ-a.e. to a function $f(x)$.

We consider the inequality

$$|f_\ell(x) - f_m(x)| \leq \varepsilon \quad \ell, m \geq k_\varepsilon,\ x \in \Omega \setminus E_0.$$

Take the limit as $m \to +\infty$ and we get

$$|f_\ell(x) - f(x)| \leq \varepsilon \quad \forall \ell \geq k_\varepsilon \quad \forall x \in \Omega \setminus E_0. \tag{3.7.16}$$

Since $\mu(E_0) = 0$, we have

$$\|f_\ell(x) - f(x)\|_{L^\infty} \leq \varepsilon \quad \forall \ell \geq k_\varepsilon \tag{3.7.17}$$

and

$$\|f\|_{L^\infty} \leq \|f_\ell - f\|_{L^\infty} + \|f_\ell\|_{L^\infty} \leq c \quad \text{if } \ell \geq k_\varepsilon. \tag{3.7.18}$$

From (3.7.18) it follows that $f \in L^\infty(\Omega)$.

$\underline{p < \infty}$:

We are going to construct a convergent subsequence $(f_{k_n})_{n\in\mathbb{N}}$. One chooses k_n so that

$$\|f_\ell - f_m\|_{L^p} < \frac{1}{2^n} \quad \forall \ell, m \geq k_n.$$

(One proceeds as follows: choose k_1 such that $\|f_\ell - f_m\|_{L^p} < \frac{1}{2}\ \forall \ell, m \geq k_1$, then choose $k_2 > k_1$ such that $\|f_\ell - f_m\|_{L^p} < \frac{1}{2^2}\ \forall \ell, m \geq k_2$, etc.). We claim that f_{k_n} converges in L^p. In order to simplify the notation we write f_n instead of f_{k_n}, so that

$$\|f_{n+1} - f_n\|_{L^p} \leq \frac{1}{2^n} \quad \forall n \geq 1. \tag{3.7.19}$$

Let $g_n(x) = \sum_{k=1}^n |f_{k+1}(x) - f_k(x)|$ so that $\|g_n\|_{L^p} \leq 1$.

As a consequence of the monotone convergence theorem, $g_n(x)$ converges to $g(x)$ μ-a.e. in Ω where $g(x) = \sum_{k=1}^\infty |f_{k+1}(x) - f_k(x)| \in L^p$. On the other hand for $m \geq \ell \geq 2$, we have

$$\begin{aligned}|f_m(x) - f_\ell(x)| &\leq |f_m(x) - f_{m-1}(x)| + \cdots + |f_{\ell+1}(x) - f_\ell(x)| \\ &= g_m(x) - g_{\ell-1}(x) \\ &\leq g(x) - g_{\ell-1}(x).\end{aligned} \tag{3.7.20}$$

It follows that μ-a.e. on Ω, $f_n(x)$ is a Cauchy sequence and converges to a limit $f(x)$. We have μ-a.e. on Ω

$$|f(x) - f_\ell(x)| \leq 2g(x) \quad \forall \ell \geq 2 \tag{3.7.21}$$

and in particular $f \in L^p$.

Finally we conclude by dominated convergence that $\|f_n - f\|_{L^p} \to 0$ since $|f_n(x) - f|^p \to 0$ and

$$|f_n - f|^p \leq 2^p\, |g|^p \in L^1.$$

We can easily see that **the entire sequence $\boldsymbol{f_n}$ converges to $\boldsymbol{f}$**.

Indeed, for all $n \in \mathbb{N}$ and $k > k_n$, we have

$$\begin{aligned} \|f_k - f\|_{L^p} &\le \|f_k - f_{k_n}\|_{L^p} + \|f_{k_n} - f\|_{L^p} \\ &\le \frac{1}{2^n} + \|f_{k_n} - f\|_{L^p}. \end{aligned} \tag{3.7.22}$$

Thus

$$\limsup_{k\to+\infty} \|f_k - f\|_{L^p} \le \lim_{n\to+\infty} \left(\frac{1}{2^n} + \|f_{k_n} - f\|_{L^p}\right) = 0. \tag{3.7.23}$$

□

The proof of Theorem 3.7.5 is concluded as well. □

A Banach space X is called **separable** if it contains a countable dense subset.

Example 3.7.14.

1. $(\mathbb{R}, |\cdot|)$. The countable dense subset is $\mathbb{Q}$.
2. $(C^0([a,b],\mathbb{R}),\ \|\cdot\|_\infty)$. The countable subset is the set of the polynomials with rational coefficients (Stone-Weierstrass Theorem).

Theorem 3.7.15. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \le p < \infty$. Then it holds*

i) *$L^p(\Omega,\mu)$ is separable.*

ii) *$C_c^0(\Omega)$ is dense in $L^p(\Omega,\mu)$.*

We recall that $C_c^0(\Omega)$ denotes the space of continuous function $f\colon \Omega \to \mathbb{R}$ with $\operatorname{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}} \subseteq \Omega$ and $\operatorname{supp} f$ is compact.

Remark 3.7.16. For $p = \infty$ the conditions of Theorem 3.7.15 do not hold in general.

1. Let $\Omega = [0,1]$, $\mu = \mathcal{L}^1$. For $0 < t < 1$ we consider $f_t(x) := \chi_{[0,t]}(x)$ with $\|f_t - f_s\|_{L^\infty} = 1$ if $t \neq s$.

   The family $(f_t)_{t>0}$ is more than countable. Suppose there is a countable set $(e_k)_k$ dense in $L^\infty((0,1])$ with

   $$\inf_k \|f_t - e_k\|_{L^\infty} < \tfrac{1}{2} \quad \forall t > 0.$$

   We would have

   $$L^\infty([0,1]) \subseteq \cup_n B(e_n, 1/2)$$

   in contrast with the fact no pair of functions of the family $(f_t)_{t>0}$ belongs to the same ball.

2. If $\Omega \subset \mathbb{R}^n$ is compact and $\mu = \mathcal{L}^n$ then $C(\Omega)$ is a closed, proper subset of $L^\infty(\Omega)$ and therefore $C^0_c(\Omega) \subsetneq L^\infty(\Omega)$. $\square$

Proof of Theorem 3.7.15: i) We suppose $\Omega = \mathbb{R}^n$. Observe that if we prove that $L^p(\mathbb{R}^n)$ is separable then $L^p(\Omega)$ is separable as well for every $\Omega \subseteq \mathbb{R}^n$. There is a canonical isometry from $L^p(\Omega)$ into $L^p(\mathbb{R}^n)$ that to every $f \in L^p(\Omega)$ associate $\tilde{f} \in L^p(\mathbb{R}^n)$ where $\tilde{f}(x) = f(x)$ if $x \in \Omega$ and $\tilde{f}(x) = 0$ is $x \in \Omega^c$. Therefore $L^p(\Omega)$ can be identified with a subset of $L^p(\mathbb{R}^n)$ and therefore it is separable.

- Let E be the countable set defined by

$$E = \Big\{ \sum_{k=1}^{N} a_k\, \chi_{Q_k},\ a_k \in \mathbb{Q},\ Q_k \subseteq \mathbb{R}^n \text{ is a cube of a dyadic decomposition of length } 2^{-\ell}, \ell \geq 0 \Big\}. \tag{3.7.24}$$

Claim 3.7.17. *E is dense in L^p.*

Proof of the Claim 3.7.17: Let $f \geq 0$. We know that f is the monotone limit of a sequence of characteristic functions

$$f_k = \sum_{j=1}^{k} \frac{1}{j}\, \chi_{A_j}.$$

$\square$

By Lebesgue Theorem $\|f_k - f\|_{L^p} \to 0$. Therefore it is enough to show the following property:

Claim 3.7.18. *$\chi_A \in \overline{E}$ for every μ-measurable $A \subseteq \mathbb{R}^n$ with $\mu(A) < \infty$.*

Proof of the Claim 3.7.18: Since μ is a Radon measure, we have

$$\mu(A) = \inf_{A \subseteq G} \mu(G) \quad G \text{ open.} \tag{3.7.25}$$

Let us choose $G_k \supseteq A$ open $G_{k+1} \subseteq G_k$ with $\mu(G_k) \leq \mu(A) + \frac{1}{k}$, $k \in \mathbb{N}$. Since $\chi_{G_k} - \chi_A \in \{0,1\}$, it follows that

$$\int_{\mathbb{R}} |\chi_{G_k} - \chi_A|^p \, d\mu = \int_{\mathbb{R}} |\chi_{G_k \setminus A}|^p \, d\mu = \mu(G_k \setminus A) \leq \frac{1}{k}. \tag{3.7.26}$$

Thus we may suppose A open. Every open set can be expressed as countable disjoint union of dyadic cubes I_ℓ: $A = \bigcup_{\ell=1}^{\infty} I_\ell$ and $\mu(A) = \sum_{\ell=1}^{\infty} \mu(I_\ell)$ (Lemma 1.3.4).

In particular:

$$\chi_{\bigcup\limits_{\ell=1}^{n} I_\ell} = \sum_{\ell=1}^{n} \chi_{I_\ell} \to \chi_A \text{ as } n \to +\infty.$$

Since $\mu(A) < \infty$ ($\Longrightarrow \chi_A \in L^p$), we have by Lebesgue Theorem

$$\|\chi_A - \sum_{\ell=1}^{n} \chi_{I_\ell}\|_{L^p} \to 0 \tag{3.7.27}$$

as $n \to +\infty$ and clearly

$$\sum_{\ell=i}^{n} \chi_{I_\ell} \in E \quad \forall n. \tag{3.7.28}$$

This conclude the proof of Claim 3.7.18 and 3.7.17. $\square$

If f is a general function, then we apply the above arguments to f^+ and f^-.

- $\mathbf{C_c^0(\mathbb{R}^n)}$ **is dense in** $\mathbf{L^p(\mathbb{R}^n)}$.

Notation:

Truncation operator

$T_n \colon \mathbb{R} \to \mathbb{R} \qquad (n \geq 1)$

$$T_n(x) = \begin{cases} x & \text{if } |x| \leq n \\ n\,\dfrac{x}{|x|} & \text{if } |x| > n \end{cases}$$

(T_n is continuous).

Claim 3.7.19. *Given $f \in L^p$ and $\varepsilon > 0$ $\exists g \in L^\infty(\mathbb{R})$ and $K \subseteq \mathbb{R}^n$ compact such that $g \equiv 0$ on K^c and*

$$\|f - g\|_p < \frac{\varepsilon}{2}.$$

Proof of Claim 3.7.19: Let $f_n = \chi_{B(0,n)}\, T_n(f)$, $n \geq 1$.

We have $f_n \to f$ and by Lebesgue Theorem $\|f_n - f\|_{L^p} \to 0$ as $n \to +\infty$. Let us choose $g = f_n$, n large enough. □

Claim 3.7.20. *Let $g \in L^\infty(\mathbb{R}^n)$ with compact support and $\delta > 0$. Them there exists $g_1 \in C_c^0(\mathbb{R}^n)$ such that*

$$\|g - g_1\|_{L^1} < \delta.$$

Proof of Claim 3.7.20: We assume $g \geq 0 \implies g$ is the pointwise limit of simple functions g_n and we can pass to the limit under integral sign by Lebesgue Theorem, namely $\|g_n - g\|_{L^1} \to 0$ as $n \to +\infty$.

Therefore we can prove Claim 3.7.20 when $g = \chi_A$, A measurable and bounded.

We know that $\exists F$ closed and G open such that

$$F \subseteq A \subseteq G \text{ and } \mu(G \setminus F)\,| < \delta$$

(we may assume that $\overline{G}$ is compact). By Urysohn Lemma there is $g_1 \colon \mathbb{R}^n \to \mathbb{R}$ continuous such that

a) $0 \leq g_1(x) \leq 1$,

b) $g_1(x) = 0$ on G^c,

c) $g_1(x) = 1$ on F;

and

$$\int_{\mathbb{R}^n} |\chi_A - g_1|\, d\mu \leq \mu(G \setminus F) < \delta.$$

We may always suppose without loss of generality that the function g_1 in Claim 3.7.20 satisfies $\|g_1\|_{L^\infty} \leq \|g\|_{L^\infty}$ (otherwise we take $\widetilde{g}_1(x) = T_n g_1$ with $n = \|g\|_{L^\infty}$).

We observe that

$$\|g - g_1\|_{L^p} \leq \|g - g_1\|_{L^1}^{\frac{1}{p}} \big(2\|g\|_{L^\infty}\big)^{\frac{p-1}{p}}.$$

If $\delta > 0$ is chosen small enough so that $(2\|g\|_{L^\infty})^{\frac{p-1}{p}} \delta^{1/p} < \frac{\varepsilon}{2}$, then $\|g - g_1\|_{L^p} < \frac{\varepsilon}{2}$. Now we have

$$\|f - g_1\|_{L^p} \leq \|f - g\|_{L^p} + \|g - g_1\|_{L^p} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad (3.7.29)$$

and we can conclude.

• Let $\Omega \subseteq \mathbb{R}^n$ be open. Then $\mathbf{C_c^0(\Omega)}$ **is dense in** $\mathbf{L^p(\Omega)}$.

Let $f \in L^p(\Omega)$ and $\tilde{f}$ be defined as $\tilde{f}(x) = f(x)$ if $x \in \Omega$ and $\tilde{f}(x) = 0$ is $x \in \Omega^c$. We have $\tilde{f} \in L^p(\mathbb{R}^n)$. For every $\varepsilon > 0$ there is $g_\varepsilon \in C_c(\mathbb{R}^n)$ such that

$$\|\tilde{f} - g_\varepsilon\|_{L^p} < \frac{\varepsilon}{2}. \qquad (3.7.30)$$

One considers a sequence of open sets O_n of $\mathbb{R}^n$ such that $\bar{O}_n$ is compact, $\bar{O}_n \subseteq O_{n+1}$ and $\cup_n O_n = \Omega$. We set for every $n \geq 1$,

$$\psi_n(x) = g_\varepsilon(x) \frac{d_{O^c_{n+1}}(x)}{d_{O^c_{n+1}(x)} + d_{O_n}(x)}.$$

We have $\psi_n = 0$ in O^c_{n+1} and $\psi_n \in C_c(\Omega)$. Moreover $\psi_n = g_\varepsilon$ in O_n, $|\psi_n| \leq |g_\varepsilon|$, and since $O_n \to \Omega$ as $n \to +\infty$ we have $\psi_n \to g_\varepsilon$ in Ω and $\|\psi_n - g_\varepsilon\|_{L^p} \to 0$ as $n \to +\infty$. Therefore for all $\varepsilon > 0$ there is $n_\varepsilon \geq 1$ such that

$$\|\psi_{n_\varepsilon} - g_\varepsilon\|_{L^p(\Omega)} < \frac{\varepsilon}{2}. \qquad (3.7.31)$$

The following estimate holds:

$$\begin{aligned}\|f-\psi_{n_\varepsilon}\|_{L^p(\Omega)} &\leq \|\tilde{f}-g_\varepsilon\|_{L^p(\Omega)}+\|\psi_{n_\varepsilon}-g_\varepsilon\|_{L^p(\Omega)}\\ &\leq \|\tilde{f}-g_\varepsilon\|_{L^p(\mathbb{R}^n)}+\|\psi_{n_\varepsilon}-g_\varepsilon\|_{L^p(\Omega)} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

We can conclude the proof. $\square$

Theorem 3.7.21. *Let $f, f_k \in L^p(\Omega,\mu)$, $k \in \mathbb{N}$, $1 \leq p < \infty$. Then following two conditions are equivalent:*

i) $\|f_k - f\|_{L^p} \to 0 \quad k \to \infty$.

ii) $f_k \xrightarrow{\mu} f$ *as* $k \to +\infty$ *and* $\|f_k\|_{L^p} \to \|f\|_{L^p}$ *as* $k \to +\infty$.

Proof.

i) $\Longrightarrow$ ii):

The fact that $f_k \xrightarrow{\mu} f$ as $k \to +\infty$ is a consequence of Tchebychev inequality as we have already observed several times.

From Minkowski inequality it follows that

$$|\,\|f\|_{L^p}-\|f_k\|_{L^p}| \leq \|f_k-f\|_{L^p}.$$

ii) $\Longrightarrow$ i): We observe that

$$2^{p-1}|f_k|^p+2^{p-1}|f|^p-|f_k-f|^p \geq 0.$$

Let (f_{k_n}) be a subsequence of (f_k) converging to f as $n \to +\infty$ μ-a.e.

By Fatou's Lemma, we get

$$\begin{aligned}
&2^p \int_\Omega |f|^p \, d\mu - \limsup_{n\to+\infty} \int_\Omega |f_{k_n} - f|^p \, d\mu = \\
&\liminf_{n\to\infty} \int 2^{p-1}|f_{k_n}|^p + 2^{p-1}|f|^p - |f_{k_n} - f|^p \, d\mu \geq \\
&\int_\Omega \liminf_{n\to\infty} (2^{p-1}|f_{k_n}|^p + 2^{p-1}|f|^p - |f_{k_n} - f|^p) \, d\mu = 2^p \int_\Omega |f|^p. \\
\implies &\limsup_{n\to+\infty} \int_\Omega |f_{k_n} - f|^p \, d\mu = 0.
\end{aligned} \tag{3.7.32}$$

It follows that

$$\lim_{n\to+\infty} \int_\Omega |f_{k_n} - f|^p \, d\mu = 0.$$

□

Claim 3.7.22. $\|f_k - f\|_{L^p} \to 0$ *as* $k \to \infty$.

Proof of Claim 3.7.22.
Suppose by contradiction that

$$\limsup_{k\to+\infty} \int_\Omega |f_k - f|^p \, d\mu > 0.$$

Then there is a subsequence f_{k_n} such that

$$\lim_{n\to+\infty} \int_\Omega |f_{k_n}(x) - f(x)|^p \, d\mu = \limsup_{k\to+\infty} \int_\Omega |f_k(x) - f(x)|^p \, d\mu > 0.$$

Since $f_{k_n} \xrightarrow{\mu} f$ as $n \to +\infty$, $\exists f_{k_{n'}} \to f$ μ-a.e. as $n' \to +\infty$. Then by the prevoius arguments we get that $\|f_{k_{n'}} - f\|_{L^p} \to 0$ which contradicts that

$$\lim_{n\to+\infty} \int_\Omega |f_{k_n} - f|^p \, d\mu > 0. \quad \square$$

Remark 3.7.23. For $p \geq 1$, the function $x \mapsto |x|^p$ is convex, therefore

$$\begin{aligned} \left|\frac{x+y}{2}\right|^p &\leq \frac{1}{2}\,|x|^p + \frac{1}{2}\,|y|^p \\ |x+y|^p &\leq 2^{p-1}|x|^p + 2^{p-1}|y|^p \quad \forall x, y \in \mathbb{R}. \end{aligned} \tag{3.7.33}$$

One can directly prove that $\forall a, b \in \mathbb{R}$

$$|a+b|^p - 2^{p-1}|a|^p - 2^{p-1}|b|^p \leq 0.$$

It is enough to consider $f(x) = |1+x|^p - 2^{p-1} - 2^{p-1}|x|^p$ and show that $f(x) \geq 0$.

$\square$

Chapter 4

Product Measures and Multiple Integrals

4.1 Fubini's Theorem

Let X, Y be sets.

Definition 4.1.1. *Let μ be a measure on X and ν be a measure on Y. We define the measure $\mu \times \nu$: $\mathcal{P}(X \times Y) \to [0, +\infty]$ by setting for each $S \subseteq X \times Y$*

$$(\mu \times \nu)(S) = \inf \Bigg\{ \sum_{i=1}^{\infty} \mu(A_i)\, \nu(B_i) \quad \begin{aligned} & S \subseteq \textstyle\bigcup_{i=1}^{\infty} A_i \times B_i \\ & A_i \subseteq X \ \ \mu\text{-measurable} \\ & B_i \subseteq Y \ \ \mu\text{-measurable},\ i \in \mathbb{N} \end{aligned} \Bigg\}. \tag{4.1.1}$$

The map $\mu \times \nu$ is called the **Product Measure**.

Remark 4.1.2. If we define for every $A \subseteq X$ μ-measurable, $B \subseteq Y$ ν-measurable

$$\lambda(A \times B) = \mu(A)\, \nu(B). \tag{4.1.2}$$

Then λ is a **pre-measure** on the algebra of finite disjoint unions. Thus $\mu \times \nu$ is the Caracthéodory extension of λ.

Exercise 4.1.3. $X = \mathbb{R}^k$, $Y = \mathbb{R}^\ell$ and $\mu = \mathcal{L}^k$, $\nu = \mathcal{L}^\ell \implies \mu \times \nu = \mathcal{L}^n$ $n = k + \ell$.

Digression

We consider for simplicity $\mathbb{R}^2$. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{L}^2$-summable function, namely a measurable function such that

$$\int_{\mathbb{R}^2} |f(x,y)| \, d\mathcal{L}^2 < +\infty.$$

We wonder if (by analogy with Riemann-integral for continuous functions) if the above integral can be splited in two integrals on $\mathbb{R}$, namely

$$\begin{aligned}\int_{\mathbb{R}^2} |f(x,y)| \, d\mathcal{L}^2 &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^1(x) \Big) \, d\mathcal{L}^1(y), \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^1(y) \Big) \, d\mathcal{L}^1(x).\end{aligned}$$

In order that the above formula is correct, it is necessary that

i) $\forall \overline{y} \in \mathbb{R}$: $x \longmapsto |f(x,\overline{y})|$ $\mathcal{L}^1$-measurable.

ii) $y \longmapsto \displaystyle\int_{\mathbb{R}} |f(x,y)| \, d\mathcal{L}^1(x)$ is $\mathcal{L}^1$-measurable and summable.

Example 4.1.4. Let P be a non-measurable subset of $\mathbb{R}$ which is contained in $[0,1]$ and let $E = P \times \mathbb{Q}$. Then E is measurable in $\mathbb{R}^2$. Indeed, $E \subseteq [0,1] \times \mathbb{Q}$ which is a rectangular of null-measure

$$\begin{aligned} \mathcal{L}^2([0,1] \times \mathbb{Q}) \ &= \mathcal{L}^1[0,1] \times \mathcal{L}^1(\mathbb{Q}) = 0 \\ \Longrightarrow \quad & E \text{ is } \mathcal{L}^2\text{-measurable.}\end{aligned}$$

Let us consider $f(x,y) = \chi_E(x,y) = \chi_P(x)\,\chi_{\mathbb{Q}}(y)$, f is $\mathcal{L}^2$-measurable and its integral is zero, since $f \equiv 0$ $\mathcal{L}^2$-a.e. Let $\overline{y} \in \mathbb{R}$ be fixed. Then $f(x,\overline{y}) = \chi_P(x)$ if $\overline{y} \in \mathbb{Q}$ and $= 0$ otherwise.

Thus if $\overline{y} \in \mathbb{Q}$, $f(x,\overline{y})$ is not measurable and it has no sense to write $\int_{\mathbb{R}} f(x,\overline{y}) \, d\mathcal{L}^1(x)$. This means that we cannot apparently split the integral of f over $\mathbb{R}^2$ into two integrals. Actually the function $f(x,y)$ is measurable for $\mathcal{L}^1$-a.e. y. Therefore we can define for $\mathcal{L}^1$-a.e. $y \in \mathbb{R}$

the function $y \longmapsto \int_{\mathbb{R}} |f(x,y)|\, d\mathcal{L}^1(x)$ which is identically zero. We can also re-define the function when $y \in \mathbb{Q}$ (the value of its integral does not change since $\mathcal{L}^1(\mathbb{Q}) = 0$) in order to obtain a $\mathcal{L}^1$-measurable function which also $\mathcal{L}^1$-summable. This example shows that even if the function obtained by freezing one of the two variables is not measurable, the values of x or y for which we get a non-measurable function have measure zero and therefore they can be neglected when we integrate.

Theorem 4.1.5. *(Fubini)*

Let μ, ν be Radon measures on $X = \mathbb{R}^k$, $Y = \mathbb{R}^\ell$ respectively, $\mu \times \nu$ be the product measure on $X \times Y = \mathbb{R}^n$, $n = \ell + k$. Then it holds

i) *For every $A \subseteq X$ μ-measurable, $B \subseteq Y$ ν-measurable, we have $A \times B$ is $\mu \times \nu$-measurable and*

$$(\mu \times \nu)(A \times B) = \mu(A)\, \nu(B). \tag{4.1.3}$$

ii) *Let $S \subseteq X \times Y$ be $(\mu \times \nu)$-measurable with $(\mu \times \nu)(S) < +\infty$. Then the set $S_y = \{x : (x, y) \in S\}$ is μ-measurable for ν a.e. y and the mapping*

$$y \longmapsto \mu(S_y) = \int_X \chi_{S_y}(x)\, d\mu$$

is ν- summable with

$$(\mu \times \nu)(S) = \int_Y \mu(S_y)\, dy = \int_Y \int_X \chi_{S_y}(x)\, d\mu\, d\nu.$$

An analogous property holds for $S_x = \{y : (x, y) \in S\}$.

iii) *$\mu \times \nu$ is a Radon measure.*

iv) *If $f\colon X \times Y \to \overline{\mathbb{R}}$ is $\mu \times \nu$-summable, then the mapping $x \longmapsto f(x, \overline{y})$ and $y \longmapsto f(\overline{x}, y)$ are μ-summable for ν-a.e. $\overline{y}$ and ν-summable for μ-a.e. $\overline{x}$ and the mappings*

$$\begin{aligned} y &\longmapsto \int_X f(x, y)\, d\mu(x) \\ x &\longmapsto \int_Y f(x, y)\, d\nu(y) \end{aligned}$$

are respectively ν- and μ-summable and

$$\begin{aligned} \int_{X \times Y} f\, d(\mu \times \nu) &= \int_Y \Big(\int_X f(x, y)\, d\mu(x) \Big)\, d\nu(x) \\ &= \int_X \Big(\int_Y f(x, y)\, d\mu(y) \Big)\, d\mu(x). \end{aligned}$$

Proof. We define the set

$$\mathcal{F} = \Big\{ S \subseteq X \times Y\colon\ S \text{ satisfies } ① \text{ and } ② \Big\} \tag{4.1.4}$$

where

1 $x \longmapsto \chi_S(x,y)$ is μ-measurable μ-a.e. y

and

2 $y \longmapsto \displaystyle\int_X \chi_S(x,y)\,d\mu(x)\ \big(= \mu(S_y)\big)$ is ν-measurable.

For $S \in \mathcal{F}$ we set

$$\rho(S) = \int_Y \Big(\int_X \chi_S(x,y)\,d\mu(x) \Big)\,d\nu(y)\Big). \tag{4.1.5}$$

Goal:

To show that every $S \subseteq X \times Y$ $\mu \times \nu$-measurable belongs to $\mathcal{F}$ and $f(S) = (\mu \times \nu)(S)$, in particular for $S = A \times B$. We define the families

$$\begin{aligned}
\mathcal{P}_0 &= \{A \times B,\ A \subseteq \mu\text{-measurable},\ B \subseteq Y\ \nu\text{-measurable}\} \\
\mathcal{P}_1 &= \Big\{ \bigcup_{j=1}^{\infty} S_j,\ S_j \in \mathcal{P}_0 \Big\} \\
\mathcal{P}_2 &= \Big\{ \bigcap_{j=1}^{\infty} R_j,\ R_j \in \mathcal{P}_1 \Big\}.
\end{aligned} \tag{4.1.6}$$

□

Claim 4.1.6. $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{F}$.

Proof of the Claim 4.1.6:

i) Clearly, $\mathcal{P}_0 \in \mathcal{F}$. Let $S = A \times B$. We have

$$S_y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B\,. \end{cases} \tag{4.1.7}$$

Thus S_y is clearly μ-measurable. Moreover, $\mu(S_y) = \mu(A)\,\chi_B$. Thus

$$y \longmapsto \mu(S_y) \tag{4.1.8}$$

is ν-measurable as well and $\rho(S) = \mu(A)\,\mu(B)$.

ii) $\mathcal{P}_1 \subseteq \mathcal{F}$. We first observe that for $A_1 \times B_1,\ A_2 \times B_2 \in \mathcal{P}_0$ it holds

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \in \mathcal{P}_0 \tag{4.1.9}$$

and

$$(A_1 \times B_1) \setminus (A_2 \times B_2) = \big[(A_1 \setminus A_2) \times B_1\big] \cup \big[(A_1 \cap A_2) \times B_1 \setminus B_2\big]. \tag{4.1.10}$$

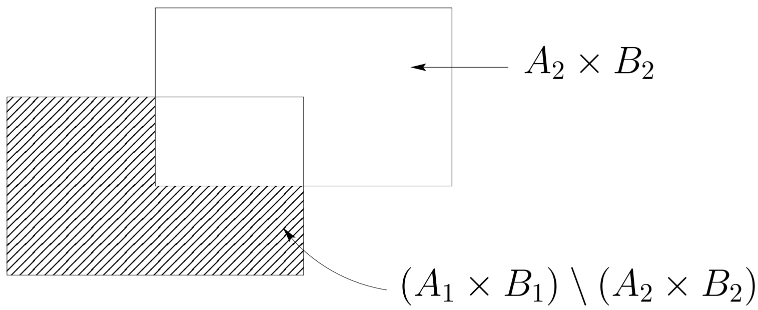


Fig. 4.1

Let now $S = \bigcup_{j=1}^{\infty} S_j$. We define

$$\widetilde{S}_1 = S_1,\ \widetilde{S}_2 = S_2 \setminus S_1,\ \widetilde{S}_n = S_n \setminus \bigcup_{k=1}^{n-1} S_k. \tag{4.1.11}$$

Therefore each member of $\mathcal{P}_1$ is a countable disjoint union of members of $\mathcal{P}_0$. If $S = \bigcup_{j=1}^{\infty} S_j$, $S_j \in \mathcal{P}_0$ mutually disjoint, then the map

$$x \longmapsto \chi_S(x,y) = \sum_0^{\infty} \chi_{S_j}(x,y) = \lim_{k \to +\infty} \sum_j^k \chi_{S_j}(x,y)$$

is measurable since it is the limit of μ-measurable functions.

We have A_J: $y \longmapsto \int_X \chi_{S_j}(x,y)\, d\mu(x)$ is ν-measurable. Thus

$$\begin{aligned} y \longmapsto \int_X \chi_S(x,y)\, d\mu(x) &= \int_X \lim_{k \to +\infty} \sum_0^k \chi_{S_j}(x,y)\, d\mu(x) \\ &= \lim_{k \to \infty} \sum_0^k \int_X \chi_{S_j}(x,y)\, d\mu(x) \end{aligned} \tag{4.1.12}$$

by Beppo-Levi Theorem

is ν-measurable. Moreover,

$$\begin{aligned}\rho(S) &= \int_Y \Big(\int_X \chi_S(x,y)\,d\mu\Big)\,d\nu \\ &= \int_Y \sum_j^{\infty} \int_X \chi_{S_j}(x,y)\,d\mu\,d\nu \\ &= \sum_{j=0}^{\infty} \int_Y \Big(\int_X \chi_{S_j}(x,y)\,d\mu\Big)\,d\nu \\ &= \sum_{j=0}^{\infty} \rho(S_j).\end{aligned} \tag{4.1.13}$$

Above we have applied two times Beppo Levi-s Theorem. □

Claim 4.1.7. *For $R = \bigcap_{j=0}^{\infty} R_j$, $R_j \in \mathcal{P}_1$ and $\rho(R_j) < +\infty\ \forall j$, then*

$$\boxed{\rho(R) = \lim_{k\to+\infty} \rho\Big(\bigcap_0^k R_j\Big).} \tag{4.1.14}$$

Proof of Claim 4.1.7: Let $R = \bigcap_{j=0}^{\infty} R_j$, $R_j \in \mathcal{P}_1$, $\rho(R_j) < +\infty\ \forall j$.

We may assume that $R_{j+1} \subseteq R_j$ (otherwise we consider the family $(\widetilde{R}_j)_j$:

$$\begin{aligned}&\widetilde{R}_1 = R_1, \quad \widetilde{R}_2 = \widetilde{R}_1 \cap R_2 \\ &\widetilde{R}_j = R_{j+1} \cap \widetilde{R}_j.\end{aligned} \tag{4.1.15}$$

Since $\chi_{R_j}(x,y) \le \chi_{R_j}(x,y)$ we apply twice the Lebesgue Theorem and get

$$\rho(R) = \int_Y \Big(\int_X \chi_R(x,y)\, d\mu\Big)\, d\nu$$
$$= \int_Y \Big(\int_X \lim_y \chi_{R_j}(x,y)\, d\mu\Big)\, d\nu$$
$$= \int_Y \lim_{j\to+\infty} \Big(\int_X \chi_{R_j}(x,y)\, d\mu\Big)\, d\nu \tag{4.1.16}$$
$$= \lim_{j\to+\infty} \int_Y \Big(\int_X \chi_{R_j}(x,y)\, d\mu)\, d\nu$$
$$= \lim_{j\to+\infty} \rho(R_j).$$

Therefore (4.1.14) holds. □

Claim 4.1.8. *For $S \subseteq X \times Y$, we have*

$$(\mu\times\nu)(S) = \inf\big\{\rho(R) \,|\, S \subseteq R,\ R \in \mathcal{P}_1\big\}. \tag{4.1.17}$$

Proof of Claim 4.1.8: First we observe that if $S \subseteq R = \bigcup_{i=1}^{\infty} A_i \times B_i$, then

$$\rho(R) \underset{\boxed{\text{Claim 4.1.6}}}{\underset{\downarrow}{\leq}} \sum_{i=1}^{n} \rho(A_i\times B_i) = \sum_{i=1}^{\infty} \mu(A_i)\,\mu(B_i) \tag{4.1.18}$$

Thus

$$\inf\{\rho(R) : S \subseteq R \in \mathcal{P}_1\} \leq (\mu\times\nu)(S^1). \tag{4.1.19}$$

Moreover, there exists a disjoint sequence $\{A'_j \times B'_j\}$ in $\mathcal{P}_0$ such that

$$R = \bigcup_{j=0}^{\infty} (A'_j \times B'_j).$$

Thus

$$\begin{aligned}\rho(R) = \sum_{j=0}^{\infty} \rho(A'_j\times B'_j) &= \sum_{j=0}^{\infty} \mu(A'_i)\,\nu(B'_j)\\ &\geq (\mu\times\nu)(S).\end{aligned} \tag{4.1.20}$$

$\square$

We prove i) of Theorem 4.1.5.

Let $A\times B \in \mathcal{P}_0$. By definition of $\mu\times\nu$ and that $\rho(A\times B) = \mu(A)\,\nu(B)$, we have

$$(\mu\times\nu)(A\times B) \le \mu(A)\,\nu(B) = \rho(A\times B) \le \rho(R)$$

$\forall R \in \mathcal{P}_1$ with $A\times B \subseteq \mathbb{R}$. Claim 4.1.8 implies that

$$(\mu\times\nu)(A\times B) = \rho(A\times B) = \mu(A)\,\nu(B).$$

Claim 4.1.9. *$A\times B$ is $\mu\times\nu$-measurable.*

Proof of Claim 4.1.9. Take $T\subseteq X\times Y$, $R\in\mathcal{P}_1$ with $T\subseteq R$. Then $R\setminus(A\times B)$, $R\cap(A\times B)$ are disjoint and belong to $\mathcal{P}_1$. Consequently,

$$\begin{aligned}&(\mu\times\nu)\big(T\setminus(A\times B)\big) + (\mu\times\nu)\big(T\cap(A\times B)\big) \le\\ &\rho(R\setminus(A+B) + \rho\big(R\cap(A\times B)\big) = \rho(R).\end{aligned}$$

According to Claim 4.1.8, we get

$$(\mu\times\nu)\big(T\setminus(A\times B)\big) + (\mu\times\nu)\big(T\cap(A\times B)\big) \le (\mu\times\nu)(T). \tag{4.1.21}$$

$\Longrightarrow A\times B$ is $\mu\times\nu$-measurable. $\square$

Claim 4.1.10. *For each $S\subseteq X\times Y$ there is a set $R\subseteq\mathcal{P}_2$ such that $S\subseteq R$ and*

$$\rho(R) = (\mu\times\nu)(S).$$

Proof of Claim 4.1.10: If $(\mu\times\nu)(S) = +\infty$, then take $R = X\times Y$. Otherwise, for $j\in\mathbb{N}$, we choose $S\subseteq R_j\in\mathcal{P}_1$ with $(\mu\times\nu)(S)\le\rho(R_j)\le(\mu\times\nu)(S)+\frac{1}{j}$.

We may suppose $R_{j+1}\subset R_j$. We set $R=\bigcap_j R_j \supseteq S$. Since by Claim 4.1.7 $\lim_{j\to\infty}\rho(R_j) = \rho(R)$, we get the property, since $R\in\mathcal{P}_2$.

We prove ii) of Theorem 4.1.5.

Let $S \subseteq X \times Y$ be $\mu \times \nu$-measurable and $(\mu \times \nu)(S) < +\infty$. We choose $R \in \mathcal{P}_2$ according to Claim 4.1.10 with $S \subseteq R$ and

$$(\mu \times \nu)(S) = \rho(R).$$

We have two cases.

a) $(\mu \times \nu)(S) = 0 \Longrightarrow \rho(R) = 0 \Longrightarrow \int_Y \mu(R_y)\, d\nu = 0 \Longrightarrow \mu(R_y) = 0 \quad \nu$-a.s. y.

Now we have

$$0 \leq X_S(x,y) \leq X_R(x,y),$$

and in particular

$$0 \leq \chi_{S_y}(x) \leq \chi_{R_y}(x).$$

Since $\mu(R_y) = 0 \Longrightarrow \mu(S_y) = 0$. Thus S_y is μ-measurable. Moreover,

$$y \longmapsto \int_y \mu(S_y)\, d\nu = 0 \tag{4.1.22}$$

and thus it is ν-measurable. In particular, $\rho(S) = 0$.

b) $(\mu \times \nu)(S) = \rho(R) > 0$. Consider $R \setminus S$:

$$\begin{aligned}(\mu \times \nu)(R \setminus S) &= (\mu \times \nu)(R) - (\mu \times \nu)(S) \\ &= \rho(R) - (\mu \times \nu)(S) = 0.\end{aligned}$$

We remark that every member of $\mathcal{P}_2$ is $\mu \times \nu$-measurable and if $R \in \mathcal{P}_2$, $(\mu \times \nu)(R) < \infty$, then $(\mu \times \nu)(R) = \rho(R)$.

Since $(\mu \times \nu)(R \setminus S) = 0$, then by a) $\rho(R \setminus S) = 0$. This means that

$$\mu(R \setminus S)_y = \mu(R_y \setminus S_y) = 0.$$

Since $R_y - S_y$ is μ-measurable (having null measure), then S_y is μ-measurable as well and $\mu(R_y) = \mu(S_y)$,

$$\begin{aligned}(\mu \times \nu)(S) = \rho(R) &= \int_Y \mu(R_y)\, dy \\ &= \int_Y \mu(S_y)\, dy = \rho(S).\end{aligned}$$

iii) No proof.

Proof of iv) of Theorem 4.1.5. If $f = \chi_S < +\infty$, then iv) follows from ii). If $f \geq 0$ we write

$$f = \sum_{j=1}^{\infty} \frac{1}{j}\, \chi_{A_j}(x, y), \tag{4.1.23}$$

with $A_j (\mu \times \nu)$-measurable.

Because of ii), the linearity of integrals, the property holds for the partial sums

$$f_k = \sum_{j=1}^{k} \frac{1}{j}\, \chi_{A_j}. \tag{4.1.24}$$

By passing to the limit as $k \to +\infty$ and applying Beppo-Levi Theorem, we get the property for f

$$\begin{aligned}\int_Y \Big(\int_X f(x, y)\, d\mu \Big)\, d\nu &= \int_Y \int_X \Big(\sum_{j=1}^{\infty} \frac{1}{j}\, X_j(x)\, d\mu(x) \Big)\, d\nu(y) \\ &= \sum_{j=1}^{\infty} \frac{1}{j} \int_Y \Big(\int_X \chi_{A_j}\, d\mu(x) \Big)\, d\nu(y) \\ &= \sum_{j=1}^{\infty} \frac{1}{j}\, (\mu \times \nu)(A_j) = \int_{X \times Y} f(x, y)\, d\mu \times \nu. \end{aligned} \tag{4.1.25}$$

For general f, consider $f = f^+ - f^-$. □

Remark 4.1.11. A set $S \subseteq X \times Y$ that satisfies 1 and 2 is not necessarily $(\mu \times \nu)$-measurable: take $X = Y = \mathbb{R}$, $\mu = \nu = \mathcal{L}^1$, $A \subseteq [0, 1]$ **not measurable** and set

$$S = \{(x, y) : |x - \chi_A(y)| < \frac{1}{2};\;\; y \in [0, 1]\}. \tag{4.1.26}$$

S is not $(\mu \times \nu)$-measurable, but $\forall y \in [0, 1]$, S_y is μ-measurable and $\mu(S_y) = 1$.

- If $f \colon X \times Y \to \mathbb{R}^+$ is $\mu \times \nu$-measurable, then f is $\mu \times \nu$-summable if one of the two iterated integrals exists finite (Tonelli 's Theorem).

- It may happen that the two iterated integrals exist without f being $\mu \times \nu$-measurable (take $f = \chi_S$, S as above) or without f being $\mu \times \nu$-summable (take $f(x,y) = \frac{\sin(y)}{x}$ with $(x,y) \in (0,1] \times [0,2\pi]$. In this case we have $\int_0^{2\pi} f(x,y)dy = 0$ for all $x \in (0,1]$ but the function is not summable).

4.2 Applications

1. Show that $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ is not Lebesgue summable in $[0,1] \times [0,1]$, namely

$$\int_{[0,1]\times[0,1]} \left| \frac{x^2-y^2}{(x^2+y^2)^2} \right| dx\, dy = +\infty.$$

1st *Proof.* We apply Fubini's Theorem: if $f \in L^1(X \times Y)$ then for μ a.e. $x \in X$, $f(x,\cdot) \in L^1(Y)$ and $\int_Y f(x,y)\, d\nu \in L^1(X)$, for ν-a.e. $y \in Y$, $f(\cdot,y) \in L^1(X)$ and $\int_X f(x,y)\, d\mu \in L^1(Y)$. Moreover,

$$\begin{aligned} \int_{X\times Y} f(x,y)\, d(\mu\times\nu) &= \int_X \Big(\int_Y f(x,y)\, d\nu \Big)\, d\mu \\ &= \int_Y \Big(\int_X f(x,y)\, d\mu \Big)\, d\nu\,. \end{aligned} \tag{4.2.1}$$

If $f \in L^1([0,1]\times[0,1])$ then the two iterated integrals should be equal. We compute separately the two iterated integrals (we observe that we can treated the two iterated integral as Riemann improper integrals):

$$\begin{aligned} \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dy &= \int_0^1 \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\, dy \\ &= \int_0^1 \frac{1}{x^2+y^2}\, dy + \int_0^1 \frac{-2y^2}{(x^2+y^2)^2}\, dy \\ &= \int_0^1 \frac{1}{x^2+y^2}\, dy + \frac{y}{x^2+y^2}\Big|_0^1 - \int_0^1 \frac{1}{x^2+y^2}\, dy \\ &= \frac{1}{x^2+1} \\ &\Longrightarrow \int_0^1 \frac{1}{x^2+1}\, dx = \arctan x \Big|_0^1 = \boxed{\frac{\pi}{4}}\,. \end{aligned}$$

On the other hand:

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx\, dy = \boxed{-\frac{\pi}{4}}.$$

2$^{\text{nd}}$ *Proof.* One computes directly

$$\int_{[0,1]^2} \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| d(\mu \times \nu) = \int_0^1 \Big[\underbrace{\int_0^y \frac{y^2-x^2}{(x^2+y^2)^2}\, dx}_{1} + \underbrace{\int_y^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dx}_{2}\Big]\, dy$$

$$\begin{aligned}
\int_0^y \frac{y^2-x^2}{(x^2+y^2)^2}\, dx &= \int_0^y \frac{1}{(x^2+y^2)}\, dx - \int_0^y \frac{2x^2}{(x^2+y^2)^2}\, dx \\
&= \int_0^y \frac{1}{(x^2+y^2)}\, dx + \frac{x}{y^2+x^2}\Big|_0^y - \int_0^y \frac{1}{(x^2+y^2)^2}\, dx \\
&= \frac{y}{2y^2} = \frac{1}{2y} \\
\int_y^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dx &= -\int_y^1 \frac{1}{(x^2+y^2)}\, dx + \int_y^1 \frac{2x^2}{(x^2+y^2)}\, dx \\
&= -\int_y^1 \frac{1}{(x^2+y^2)}\, dx - \frac{x}{x^2+y^2}\Big|_y^1 + \int_y^1 \frac{1}{x^2+y^2}\, dx \\
&= \frac{1}{1+y^2} + \frac{y}{2y^2} = -\frac{1}{1+y^2} + \frac{1}{2y}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^1 \Big[\int_0^y \frac{y^2-x^2}{(x^2+y^2)^2}\, dx + \int_y^1 \frac{x^2-y^2}{(x^2+y^2)^2}\Big]\, dy \;&=\; \int_0^1 \frac{1}{y}\, dy + \int_0^1 -\frac{1}{1+y^2}\, dy \\
&= \log|y|\Big|_0^1 - \arctan y\Big|_0^1 = +\infty.
\end{aligned}$$

2. **Gaussian Integral**

We want to compute

$$\int_{-\infty}^{+\infty} e^{-x^2} dx.$$

There is no elementary indefinite integral for $\int e^{-x^2} dx$ but the definite integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$ can be evaluated:

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-(x^2+y^2)}\, dx \Big)\, dy \\ &= \Big(\int_{-\infty}^{+\infty} e^{-x^2} dx \Big)\Big(\int_{-\infty}^{+\infty} e^{-y^2} dy \Big) \\ &= \Big(\int_{-\infty}^{+\infty} e^{-x^2}\, dx \Big)^2 .\end{aligned}$$

On the other hand:

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r\, dr\, d\theta \\ &= 2\pi \int_0^{+\infty} r e^{-r^2} dr \underset{\substack{\downarrow \\ s=-r^2}}{=} \pi \int_0^{\infty} e^{-s} ds = \pi\end{aligned}$$

Therefore $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. $\square$

4.3 Convolution

We suppose $\mu = \mathcal{L}^n$ is the n-dimensional Lebesgue measure. For $\Omega \subseteq \mathbb{R}^n$, $1 \leq p \leq \infty$ we will write for simplicity $L^p(\Omega)$ or L^p instead of $L^p(\Omega, \mathcal{L}^n)$.

Lemma 4.3.1. *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be $\mathcal{L}^n$-measurable. Then the function*

$$F(x, y) = f(x - y) : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$$

is $\mathcal{L}^{2n}$-measurable.

Proof. With the substitution $z := x - y$ we obtain

$$\begin{aligned} F^{-1}\big([-\infty, a)\big) = \{(x,y) : f(x-y) < a\} &= \{(x, x-z) : f(z) < a\} \\ &= T(\mathbb{R}^n \times \{z : f(z) < a\}) \end{aligned}$$

where $T\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $(x,z) \mapsto (x, x-z)$. ☐

T is Lipschitz continuous and $\mathbb{R}^n \times \{z : f(z) < a\}$ is $\mathcal{L}^{2n}$-measurable. The claim follows from the following lemma:

Lemma 4.3.2. *(no proof)*
Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous with

$$|T(x) - T(y)| \leq L\,|x - y| \quad \forall x, y \in \mathbb{R}^n$$

and let $A \subseteq \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then $T(A)$ is $\mathcal{L}^n$-measurable.

Definition 4.3.3. *Let $f, g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be $\mathcal{L}^n$-measurable functions such that for a.e $x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is $\mathcal{L}^n$-integrable. We define the convolution product of f and g by*

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \quad \text{for a.e } x \in \mathbb{R}^n. \tag{4.3.1}$$

Remark 4.3.4.

1. *If $f, g\colon \mathbb{R}^n \to [0, +\infty]$ are $\mathcal{L}^n$-measurable functions, then, since the function $f(x-y)\,g(y)$ is positive and $\mathcal{L}^n$-measurable, $(f \star g)\colon \mathbb{R}^n \to [0, +\infty]$ is well defined for every $x \in \mathbb{R}^n$.*

2. *By making in (4.3.1) the change of variable $z = x - y$ and using the translation invariance of the Lebesgue measure we obtain that the function $f(x-y)g(y)$ is $\mathcal{L}^n$-integrable iff the function $f(z)g(x-z)$ is $\mathcal{L}^n$-integrable and $(f \star g)(x) = (g \star f)(x)$. This proves that the convolution is commutative.*

Theorem 4.3.5. *Let $f, g \in L^1(\mathbb{R}^n)$. Then for a.e $x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is $\mathcal{L}^n$-summable and $f \star g \in L^1(\mathbb{R}^n)$ with*

$$\|f \star g\|_{L^1} \leq \|f\|_{L^1}\,\|g\|_{L^1}.$$

Proof of Theorem 4.3.5. By Lemma 4.3.1 the function $F(x, y) = f(x-y)\,g(y)$ is $\mathcal{L}^{2n}$-measurable.

i) We suppose $f, g \geq 0$. Then F is $\mathcal{L}^{2n}$-integrable. By Tonelli's Theorem, F is $\mathcal{L}^{2n}$-summable if and only if one of the iterated integrals exist finite

$$\begin{aligned}
\|f\star g\|_{L^1} &= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy\Big)\,dx\\
&= \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dx\Big)\,dy\\
&= \int_{\mathbb{R}^n} g(y)\Big(\int_{\mathbb{R}^n} f(x-y)\,dx\Big)\,dy\\
&= \|f\|_{L^1}\,\|g\|_{L^1} < +\infty\,.
\end{aligned}$$

ii) For general f, g it follows from i) that

$$|F(x,y)| = |f(x-y)|\;|g(y)| \in L^1(\mathbb{R}^{2n}).$$

Thus by Fubini's Theorem $|f|\star|g| \in L^1(\mathbb{R}^n)$. We have

$$\begin{gathered}
\|f\star g\|_{L^1} = \int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy\Big|\,dx \leq\\
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|\;|g(y)|\,dy\,dx = \\
\int_{\mathbb{R}^n} |f|\star|g|(x)\,dx = \|\;|f|\star|g|\;\|_{L^1} = \|f\|_{L^1}\,\|g\|_{L^1}.
\end{gathered} \tag{4.3.2}$$

In particular it follows that for a.e $x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is in $L^1(\mathbb{R}^n)$. □

Corollary 4.3.6. *Let $1 \leq p \leq +\infty$, $1 \leq q \leq +\infty$.*

i) $f \in L^1$, $g \in L^p$ *Then for a.e* $x \in \mathbb{R}^n$ *the function* $y \mapsto f(x-y)g(y)$ *is* $\mathcal{L}^n$*-summable and* $f \star g \in L^p$ *with* $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$.

ii) $f \in L^p$, $g \in L^q$, $1 \leq \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. *Then for a.e* $x \in \mathbb{R}^n$ *the function* $y \mapsto f(x-y)g(y)$ *is* $\mathcal{L}^n$*-summable and* $f \star g \in L^r$ *with*

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof of Corollary 4.3.6.

i) The case $p = 1$ follows from Theorem 4.3.5. For $1 < p < +\infty$, let $1 < q < \infty$ be the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$):

$$\begin{aligned}\int_{\mathbb{R}^n} |f \star g|^p dx &= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \Big|^p dx \\ &\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{q}}\, |f(x-y)|^{\frac{1}{p}} |g(y)|\, dy \Big)^p dx.\end{aligned}$$

From the case $p = 1$, it follows that for a.e. $x \in \mathbb{R}^n$, $y \longmapsto |f(x-y)|\, |g(y)|^p$ is μ-summable and thus

$$y \longmapsto |f(x-y)|^{\frac{1}{p}}\, |g(y)| \in L^p(\mathbb{R}^n). \tag{4.3.3}$$

Moreover, $y \longmapsto |f(x-y)|^{\frac{1}{q}} \in L^q$. From Hölder inequality it follows

$$|f(x-y)|\, |g(y)| = |f(x-y)|^{\frac{1}{q}} |f(x-y)|^{\frac{1}{p}} |g(y)| \in L^1(\mathbb{R}^n) \text{ for a.e } x \in \mathbb{R}^n. \tag{4.3.4}$$

We have

$$\int_{\mathbb{R}^n} |f \star g|^p dx = \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \Big|^p dx$$

$$\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{q}} \, |f(x-y)|^{\frac{1}{p}} |g(y)|\, dy \right)^p dx.$$

$$\int_{\mathbb{R}^n} |f \star g|^p dx \leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y)|\, dy \Big)^{\frac{p}{q}} \Big(\int_{\mathbb{R}^n} |f(x-y)| \, |g(y)|^p \Big)\, dx$$

$$\leq \|f\|_{L^1}^{\frac{p}{q}} \, \| \, |f| \star |g|^p \|_{L^1}$$

$$\leq \|f\|_{L^1}^{\frac{p}{q}} \, \|f\|_{L^1} \, \| |g|^p \|_{L^1}$$

$$\leq \|f\|_{L^1}^{\frac{p}{q}+1} \, \|g\|_{L^p}^p$$

$$\Longrightarrow \|f \star g\|_{L^p} \leq \|f\|_{L^1} \, \|g\|_{L^p}. \tag{4.3.5}$$

The case $p = \infty$ follows form Hölder inequality.

ii) We assume without loss of generality that $p \leq q$. Since $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{q}$, we get $p \leq q \leq r$.

- The case $\boxed{r = \infty}$ follows from Hölder inequality.

- The case $\boxed{p = 1}$ follows from i) $(q = r)$.

Thus $1 < p \leq q < r < \infty$. We write

$$\int_{\mathbb{R}^n} |f(x-y)| \, |g(y)|\, dy = \int_{\mathbb{R}^n} |f(x-y)|^{1-\frac{p}{r}} |g(y)|^{1-\frac{q}{r}} |f(x-y)|^{\frac{p}{r}} |g(y)|^{\frac{q}{r}} dy. \tag{4.3.6}$$

We have

$$\begin{aligned} &|f(x-y)|^{1-\frac{p}{r}} \in L^s, \ s = \frac{p}{1-\frac{p}{r}} = \frac{pr}{r-p} \\ &|g(y)|^{1-\frac{q}{r}} \in L^t, \ \ t = \frac{q}{1-\frac{q}{r}} = \frac{rq}{r-q} \\ &|f(x-y)|^{\frac{p}{r}}\ |g(y)|^{\frac{q}{r}} \in L^r \end{aligned} \tag{4.3.7}$$

(observe that $|f|^p \in L^1$, $|g|^q \in L^1$ and from Theorem 4.3.5 $|f|^p \star |g|^q \in L^1$). We have

$$\frac{1}{r}+\frac{1}{s}+\frac{1}{t} = \frac{1}{r} + \Big(\frac{1}{p}-\frac{1}{r}\Big) + \Big(\frac{1}{q}-\frac{1}{r}\Big) = 1. \tag{4.3.8}$$

We apply the *generalized Hölder inequality* (see Corollary 3.7.8) in (4.3.6) and get that for a.e $x \in \mathbb{R}^n$, the function $y \mapsto f(x-y)g(y)$ is in $L^1(\mathbb{R}^n)$. Moreover the following estimate holds:

$$\begin{aligned} &|f\star g|^r \leq \|\ |f|^{1-\frac{p}{r}}\|_{L^s}^r \|\ |g|^{1-\frac{q}{r}}\|_{L^t}^r |f|^p \star |g|^q \\ &= \Big(\int |f|^p dx\Big)^{r\,(\frac{r-p}{rp})}\Big(\int |g|^q dx\Big)^{r(\frac{r-q}{rq})} |f|^p\star|g|^q \\ &= \|f\|_{L^p}^{r(1-\frac{p}{r})}\ \|g\|_{L^q}^{r(1-\frac{q}{r})}\ |f|^p \star |g|^q \end{aligned} \tag{4.3.9}$$

Therefore:

$$\|f\star g\|_{L^r} \leq \Big(\|f\|_{L^p}^{r(1-\frac{p}{r})}\ \|f\|_{L^p}^{p}\ \|g\|_{L^q}^{r(1-\frac{q}{r})}\ \|g\|_{L^q}^{q}\Big)^{\frac{1}{r}} = \|f\|_{L^p}\ \|g\|_{L^q}\,.$$

□

Other properties of convolutions

Definition 4.3.7. *Let $1 \leq p \leq \infty$. We say that a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ if $f\,\chi_K \in L^p(\mathbb{R}^n)$ for every compact $K \subseteq \mathbb{R}^n$.*

Note that if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proposition 4.3.8. *Let $f \in C_c^0(\mathbb{R}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $(f \star g)(x)$ is well defined for every $x \in \mathbb{R}^n$ and moreover $f \star g \in C^0(\mathbb{R}^n)$.*

Proof. Note that for every $x \in \mathbb{R}^n$ the function $y \longmapsto f(x-y)\, g(y)$ is $\mathcal{L}^n$-summable on $\mathbb{R}^n$ and therefore $(f \star g)(x)$ is defined for every $x \in \mathbb{R}^n$.

Let $x_n \to x$ and let K be a fixed compact set in $\mathbb{R}^n$ such that $x_n - \operatorname{supp} f \subseteq K\ \forall n$. Therefore, we have $f(x_n - y) = 0\ \forall n$ and $y \notin K$ ($y \notin K \implies y \notin x_n - \operatorname{supp} f \implies y - x_n \notin -\operatorname{supp} f$ and $y - x \notin -\operatorname{supp} f$).

We deduce that

$$|f(x_n - y) - f(x-y)| \le w_f(|x_n - x|)\, \chi_K(y)$$

where w_f is the uniform modulus of continuity of f:

$$\begin{aligned}
&|(f \star g)(x_n) - (f \star g)(y)| \\
&= \Big| \int_{\mathbb{R}^n} \big(f(x_n - y) - f(x-y)\big)\, g(y)\, dy \Big| \\
&\le \int_{\mathbb{R}^n} w_f(|x_n - x|)\, \chi_K(y)\, |g(y)|\, dy \\
&\le w_f(|x_n - x|)\, \|g\|_{L^1(K)} \to 0 \ \text{ as } n \to +\infty.
\end{aligned}$$

□

Proposition 4.3.9. *(no proof)*

Let $f \in C_c^k(\mathbb{R}^n)$ $k \ge 1$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $f \star g \in C^k(\mathbb{R}^n)$ and

$$D^\alpha (f \star g) = D^\alpha f \star g \quad \forall |\alpha| \le k.$$

In particular, if $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $f \star g \in C^\infty(\mathbb{R}^n)$.

4.4 Application: Solution of the Laplace Equation

For $u \in C^2(\mathbb{R}^n)$ let

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \operatorname{div}(\nabla u).$$

Given $f \in C_c^\infty(\mathbb{R}^n)$ we look for a solution $u \in C^\infty(\mathbb{R}^n)$ of the Laplace equation

$$-\Delta u = f \;\text{ in }\; \mathbb{R}^n.$$

We consider first the problem to determine a function $u \in C_c^\infty(\mathbb{R}^n)$ from Δu. We recall the **Gauss Theorem**.

Theorem 4.4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, $\varphi, \psi \in C^2(\overline{\Omega})$. Then it holds*

$$\int_\Omega \varphi \Delta \psi - \psi \Delta \varphi = \int_{\partial\Omega} \varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\, d\sigma \tag{4.4.1}$$

where $\vec{n}$ is the outward normal vector to $\partial\Omega$ and

$$\frac{\partial \varphi}{\partial n} = \nabla \varphi \cdot \vec{n}.$$

We choose $\psi = u \in C_c^\infty(\mathbb{R}^n)$ and Ω **open set** containing $\operatorname{supp}(u) = \overline{\{x : u(x) \neq 0\}}$. For every $\varphi \in C^2(\mathbb{R}^n)$ it follows from **Theorem 4.4.1**

$$-\int_\Omega u\, \Delta \varphi \, dx = -\int_\Omega \varphi\, \Delta u \, dx.$$

If we succeed in finding $\overline{\varphi}$ that solves in a suitable sense the equation:

$$-\Delta \overline{\varphi} = \delta_0$$

(δ_0 is the Delta-Dirac distribution with $\delta_0(u) = u(0)\ \forall u \in C_c^\infty(\mathbb{R}^n)$), then we expect a representation of the form

$$\begin{aligned} u(n) &= \overline{\varphi} \star (-\Delta u) \\ &= \int_{\mathbb{R}^n} \overline{\varphi}(x-y)(-\Delta u)(y)\, dy\,. \end{aligned}$$

$\boxed{n=1}$ $\quad \overline{\varphi}(x) = -\dfrac{1}{2}\,|x| \quad \forall x \in \mathbb{R}.$

In this case, we have

$$\Delta\overline{\varphi}(x) = \overline{\varphi}''(x) = 0,\ x \neq 0.$$

Moreover,

$$\begin{aligned}
&\overline{\varphi} \star (-\Delta u)(x_0) = -\int_{\infty}^{+\infty} u''(x)\,\overline{\varphi}(x_0 - x)\,dx \\
&= -\lim_{\varepsilon\to 0}\Big[\int_{-\infty}^{x_0-\varepsilon} u''(x)\,\overline{\varphi}(x_0-x)\,dx + \int_{x_0,+\varepsilon}^{+\infty} u''(x)\,\overline{\varphi}(x_0-x)\,dx\Big] \\
&= -\lim_{\varepsilon\to 0}\big[(u'(x)\varphi(x_0-x))\big|_{-\infty}^{x_0-\varepsilon} + (u'(x)\varphi(x_0-x))\big|_{x_0+\varepsilon}^{+\infty}\big] \\
&-\lim_{\varepsilon\to 0}\Big[\int_{-\infty}^{x_0-\varepsilon} u'(x)\,\overline{\varphi}'(x_0-x)\,du + \int_{x_0+\varepsilon}^{+\infty} u'(x)\,\overline{\varphi}'(x_0-x)\,dx\Big] \\
&= -\lim_{\varepsilon\to 0}\Big[u(x)\,\overline{\varphi}'(x_0-x)\Big|_{-\infty}^{x_0-\varepsilon} + u(x)\,\overline{\varphi}'(x_0-x)\Big|_{x_0+\varepsilon}^{+\infty} \\
&\quad -\lim_{\varepsilon\to 0}(\int_{-\infty}^{x_0-\varepsilon} u(x)\overline{\varphi}''(x_0-x)\,dx + \int_{x_0+\varepsilon}^{+\infty} u(x)\overline{\varphi}''(x_0-x)\,dx) \\
&= -\lim_{\varepsilon\to 0}\big[u(x_0-\varepsilon)\,\varphi'(\varepsilon) - u(x_0+\varepsilon)\,\varphi'(-\varepsilon)\big] \\
&= u(x_0)\ \lim_{\varepsilon\to 0}\,[\varphi'(-\varepsilon) - \varphi'(\varepsilon)] \\
&= u(x_0).
\end{aligned} \tag{4.4.2}$$

$\boxed{n=2}$ $\quad \overline{\varphi}(x) = \dfrac{1}{2\pi}\ \log\Big(\dfrac{1}{|x|}\Big).$

$\boxed{n=3}$

$$\overline{\varphi}(x) = \frac{1}{(n-2)\,\alpha_{n-1}\,|n|^{n-2}} := C_n\,|n|^{2-n}$$

$\alpha_{n-1} = \mathrm{vol}(S^{n-1})$. $\overline{\varphi}$ is called the fundamental solution of the Laplace equation.

Theorem 4.4.2. *Let* $f \in C_c^\infty(\mathbb{R}^n)$*. Then* $u = \overline{\varphi} \star f \in C^\infty(\mathbb{R}^n)$ *and solves* $-\Delta u = f$*.*

Proof. See Proof of Theorem 4.2.3 in Struwe's Notes. $\square$

Chapter 5

Differentiation of Measures

5.1 Differentiability of Lebesgue Integrals

We start with the following example.

Example 5.1.1.

i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, $x_0 \in \mathbb{R}$ and let $F(x) := \int_{x_0}^{x} f(t)\,dt$ be a primitive of f. Then it holds $F \in C^1(\mathbb{R})$ and

$$F'(x) = \lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} f(t)\,dt = f(x).$$

ii) In general, for $f \in C(\mathbb{R}^n)$, it holds

$$f(x) = \lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)\,dy. \tag{5.1.1}$$

Question: Do similar formulas hold when $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ or if we replace $\mathcal{L}^n$ with a general Radon measure μ?

In this section we suppose $\mu = \mathcal{L}^n$.

Notation: We denote the average of f over the set E by

$$\fint_E f\,d\mu := \frac{1}{\mu(E)} \int_E f\,d\mu$$

provided $0 < \mu(E) < +\infty$ and the integral on the right-hand side is defined.

Theorem 5.1.2. *(Lebesgue Differentiation Theorem)*

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for μ- a.e. $x \in \mathbb{R}^n$

$$\lim_{r\to 0} \fint_{B(x,r)} f \, d\mu = f(x). \tag{5.1.2}$$

Remark 5.1.3. In Theorem 5.1.2 we can replace balls with cubes. To prove Theorem 5.1.2 we need to associate to $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the function

$$f^*(x) = \sup_{r>0} \fint_{B(x,r)} |f| \, d\mu. \tag{5.1.3}$$

Definition 5.1.4. *f^* is called Hardy-Littlewood Maximal function.*

Remark 5.1.5. We showed that if $f \in L^1(\mathbb{R}^n)$ then $\forall \varepsilon > 0$ $\exists \delta > 0$: $\forall A \subseteq \mathbb{R}^n$, $\mu(A) < \delta$, A μ-measurable and we have $\int_A |f| \, d\mu < \varepsilon$.

It follows that for $r > 0$ fixed the mapping

$$x \longmapsto \fint_{B(x,r)} |f(y)| \, d\mu$$

is continuous. $\square$

Thus f^* is lower semicontinuous and thus measurable. However, $f^* \in L^1(\mathbb{R}^n)$ only is the trivial case $f = 0$. Indeed, if f is not zero up to a set of measure zero, then $\exists R$, $C > 0$ such that $f^*(x) \geq c|x|^{-n}$ $\forall\, |x| \geq R$. (Exercise).

We next show that it is "almost in L^1".

Proposition 5.1.6. *Let $f \in L^1(\mathbb{R}^n)$. It holds for all $a > 0$*

$$\mu\{x : f^*(x) > a\} \leq \frac{5^n}{a} \|f\|_{L^1}. \tag{5.1.4}$$

Proof of Theorem 5.1.2. Suppose $f \in L^1(\mathbb{R}^n)$ (otherwise we consider for a given $x \in \mathbb{R}^n$ the function $g(x) = f(x)\,\chi_{B_1(x)} \in L^1(\mathbb{R})$). Then $\exists f_k \in C_c^0(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ as $k \to \infty$. We have

$$\begin{aligned}
&\left|\limsup_{r\to 0} ⨍_{B(x,r)}[f(y) - f(x)]\,dy\right| \leq \limsup_{r\to 0} ⨍_{B(x,r)} |f(y) - f(x)|\,dy \\
&\leq \limsup_{r\to 0} ⨍_{B(x,r)} |f_k(y) - f(y)|\,dy + \limsup_{r\to 0} ⨍_{B(x,r)} |f_k(y) - f_k(x)|\,dy + |f_k(x) - f(x)| \\
&\leq \underbrace{(f - f_k)^*(x)}_{1} + \underbrace{\limsup_{r\to 0} ⨍_{B(x,r)} |f_k(y) - f_k(x)|\,dy}_{2} + \underbrace{|f_k(x) - f(x)|}_{3}.
\end{aligned} \tag{5.1.5}$$

- Since $\{f_k\}$ are continuous $\forall k \geq 1$, we have 2 $= 0$.
- For $\varepsilon > 0$ we set

$$A_\varepsilon := \Big\{x : \limsup_{r\to 0} ⨍_{B(x,r)} |f(y) - f(x)|\,dy > 2\varepsilon\Big\}. \tag{5.1.6}$$

We have

$$A_\varepsilon \subseteq \{x : (f - f_k)^*(x) > \varepsilon\} \cup \{x : (f_k(x) - f_k(x)| > \varepsilon\}. \tag{5.1.7}$$

This inclusion holds for all $\varepsilon > 0$ and for all $k \geq 1$. Thus from Proposition 5.1.6 and Tchebychev Inequality, we get

$$\begin{aligned}
\mu(A_\varepsilon) &\leq \mu\{x : (f - f_k)^*(x) > \varepsilon\} + \mu\{x : |f(x) - f_k(x)| > \varepsilon\} \\
&\leq \frac{C}{\varepsilon} \|f_k - f\|_{L^1} \to \quad \text{as} \quad k \to +\infty.
\end{aligned} \tag{5.1.8}$$

We have

$$A := \Big\{x : \limsup_{r\to 0} ⨍_{B(x,r)} f(y) - f(x)|\,dy > 0\Big\} = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}. \tag{5.1.9}$$

Thus $\mu(A) = 0$. It follows that for μ-a.e $x \in \mathbb{R}^n$ we have

$$|f(x) - \lim_{r\to 0} \fint_{B(x,r)} f(y)\, dy| \le \lim_{r\to 0} \fint_{B(x,r)} |f(x) - f(y)|\, dy = 0$$

for μ a.e. x (namely $\forall x \notin A$). $\square$

Actually, we have proved the following strong property:

Corollary 5.1.7. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for μ-a.e. $x \in \mathbb{R}^n$, we have*

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = 0\,. \tag{5.1.10}$$

Definition 5.1.8. *A point $x \in \mathbb{R}^n$ which satisfies (5.1.10) is called a Lebesgue point of f.*

In order to prove Proposition 5.1.6, we need the following result.

Theorem 5.1.9. *(Vitali's covering theorem)*

Let $\mathcal{F}$ be a family of nondegenerate closed balls $B = B(x,r) \subseteq \mathbb{R}^n$ with $d_0 = \sup\{\operatorname{diam} B, B \in \mathcal{F}\} < \infty$. For each B, we consider $\widehat{B} = B(x, 5r)$ the concentric ball with radius 5 times the radius of B. Then there is a countable family $\mathcal{G} \subseteq \mathcal{F}$ of mutually disjoint balls such that

$$\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} \widehat{B}.$$

Proof of Theorem 5.1.9.

1. We write $d_0 = \sup\{\operatorname{diam} B \,|\, B \in \mathcal{F}\}$. Set $\mathcal{F}_j = \{B \in \mathcal{F} \colon \frac{d_0}{2^j} < \operatorname{diam} B \le \frac{d_0}{2^{j-1}}\}$. We define inductively $\mathcal{G}_j \subseteq \mathcal{F}_j$ as follows:

   a) $\mathcal{G}_1$ is any maximal disjoint collection of balls in $\mathcal{F}_1$.

b) Assuming $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_{k-1}$ have been selected, we choose $\mathcal{G}_k$ to be any maximal disjoint subcollection of

$$\Big\{B \in \mathcal{F}_k \,|\, B \cap B' = \emptyset \quad \forall B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j\Big\}.$$

Finally define $\mathcal{G} \equiv \bigcup_{j=1}^{\infty} \mathcal{G}_j$. Clearly, $\mathcal{G}$ is a collection of disjoint balls and $\mathcal{G} \subseteq \mathcal{F}$.

2. **Claim.** For each $B \in \mathcal{F}$, there exists a ball $B' \in \mathcal{G}$ so that $B \cap B' \neq \emptyset$ and $B \subseteq \widehat{B'}$.

Proof of the Claim. Fix $B \in \mathcal{F}$. Then there exists an index j such that $B \in \mathcal{F}_j$. By the maximality of $\mathcal{G}_j$, there exists a ball $B' \in \bigcup_{k=1}^{j} \mathcal{G}_k$, with $B \cap B' \neq \emptyset$. But $\operatorname{diam} B' > \frac{d_0}{2^j}$ and $\operatorname{diam} B \leq \frac{d_0}{2^{j-1}} \implies \operatorname{diam} B \leq 2 \operatorname{diam} B'$. Therefore $B \subseteq \widehat{B'}$. $\square$

$(B = B(x_1, r_1),\ B' = B(x_2, r_2)$: $r_1 \leq 2r_2$. If $y \in B \implies |x_1 - y| < r_1 \implies |y - x_2| \leq |y - x_1| + |x_1 - x_2| \leq r_1 + r_1 + r_2 \leq 2r_1 + r_2 \leq 5r_2)$.
$\square$

Proof of Proposition 5.1.6. Let $a > 0$. Set $A = \{x : f^*(x) > a\}$. For $x \in A$, choose $r = r(x) > 0$ with $\int_{B(x,r)} |f(y)|\,dy \geq a\mathcal{L}^n\big(B(x,r)\big)$. We set $w_n := \mathcal{L}^n(B(0,1))$. It follows that

$$a\,w_n\,r^n = a\,\mathcal{L}^n\big(B(0,r)\big) \leq \int_{B(x,r)} |f|\,dy \leq \|f\|_{L^1}.$$

We obtain in particular the uniform condition

$$r(x) \leq r_0 = \Big(\frac{\|f\|_{L_1}}{a\,w_n}\Big)^{\frac{1}{n}}\ \forall x \in A.$$

We consider the family

$$\mathcal{F} = \big\{B\big(x, r(x)\big);\ x \in A\big\}.$$

It holds $A \subseteq \bigcup_{B\in\mathcal{F}} B$. We choose a sub-family of disjoint balls according to Theorem 5.1.9 with

$$A \subseteq \bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} \widehat{B}.$$

We have

$$\begin{aligned}\mathcal{L}^n(A) &\leq \sum_{B\in\mathcal{G}} \mathcal{L}^n(\widehat{B}) = 5^n \sum_{B\in\mathcal{G}} \mathcal{L}^n(B)\\ &\leq \frac{5^n}{a} \sum_{B\in\mathcal{G}} \int_B |f|\,dy = \frac{5^n}{a} \int_{\bigcup\limits_{B\in\mathcal{G}} B} |f|\,dy \leq \frac{5^n}{a}\,\|f\|_{L^1}\end{aligned}$$

and we can conclude. □

5.2 Differentiation of Radon measures

We would like to extend the result of Theorem 5.1.2 to other measures on $\mathbb{R}^n$. We first must find a replacement for Vitali's Theorem (Theorem 5.1.9). This theorem relies heavily on the fact that expanding a ball concentrically by a factor only enlarge its Lebesgue measure proportionally whereas no such relation may hold for general measures. In order to bypass this difficulty, we shall present a covering lemma that is purely geometric in nature, that is, which makes no mention of measure.

Theorem 5.2.1. *(Besicovitch's Covering Theorem)*

There exists a constant N_n depending only on n with the following property: If $\mathcal{F}$ is any collection of nondegenerate **closed** *balls in $\mathbb{R}^n$ with*

$$\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < +\infty$$

and if A is the set of centers of balls in $\mathcal{F}$, then there exists $\mathcal{G}_1, \ldots, \mathcal{G}_{N_n} \in \mathcal{F}$ such that $\mathcal{G}_i$ is the countable collection of disjoint balls in $\mathcal{F}$ and

$$A \subseteq \bigcup_{i=1}^{N_n} \Big(\bigcup_{B\in\mathcal{G}_i} B \Big).$$

Note that the number of overlapping of the balls belonging to different $\mathcal{G}_j$ is controlled by N_n.

Let μ and ν be Radon measures on $\mathbb{R}^n$.

Definition 5.2.2. *For each point $x \in \mathbb{R}^n$, define*

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup_{r\to 0} \dfrac{\nu\big(\overline{B}(x,r)\big)}{\mu\big(\overline{B}(x,r)\big)} & \textit{if } \mu\big(\overline{B(x,r)}\big) > 0 \quad \forall r > 0 \\ +\infty, & \textit{if } \mu\big(B(x,r)\big) = 0 \textit{ for some } r > 0 \end{cases}$$

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf_{r\to 0} \dfrac{\nu\big(\overline{B}(x,r)\big)}{\mu\big(\overline{B}(x,r)\big)} & \textit{if } \mu\big(\overline{B(x,r)}\big) > 0 \quad \forall r > 0 \\ +\infty, & \textit{if } \mu\big(\overline{B(x,r)}\big) = 0 \textit{ for some } r > 0. \end{cases}$$

Definition 5.2.3. *If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < +\infty$, we say ν is differentiable with respect to μ at x and write*

$$D_\mu \nu(x) \equiv \overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x).$$

$D_\mu \nu$ is the derivative of ν with respect to μ. We also call $D_\mu \nu$ the density of ν with respect to μ.

Our goals are to study a) when $D_\mu \nu$ exists and b) when ν can be recovered by integrating $D_\mu \nu$.

Theorem 5.2.4. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Then $D_\mu \nu$ exists and is finite μ-a.e. Furthermore, $D_\mu \nu$ is μ-measurable.*

Definition 5.2.5. *The measure ν is absolutely continuous with respect to μ, written*

$$\nu \ll \mu,$$

provided $\mu(A) = 0$ implies $\nu(A) = 0 \ \ \forall A \subseteq \mathbb{R}^n$.

Definition 5.2.6. *The measure ν and μ are mutually singular, written*

$$\nu \perp \mu$$

if there exists a Borel subset $B \subseteq \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus B) = \nu(B) = 0$.

Theorem 5.2.7. *(Differentiation theorem for Radon measure) Let μ, ν be Radon measures on $\mathbb{R}^n$ with $\nu \ll \mu$. Then*

$$\nu(A) = \int_A D_\mu \nu \, d\mu$$

for all μ-measurable sets $A \subseteq \mathbb{R}^n$.

Remark 5.2.8. This is a version of the Radon-Nikodym Theorem. Observe, we prove not only that ν has a density with respect to μ, but also that this density $D_\mu \nu$ can be computed by "differentiating" ν with respect to μ. These assertions comprise in effect the fundamental Theorem of Calculus for Radon measures on $\mathbb{R}^n$.

$\square$

Next, we generalize Theorem 5.1.2 to general Radon measures by applying Theorem 5.2.7.

Theorem 5.2.9. *Let μ be a Radon measure on $\mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$. Then for μ-a.e. $x \in \mathbb{R}^n$ it holds*

$$f(x) = \lim_{r \to 0} \fint_{B(x,r)} f \, d\mu.$$

Proof of Theorem 5.2.9. We suppose without restriction that $f \in L^1(\mathbb{R}^n)$. We write

$$f = f^+ - f^-.$$

For $B \subseteq \mathbb{R}^n$ μ-measurable, we set

$$\nu^{\pm}(B) = \int_B f^{\pm} d\mu$$

and for a general $A \subseteq \mathbb{R}^n$, we set

$$\nu^{\pm}(A) := \inf\{\nu^{\pm}(B);\ A \subseteq B,\ B\ \mu\text{-measurable}\}.$$

By Theorem 3.5.3 (Satz 3.3.1 in Struwe's Notes) and Theorem 1.4, $\nu^{\pm}$ define Radon measures with $\nu^{\pm} \ll \mu$. From Theorem 5.2.4 it follows

$$\nu^{\pm}(A) := \int_A D_\mu \nu^{\pm}\, d\mu = \int_A f^{\pm}\, d\mu \quad \forall A\ \mu\text{-measurable.}$$

This implies that

$$D_\mu \nu^{\pm} = f^{\pm}\ \mu\text{-a.e.}.$$

Consequently,

$$\begin{aligned}
\lim_{r\to 0} ⨍_{B(x,r)} f(y)\, dy &= \lim_{r\to 0} \frac{\nu^+\big(\overline{B}(x,r)\big) - \nu^-\big(\overline{B}(x,r)\big)}{\mu\big(B(x,r)\big)} \\
&= D_\mu \nu^+(x) - D_\mu \nu^-(x) \\
&= f^+(x) - f^-(x) \\
&= f(x) \ \text{ for } \mu\text{-a.e.}.
\end{aligned}$$

□

Corollary 5.2.10. *Let μ be a Radon measure on $\mathbb{R}^n$, $1 \le p < \infty$. Then*

$$\lim_{r\to 0} ⨍_{B(x,r)} |f(x) - f(y)|^p\, d\mu = 0 \tag{5.2.11}$$

for μ-a.e. x

Definition 5.2.11. *A point x for which (5.2.11) holds is called a Lebesgue point of f with respect to μ.*

Proof of Corollary 5.2.10. Let $\{r_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathbb{R}$. By Theorem 5.2.9 we have:

$$\lim_{r\to 0} \int_{B(x,r)} |f - r_i|^p d\mu = |f(x) - r_i|^p$$

for μ-a.e. x and $i = 1, 2, \ldots$. Thus there exists a set $A \subseteq \mathbb{R}^n$ such that $\mu(A) = 0$ and $x \in \mathbb{R}^n \setminus A$ implies

$$\lim_{r\to 0} \int_{B(x,r)} |f - r_i|^p \, d\mu = |f(x) - r_i|^p \quad \forall i.$$

Fix $x \in \mathbb{R}^n \setminus A$ and $\varepsilon > 0$. Let us choose r_i such that $|f(x) - r_i|^p < \frac{\varepsilon}{2^p}$. Then

$$\begin{aligned}\limsup_{r\to 0} \int_{B(x,r)} |f - f(x)|^p d\mu &\le 2^{p-1}\Big[\limsup_{r\to 0} \Big(\int_{B(x,r)} |f - r_i|^p d\mu\Big) \\ &\qquad + \limsup_{r\to 0} \int_{B(x,r)} |f(x) - r_i|^p d\mu\Big] \\ &= 2^{p-1}\big[|f(x) - r_i|^p + |f(x) - r_i|^p\big] \\ &= 2^p |f(x) - r_i|^p < \varepsilon.\end{aligned}$$

□

5.3 Differentiation of absolutely continuous functions

Let $F \colon \mathbb{R} \to \mathbb{R}$ be non-decreasing and continuous from the left

$$F(x_0) = \lim_{x \uparrow x_0} F(x) \quad \forall x_0 \in \mathbb{R}.$$

For $a, b \in \mathbb{R}$ we set

$$\lambda_F(a, b) = \begin{cases} F(b) - F(a) & \text{if } a \le b \\ 0 & \text{otherwise}. \end{cases}$$

Consider the measure Λ_F induce by λ_F

$$\Lambda_F(A) = \inf\Big[\sum_{k=1}^{\infty} \lambda_F(a_k, b_k);\ A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k]\Big].$$

Λ_F is the Lebesgue-Stiltjes measure associated to F. It is a Borel regular measure. Moreover,

$$\Lambda_F\big((a,b)\big) = F(b) - F(a) = \lambda_F[a,b]$$

if $a < b$ (see Section 1.6).

Definition 5.3.1. *F is called* **absolutely continuous** *if it holds:*

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall (a_k, b_k)_{k\in\mathbb{N}}, a_k \le b_k \le a_{k+1},$$
$$\sum_{k=1}^{\infty} |b_k - a_k| < \delta \Longrightarrow \sum_{k=1}^{\infty} |F(b_k) - F(a_k)| < \varepsilon.$$

Remark 5.3.2.

i) F absolutely continuous $\Longrightarrow$ F uniformly continuous.

ii) F Lipschitz continuous $\Longrightarrow$ F absolutely continuous.

iii) A monotone, continuous function is not necessary absolutely continuous (see for instance Cantor-Lebesgue Function)

.

Theorem 5.3.3. *The following two conditions are equivalent:*

i) *Λ_F is absolutely continuous with respect to $\mathcal{L}^1$.*

ii) *F is absolutely continuous in $\mathbb{R}$.*

Proof of Theorem 5.3.3.

i) $\Longrightarrow$ ii): Since Λ_F is absolutely continuous with respect to $\mathcal{L}^1$, we have the representation

$$\Lambda_F(A) = \int_A f\, d\mathcal{L}^1$$

for every $A - \mathcal{L}^1$-measurable with $f = D_{\mathcal{L}^1} \Lambda_F \in L^1_{\text{loc}}(\mathbb{R}, \mathcal{L}^1)$. It follows that for $\{(a_k, b_k)\} k \in \mathbb{N}$ with $a_k \le b_k \le a_{k+1}$, we have

$$\sum_{k=1}^{\infty} |F(b_k) - F(a_k)| = \Lambda_F\Big(\bigcup_{k=1}^{\infty} [a_k, b_k)\Big) \to 0 \quad \text{if}$$

$$\mathcal{L}^1\big(\bigcup [a_k, b_k)\big) = \sum_{k=1}^{\infty} |b_k - a_k| \to 0.$$

ii) $\Longrightarrow$ i): Let $A \subseteq \mathbb{R}^n$ be a Lebesgue null set ($\mathcal{L}^1(A) = 0$). We recall that

$$\mathcal{L}^1(A) = \inf\{\mathcal{L}^1(F) : \ F \supseteq A \text{ open}\}.$$

Thus for $\delta > 0$ let $B \subseteq A$ open such that $\mathcal{L}^1(B) < \delta$. B can be written $B = \bigcup_{k=1}^{\infty} I_k$, $I_k = [a_k, b_k)$ mutually disjoint. We may suppose that $a_k < b_k \le a_{k+1}$ $\forall k \in \mathbb{N}$. Therefore

$$\sum_{k=1}^{\infty} |b_k - a_k| = \sum_{k=1}^{\infty} \mathcal{L}^1(I_k) = \mathcal{L}^1(B) < \delta.$$

Since F is absolutely continuous, we have

$$\begin{aligned} \Lambda_F(A) \le \Lambda_F(B) &= \sum \Lambda_F(I_k) \\ &= \sum_{k=1}^{\infty} |F(b_k) - F(a_k)| < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\Lambda_F(A) = 0$, namely $\Lambda_F \ll \mathcal{L}^1$.

From Theorem 5.2.7 and 5.3.3 it follows:

Theorem 5.3.4. *(Rademacher)*

Let F be nondecreasing and absolutely continuous. Then F is differentiable $\mathcal{L}^1$-a.e. and

$$0 \le F'(x) = D_{\mathcal{L}^1}\Lambda_F(x) := f(x) \in L^1(\mathbb{R})$$

and for all x_0, $x \in \mathbb{R}$ it holds:

$$F(x) - F(x_0) = \Lambda_F\big([x_0, x)\big) = \int_{x_0}^{x} f(y)\, dy.$$

Proof. We estimate

$$\begin{aligned}\limsup_{r\to 0}\left|\frac{F(x+r)-F(x)}{r}-f(x)\right| &= \limsup_{r\to 0}\left|\frac{1}{r}\int_x^{x+r} f(y)-f(x)\,dy\right| \\ &\le \limsup_{r\to 0}\frac{1}{r}\int_x^{x+r}|f(y)-f(x)|\,dy \\ &\le 2\limsup_{r\to 0}\frac{1}{2r}\int_{x-r}^{x+r}|f(x)-f(x)|\,dy = 0\end{aligned}$$

for $\mathcal{L}^1$-a.e. x. ☐

Remark 5.3.5. Conversely, if $f \in L^1(\mathbb{R})$, $f \ge 0$, then the function

$$F(x) = \int_{x_0}^{x} f(y)\,dy \quad (x_0 \text{ is given})$$

is absolutely continuous and non-decreasing. Thus by Theorem 5.3.4 it is differentiable $\mathcal{L}^1$-a.e. with $F' = f$.

The Cantor Function

Consider the standard triadic Cantor set $\mathcal{C} \in [0,1]$. We recall that $\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$, where C_k is the disjoint union of 2^k closed intervals.

If $D_k = [0,1] \setminus C_k$, then D_k consists of $2^k - 1$ intervals I_j^k (ordered from left to right) removed in the k-step of the construction of the Cantor set.

Let f_k be the continuous function on $[0,1]$ which satisfies

1. $f_k(0) = 0$.
2. $f_k(1) = 1$.
3. $f_k(x) = j2^{-k}$ on I_j^k, $j = 1, \dots 2^{k-1}$.
4. f_k is linear on each interval of C_k.

For instance: $\boxed{k=1}$, $f_1(0) = 0$, $f_1(1) = 1$, $f_1(x) = \frac{1}{2}$ if $\frac{1}{3} \le x \le \frac{2}{3}$ and f_1 is linear on C_1.

$\boxed{k = 2}$:

$$f_2(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{4} & \frac{1}{9} \leq x \leq \frac{2}{9} \\ \frac{1}{2} & \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{3}{4} & \frac{7}{9} \leq x \leq \frac{8}{9} \\ 1 & x = 1\,. \end{cases}$$

By construction each f_k is monotone increasing, $f_{k+1} = f_k$ on I_j^k, $j = 1, 2, \ldots,\ 2^k - 1$ and $|f_k - f_{k+1}| \leq 2^{-k-1}$. Hence $\sum_k (f_k - f_{k+1})$ converges uniformly on $[0, 1]$.

Let $f = \lim_{k\to\infty} f_k$. Then f is non-decreasing, continuous, $f(0) = 0$, $f(1) = 1$, f is constant $\mathcal{L}^1$-a.e., in particular $f'(x) \geq 0$, $\mathcal{L}^1$-a.e. But f is not absolutely continuous.