Mathematical Tripos Part IB: Lent 2010

Numerical Analysis – Lecture 5[1]

We commence our construction of *Gaussian quadrature* by choosing pairwise-distinct nodes $c_1, c_2, \ldots, c_\nu \in [a,b]$ and define the *interpolatory weights*

$$b_k := \int_a^b w(x) \prod_{\substack{j=1\\ j\neq k}}^{\nu} \frac{x-c_j}{c_k-c_j}\,\mathrm{d}x, \qquad k=1,2,\ldots,\nu.$$

Theorem The quadrature formula with the above choice is exact for all $f \in \mathbb{P}_{\nu-1}[x]$. Moreover, if $c_1, c_2, \ldots, c_\nu$ are the zeros of p_ν then it is exact for all $f \in \mathbb{P}_{2\nu-1}[x]$.

Proof. Every $f \in \mathbb{P}_{\nu-1}[x]$ is its own interpolating polynomial, hence by Lagrange's formula

$$f(x) = \sum_{k=1}^{\nu} f(c_k) \prod_{\substack{j=1\\ j\neq k}}^{\nu} \frac{x-c_j}{c_k-c_j}. \tag{2.7}$$

The quadrature is exact for all $f \in \mathbb{P}_{\nu-1}[x]$ if $\int_a^b w(x)f(x)\,\mathrm{d}x = \sum_{k=1}^{\nu} b_k f(c_k)$, and this, in tandem with the interpolating-polynomial representation, yields the stipulated form of $b_1, \ldots, b_\nu$.

Let $c_1, \ldots, c_\nu$ be the zeros of p_ν. Given any $f \in \mathbb{P}_{2\nu-1}[x]$, we can represent it uniquely as $f = qp_\nu + r$, where $q, r \in \mathbb{P}_{\nu-1}[x]$. Thus, by orthogonality,

$$\begin{aligned}\int_a^b w(x)f(x)\,\mathrm{d}x &= \int_a^b w(x)[q(x)p_\nu(x)+r(x)]\,\mathrm{d}x = \langle q, p_\nu\rangle + \int_a^b w(x)r(x)\,\mathrm{d}x\\ &= \int_a^b w(x)r(x)\,\mathrm{d}x.\end{aligned}$$

On the other hand, the choice of quadrature knots gives

$$\sum_{k=1}^{\nu} b_k f(c_k) = \sum_{k=1}^{\nu} b_k[q(c_k)p_\nu(c_k)+r(c_k)] = \sum_{k=1}^{\nu} b_k r(c_k).$$

Hence the integral and its approximation coincide, because $r \in \mathbb{P}_{\nu-1}[x]$ and the quadrature is exact for all polynomials in $\mathbb{P}_{\nu-1}[x]$. $\square$

Example Let $[a,b] = [-1,1]$, $w(x) \equiv 1$. Then the underlying orthogonal polynomials are the *Legendre polynomials:* $P_0 \equiv 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$ (it is customary to use this, non-monic, normalisation). The nodes of Gaussian quadrature are

$n=1$: $c_1 = 0$;

$n=2$: $c_1 = -\frac{\sqrt{3}}{3}$, $c_2 = \frac{\sqrt{3}}{3}$;

$n=3$: $c_1 = -\frac{\sqrt{15}}{5}$, $c_2 = 0$, $c_3 = \frac{\sqrt{15}}{5}$;

$n=4$: $c_1 = -\sqrt{\frac{3}{7}+\frac{2}{35}\sqrt{30}}$, $c_2 = -\sqrt{\frac{3}{7}-\frac{2}{35}\sqrt{30}}$, $c_3 = \sqrt{\frac{3}{7}-\frac{2}{35}\sqrt{30}}$, $c_4 = \sqrt{\frac{3}{7}+\frac{2}{35}\sqrt{30}}$.

3 The Peano kernel theorem

3.1 The theorem

Our point of departure is the *Taylor formula with an integral remainder term,*

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^k}{k!}f^{(k)}(a)+\frac{1}{k!}\int_a^x (x-\theta)^k f^{(k+1)}(\theta)\,\mathrm{d}\theta, \tag{3.1}$$

[1]Corrections and suggestions to these notes should be emailed to `A.Iserles@damtp.cam.ac.uk`. All handouts are available on the WWW at the URL `http://www.damtp.cam.ac.uk/user/na/PartIB/Handouts.html`.

which can be verified by integration by parts for functions $f \in \mathrm{C}^{k+1}[a,b]$, $a < b$. Suppose that we are given an approximation (e.g. to a function, a derivative at a given point, an integral etc.) which is *exact* for all $f \in \mathbb{P}_k[x]$. The Taylor formula produces an expression for the error that depends on $f^{(k+1)}$. This is the basis for the *Peano kernel theorem*.

Formally, let $L(f)$ be the approximation error. Thus, L maps $\mathrm{C}^{k+1}[a,b]$, say, to $\mathbb{R}$. We assume that it is *linear*, i.e. $L(\alpha f+\beta g) = \alpha L(f)+\beta L(g)\ \forall \alpha,\beta \in \mathbb{R}$, and that $L(f) = 0$ for all $f \in \mathbb{P}_k[x]$. In general, a linear mapping from a function space (e.g. $\mathrm{C}^{k+1}[a,b]$) to $\mathbb{R}$ is called a *linear functional.* The formula (3.1) implies

$$L(f) = \frac{1}{k!} L\left\{ \int_a^x (x-\theta)^k f^{(k+1)}(\theta)\,\mathrm{d}\theta \right\}, \qquad a \le x \le b.$$

To make the range of integration independent of x, we introduce the notation

$$(x-\theta)_+^k := \begin{cases} (x-\theta)^k, & x \ge \theta, \\ 0, & x \le \theta, \end{cases} \qquad \text{whence} \qquad L(f) = \frac{1}{k!} L\left\{ \int_a^b (x-\theta)_+^k f^{(k+1)}(\theta)\,\mathrm{d}\theta \right\}.$$

Let $K(\theta) := L[(x-\theta)_+^k]$ for $x \in [a,b]$. *[Note: K is independent of f.]* The function K is called the *Peano kernel* of L. ***Suppose that it is allowed to exchange the order of action of $\int$ and L.*** Because of the linearity of L, we then have

$$L(f) = \frac{1}{k!} \int_a^b K(\theta) f^{(k+1)}(\theta)\,\mathrm{d}\theta. \tag{3.2}$$

The Peano kernel theorem Let L be a linear functional such that $L(f) = 0$ for all $f \in \mathbb{P}_k[x]$. Provided that $f \in \mathrm{C}^{k+1}[a,b]$ and the above exchange of L with the integration sign is valid, the formula (3.2) is true. $\square$

3.2 An example and few useful formulae

We approximate a derivative by a linear combination of function values, $f'(0) \approx -\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2)$. Therefore, $L(f) := f'(0) - [-\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2)]$ and it is easy to check that $L(f) = 0$ for $f \in \mathbb{P}_2[x]$. (Verify by trying $f(x) = 1, x, x^2$ and using linearity of L.) Thus, for $f \in \mathrm{C}^3[0,2]$ we have

$$L(f) = \tfrac{1}{2} \int_0^2 K(\theta) f'''(\theta)\,\mathrm{d}\theta.$$

To evaluate the Peano kernel K, we fix θ. Letting $g(x) := (x-\theta)_+^2$, we have

$$\begin{aligned} K(\theta) = L(g) &= g'(0) - \left[-\tfrac{3}{2}g(0) + 2g(1) - \tfrac{1}{2}g(2)\right] \\ &= 2(0-\theta)_+ - \left[-\tfrac{3}{2}(0-\theta)_+^2 + 2(1-\theta)_+^2 - \tfrac{1}{2}(2-\theta)_+^2\right] \\ &= \begin{cases} -2\theta + \frac{3}{2}\theta^2 + (2\theta - \frac{3}{2}\theta^2) \equiv 0, & \theta \le 0, \\ -2(1-\theta)^2 + \frac{1}{2}(2-\theta)^2 = 2\theta - \frac{3}{2}\theta^2, & 0 \le \theta \le 1, \\ \frac{1}{2}(2-\theta)^2, & 1 \le \theta \le 2, \\ 0, & \theta \ge 2. \end{cases} \end{aligned}$$

[Note: It is obvious that $K(\theta) = 0$ for $\theta \notin [0,2]$, since then L acts on a quadratic polynomial.] This gives the form of the Peano kernel for our example.