Highly-efficient Incomplete Large-scale Multi-view Clustering with Consensus Bipartite Graph

Siwei Wang[1], Xinwang Liu[1*], Li Liu[2], Wenxuan Tu[1], Xinzhong Zhu[4], Jiyuan Liu[1], Sihang Zhou[3] and En Zhu[1]
[1]College of Computer, National University of Defense Technology,
[2] College of System Engineering, National University of Defense Technology,
[3]College of Intelligence Science and Technology, National University of Defense Technology,
[4]Zhejiang Normal University.
{wangsiwei13,xinwangliu,twx,liujiyuan13,enzhu}@nudt.edu.cn,Li.Liu@oulu.fi

Abstract

Multi-view clustering has received increasing attention due to its effectiveness in fusing complementary information without manual annotations. Most previous methods hold the assumption that each instance appears in all views. However, it is not uncommon to see that some views may contain some missing instances, which gives rise to incomplete multi-view clustering (IMVC) in literature. Although many IMVC methods have been recently proposed, they always encounter high complexity and expensive time expenditure from being applied into large-scale tasks. In this paper, we present a flexible highly-efficient incomplete large-scale multi-view clustering approach based on bipartite graph framework to solve these issues. Specifically, we formalize multi-view anchor learning and incomplete bipartite graph into a unified framework, which coordinates with each other to boost cluster performance. By introducing the flexible bipartite graph framework to handle IMVC for the first practice, our proposed method enjoys linear complexity respecting to instance numbers, which is more applicable for large-scale IMVC tasks. Comprehensive experimental results on various benchmark datasets demonstrate the effectiveness and efficiency of our proposed algorithm against other IMVC competitors. The code is available at [1].

1. Introduction

Multi-view clustering (MVC) comprehensively integrates diverse source information to divide data samples with similar structures or patterns into the same cluster without stressful label annotations [4, 10, 12, 14, 17, 19, 20, 24, 30, 44, 46, 48]. In recent years, numerous multi-view clustering algorithms have been proposed in computer vision and machine learning communities, which can basically be categorized into the following four categories: co-training style, multiple kernel clustering (MKC), non-negative matrix factorization (NMF) and graph clustering. The co-training strategy for MVC proposes to alternately combine multiple clustering results which provide predictions for the unlabeled data of other views [8]. By this way, besides extracting the specific information from the corresponding view, the clustering labels can reach an agreement on various views. By following the multiple kernel learning framework, MKC methods unitize kernel matrix from a group of pre-defined kernel matrices in Hilbert space [4, 17, 19, 21, 33, 42, 43]. Besides, the NMF applies matrix factorization into multi-view clustering with unified latent space and graph clustering approaches optimize a unified graph structure from multi-view data representation [2, 3, 25, 27, 28, 50]. In addition, deep multi-view clustering (DMVC) has also been intensively studied [13, 31, 38, 40].

Although existing MVC algorithms have been proposed to improve performances in various application tasks, most of them hold a common assumption: *all of the views are complete*. Unfortunately, it is not uncommon to see that some instances may be inaccessible from some views. For example, for web-page analysis, some have text, audio, and picture features while others may contain only one or two kinds [45]. Besides, patients may fail to do all kinds of medical testing due to financial inconvenience which leads to sample incompleteness in corresponding views. The incompleteness causes heavy performance degeneration or even execution failure for existing MVC algorithms, which makes incomplete multi-view clustering (IMVC) a challenging problem.

To deal with IMVC problem, a series of pioneer work have been proposed in literature. Existing incomplete multi-

*Corresponding author
[1]https://github.com/wangsiwei2010/CVPR22-IMVC-CBG

978-1-6654-6946-3/22/$31.00 ©2022 IEEE
DOI 10.1109/CVPR52688.2022.00955

view clustering can be divided into three categories: matrix factorization (MF)-based [6, 9, 26], graph or kernel-based [18, 21, 36, 37, 39] and deep learning [13, 31, 38, 40]. MF-based IMVC approaches attempt to establish a unified representation with incomplete data across various views. For example, PVC is the first attempt to adopt the NMF to derive the consensus latent representation of incomplete data [9]. Following PVC, MIC [26] then introduces the weighted NMF and $\ell_{2,1}$ regularizer to boost performance. Moreover, Hu *et al*. propose to align multi-view information based on semi-NMF representation alignment and basic space alignment [6]. The graph or kernel-based IMC approach attempts to recover the similarity graph or kernel matrix that is consistent across multiple views in various perspectives. Liu *et al*. propose to unify incomplete filling and clustering into one framework to boost the clustering performance [21]. With the widespread development of deep learning, many researchers extract deep information via deep neural networks to address the IMVC problem. Li *et al*. design a deep IMVC model which could simultaneously learn the representation and recovery data by contrastive learning [13]. However, the intensive computational complexity, complicated optimization process and high time expenditure prevent all the existing algorithms from being applied into large-scale tasks. Moreover, more flexible IMVC framework should be developed to handle arbitrary view incomplete other than just two.

In order to solve the aforementioned issues, we present a flexible and highly-efficient incomplete large-scale multi-view clustering approach based on consensus bipartite graph framework termed as IMVC-CBG in this paper. Firstly, we formalize IMVC problem under incomplete bipartite graph framework, which is the first practice for this community. Instead of fusing large pair-wise similarity matrices from multiple views in the previous study, we unify anchor learning and incomplete bipartite graph construction together to make instances share the consensus bipartite graph and ensure structure consistency across views. We also provide the interpretation of our method with probability model. Then, a four-step alternative optimization algorithm with proved convergence is proposed to efficiently solve the resultant optimization problem. By virtue of the proposed novel framework, our proposed method enjoys linear complexity respecting to instance numbers, which is more applicable for large-scale clustering tasks. Comprehensive experimental results on various benchmark datasets demonstrate the effectiveness and efficiency of our proposed algorithm against other IMVC competitors. Remarkably, ours is the first algorithm that can efficiently and effectively run on datasets with more than 100000 samples with 64G RAM in IMVC community. The main contributions of this paper can be summarized as follows,

- We propose a flexible IMVC approach (IMVC-CBG) with incomplete bipartite graph fusion to handle scalability issues in the previous study. By introducing a flexible framework to handle arbitrary view incompleteness for the first practice, IMVC-CBG is a pioneering work that integrates IMVC and bipartite graph into a joint framework.
- Comparing to the existing IMVC approaches, we simultaneously utilize the unified bipartite graph to capture the complementary information and learn to recover clustering structures across incomplete views.
- We design a four-step alternative optimization algorithm to effectively and efficiently solve IMVC-CBG with proven convergence. Our proposed method enjoys linear complexity respecting to instance numbers, which is more applicable for large-scale IMVC tasks. Comprehensive experimental results clearly demonstrate the effectiveness and efficiency of our proposed approach.

Table 1. Main notations used throughout the paper.

Notation	Meaning
n	number of samples
v	number of views
k	number of clusters
m	number of selected anchors
$\boldsymbol{\alpha} \in \mathbb{R}^{v\times 1}$	view coefficient vector
d_i	dimension for the i-th view
d	$\sum_{i=1}^{v} d_i$
$\mathbf{X}^{(i)} \in \mathbb{R}^{d_i\times n}$	data matrix for the i-th view
$\mathbf{A}^{(i)} \in \mathbb{R}^{n\times n_i}$	missing index for i-th view
$\mathbf{W}^{(i)} \in \mathbb{R}^{k\times d_i}$	projection matrix for the i-th view
$\mathbf{S} \in \mathbb{R}^{n\times n}$	similarity matrix
$\mathbf{C} \in \mathbb{R}^{k\times m}$	consensus anchor matrix
$\mathbf{Z} \in \mathbb{R}^{m\times n}$	consensus anchor graph

2. Related Work

In this section, we briefly introduce some related works in the following subsection, namely, IMVC and MVC with bipartite graphs. Table 1 lists main notations used throughout the paper.

2.1. Incomplete Multi-view Clustering

The incomplete multi-view clustering (IMVC) problem has gotten increased attention in recent years. We divide the incomplete multi-view clustering algorithms into three categories: matrix factorization (MF)-based algorithms [6, 9, 15, 26, 49], graph or kernel-based algorithms [21, 32, 47, 49] and deep algorithms [13, 31, 40, 41]. MF-based IMVC approaches attempt to establish a representation unifying the incomplete data across various views with

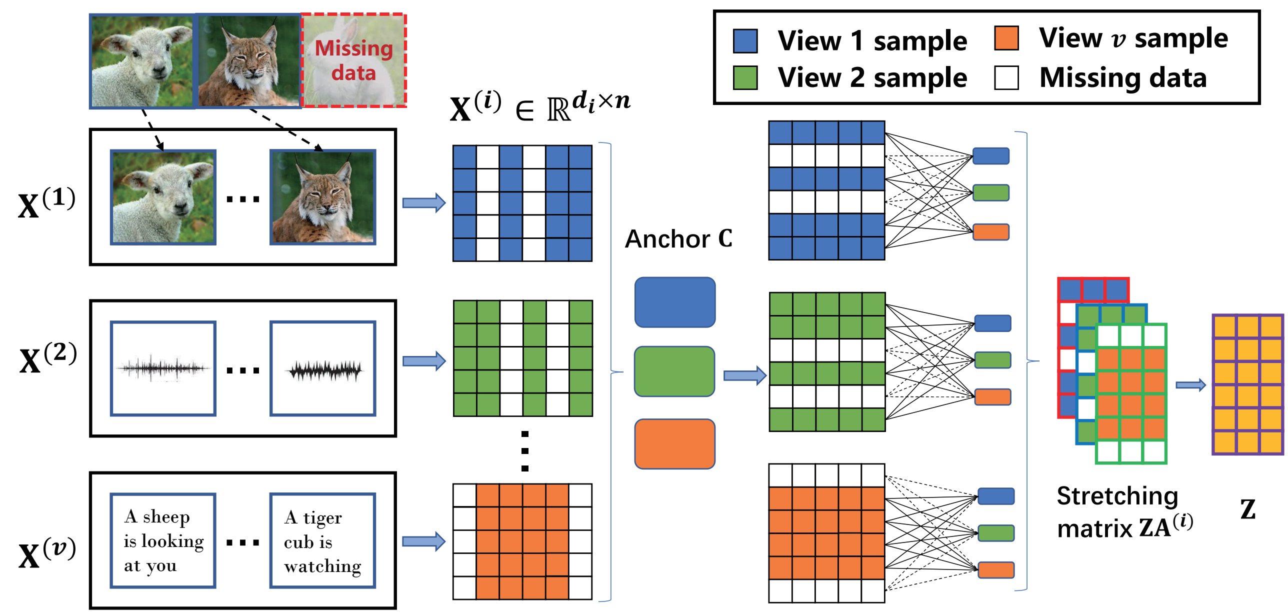


Figure 1. The framework of our proposed IMVC-CBG. We project the multi-view incomplete data into a shared latent space and construct the respective incomplete bipartite graph for each view. By regularizing consistent anchors, all views contribute to filling in the consensus bipartite graph with respective stretching matrix $\mathbf{A}^{(i)}$. Finally, we directly preform clustering algorithm with the complete consensus bipartite graph $\mathbf{Z}$. IMVC-CBG is proved to have $\mathcal{O}(n)$ complexity which both effectively and efficiently handle large-scale IMVC problems.

latent subspace . In this category, partial multi-view clustering (PVC) [9] is a pioneer work which uses the Non-negative Matrix Factorization (NMF) to derive the consensus latent representation of incomplete data. Following PVC, Shao *et al.* [26] then introduce the weighted NMF and $\ell_{2,1}$ regularizer to boost performance. Furthermore, Hu *et al.* propose DAIMC method [6] to achieve consensus information across multiple views based on semi-NMF representation alignment and basic space alignment. The graph-based or kernel-based IMC methods attempt to recover the similarity graph or kernel matrix that is consistent across multiple views in various ways. Considering a graph laplacian term, IMG [49] extends PVC by exploring the compact global structure over the multiple views. Liu *et al.* [21] combine filling and clustering into one framework to boost the clustering performance. These two categories mentioned above are all shallow algorithms. With the widespread use of deep learning, many researchers extract high-level information via deep neural networks to address the IMC problem. Cai *et al.* propose an adversarial incomplete multi-view clustering method [41] to extend it to multiple views. After that, Wen *et al.* propose a DIMC-net [40], which adaptively fuses the view-specific high-level information to seek a consensus representation. Based on DIMC-net, CDIMC-net [38] additionally introduces a self-paced strategy to reduce the negative influence of outliers.

2.2. Bipartite Graph for Clustering

Bipartite graph has been widely regarded as an effective strategy to deal with large-scale datasets in multi-view spectral clustering [7, 10, 11, 28, 29, 35]. The main advantage of bipartite graph is to select/sample a relative small proportion of representative landmarks and explore the relationship between anchors and original samples. The traditional framework of multi-view bipartite graph can be written as follows,

$$\min_{\mathbf{Z}^{(i)},\mathbf{Z}} \left\|\mathbf{X}^{(i)} - \mathbf{C}^{(i)}\mathbf{Z}^{(i)}\right\|_{\mathbf{F}}^2 + \Omega(\mathbf{Z}^{(i)}, \mathbf{Z}), \text{s.t. } \mathbf{Z}^{(i)} \geq 0, \mathbf{Z}^{(i)\top}\mathbf{1} = \mathbf{1}, \tag{1}$$

where $\mathbf{C}^{(i)} \in \mathbb{R}^{d_i \times m}$ denotes the m selected or sampled instances in i-th view. Then the size of respective graph has been reduced from $\mathbb{R}^{n\times n}$ to $\mathbf{Z} \in \mathbb{R}^{m\times n}$. Following this line, Li et al. propose an alternate anchor sampling strategy to build individual anchor graphs and then combine them into the consensus graph [10]. As demonstrated by former anchor subspace methods, the anchor graph can help reduce both storage and computational time while providing comparable clustering performance. However, existing bipartite graph framework cannot handle IMVC circumstances. In the next section, we will introduce IMVC-CBG for the first practice on large-scale IMVC with bipartite graph.

3. Incomplete Large-scale Multi-view Clustering with Consensus Bipartite Graph (IMVC-CBG)

3.1. Problem Formulation

To handle IMVC problem with bipartite graph, we firstly define the incompleteness of $\mathbf{X}^{(i)}$. Given the indicator vector $\mathbf{h}^{(i)} \in \mathbb{R}^{n_i}$ containing the index for n_i existing samples for i-th view in sort, we can define the index matrix

$\mathbf{A}^{(i)} \in \mathbb{R}^{n \times n_i}$ for i-th view as follows,

$$\mathbf{A}_{pq}^{(i)} = \begin{cases} 1, & \text{if the entry } h_q^{(i)} = p, \forall q = 1, 2 \cdots n_i, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It is easy to verify that $\mathbf{X}^{(i)}\mathbf{A}^{(i)} \in \mathbb{R}^{d_i \times n_i}$ is the sorted complete data matrix in the i-th view. Thus we can adopt Eq. (1) with single-view bipartite graph construction as,

$$\min_{\mathbf{Z}^{(i)}} \left\| \mathbf{X}^{(i)}\mathbf{A}^{(i)} - \mathbf{C}^{(i)}\mathbf{Z}^{(i)}\mathbf{A}^{(i)} \right\|_{\mathbf{F}}^2 + \Omega(\mathbf{Z}^{(i)}), \text{s.t. } \mathbf{Z}^{(i)} \geq 0, \mathbf{Z}^{(i)\top}\mathbf{1} = \mathbf{1}, \tag{3}$$

However, if the incomplete ratio is relatively large (very small n_i), the k-means or uniform sampling based anchor selection strategy will heavily affect the anchor quality with insufficient available instance information.

Different from existing complete bipartite graph strategy, we decide to iteratively learn anchors based on all the available instances among views. IMVC-CBG seeks to construct consensus bipartite graph structure with anchors. Firstly, we project various dimensional complete data $\mathbf{X}^{(i)}\mathbf{A}^{(i)} \in \mathbb{R}^{d_i}$ into a common latent space. With the consensus latent anchor $\mathbf{C}$, we define the optimization goal for IMVC-CBG as follows,

$$\min_{\boldsymbol{\alpha}, \{\mathbf{W}^{(i)}\}_{i=1}^{v}, \mathbf{C}, \mathbf{Z}} \sum_{i=1}^{v} \alpha_i^2 \left\| \mathbf{W}^{(i)}\mathbf{X}^{(i)}\mathbf{A}^{(i)} - \mathbf{CZA}^{(i)} \right\|_{\mathbf{F}}^2 + \lambda \left\| \mathbf{Z} \right\|_{\mathbf{F}}^2,$$
$$\text{s.t. } \boldsymbol{\alpha}^\top \mathbf{1} = 1, \mathbf{W}^{(i)}\mathbf{W}^{(\mathbf{i})\top} = \mathbf{I}_k, \mathbf{CC}^\top = \mathbf{I}_k, \mathbf{Z} \geq 0, \mathbf{Z}^\top \mathbf{1} = \mathbf{1}. \tag{4}$$

where $\mathbf{X}^{(i)} \in \mathbb{R}^{d_i \times n}$ is the i-th view of original data and $\mathbf{C} \in \mathbb{R}^{k \times m}$ is the unified anchor matrix with m selected anchors. λ is the balanced hyper-parameter for consensus bipartite graph construction and regularization term.

Although bipartite graph learning works have been widely adopted to reduce complexity in multi-view clustering, **none** of the existing works has successfully adopted in IMVC tasks. As can be seen from Eq. (4), simply introducing the incomplete indicator matrix $\mathbf{A}^{(i)} \in \mathbb{R}^{n \times n_i}$ will bring $\mathcal{O}(n^2)$ space complexity and $\mathcal{O}(n^3)$ time complexity, which become a serious issue for large-scale data.

We summarize our contributions as:(i)novelly solve large-scale IMVC issues by revealing $\mathbf{X}^{(i)}\mathbf{A}^{(i)}\mathbf{A}^{(i)\top} = \mathbf{X}^{(i)} \otimes \mathbf{P}^{(i)}$ where $\mathbf{P}^{(i)} = \mathbf{1}_{d_i}\mathbf{a}^{(i)} \in \mathbb{R}^{d_i \times n}$ and $\mathbf{a}^{(i)} = [a_1^{(i)}, ..., a_n^{(i)}]^\top$ with $a_j^{(i)} = \sum_{l=1}^{n_i} \mathbf{A}_{j,l}^{(i)}$. With the carefully designed objective in Eq. (4), we reduce the space complexity from $\mathcal{O}(vn^2)$ to $\mathcal{O}(dn)$ and the time complexity to $\mathcal{O}(dn)$. (ii)IMVC-CBG is a pioneering work that integrates IMVC and bipartite graph for the **first successful practice** for large-scale IMVC tasks.(iii) By following our proposed framework, new bipartite graph methods for IMVC can be easily proposed and benefit the community.

3.2. Interpretation with Probability Model

In this section, we provide a deep theoretical interpretation for our IMVC-CBG in Figure 2. Followed [16], we

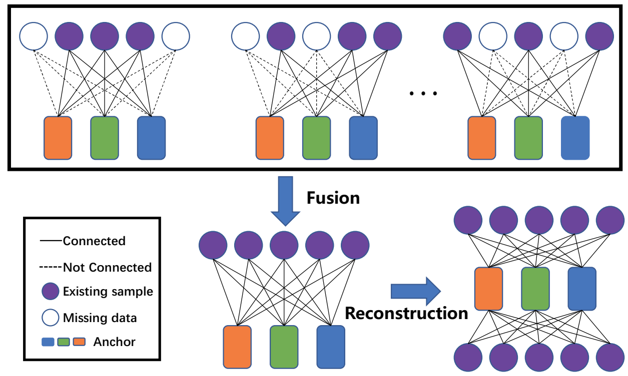


Figure 2. Interpretation of our framework with probability model.

establish the one-step stationary Markov random-walk with transition probability matrix $\mathbf{M} = \mathbf{Z}^\top$ where $\mathbf{M1} = \mathbf{1}$. Then the transition probability in one-step for i-th sample and j-th anchor as follows,

$$\mathbf{M}^{(1)}(x_i|c_j) = \frac{\mathbf{Z}_{ji}}{\sum_{j=1}^{m} \mathbf{Z}_{ji}}, \mathbf{M}^{(1)}(c_j|x_i) = \frac{\mathbf{Z}_{ji}}{\sum_{i=1}^{n} \mathbf{Z}_{ji}}. \tag{5}$$

For multiple bipartite graph across views in Figure 2, the incomplete sample can regraded as no connection with anchors (dotted lines) and the respective transition probability is 0. By fusing them into the complete bipartite graph $\mathbf{Z}$ with proper stretching, the double-step transition probability matrix can be reconstructed as $\mathbf{S} = \mathbf{Z}^\top \Lambda^{-1}\mathbf{Z}$ where $\Lambda_{ii} = \sum_{j=1}^{n} \mathbf{Z}_{ij}$. It is easy to prove that $\mathbf{S}$ is a doubly stochastic matrix and thus we can simply perform SVD on $\mathbf{Z}$ to get clustering labels. Details of derivation can be found in supplementary materials.

3.3. Optimization

The optimization problem in Eq. (4) is not jointly convex when all variables are considered simultaneously. Therefore, we propose an alternating optimization algorithm to optimize each variable with the other variables been fixed.

3.3.1 Update projection matrix $\{\mathbf{W}^{(i)}\}_{i=1}^{v}$.

When $\mathbf{C}$, $\mathbf{Z}$ and α_i are fixed, the objective function w.r.t. $\mathbf{W}^{(i)}$ can be formulated as

$$\min_{\mathbf{W}^{(i)}} \sum_{i=1}^{v} \alpha_i^2 \| \mathbf{W}^{(i)}\mathbf{X}^{(i)}\mathbf{A}^{(i)} - \mathbf{CZA}^{(i)} \|_{\mathbf{F}}^2, \ s.t. \ \mathbf{W}^{(i)}\mathbf{W}^{(i)\top} = \mathbf{I}_k. \tag{6}$$

Since each $\mathbf{W}^{(i)}$ is separated from each other, we can transform Eq. (6) into the following equivalent problem by expanding the Frobenius norm by trace and removing the items that are not related to $\mathbf{W}^{(i)}$,

$$\max_{\mathbf{W}^{(i)}} \mathrm{Tr}(\mathbf{W}^{(i)}\mathbf{B}^{(i)}), \text{ s.t. } \mathbf{W}^{(i)}\mathbf{W}^{(i)\top} = \mathbf{I}_k, \tag{7}$$

where $\mathbf{B}^{(i)} = \left(\mathbf{X}^{(i)} \otimes \mathbf{P}^{(i)}\right) \mathbf{Z}^\top \mathbf{C}^\top$. Supposing the Singular value decomposition (SVD) result of $\mathbf{B}^{(i)}$ is $\mathbf{U\Sigma V}^\top$, the optimal $\mathbf{W}^{(i)}$ can be easily obtained by calculating $\mathbf{U}_k\mathbf{V}_k^\top$ according to [34]. The total time complexity of calculating $\left\{\mathbf{W}^{(i)}\right\}_{i=1}^{v}$ needs $\mathcal{O}(d(nm + km + k^2))$ per iteration.

3.3.2 Update consensus anchor matrix C.

With $\mathbf{W}^{(i)}$, $\mathbf{Z}$ and α_i being fixed, the consensus anchor matrix $\mathbf{C}$ can be updated as follows,

$$\min_{\mathbf{A}} \sum_{i=1}^{v} \alpha_i^2 \left\|\mathbf{W}^{(i)}\mathbf{X}^{(i)}\mathbf{A}^{(i)} - \mathbf{CZA}^{(i)}\right\|_{\mathbf{F}}^2, \text{ s.t. } \mathbf{CC}^\top = \mathbf{I}_k. \tag{8}$$

Similar to the optimization of $\mathbf{W}^{(i)}$, the optimization of $\mathbf{C}$ in Eq. (8) equals to the following form

$$\max_{\mathbf{C}} \operatorname{Tr}(\mathbf{C}^\top \mathbf{G}), \text{ s.t. } \mathbf{CC}^\top = \mathbf{I}_k, \tag{9}$$

where $\mathbf{G} = \sum_{i=1}^{v} \alpha_i^2 \mathbf{W}^{(i)} \left(\mathbf{X}^{(i)} \otimes \mathbf{P}^{(i)}\right) \mathbf{Z}^\top$. Similarly, the optimal solution of updating variable $\mathbf{A}$ can be attained the product of left singular matrix and the right singular matrix of $\mathbf{C}$. The total time complexity of calculating $\mathbf{C}$ needs $\mathcal{O}(ndk + nmk + m^2k)$ per iteration.

3.3.3 Update consensus anchor graph Z.

Fixing other variables $\mathbf{W}^{(i)}$, $\mathbf{C}$ and α_i, the optimization problem for updating variable $\mathbf{Z}$ can be rewritten as,

$$\begin{aligned} \min_{\mathbf{Z}} \sum\nolimits_{i=1}^{v} \alpha_i^2 \left\|\mathbf{W}^{(i)}\mathbf{X}^{(i)}\mathbf{A}^{(i)} - \mathbf{CZA}^{(i)}\right\|_{\mathbf{F}}^2 + \lambda \left\|\mathbf{Z}\right\|_{\mathbf{F}}^2, \\ \text{s.t. } \mathbf{Z} \geq 0,\ \mathbf{Z}^\top \mathbf{1} = \mathbf{1}. \end{aligned} \tag{10}$$

Denoting $\mathbf{Z}_{:,j}$ as a vector with the i-th element to be $\mathbf{z}_{ij}$, we optimize $\mathbf{Z}$ by column:

$$\min_{\mathbf{Z}_{:,j}} \left\|\mathbf{Z}_{:,j} - \mathbf{f}_j\right\|_{\mathbf{F}}^2, \text{ s.t. } \mathbf{Z}_{:,j}^\top \mathbf{1} = 1,\ \mathbf{Z} \geq 0, \tag{11}$$

where $\mathbf{f}_j^\top = \frac{\sum_{i=1}^{v} \alpha_i^2 {\mathbf{X}_{:,j}^{(i)}}^\top {\mathbf{W}^{(i)}}^\top \mathbf{C}}{\lambda + \sum_{i=1}^{v} \alpha_i^2 \mathbf{P}_{1j}^{(i)}}$. We write the Lagrangian function of Eq. (11) as,

$$\mathcal{L}\left(\mathbf{Z}_{:,j}, \beta, \eta\right) = \left\|\mathbf{Z}_{:,j} - \mathbf{f}_j\right\|_{\mathbf{F}}^2 - \beta_j \left(\mathbf{Z}_{:,j}^\top \mathbf{1} - 1\right) - \eta_j^\top \mathbf{Z}_{:,j}, \tag{12}$$

where β and η_j are the respective Lagrangian multipliers. Then the KKT conditions are written as,

$$\begin{cases} \mathbf{Z}_{:,j} - \mathbf{f}_j - \beta_j \mathbf{1} - \eta_j = 0, \\ \eta_j \bigodot \mathbf{Z}_{:,j} = 0, \end{cases} \tag{13}$$

Together with $\mathbf{Z}_{:,j}^\top \mathbf{1} = 1$, we can easily obtain that

$$\mathbf{Z}_{:,j} = \max\left(\mathbf{f}_j + \beta_j \mathbf{1}, 0\right), \beta_j = \frac{1 + \mathbf{f}_j^\top \mathbf{1}}{m}. \tag{14}$$

Since the optimum value of $\mathbf{Z}$ can be analytically obtained, the time complexity is $\mathcal{O}(nk(d + m))$.

3.3.4 Update view coefficient α_i.

Fixing the irrelevant variables, we can obtain the optimization problem for updating α_i.

$$\min_{\alpha_i} \sum\nolimits_{i=1}^{v} \alpha_i^2 \tau_i, \text{ s.t. } \boldsymbol{\alpha}^\top \mathbf{1} = 1, \tag{15}$$

Algorithm 1 IMVC-CBG

Input: Input v views incomplete dataset $\{\mathbf{X}^{(i)}\}_{i=1}^{v}$, the missing index $\{\mathbf{P}^{(i)}\}_{i=1}^{v}$ and the number of cluster k.
Initialize: Initialize $\mathbf{C}$, $\mathbf{Z}$ and α_i with $\frac{1}{v}$.

1: **while** not converged **do**
2: Update $\mathbf{W}^{(i)}$ by solving the problem in Eq. (7).
3: Update $\mathbf{C}$ by solving the problem in Eq. (9).
4: Update $\mathbf{Z}$ by solving the problem in Eq. (11).
5: Update α_i by solving Eq. (15).
6: **end while**
7: Obtain $\mathbf{U}$ by performing SVD on $\mathbf{Z}$.
8: **Output:** Perform k-means on $\mathbf{U}$ to achieve the final clustering result.

where $\tau_i = \left\|\mathbf{W}^{(i)}\mathbf{X}^{(i)}\mathbf{A}^{(i)} - \mathbf{CZA}^{(i)}\right\|_{\mathbf{F}}^2$. According to Cauchy-Buniakowsky-Schwarz inequality, the optimal α_i can be directly obtained by $\alpha_i = \frac{e}{\tau_i}$, where $e = \frac{1}{\frac{1}{\tau_1} + \frac{1}{\tau_2} + \cdots + \frac{1}{\tau_v}}$. the time complexity is $\mathcal{O}(nk(d + m))$.

As the iteration proceeds, the four variables in the above optimization are solved separately with other variables fixed. Since each sub-problem can reach its global optimum, the objective value will monotonically decrease until the convergence condition is reached and the lower bound of the objective function can be easily proven to be zero. The entire procedures of the above optimization are listed in the following Algorithm 1.

3.4. Discussions

In this subsection, we will analyze our proposed algorithm in terms of space/time complexity and convergence. Then a theoretical comparison is conducted between SOTA IMVC method in Table 2.

Space Complexity: In this paper, the major memory costs of our method are matrices $\mathbf{W}^{(i)} \in \mathbb{R}^{k \times d_i}$, $\mathbf{P}^{(i)} \in \mathbb{R}^{d_i \times n}$, $\mathbf{C} \in \mathbb{R}^{k \times m}$ and $\mathbf{Z} \in \mathbb{R}^{m \times n}$. Thus the space complexity of our IMVC-CBG is $(d + m)(n + k)$. In our algorithm, $m \ll n$ and $k \ll n$. Therefore, the space complexity of IMVC-CBG is $\mathcal{O}(n)$. We summarize the memory cost of compared algorithms in the following Table 2.

Time complexity: The computational complexity of IMVC-CBG is composed of four optimization steps as mentioned before. When updating $\left\{\mathbf{W}^{(i)}\right\}_{i=1}^{v}$, it costs $\mathcal{O}(d(nm + km + k^2))$ to get the optimal value. Similar to updating $\left\{\mathbf{W}^{(i)}\right\}_{i=1}^{v}$, updating $\mathbf{C}$ needs $\mathcal{O}(ndk + nmk + m^2k)$. When analytically obtaining $\mathbf{Z}$, it costs $\mathcal{O}(nk(d+m))$ for all columns. The time cost of calculating $\boldsymbol{\alpha}$ is $\mathcal{O}(nk(d + m))$. Therefore, the total time cost of the optimization process is $\mathcal{O}(n(dk+mk+dm)+mdk)$. Consequently, the computational complexity of our proposed optimization algorithm is linear complexity $\mathcal{O}(n)$.

After the optimization, we perform SVD on $\mathbf{Z}$ to obtain the spectral embedding and output the discrete clustering labels by k-means [35]. The post-process needs $\mathcal{O}(nm^2)$, which is also linear complexity respecting to samples. In total, our algorithm achieves IMVC with linear time complexity, which demonstrates the efficiency of IMVC-CBG.

Table 2. Complexity analysis on SOTA IMVC methods

Method	Memory Cost	Time Complexity
BSV [23]	vn^2	$\mathcal{O}(n^3)$
MIC [26]	$vn^2 + nd + nvk + vdk$	$\mathcal{O}(n^3 + n^2dk)$
MKKM-IK [17]	$vn^2 + vnk$	$\mathcal{O}(vn^3)$
DAIMC [6]	$vn^2 + nd + nk + dk$	$\mathcal{O}(nd^3 + ndk)$
APMC [5]	$nd + vnm + nk$	$\mathcal{O}(n^3 + nmd + m^3)$
UEAF [37]	$vn^2 + dn + nvk + dk$	$\mathcal{O}(n^3 + dk^2)$
MKKM-IK-MKC [22]	$vn^2 + vnk$	$\mathcal{O}(vn^3)$
EEIMVC [18]	$vn^2 + vnk + vk^2$	$\mathcal{O}(nk^2 + vk^3)$
FLSD [39]	$vn^2 + dnk + nk$	$\mathcal{O}(nd^2)$
Ours	$mn + (d+m)k$	$\mathcal{O}(ndk + nmd + mdk)$

4. Experiments

4.1. Datasets

We perform experiments on nine widely used multi-view benchmark datasets: NGs, Caltech101-7, Caltech101-20, BDGP, Caltech101-all, NUSWIDE, MNIST and YoutubeFace. The detail of them are shown in Table 3. Specifically, Caltech101-7 and Caltech101-20 are both subsets of the image dataset Caltech101[2]. BDGP[3] contains 2500 Drosophila embryo samples in 5 classes. NUSWIDE[4] is an object recognition dataset. MNIST[5] consists of handwritten digits of 0 to 9. YoutubeFace[6] is a video dataset produced from YouTube with 101499 instances. These datasets can be downloaded at [7].

4.2. Compared Methods and Experimental Setting

The following state-of-the-art multi-view clustering methods are compared with our proposed algorithm in the experiment. **Best Single-view Spectral Clustering (BSV) [23]** fills in the missing parts with the mean values. **Multiple incomplete views clustering via weighted non-negative matrix factorization with $\ell_{2,1}$ regularization (MIC) [26]** generates a consensus clustering representation with the latent feature matrices learned from all incomplete views. **Multiple kernel k-means with incomplete kernels (MKKM-IK) [17]** proposes to impute incomplete kernels and perform kernel k-means in a unified framework. **Doubly aligned incomplete multi-view clustering (DAIMC) [6]** learns a common representation with aligned samples based on semi-NMF and aligns the base matrices with $\ell_{2,1}$ regularized regression. **Anchors bring ease: An embarrassingly simple approach to partial multi-view clustering (APMC)** [5] fuses constructed anchor similarity matrices and is restricted to use on datasets with two views or three views. **Unified embedding alignment with missing views inferring for incomplete multi-view clustering (UEAF) [37]** jointly imputes missing samples by using locality preservation and learns latent representation with embedding alignment. **Multiple kernel k k-means with incomplete kernels (MKKM-IK-MKC) [22]** unifies kernel imputation and kernel k-means clustering into a joint framework. **Efficient and effective regularized incomplete multi-view clustering (EEIMVC) [18]** jointly imputes the low dimensional base feature matrices and seeks a consensus feature matrix for clustering. **Generalized incomplete multi-view clustering with flexible locality structure diffusion (FLSD) [39]** jointly learns view-specific latent representations and the shared representation.

In our experiments, we randomly select n_p paired samples which are observed in all views. For the rest $n - n_p$ samples, a random matrix $\mathbf{H} = \left[\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{(n-n_p)}\right] \in \{0,1\}^{(n-n_p)\times v}$ is generated, $0 < \sum_{i=1}^{v} h_{li} < v$. The j-th sample is observed in the r-th view with $h_{jr} = 1$, otherwise the j-th sample is missing in the r-th view. With the paired ratio $\epsilon = \frac{n_p}{n}$ setting as $[0.1 : 0.1 : 0.9]$, we generate incomplete multi-view datasets.

For all the above-mentioned algorithms, we have downloaded their public Matlab code implementations from original websites. The parameters of the compared methods are set according to the suggestions of the corresponding literature. In the proposed method, we tune λ in $[0.001, 0.01, 1, 10]$ and the number of anchor points in $[k, 2k, 3k, 5k]$ with a grid search scheme. Moreover, we repeat each experiment 20 times for average performance and standard deviation. To evaluate the clustering performance, we employ three widely used criteria including accuracy (ACC), normalized mutual information (NMI), Purity and Fscore. All the experiments are performed on a desktop with Intel core i9-10900X CPU and 64G RAM, MATLAB 2020b(64-bit).

Table 3. Incomplete multi-view datasets in experiments.

Dataset	Size	Classes	Views	Dimensionality
NGs	500	5	3	500/500/500
Caltech101-7	1474	7	6	48/40/254/198/512/928
Caltech101-20	2386	20	6	48/40/254/198/512/928
BDGP	2500	5	3	1000/500/250
Caltech101-all	9144	102	5	48/40/254/512/928
NUSWIDE	30000	31	5	65/226/145/74/129
MNIST	60000	10	3	342/1024/64
YoutubeFace	101499	31	5	64/512/64/647/838

[2] http://www.vision.caltech.edu/Image_Datasets/Caltech101/
[3] https://www.fruitfly.org/
[4] https://lms.comp.nus.edu.sg/wp-content/uploads/2019/research/nuswide/NUS-WIDE.html
[5] http://yann.lecun.com/exdb/mnist/
[6] https://www.cs.tau.ac.il/ wolf/ytfaces/
[7] https://github.com/wangsiwei2010/Incomplete_multi-view_Datasets

Table 4. The aggregated ACC, NMI, Purity and Fscore comparison (mean±std) of different algorithms on benchmark datasets. '-' means out of the CPU memory. The detailed results are omitted due to space limit and provided in supplementary materials.

Datasets	BSV	MIC	MKKM-IK	DAIMC	APMC	UEAF	MKKM-IK-MKC	EEIMVC	FLSD	Proposed
					ACC					
NGs	41.13±2.03	20.92±0.46	79.63±0.09	74.53±9.27	**89.41±0.01**	**89.49±0.03**	49.87±0.13	77.84±0.12	84.19±0.03	**88.95±0.02**
Caltech101-7	54.63±0.07	37.33±1.97	38.08±0.33	42.10±3.72	-	40.65±0.34	43.64±2.02	40.1±0.67	**55.08±0.22**	**60.40±0.88**
Caltech101-20	39.16±0.35	26.53±1.55	32.70±1.37	**45.68±2.10**	-	39.55±1.54	31.95±1.78	41.59±1.69	41.78±1.18	**50.25±2.61**
BDGP	34.96±1.06	25.37±0.61	32.17±0.24	28.12±0.05	28.12±0.01	**44.88±0.02**	40.77±0.20	**44.00±0.05**	42.96±0.03	**47.81±0.12**
Caltech101-all	10.95±0.26	12.65±0.45	19.54±0.72	-	-	17.28±0.46	15.45±0.53	**23.07±0.77**	15.73±0.46	**24.23±1.04**
NUSWIDE	12.05±0.03	-	-	**13.79±0.37**	-	-	-	**12.73±0.16**	-	**12.53 ± 0.11**
MNIST	-	-	-	**97.57±0.31**	-	-	-	-	-	**98.10±0.06**
YoutubeFace	-	-	-	-	-	-	-	-	-	**22.96±2.83**
					NMI					
NGs	20.24±1.37	2.37±0.49	63.16±0.11	59.97±6.48	**73.36±0.03**	**73.6±0.08**	32.76±0.11	57.23±0.18	64.22±0.04	**73.05±0.06**
Caltech101-7	15.93±0.60	24.8±1.13	31.32±0.25	**45.45±2.12**	-	39.98±0.22	31.26±0.53	42.96±0.33	37.20±0.35	**43.58±0.58**
Caltech101-20	25.26±0.66	30.02±1.31	40.18±0.88	**55.56±1.18**	-	50.37±0.84	37.39±1.16	**55.17±0.82**	50.90±0.7	52.92±1.12
BDGP	12.88±0.94	4.47±0.70	7.41±0.16	8.68±0.01	8.68±0.01	**23.77±0.04**	16.35±0.13	19.91±0.09	18.95±0.06	**22.39±0.09**
Caltech101-all	21.16±0.26	28.7±0.33	38.09±0.37	-	-	35.87±0.21	33.38±0.31	**43.74±0.30**	34.18±0.21	**39.64±0.51**
NUSWIDE	2.68±0.03	-	-	**11.84±0.36**	-	-	-	10.31±0.16	-	**10.42 ± 0.03**
MNIST	-	-	-	**93.89±0.53**	-	-	-	-	-	**94.94±0.13**
YoutubeFace	-	-	-	-	-	-	-	-	-	**14.12±1.52**
					Purity					
NGs	43.15±1.51	21.3±0.42	79.63±0.09	75.56±9.27	**89.41±0.01**	**89.49±0.03**	50.57±0.10	77.84±0.12	84.19±0.03	**88.95±0.02**
Caltech101-7	64.7±0.61	71.58±1.24	76.36±0.20	**81.12±1.08**	-	80.91±0.10	77.52±0.57	80.6±0.25	80.03±0.27	**80.96±0.08**
Caltech101-20	47.78±0.72	54.31±1.2	62.66±1.01	**74.92±0.93**	-	70.23±0.83	60.33±1.42	**74.75±0.74**	71.08±0.63	70.3±0.87
BDGP	36.75±0.89	25.67±0.59	33.42±0.18	28.46±0.01	28.46±0.01	**45.92±0.01**	41.1±0.13	**46.4±0.05**	44.41±0.03	**48.2±0.10**
Caltech101-all	17.88±0.22	27.27±0.46	36.26±0.53	-	-	34.02±0.33	30.9±0.44	**39.16±0.42**	32.54±0.32	**37.84±0.49**
NUSWIDE	13.72±0.04	-	-	**23.41±0.63**	-	-	-	21.83±0.21	-	**22.17 ± 0.19**
MNIST	-	-	-	**97.57±0.31**	-	-	-	-	-	**98.1±0.16**
YoutubeFace	-	-	-	-	-	-	-	-	-	**27.11±0.39**
					Fscore					
NGs	32.39±1.08	32.9±0.19	68.72±0.07	64.83±7.49	**80.28±0.02**	**80.55±0.06**	42.28±0.09	63.75±0.14	71.8±0.05	**79.49±0.03**
Caltech101-7	56.03±0.02	37.74±1.35	39.57±0.28	49.44±2.93	-	44.48±0.19	43.6±1.66	44.65±0.34	**55.63±0.03**	**59.28±0.98**
Caltech101-20	32.34±0.32	23.78±1.38	25.63±0.91	**40.34±2.41**	-	36.11±1.83	26.82±1.55	35.22±1.32	38.07±1.61	**44.38±2.94**
BDGP	28.76±0.61	29.88±0.06	25.25±0.08	31.21±0.06	31.21±0.04	33.69±0.03	30.15±0.10	32.93±0.04	**34.29±0.01**	**35.76±0.06**
Caltech101-all	6.02±0.07	9.19±0.41	13.53±0.62	-	-	12.80±0.53	10.66±0.52	**16.37±0.83**	11.96±0.49	**15.97±1.13**
NUSWIDE	**10.95±0.00**	-	-	**8.58±0.19**	-	-	-	7.81±0.08	-	7.70 ± 0.03
MNIST	-	-	-	**95.28±0.57**	-	-	-	-	-	**96.29±0.28**
YoutubeFace	-	-	-	-	-	-	-	-	-	**15.77±0.49**

Table 5. Time-consuming comparison of different IMVC algorithms on benchmark datasets (in seconds). '-' means out of CPU memory.

Datasets	Num	BSV	MIC	MKKM-IK	DAIMC	APMC	UEAF	MKKM-IK-MKC	EEIMVC	FLSD	Proposed
NGs	500	0.06	143.83	0.50	42.50	0.18	2.25	1.36	0.14	3.30	1.31
Caltech101-7	1474	0.50	1298.58	14.47	38.05	-	7.22	32.95	1.20	18.39	2.59
Caltech101-20	2386	2.21	2847.34	81.01	69.77	-	26.63	126.15	3.96	54.04	5.28
BDGP	2500	0.92	1146.74	34.53	25.46	18.46	39.76	42.43	1.30	32.41	2.73
Caltech101-all	9144	160.38	30672.67	1961.33	-	-	923.26	3507.48	111.25	1216.41	46.40
NUSWIDE	30000	8281.38	-	-	1667.67	-	-	-	1872.44	-	22.43
MNIST	60000	-	-	-	5588.82	-	-	-	-	-	549.08
YoutubeFace	101499	-	-	-	-	-	-	-	-	-	1168.35

4.3. Experimental Results

Table 4 reports the ACC, NMI, Purity and Fscore comparison of the above baseline algorithms. Table 5 presents the operational time of the aforementioned algorithms. From these two tables, we have the following observations.

- Our proposed algorithm achieves the best performance in terms of four metrics against all compared algorithm in most circumstances. Taking the results on DAIMC for instance, DAIMC has been considered as the strongest incomplete multi-view algorithm, our IMVC-CBG further exceeds it by 19.3%, 43.4%, 10.0%, 70.0% and 0.5% increment in terms of ACC on NGs, Caltech101-7, Caltech101-20, BDGP and MNIST. Moreover, IMVC-CBG also achieves considerable results over other metrics.
- In comparison with existing subspace-based IMVC method (BSV, MIC, MKKM-IK, DAIMC, UEAF, MIKKM-IK-MKC, EEIMVC, FLSD), our bipartite graph framework shows effectiveness and efficiency and therefore more applicable on large-scale IMVC datasets.
- Our IMVC-CBG shows substantially shorter running time on NGs, Caltech101-7, Caltech101-20, BDGP and Caltech101-all when compared to other incomplete multi-view algorithms, demonstrating its computational efficiency. Moreover, the proposed IMVC-CBG is capable of handling over 100,000 instances of large-scale datasets (i.e., YoutubeFace) while all other compared baselines suffer from out-of-memory errors.

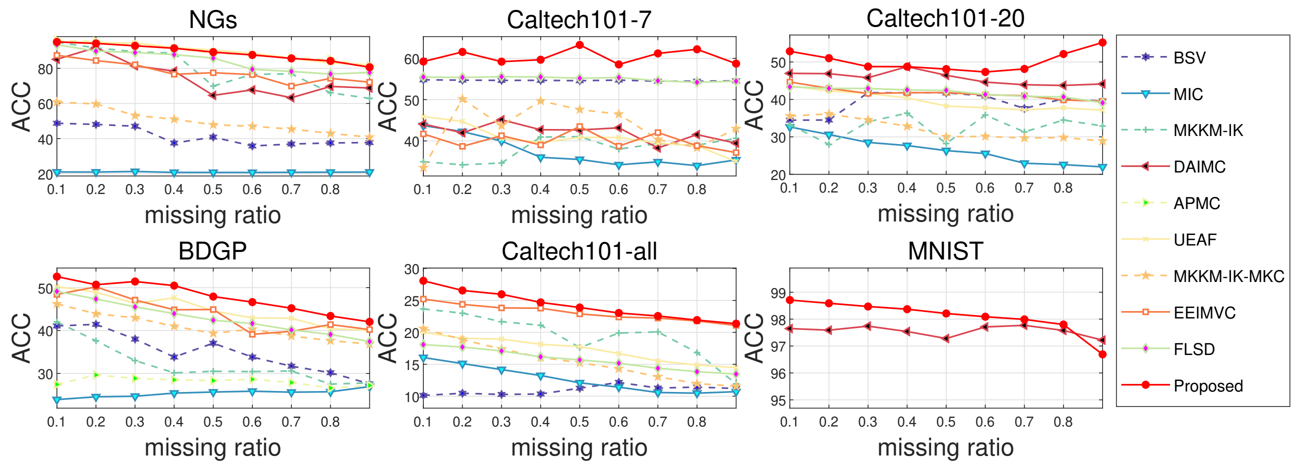
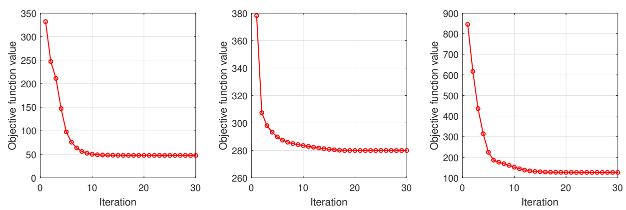
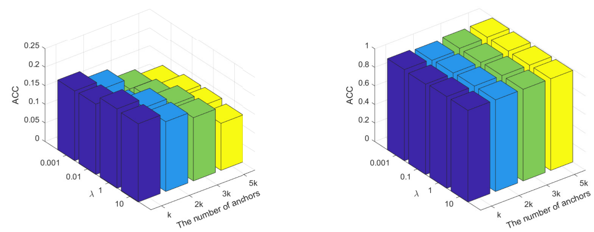


Figure 3. The clustering results of ACC metric on benchmark datasets with different incomplete ratios. Only ours can run YoutubeFace so it is omitted. The results of other metrics are provided in supplementary materials due to space limit.



(a) Caltech101-20 (b) BDGP (c) Caltech101-all

Figure 4. Variation of the objective function values on three benchmark datasets. Others are displayed in supplementary materials.



(a) Caltech101-all (b) MNIST

Figure 5. Sensitivity analysis of λ and anchors number for our method over Caltech101-all and MNIST.

In summary, IMVC-CBG demonstrates superior clustering performance over the alternatives on all datasets and can effectively handle large-scale datasets with more than 100000 samples. We expect that the effectiveness and high efficiency of IMVC-CBG make it a well-suited consideration for applications in practical clustering.

4.4. Qualitative Study

To further illustrate the effectiveness of IMVC-CBG handling in incomplete data, we evaluate the curves of the clustering results on the datasets respecting to various missing ratio in Figure 3. As can be seen, the performance of all algorithms drops as missing ratio increases. Compared to other missing clustering algorithms, our proposed algorithm is able to maintain the best ACC over all datasets even at high missing ratio. This is a piece of evidence that IMVC-CBG can efficiently handle incomplete data.

4.5. Sensitivity and Convergence

To analyze the influence of parameters on the efficiency of the algorithm, we conducted a comparative experiment on Caltech101-all and MNIST with different settings of λ and anchors numbers vary. As shown in Figure 5, our algorithm is not greatly affected by λ with fixed anchors number. According to [1], our algorithm is theoretically guaranteed to converge to a local minimum. The examples of the evolution of the objective value on the experimental results are shown in Figure 4. From these experiments, we observe that the objective values of our algorithm monotonically decrease at each iteration. These results clearly verify our algorithm's convergence.

5. Conclusion

In this paper, we propose a novel consensus bipartite graph fusion framework for incomplete multi-view clustering termed IMVC-CBG. Different from existing IMVC methods, the proposed framework can flexibly and efficiently handle arbitrary view incompleteness for large-scale IMVC tasks. IMVC-CBG is the first practice to introduce bipartite graph into IMVC community for large-scale tasks. In the future, we will explore the influence of various anchor selection strategies on the clustering quality.

Acknowledgments

This work was supported by the National Key R&D Program of China 2020AAA0107100, the Natural Science Foundation of China (project no. 61922088, 61773392, 62006237 and 61976196).

References

[1] James C. Bezdek and Richard J. Hathaway. Convergence of alternating optimization. *Neural, Parallel Sci. Comput.*, 11(4):351–368, 2003. 8

[2] Xiao Cai, Feiping Nie, and Heng Huang. Multi-view k-means clustering on big data. In *Twenty-Third International Joint conference on artificial intelligence*. Citeseer, 2013. 1

[3] Xiaochun Cao, Changqing Zhang, Huazhu Fu, Si Liu, and Hua Zhang. Diversity-induced multi-view subspace clustering. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 586–594, 2015. 1

[4] Mehmet Gönen and Adam A Margolin. Localized data fusion for kernel k-means clustering with application to cancer biology. In *Advances in Neural Information Processing Systems*, pages 1305–1313, 2014. 1

[5] Jun Guo and Jiahui Ye. Anchors bring ease: An embarrassingly simple approach to partial multi-view clustering. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 118–125, 2019. 6

[6] Menglei Hu and Songcan Chen. Doubly aligned incomplete multi-view clustering. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence*, pages 2262–2268. 2, 3, 6

[7] Zhao Kang, Wangtao Zhou, Zhitong Zhao, Junming Shao, Meng Han, and Zenglin Xu. Large-scale multi-view subspace clustering in linear time. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 4412–4419, 2020. 3

[8] Abhishek Kumar, Piyush Rai, and Hal Daume. Co-regularized multi-view spectral clustering. In *Advances in neural information processing systems*, pages 1413–1421, 2011. 1

[9] Shao-Yuan Li, Yuan Jiang, and Zhi-Hua Zhou. Partial multi-view clustering. In *Proceedings of the AAAI conference on artificial intelligence*, volume 28, 2014. 2, 3

[10] Xuelong Li, Han Zhang, Rong Wang, and Feiping Nie. Multi-view clustering: A scalable and parameter-free bipartite graph fusion method. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2020. 1, 3

[11] Yeqing Li, Feiping Nie, Heng Huang, and Junzhou Huang. Large-scale multi-view spectral clustering via bipartite graph. In *Twenty-Ninth AAAI Conference on Artificial Intelligence*, 2015. 3

[12] Weixuan Liang, Sihang Zhou, Jian Xiong, Xinwang Liu, Siwei Wang, En Zhu, Zhiping Cai, and Xin Xu. Multi-view spectral clustering with high-order optimal neighborhood laplacian matrix. *IEEE Transactions on Knowledge and Data Engineering*, 2020. 1

[13] Yijie Lin, Yuanbiao Gou, Zitao Liu, Boyun Li, Jiancheng Lv, and Xi Peng. Completer: Incomplete multi-view clustering via contrastive prediction. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, pages 11174–11183, 2021. 1, 2

[14] Bao-Yu Liu, Ling Huang, Chang-Dong Wang, Jian-Huang Lai, and Philip Yu. Multi-view consensus proximity learning for clustering. *IEEE Transactions on Knowledge and Data Engineering*, 2020. 1

[15] Jiyuan Liu, Xinwang Liu, Yi Zhang, Pei Zhang, Wenxuan Tu, Siwei Wang, Sihang Zhou, Weixuan Liang, Siqi Wang, and Yuexiang Yang. Self-representation subspace clustering for incomplete multi-view data. In *Proceedings of the 29th ACM International Conference on Multimedia*, pages 2726–2734, 2021. 2

[16] Wei Liu, Junfeng He, and Shih-Fu Chang. Large graph construction for scalable semi-supervised learning. In *ICML*, 2010. 4

[17] Xinwang Liu, Yong Dou, Jianping Yin, Lei Wang, and En Zhu. Multiple kernel k -means clustering with matrix-induced regularization. In *Thirtieth AAAI Conference on Artificial Intelligence*, pages 1888–1894, 2016. 1, 6

[18] X Liu, M Li, C Tang, J Xia, J Xiong, L Liu, M Kloft, and E Zhu. Efficient and effective regularized incomplete multi-view clustering. *IEEE transactions on pattern analysis and machine intelligence*, 2020. 2, 6

[19] Xinwang Liu, Sihang Zhou, Yueqing Wang, Miaomiao Li, Yong Dou, En Zhu, and Jianping Yin. Optimal neighborhood kernel clustering with multiple kernels. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 31, 2017. 1

[20] Xinwang Liu, Xinzhong Zhu, Miaomiao Li, Lei Wang, Chang Tang, Jianping Yin, Dinggang Shen, Huaimin Wang, and Wen Gao. Late fusion incomplete multi-view clustering. *IEEE transactions on pattern analysis and machine intelligence*, 41(10):2410–2423, 2018. 1

[21] Xinwang Liu, Xinzhong Zhu, Miaomiao Li, Lei Wang, En Zhu, Tongliang Liu, Marius Kloft, Dinggang Shen, Jianping Yin, and Wen Gao. Multiple kernel k k-means with incomplete kernels. *IEEE transactions on pattern analysis and machine intelligence*, 42(5):1191–1204, 2019. 1, 2, 3

[22] Xinwang Liu, Xinzhong Zhu, Miaomiao Li, Lei Wang, En Zhu, Tongliang Liu, Marius Kloft, Dinggang Shen, Jianping Yin, and Wen Gao. Multiple kernel k k -means with incomplete kernels. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 42(5):1191–1204, 2020. 6

[23] Andrew Y Ng, Michael I Jordan, and Yair Weiss. On spectral clustering: Analysis and an algorithm. In *Advances in neural information processing systems*, pages 849–856, 2002. 6

[24] Feiping Nie, Guohao Cai, Jing Li, and Xuelong Li. Auto-weighted multi-view learning for image clustering and semi-supervised classification. *IEEE Transactions on Image Processing*, 27(3):1501–1511, 2017. 1

[25] Feiping Nie, Jing Li, and Xuelong Li. Self-weighted multi-view clustering with multiple graphs. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*, pages 2564–2570. ijcai.org, 2017. 1

[26] Weixiang Shao, Lifang He, and S Yu Philip. Multiple incomplete views clustering via weighted nonnegative matrix factorization with $\ell_{2,1}$ regularization. In *Joint European conference on machine learning and knowledge discovery in databases*, pages 318–334. Springer, 2015. 2, 3, 6

[27] S. Shi, F. Nie, R. Wang, and X. Li. Fast multi-view clustering via prototype graph. *IEEE Transactions on Knowledge and Data Engineering*, PP(99):1–1, 2021. 1

[28] Mengjing Sun, Siwei Wang, Pei Zhang, Xinwang Liu, Sihang Zhou, Xifeng Guo, and En Zhu. Projective multiple kernel subspace clustering. *IEEE Transactions on Multimedia*, 2021. 1, 3

[29] Mengjing Sun, Pei Zhang, Siwei Wang, Sihang Zhou, Wenxuan Tu, Xinwang Liu, En Zhu, and Changjian Wang. Scalable multi-view subspace clustering with unified anchors. In *Proceedings of the 29th ACM International Conference on Multimedia*, pages 3528–3536, 2021. 3

[30] Chang-Dong Wang, Jian-Huang Lai, and S Yu Philip. Multi-view clustering based on belief propagation. *IEEE Transactions on Knowledge and Data Engineering*, 28(4):1007–1021, 2015. 1

[31] Qianqian Wang, Zhengming Ding, Zhiqiang Tao, Quanxue Gao, and Yun Fu. Partial multi-view clustering via consistent gan. In *2018 IEEE International Conference on Data Mining (ICDM)*, pages 1290–1295. IEEE, 2018. 1, 2

[32] Qianqian Wang, Zhengming Ding, Zhiqiang Tao, Quanxue Gao, and Yun Fu. Generative partial multi-view clustering with adaptive fusion and cycle consistency. *IEEE Transactions on Image Processing*, 30:1771–1783, 2021. 2

[33] Siwei Wang, Xinwang Liu, Li Liu, Sihang Zhou, and En Zhu. Late fusion multiple kernel clustering with proxy graph refinement. *IEEE Transactions on Neural Networks and Learning Systems*, pages 1–12, 2021. 1

[34] Siwei Wang, Xinwang Liu, En Zhu, Chang Tang, Jiyuan Liu, Jingtao Hu, Jingyuan Xia, and Jianping Yin. Multi-view clustering via late fusion alignment maximization. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*, pages 3778–3784, 7 2019. 4

[35] Siwei Wang, Xinwang Liu, Xinzhong Zhu, Pei Zhang, Yi Zhang, Feng Gao, and En Zhu. Fast parameter-free multi-view subspace clustering with consensus anchor guidance. *IEEE Trans. Image Process.*, 31:556–568, 2022. 3, 6

[36] Jie Wen, Yong Xu, and Hong Liu. Incomplete multiview spectral clustering with adaptive graph learning. *IEEE Trans. Cybern.*, 50(4):1418–1429, 2020. 2

[37] Jie Wen, Zheng Zhang, Yong Xu, Bob Zhang, Lunke Fei, and Hong Liu. Unified embedding alignment with missing views inferring for incomplete multi-view clustering. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 5393–5400, 2019. 2, 6

[38] Jie Wen, Zheng Zhang, Yong Xu, Bob Zhang, Lunke Fei, and Guo-Sen Xie. Cdimc-net: Cognitive deep incomplete multi-view clustering network. In *IJCAI*, pages 3230–3236, 2020. 1, 2, 3

[39] Jie Wen, Zheng Zhang, Zhao Zhang, Lunke Fei, and Meng Wang. Generalized incomplete multiview clustering with flexible locality structure diffusion. *IEEE transactions on cybernetics*, 51(1):101–114, 2020. 2, 6

[40] Jie Wen, Zheng Zhang, Zhao Zhang, Zhihao Wu, Lunke Fei, Yong Xu, and Bob Zhang. Dimc-net: Deep incomplete multi-view clustering network. In *Proceedings of the 28th ACM International Conference on Multimedia*, pages 3753–3761, 2020. 1, 2, 3

[41] Cai Xu, Ziyu Guan, Wei Zhao, Hongchang Wu, Yunfei Niu, and Beilei Ling. Adversarial incomplete multi-view clustering. In *IJCAI*, pages 3933–3939, 2019. 2, 3

[42] Jinglin Xu, Junwei Han, Feiping Nie, and Xuelong Li. Re-weighted discriminatively embedded k-means for multi-view clustering. *IEEE Transactions on Image Processing*, 26(6):3016–3027, 2017. 1

[43] Shi Yu, Leon Tranchevent, Xinhai Liu, Wolfgang Glanzel, Johan AK Suykens, Bart De Moor, and Yves Moreau. Optimized data fusion for kernel k-means clustering. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 34(5):1031–1039, 2012. 1

[44] Kun Zhan, Feiping Nie, Jing Wang, and Yi Yang. Multiview consensus graph clustering. *IEEE Transactions on Image Processing*, 28(3):1261–1270, 2018. 1

[45] Changqing Zhang, Zongbo Han, Yajie Cui, Huazhu Fu, Joey Tianyi Zhou, and Qinghua Hu. Cpm-nets: cross partial multi-view networks. In *Proceedings of the 33rd International Conference on Neural Information Processing Systems*, pages 559–569, 2019. 1

[46] P. Zhang, X. Liu, J. Xiong, S. Zhou, W. Zhao, E. Zhu, and Z. Cai. Consensus one-step multi-view subspace clustering. *IEEE Transactions on Knowledge and Data Engineering*, pages 1–1, 2020. 1

[47] Pei Zhang, Siwei Wang, Jingtao Hu, Zhen Cheng, Xifeng Guo, En Zhu, and Zhiping Cai. Adaptive weighted graph fusion incomplete multi-view subspace clustering. *Sensors*, 20(20):5755, 2020. 2

[48] Zheng Zhang, Li Liu, Fumin Shen, Heng Tao Shen, and Ling Shao. Binary multi-view clustering. *IEEE transactions on pattern analysis and machine intelligence*, 41(7):1774–1782, 2018. 1

[49] Handong Zhao, Hongfu Liu, and Yun Fu. Incomplete multi-modal visual data grouping. In *IJCAI*, pages 2392–2398, 2016. 2, 3

[50] Xiaofeng Zhu, Shichao Zhang, Wei He, Rongyao Hu, Cong Lei, and Pengfei Zhu. One-step multi-view spectral clustering. *IEEE Transactions on Knowledge and Data Engineering*, 31(10):2022–2034, 2018. 1