Noesis : Projectile Motion Documentation

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1 Introduction

Projectile motion can be explained using the principle of least action, leading to parabolic motion.

2 The Lagrangian

For 2 dimensional projectile motion, the velocity of the body can be broken down into its x and y components, as in

$$v^2 = \dot{x}^2 + \dot{y}^2 \tag{1}$$

The potential energy of the body is only due to its height; hence it is a function of y.

$$V(y) = mgy \tag{2}$$

The Lagrangian, written for one dimension,

$$\begin{aligned} L &= T(\dot{x}) - V(x) & (3) \\ &= \frac{1}{2}m\dot{x}^2 - V(x) & (4) \end{aligned}$$

can be rewritten for this problem as

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy \tag{5}$$

3 Equations Of Motion

The equations of motion can be derived from the Euler-Lagrange equations for 2 dimensions,

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{6}$$
$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \tag{7}$$

For the problem of projectile motion, these reduce to

$$m\ddot{x} = 0 \tag{8}$$
$$m\ddot{y} + mg = 0 \tag{9}$$

This gives us the functions of $x(t)$ and $y(t)$ as

$$\begin{aligned} x(t) &= c_1 t + c_2 \\ y(t) &= -\frac{gt^2}{2} + c_3 t + c_4 \end{aligned} \tag{10}$$

4 Final Equation of motion

Since we assume the motion to start from the origin, $x(0) = y(0) = 0$, we can rewrite the equations in (10) to

$$x(t) = c_1 t \tag{11}$$
$$y(t) = -\frac{gt^2}{2} + c_3 t \tag{12}$$

We also know that the maximum height H is achieved at $t = \frac{T}{2}$ and that the range of motion is denoted by R.

$$x(T) = R \implies x(t) = \frac{R}{T}t \tag{13}$$
$$y(\frac{T}{2}) = H \implies y(t) = -\frac{g}{2}t^2 + \frac{gT}{2}t \tag{14}$$

Therefore, the final equation of motion can be written as follows

$$\boxed{y(t) = -\frac{gT^2}{2R^2}\left(x^2(t) - Rx(t)\right)} \tag{15}$$