

# Noesis : Pendulum Documentation

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## 1 Introduction

A simple pendulum follows periodic, sinusoidal motion. The equation of motion can be derived from the principle of least action.

## 2 Derivation Of The Equations Of Motion

From a Lagrangian mechanics point of view, the simple pendulum has only one degree of freedom, and therefore needs one dimension, which is the angle described by the mass ( $\theta$ ).

We can write the Lagrangian (in terms of displacement from the mean position  $x$ ) as the difference between the kinetic and potential energy, which are given here.

$$\begin{aligned} L &= T - V \\ L &= \frac{1}{2}m\dot{x}^2 - mgh \end{aligned}$$

where  $h$  is the vertical height of the mass from the mean position.

We can rewrite the Lagrangian in terms of  $\theta$  using  $x = l \cdot \theta$ , where  $l$  is the length of the pendulum. We also change  $h$  to  $mgl(1 - \cos \theta)$  from trigonometric analysis.

$$L = \frac{1}{2}ml\dot{\theta}^2 - mgl(1 - \cos \theta) \quad (1)$$

We also need to calculate the damping, and add the functions to the generalized forces ( $\Theta$ ).

$$\Theta = \sum_i F_i \frac{dx_i}{d\theta} \quad (2)$$

$$= F_x \frac{dx}{d\theta} + F_y \frac{dy}{d\theta} \quad (3)$$

The damping force is assumed to be  $f(\dot{\theta}) = b\dot{\theta}$ , where  $b$  is the damping coefficient. Therefore the  $F_x$  and  $F_y$  components are the resolved vectors of the same function.

$$\Theta = b\dot{\theta}\cos\theta \frac{dx}{d\theta} - b\dot{\theta}\sin\theta \frac{dy}{d\theta} \quad (4)$$

To convert to standard rectangular coordinates, the transforms are  $x = l\sin\theta$  and  $y = l\cos\theta$ . Therefore,  $\Theta$  becomes

$$\Theta = b\dot{\theta}\cos\theta(l\cos\theta) + b\dot{\theta}\sin\theta(l\sin\theta) \quad (5)$$

$$= b\dot{\theta}l\cos^2\theta + b\dot{\theta}l\sin^2\theta \quad (6)$$

$$= bl\dot{\theta} \quad (7)$$

Plugging the Lagrangian into the Euler-Lagrange equation,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \Theta \quad (8)$$

$$-ml^2\ddot{\theta} - mgl\sin\theta = bl\dot{\theta} \quad (9)$$

This is a difficult equation to solve, therefore, we assume  $\theta$  to be a small angle, and approximate  $\sin\theta \approx \theta$ . The equation reduces to

$$ml^2\ddot{\theta} - bl\dot{\theta} + mgl\theta = 0 \quad (10)$$

$$\ddot{\theta} - \frac{b}{ml}\dot{\theta} + \frac{g}{l}\theta = 0 \quad (11)$$

which is our desired equation of motion.

### 3 Application Of Equation Of Motion

To apply the equation of motion to Noesis, the function  $\theta(t)$  must be found by solving the derived differential equation, which can be solved using the Laplace transform.

The first and second derivatives of  $\theta(t)$  transform under the Laplace transform as

$$\mathcal{L}(\dot{\theta}) = s\mathcal{L}\{\theta(t)\} - \theta(0) \quad (12)$$

$$\mathcal{L}(\ddot{\theta}) = s^2\mathcal{L}\{\theta(t)\} - s\theta(0) - \dot{\theta}(0) \quad (13)$$

For the boundary conditions, We know that the angular velocity ( $\dot{\theta}(0) = \omega$ ) is maximum at the mean position ( $\theta(0) = 0$ ). Hence, the transforms convert to

$$\mathcal{L}\{\dot{\theta}\} = s\mathcal{L}\{\theta(t)\} \quad (14)$$

$$\mathcal{L}\{\ddot{\theta}\} = s^2\mathcal{L}\{\theta(t)\} - \omega \quad (15)$$

Applying the Laplace transform to equation (11), we get

$$\mathcal{L}\{\ddot{\theta}(t)\} - \frac{b}{ml}\mathcal{L}\{\dot{\theta}(t)\} + \frac{g}{l}\mathcal{L}\{\theta(t)\} = 0 \quad (16)$$

Substituting the transforms (14) and (15) into the previous equation,

$$s^2\mathcal{L}\{\theta(t)\} - \omega - \frac{b}{ml}s\mathcal{L}\{\theta(t)\} + \frac{g}{l}\mathcal{L}\{\theta(t)\} = 0 \quad (17)$$

Solving for  $\mathcal{L}\{\theta(t)\}$ ,

$$\mathcal{L}\{\theta(t)\} = \frac{\omega}{s^2 - \frac{b}{ml}s + \frac{g}{l}} \quad (18)$$

Taking Inverse Laplace transform to obtain  $\theta(t)$ ,

$$\theta(t) = \frac{2ml\omega}{\sqrt{4m^2lg - b^2}} \sin\left(\frac{\sqrt{4m^2lg - b^2}}{2ml}t\right) \quad (19)$$

For the sake of simplicity, we denote the constant term  $A = \frac{2ml\omega}{\sqrt{4m^2lg - b^2}}$ , such that the function  $\theta(t)$  becomes

$$\theta(t) = A \sin\frac{\omega}{A}t \quad (20)$$