Real Analysis (UG1, Monsoon 2022)

Midsem Exam [15 marks]

A Wednesday!

1 Instructions

- You are allowed to bring at most one A4 sheet with only "handwritten" notes (no xerox, e-print, etc.).
- Give satisfactory reasoning. State clearly which theorem or axioms you are using.
- Please read questions carefully before you begin to answer. Turn both sides of the question paper.

Question A [$3 \times 2.5 = 7.5$ marks]

Let $C[0,1]$ be the set of functions $f:[0,1] \to \mathbb{R}$ such that functions f are continuous over $[0,1]$. In other words, $C[0,1]$ is the set of continuous functions $f:[0,1]\to\mathbb{R}$. Consider the set $C[0,1]$ to be equipped with metric d_p defined for $p \geq 1$ as

$$d_p(f,g) = \left(\int_0^1 |f(x)-g(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Answer any 3 of the following questions.

1. Show that $(C[0,1], d_p)$ is a metric space (for $p \geq 1$).
2. Consider $p, q \geq 1$ and $p \neq q$. Are metrics d_p and d_q equivalent over $C[0,1]$? Provide satisfactory justification.
3. State a necessary and sufficient condition for a subset $S \subset C[0,1]$ to be compact.
4. Consider a mapping $G: C[0,1] \to \mathbb{R}$ defined by

$$G(f) = \int_0^1 |f(x)|\mathrm{d}x.$$

Is G continuous? Provide satisfactory explanation.

Question B [$2.5 \times 3 = 7.5$ marks]

Answer any 3 of the following.

1. Prove that no order can be defined in the complex field $(\mathbf{C}, +, \cdot)$ (the set of complex numbers with conventional addition and multiplication rules) that turns it into an ordered field.

2. Let A be a nonempty set of real numbers which is bounded from below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

3. Construct a bounded set of real numbers with exactly three limit points.

4. Prove that the convergence of a series $\sum_{n=1}^{\infty} a_n$, where $a_n \in \mathbf{R}$ for all $n \in \mathbf{N}$, implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \quad \text{if} \quad a_n \geq 0.$$

5. If a series $\sum_{n=1}^{\infty} a_n$ of real numbers converges, and if a sequence $\{b_n\}$ of real numbers is monotonic and bounded, prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.