Quiz: Probability and Statistics (30 Marks)

Instruction:

- Please state reasons wherever applicable.

- Use precise mathematical arguments, no speeches.

Each question: 5 marks

1. Consider a discrete random variable X. Prove that for $a, b \in \mathbb{R}$ and $a < b$, $P(a < X \leq b) = F_X(b) - F_X(a)$ where $F_X(\cdot)$ denotes the CDF of X.

2. A geometric random variable X with parameter p has PMF given by $p_X(k) = (1 - p)^{k-1}p$. Derive the expression for its mean and variance.

3. In a game show, you have to choose one of three doors. Behind one door is a car and behind the other two doors is a goat. Once you choose a door, the presenter will open one of the other doors that has a goat. He offers you an opportunity to change your choice to the third unopened and unchosen door. Will you do so? Why? (Give mathematical/probabilistic proof justifying your arguments using Bayes rules/conditioning etc.)

4. Using probability axioms, prove the following

   - $P(\cap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$.
   - Probability that exactly one of the two events A and B occurs is $P(A) + P(B) - 2P(A \cap B)$

5. Give the definition of a sigma algebra. Prove that it is closed under countable intersections.

6. Airlines find that any passenger who books a seat, often fails to turn up with probability $\frac{1}{10}$. Indigo always sells 10 tickets for their 9 seat aeroplane. AirIndia sells 20 tickets for their 18 seat aeroplane. Which airline is more overbooked. Justify your answer. (Hint: Invoke Binomial $B(n, p)$ random variable)