**Question:** Verify that:

$$P(C|A) = P(C|BA)P(B|A) + P(C|B^cA)P(B^c|A)$$

Where, $AB$ means $A \cap B$.

**Answer:**

Take the RHS,

We can expand $P(C|BA)$ as:

$$P(C \cap B \cap A)/P(B \cap A)$$

Similarly, we can write $P(C|B^cA)$ as:

$$P(C \cap B^c \cap A)/P(B^c \cap A)$$

We can expand it as:

$$\frac{P(C \cap B \cap A)}{P(B \cap A)} \times \frac{P(B \cap A)}{P(A)} + \frac{P(C \cap B^c \cap A)}{P(B^c \cap A)} \times \frac{P(B^c \cap A)}{P(A)}$$

Which gives:

$$\frac{P(C \cap B \cap A)}{P(A)} + \frac{P(C \cap B^c \cap A)}{P(A)}$$

$P(CBA)$ and $P(CB^cA)$ are the probabilities of mutually exclusive events.

$$= \frac{1}{P(A)} \times P(CBA \cup CB^cA)$$
$$= \frac{P(CA)}{P(A)} or \frac{P(C \cap A)}{P(A)}$$

Thus, proved.

# Quiz-2: Probability and Statistics (30 Marks)

[Instruction: Please state reasons wherever applicable.]

## 5 Marks

**Find the stationary distribution $\pi$ for Markov Chains with the following transition probability matrix (3 marks). State if $\pi$ is unique in each case (1 mark). Also which of the two chains are irreducible? Give reasons (1 mark).**

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \textbf{ and } Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Solution**

Stationary distribution $\pi$ for a Discrete Markov Chain, given it's transition probability matrix $P$, is given as:

$$\pi P = \pi$$

**Stationary Distribution for P (1.5 marks)**

Given $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} \pi P &= \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \\ &= \pi \end{aligned}$$

This, holds for all $\pi$ (since $P$ is an identity matrix).

$\therefore$ Stationary distribution for $P$

$$= \begin{bmatrix} p & 1-p \end{bmatrix} \text{ where } 0 \le p \le 1,\ p \in \mathbb{R}$$

**Stationary Distribution for Q (1.5 marks)**

Given $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{aligned} \pi Q &= \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \pi_2 & \pi_1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \pi Q &= \pi \\ \implies \begin{bmatrix} \pi_2 & \pi_1 \end{bmatrix} &= \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \\ \implies \pi_1 = \pi_2 &= \frac{1}{2} \qquad (\because \pi_1 + \pi_2 = 1) \end{aligned}$$

$\therefore$ Stationary distribution for $Q = \left[ \frac{1}{2} \; \frac{1}{2} \right]$.

**State if $\pi$ is unique (0.5 + 0.5 mark)**

- Stationary distribution for $P$ is not unique. (since any $\pi$ can be its stationary distribution).
- Stationary distribution for $Q$ is unique.

**Which of the two chains are irreducible? (1 mark)**

We know $P = I \implies P^n = I^n = I$ where $I$ is the Identity matrix of order 2.

$$\implies P_{12}^n = 0$$

Thus, state 2 is not accessible from state 1. This is sufficient to show that $P$ is not an irreducible chain.

$$Q^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{when } n \text{ is odd} \\ \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{when } n \text{ is even} \end{cases}$$

We see that states 1 and 2 communicate with each other *i.e.,* $P_{12}^n > 0$ and $P_{21}^n > 0$ (when $n$ is odd). Thus, $Q$ is an irreducible markov chain.

# Quiz: Probability and Statistics

October 28, 2022

## Question 3

Suppose $X$ is an exponential random variable with parameter $\lambda$ and CDF denoted by $F_X(.)$. $U$ is a uniform random variable over the interval $[0, 1]$. Now consider another random variable $Y = F_X^{-1}(U)$. Then derive the expression for the CDF $F_Y(y)$.

### Solution

$$\begin{aligned}
F_Y(y) &= P(F_X^{-1}(U) \leq y) \\
&= P(U \leq F_X(y)) \\
&= P(U \leq 1 - e^{-\lambda y}) \\
&= F_U(1 - e^{-\lambda y}) \\
&= 1 - e^{-\lambda y}
\end{aligned}$$

# Quiz 2: Probability and Statistics

[Instruction: Please state reasons wherever applicable.]

## 1 5 Marks

1. Consider a sequence of random variables $\{X_n\}$ where $X_n \sim Exponential(n)$. Show that $X_n$ converges to X in probability where $X = 0$ with probability 1. Also show that $X_n$ converges to X in distribution (without using the fact that convergence in probability implies convergence in distribution).

### Solution:

**a.**

$$\begin{aligned}
\lim_{n\to\infty} P(|X_n - 0| \geq \epsilon) &= \lim_{n\to\infty} P(X_n \geq \epsilon) && [\because X_n \geq 0]\ \{\textit{0.5 Marks}\} \\
&= \lim_{n\to\infty} e^{-n\epsilon} && [\because X_n \sim Exponential(n)]\ \{\textit{1 Mark}\} \\
&= 0 && \{\textit{1 Mark}\}
\end{aligned}$$

*Hence Proved.*

**b.**

$$\begin{aligned}
\lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty} 1 - e^{-nx} && \{\textit{1 Mark}\} \\
&= 1 && \{\textit{0.5 Mark}\} \\
&= F_X(x) && [\forall x > 0][\because P_X(0) = 1]
\end{aligned}$$

*Note that at $x = 0$, $F_X(x)$ is discontinuous*

$\implies$ Convergence in distribution doesn't take place at $x = 0$ $\{$*0.5 Mark*$\}$

$\implies$ Convergence in distribution takes place $\forall x > 0$ $\{$*0.5 Mark*$\}$

*Hence Proved.*

**Marks Division**

(a) Convergence in probability (2.5M)

   i. 0.5M for identifying $P(|X_n - 0| \geq \epsilon) = P(X_n \geq \epsilon)$

   ii. 1M for getting to $e^{-n\epsilon}$

   iii. 1M for getting to final step

(b) Convergence in distribution (2.5M)

   i. 1M for writing CDF of $X_n$

   ii. 0.5M for getting to 1

   iii. 0.5M for accounting for discontinuity

   iv. 0.5M for getting $F_X(x) = 1 \quad \forall x > 0$

**Note:** Simple stating of final answers without any logical approach will be given 0.

# Probability and Statistics: Quiz 2

October 2022

## Question 1 (10 Marks)

**Given:** Three samples $u_1 = 0.23$, $u_2 = 0.73$ and $u_3 = 0.5$ from uniform random variable. We will use the inverse transform method in all the following parts to convert the given sample to the required.

1. Let $X$ be the random variable denoting the outcome of a fair dice. Now

$$\mathrm{F}_x(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/6 & \text{if } x \geq 1 \; and \; x < 2 \\ 2/6 & \text{if } x \geq 2 \; and \; x < 3 \\ 3/6 & \text{if } x \geq 3 \; and \; x < 4 \\ 4/6 & \text{if } x \geq 4 \; and \; x < 5 \\ 5/6 & \text{if } x \geq 5 \; and \; x < 6 \\ 1 & \text{if } x \geq 6 \end{cases}$$

Now we know by the Lemma of inverse transform method that if:
$X := F^{-1}(U)$
Then the cdf of $X$ is $F$. Hence, applying the inverse transform method, we get:

$$\mathrm{X} = \begin{cases} 1 & \text{if } p < 1/6 \\ 2 & \text{if } p > 1/6 \; and \; p \leq 2/6 \\ 3 & \text{if } p > 2/6 \; and \; p \leq 3/6 \\ 4 & \text{if } p > 3/6 \; and \; p \leq 4/6 \\ 5 & \text{if } p > 4/6 \; and \; p \leq 5/6 \\ 6 & \text{if } p > 5/6 \; and \; p \leq 6/6 \end{cases}$$

Where $p$ is a realization from uniform random variables. Hence, applying this, we get the following samples for $X$.

- $u_1 = 0.23$ generates $x = 2$ as a sample.
- $u_2 = 0.78$ generates $x = 5$ as a sample.
- $u_3 = 0.5$ generates $x = 3$ *or* $x = 4$ as a sample.

2. Let $X$ be a random variable with 0.7 as a probability of getting a head. Let head be $X = 0$ and tail be $X = 1$

$$\mathrm{F}_x(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.7 & \text{if } x \geq 0 \ and \ x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Applying the inverse transform method, we get:

$$\mathrm{X} = \begin{cases} 0 & \text{if } p \leq 0.7 \\ 1 & \text{if } p > 0.7 \ and \ p \leq 1 \end{cases}$$

Where $p$ is a realization from uniform random variables. Hence, applying this, we get the following samples for $X$.

- $u_1 = 0.23$ generates $x = 0$ i.e Heads as a sample.
- $u_2 = 0.78$ generates $x = 1$ i.e Tails as a sample.
- $u_3 = 0.5$ generates $x = 0$ i.e Heads as a sample.

3. Let $X$ be the exponential random variable with parameter $\lambda = 1$. We know

   $f_x(x) = \lambda e^{-\lambda x}$
   Also, cdf of $f_x(x)$ is written as
   $F_x(x) = 1 - e^{-\lambda x}$
   $F_x(x) = 1 - e^{-x}$ as $\lambda = 1$

   Using the lemma of inverse transform method, we substitute $F_x(x)$ with $U$ and thus, we have
   $U = 1 - e^{-x}$
   $x = -ln(1 - U)$

   Since, $U$ and $1-U$ are equivalent, since both are uniform random variable over $(0, 1)$, we can replace $U$ and $1 - U$.

   $X = -ln(U)$

   Where $U$ is a realization from uniform random variables. Hence, applying this, we get the following samples for $X$.

   - $u_1 = 0.23$ generates $x = -ln(0.23)$ or $x = -ln(0.77)$ as a sample.
   - $u_2 = 0.78$ generates $x = -ln(0.78)$ or $x = -ln(0.22)$ as a sample.
   - $u_3 = 0.5$ generates $x = -ln(0.5)$ as a sample.

4. Let $X$ be the inform random variable in the interval $[5, 10]$
   $f_x(x) = \frac{1}{10-5} = 1/5$

   $$\mathrm{F}_x(x) = \begin{cases} 0 & \text{if } x < 5 \\ \frac{x-5}{5} & \text{if } x \geq 5 \ and \ x \leq 10 \\ 1 & \text{if } x > 10 \end{cases}$$

   Using the lemma of inverse transform method, we substitute $F_x(x)$ with $U$ and thus, we have

$U = \frac{X-5}{5}$
$X = 5U + 5$

- $u_1 = 0.23$ generates $x = 6.15$ as a sample.
- $u_2 = 0.78$ generates $x = 8.9$ as a sample.
- $u_3 = 0.5$ generates $x = 7.5$ as a sample.

## Marking Scheme

- Part 1: 2 Marks
- Part 2: 2 Marks
- Part 3: 3 Marks
- Part 4: 3 Marks

**Note:** Marks will be deducted if the inequalities are incorrect.