# Digital Signal Analysis - Midsem Solutions

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## SET-1

### Q1

(a) State whether below signals are periodic or not. Justify:

(i) $x(n) = \cos^2(2n + \frac{\pi}{2})$

(ii) $x(n) = u(n) + u(n-1)$

(iii) $x(n) = \delta(n)$

**Solution:**

a) $x(n) = \cos^2(2n + \frac{\pi}{2})$
Let $N$ be the fundamental period of $x[n]$. Then,

$$\begin{aligned} x[N+n] &= x[n] \\ \cos^2\left(2n + \frac{\pi}{2}\right) &= \cos^2\left(2(N+n) + \frac{\pi}{2}\right) \\ \cos(4n+\pi) &= \cos(4(n+N)+\pi) \\ 4n+\pi &= 4(N+n) \pm 2k\pi \\ N &= (2k-1)\frac{\pi}{4} \end{aligned}$$

But $N$ is an irrational value, and hence cannot be a period as it is supposed to be taking only integral values for a discrete signal.

b) To check if this signal is periodic, we need to verify if there exists a positive integer $N$ such that:

$$x(n) = x(n+N)$$

for all values of $n$.

Let's analyze the signal:

$$x(n) = u(n) + u(n-1)$$

Given signal: $x(n) = \begin{cases} 2 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

To check if this signal is periodic, we need to verify if there exists a positive integer $N$ such that:

$$x(n) = x(n + N)$$

for all values of $n$.

Let's specifically check the condition for $n = 0$:

$$x(0) = x(0 + N)$$

$$1 \neq x(N) = 2$$

Since $1 \neq 2$ for any positive integer $N$, the signal $x(n)$ is not periodic.

c) To check if $x[n] = \delta(n)$ is periodic, we need to verify if there exists a positive integer $N$ such that $x[n] = x[n+N]$ for all $n$. However, since $\delta(n)$ is only nonzero at $n = 0$, we have $x[n] = 1$ at $n = 0$ and $x[n + N] = 0$ for $N > 0$. Thus, $x[n]$ does not satisfy the periodicity condition for any $N > 0$. Hence, $x[n] = \delta(n)$ is not periodic.

(b) State whether below signal is energy signal or power signal or neither : $x[n] = -a^n u(-n-1)$
**Solution:** Energy of the signal is given by

$$\begin{aligned}
E &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |(-a^n u(-n-1))^2| \\
&= \sum_{n=-\infty}^{\infty} |(a^n u(-n-1))^2| \\
&= \sum_{n=-\infty}^{-1} |(-a^n \cdot 1)^2| + \sum_{n=0}^{\infty} |(-a^n \cdot 0)^2| \\
&= \sum_{n=-\infty}^{-1} |(-a^{2n})| \\
&= \begin{cases} \frac{1}{a^2-1} & \text{if } |a| > 1 \\ \infty & \text{otherwise} \end{cases}
\end{aligned}$$

Power of the signal is given by

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{-1} |(a^n)^2| = \frac{1}{a^{-2}-1} \lim_{N\to\infty} \frac{a^{-2N-2} - a^{-2}}{2N+1} = \begin{cases} \infty & \text{if } |a| < 1 \\ \frac{1}{2} & \text{if } |a| = 1 \\ 0 & \text{if } |a| > 1 \end{cases}$$

When $|a| < 1$ - Energy and power are infinite, so it's neither power nor energy signal.
When $|a| = 1$ - Energy is infinite, power is finite. Hence it's a power signal.
When $|a| > 1$ - Energy is finite, power is 0. Hence it's an energy signal.

## Q2

State and Prove Convolution property of Fourier Transform

**Solution:**

Let $f(t)$ and $g(t)$ be two functions with Fourier transforms $F(\omega)$ and $G(\omega)$ respectively. Then, the convolution of $f(t)$ and $g(t)$ is given by:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$$

The Fourier transform of the convolution $f * g$ is given by:

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} (f * g)(t)e^{-i\omega t}\,dt$$

Using the definition of convolution, we have:

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau\right) e^{-i\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-i\omega t}\,d\tau dt$$

Now, we change the order of integration:

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-i\omega t}\,dt d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)e^{-i\omega(t-\tau)}e^{-i\omega\tau}\,dt d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} g(t-\tau)e^{-i\omega(t-\tau)}\,dt\right) e^{-i\omega\tau}\,d\tau$$

Now, let $v = t - \tau$, then $dt = dv$ and when $t = -\infty$, $v = -\infty$, and when $t = \infty$, $v = \infty$.

$$= \int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} g(v)e^{-i\omega v}\,dv\right) e^{-i\omega\tau}\,d\tau$$

$$= \left(\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,d\tau\right)\left(\int_{-\infty}^{\infty} g(v)e^{-i\omega v}\,dv\right)$$

$$= F(\omega)\cdot G(\omega)$$

Hence, the Fourier transform of the convolution of $f(t)$ and $g(t)$ is equal to the pointwise product of their individual Fourier transforms.

## Q3

A signal has amplitude of -5V to 5V. If maximum quantisation error should be less than 0.1 , how many bits are required for Quantisation ?

**Solution:** To find the number of bits required for quantization such that the maximum quantization error is less than 0.1, we can use the formula:

$$\text{Maximum Quantisation Error} = \frac{V_{\text{max}} - V_{\text{min}}}{2^{N+1}}$$

Given that $V_{\text{max}} = 5$ V, $V_{\text{min}} = -5$ V, and the maximum quantization error should be less than 0.1 V, we can rearrange the formula to solve for $N$, the number of bits:

$$N = \log_2 \left( \frac{V_{\text{max}} - V_{\text{min}}}{\text{Maximum Quantisation Error}} \right) - 1$$

Substituting the given values:

$$N = \log_2 \left( \frac{5 - (-5)}{0.1} \right) - 1$$
$$N \geq \log_2(100) - 1$$
$$N = \log_2(2^7) - 1$$
$$N = 7 - 1$$
$$N = 6$$

So, 6 bits are required for quantization.

## Q4

Let $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$. Find the Discrete Fourier Transform (DFT).

**Solution:** To find the Discrete Fourier Transform (DFT) of the given sequence $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$, we'll use the formula for the DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i2\pi \frac{kn}{N}}, \quad k = 0, 1, \ldots, N-1$$

Where $N$ is the length of the sequence, $x[n]$ is the sequence, and $X[k]$ is the DFT coefficient at frequency bin $k$.

Given $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$, and $N = 8$ (since the sequence has 8 elements), let's compute $X[k]$ for $k = 0, 1, \ldots, 7$.

For $k = 0$:

$$
\begin{aligned}
X[0] &= \sum_{n=0}^{7} x[n] \cdot e^{-i2\pi \frac{0 \cdot n}{8}} \\
&= \sum_{n=0}^{7} x[n] \cdot e^{0} \\
&= \sum_{n=0}^{7} x[n] \\
&= 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 \\
&= 4
\end{aligned}
$$

Similarly , X[4] = 4

For $k = 1$:

$$
\begin{aligned}
X[1] &= \sum_{n=0}^{7} x[n] \cdot e^{-i2\pi \frac{1 \cdot n}{8}} \\
&= \sum_{n=0}^{7} x[n] \cdot e^{-i \frac{\pi n}{4}} \\
&= 1 \cdot e^{-i \frac{\pi \cdot 0}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 1}{4}} + 1 \cdot e^{-i \frac{\pi \cdot 2}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 3}{4}} + 1 \cdot e^{-i \frac{\pi \cdot 4}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 5}{4}} + 1 \cdot e^{-i \frac{\pi \cdot 6}{4}} + 0 \cdot e^{-i \frac{\pi \cdot 7}{4}} \\
&= 0
\end{aligned}
$$

For $k = 2, 3, \ldots, 7$:

$$X[k] = 0$$

So, the DFT of the sequence $x[n] = \{1, 0, 1, 0, 1, 0, 1, 0\}$ is $X[k] = \{4, 0, 0, 0, 4, 0, 0, 0\}$.

## Q5

Let $x_1[n] = \{4, 5, 6\}$ and $x_2[n] = \{1, 2, 3, 4\}$. Calculate Linear convolution using Circular convolution.
**Solution:**

$$\text{Length of linear convolution} = l_1 + l_2 - 1 = 4 + 3 - 1$$

$$\text{Length of circular convolution} = \max(l_1, l_2) = 4$$

In order to calculate linear convolution from circular convolution, we pad $x_2(n)$ with 2 zeroes and $x_1(n)$ with 3 zeroes:

$$
\begin{bmatrix}
1 & 0 & 0 & 4 & 3 & 2 \\
2 & 1 & 0 & 0 & 4 & 3 \\
3 & 2 & 1 & 0 & 0 & 4 \\
4 & 3 & 2 & 1 & 0 & 0 \\
0 & 4 & 3 & 2 & 1 & 0 \\
0 & 0 & 4 & 3 & 2 & 1
\end{bmatrix}
\times
\begin{bmatrix} 4 \\ 5 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix} 4 \\ 13 \\ 28 \\ 43 \\ 38 \\ 24 \end{bmatrix}
$$

## Q6

(a) Check whether the below system is causal or not. Justify.

$$y(n+4) = x(n+3) + x(n+2) + y(n+3) + x(n-4)$$

**Solution:** Causal system is a system in which the output depends only on present-past inputs and past outputs.

Re-adjusting the above equation by replacing $n$ with $n-4$, the equation becomes:

$$y((n-4)+4) = x((n-4)+3) + x((n-4)+2) + y((n-4)+3) + x((n-4)-4)$$

$$\Rightarrow y(n) = x(n-1) + x(n-2) + y(n-1) + x(n-8)$$

Since the equation $y(n) = x(n-1) + x(n-2) + y(n-1) + x(n-8)$ satisfies the condition where the output depends only on present and past inputs and past outputs, it is a causal system.

(b) Check whether the below system is LTI or not. Justify.

(i) $y(n) = x(n-2) + y(2n-1)$

(ii) $y(n) = x(n-1) + y(n-1) + 4$

**Solution:**

(i) Let $x_1(n-2)$ be the input for $y_1(n)$ and $x_2(n-2)$ be the input for $y_2(n)$.

Let $y'$ be the output for $a_1x_1 + a_2x_2$.

We have:

$$y'(n) = a_1x_1 + a_2x_2 + y'(2n-1)$$

We'll show that:

$$y'(n) - y'(2n-1) = a_1(y_1(n) - y_1(2n-1)) + a_2(y_2(n) - y_2(2n-1))$$

$$\begin{aligned} y'(n) - y'(2n-1) &= a_1(y_1(n) - y_1(2n-1)) + a_2(y_2(n) - y_2(2n-1)) \\ &= a_1y_1(n) + a_2y_2(n) \end{aligned}$$

Thus, $y'(n) = a_1y_1(n) + a_2y_2(n)$.

**Hence, the system is linear.**

Let $x_1(n) = x(n-k) \to y_1(n)$.

We have:

$$\begin{aligned} &\text{Equation - 1: } y_1(n) = x_1(n-2) + y_1(2n-1) \\ &\text{Equation - 2: } y(n-k) = x(n-k-2) + y(2n-2k-1) \end{aligned}$$

But $x_1(n-2) = x(n-k-2)$.

So, we have:
$$y_1(n) - y_1(2n-1) = y(n-k) - y(2n-2k-1)$$

If $y_1(n) = y(n-k)$, but $y_1(2n-1) = y(2n-k-1) \neq y(2n-2k-1)$.

**Hence, the system is time variant.**

**Therefore , the system is not LTI**

(ii) Let $x_1(n-1)$ be the input for $y_1(n)$ and $x_2(n-1)$ be the input for $y_2(n)$.

Let $y'$ be the output for $a_1x_1 + a_2x_2$.

We have:

$$y'(n) = a_1x_1 + a_2x_2 + y'(n-1) + 4$$

We'll show that:

$$y'(n) - y'(n-1) = a_1(y_1(n) - y_1(n-1) - 4) + a_2(y_2(n) - y_2(n-1) - 4) + 4$$

$$y'(n) - y'(n-1) = a_1(y_1(n) - y_1(n-1)) + a_2(y_2(n) - y_2(n-1)) - 4(a_1 + a_2 - 1)$$

Thus, $y'(n) \neq a_1y_1(n) + a_2y_2(n)$.

Since the expression $y'(n)$ does not match the expected linear combination of $y_1(n)$ and $y_2(n)$, the system is non-linear, and this is due to the presence of the constant term 4.

**Hence, the system is non-linear.**

Let $x_1(n) = x(n-k) \to y_1(n)$.

We have:

$$\begin{aligned}
&\text{Equation - 1: } y_1(n) = x_1(n-1) + y_1(n-1) + 4 \\
&\text{Equation - 2: } y(n-k) = x(n-k-1) + y(n-k-1) + 4
\end{aligned}$$

But $x_1(n-1) = x(n-k-1)$.

So, we have:
$$y_1(n) = y(n-k)$$

This indicates that the output for a delayed input in the first equation is equal to the output for the corresponding delayed input in the second equation, satisfying the time-invariance property.

**Thus, the system is time-invariant.**

**Therefore , the system is not LTI**