Midsem: Probability and Statistics (50 Marks)

[Instruction: Please state reasons wherever applicable.]

5 marks

1. For a continuous non-negative random variable X prove that $E[X^2] = 2\int_x x\bar{F}_X(x)dx$ where $\bar{F}_X(x) = 1 - F_X(x)$.

2. Let X be a continuous random variable with distribution $F_X(\cdot)$ and density $f_X(x)$. Find the density and distribution for $Z = \sqrt{X}$.

3. Consider two exponential random variables X and Y with parameters λ_1 and λ_2 respectively. Consider $Z = min(X, Y)$ (min stands for minimum). Find the probability density and cumulative distribution of Z.

4. Let X and Y denote Gaussian random variables with mean μ_1 and μ_2 and standard deviation σ_1 and σ_2 respectively. Consider $Z = X + Y$. Using Moment generating functions, show that Z is also a Gaussian random variable, with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

5. Let U_1 and U_2 be two independent Uniform random variables with support $[0, 1]$. Then find the cdf or pdf of $U_1 + U_2$.

6. If X and Y are independent random variables, prove that $Var(X + Y) = Var(X) + Var(Y)$. (Recall that $Var(X) = E[(X - E[X])^2]$)

Please turn over for 10 marks questions

10 marks

1. $X_1, X_2, \ldots, X_n$ are independent and identically dsitributed $Bernoulli(p)$ random variables (i.e., they take the value 1 with probability p and 0 otherwise). Consider $S_n = \sum_{i=1}^{n} X_n$. Find the PMF, MGF, mean and variance of S_n.

2a (5 marks) The joint probability mass function of the discrete random variables X and Y are given by $p_{X,Y}(x, y) = \frac{1}{2^{x+y}}$, $x = 1, 2., \ldots$ and $y = 1, 2, \ldots$.

   (a) Find the expression for the marginal pmf $p_X(x)$ and $p_Y(y)$ and the conditional pmf $p_{X|Y}(x|y)$.

   (b) Find $E[XY]$ and determine if the RV X and Y are independent.

2b (5 marks) The joint pdf of random variables X and Y is given by $f_{X,Y}(x, y) = \lambda\mu e^{-\lambda x - \mu y}$, $x \geq 0, y \geq 0, \lambda > 0, \mu > 0$.

   (a) Find the expression for the marginal pdf's $f_X(x)$ and $f_Y(y)$ and the joint CDF $F_{X,Y}(x, y)$

   (b) Are the RV X and Y independent? Give reasons.