MA2.101: Linear Algebra (Spring 2022)

Exam

Wednesday, 28 March 2024

Course outcomes: **CO1, CO3, CO6.**

1. ([4 marks]) Solve one of the following.

   - The system of equations

$$
\begin{aligned}
x + y + z &= 6 \\
x + 4y + 6z &= 20 \\
x + 4y + \lambda z &= \phi.
\end{aligned}
$$

     Find the values of λ and ϕ for which this system of equations has no solutions.

   - If $A\mathbf{x} = \mathbf{b}$ always has at least one solution, show that the only solution to $A^T\mathbf{y} = \mathbf{0}$ is $\mathbf{y} = \mathbf{0}$. Here A^T denotes the transposition of matrix A.

2. ([3 marks]) V is a finite-dimensional vector space and let $T : V \to V$ be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and nullspace of T have only the zero vector $\mathbf{0}$ in common.

3. ([4 marks]) Two vector spaces are called *isomorphic* if there exists an invertible linear transformation from one vector space onto the other one. Prove that two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

4. ([4 marks]) Solve one of the following.

   (a) Prove both of the following statements.

   - The image or the range of a linear transformation $T : V \to W$ is a subspace of W.
   - A linear transformation $T : V \to W$ is one-to-one if and only if the nullspace of T only contains $\mathbf{0} \in V$.

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(b) Consider the ordered bases $\mathcal{A} = \{(1, 2), (-2, -3)\}$ and $\mathcal{B} = \{(2, 1), (1, 3)\}$ for a vector space $\mathbf{V}$. Then find the following

- Matrix P that changes coordinates of any vector $\vec{\alpha} \in \mathbf{V}$ w.r.t. the ordered basis $\mathcal{A}$ to coordinates w.r.t. the ordered basis $\mathcal{B}$.
- Matrix Q that changes coordinates of any vector $\vec{\alpha} \in \mathbf{V}$ w.r.t. the ordered basis $\mathcal{B}$ to coordinates w.r.t. the ordered basis $\mathcal{A}$.