SC3.316: Mathematical Methods in Biology

Final (Spring 2024)

Duration: 3 hrs

Definitions/notations:

- Consider a reaction network $G = (V, E)$. Let $\{\omega_y\}_{y \in \mathcal{C}}$ be the standard basis for $\mathbb{R}^{\mathcal{C}}$. Then
  1. $\Delta_{\rightarrow} = \{\omega_{y'} - \omega_y \in \mathbb{R}^{\mathcal{C}} : y \rightarrow y' \in E\}$.
  2. $\Delta = \{\omega_{y'} - \omega_y \in \mathbb{R}^{\mathcal{C}} : y \sim y' \in E\}$. (Here $y \sim y'$ means that y and y' are in the same linkage class.)
- $\mathrm{supp}(y) = \{i \mid y_i \neq 0\}$.

Questions

1. (10 points) Show that complex balanced dynamical systems are quasi-thermodynamic.
2. (5 points) Consider the following dynamical system

$$3X + 2Y \xrightarrow{k_1} 4X + 3Z$$
$$2X + Y \xrightarrow{k_2} 3Y$$
$$2X + Z \xrightarrow{k_3} 4X + Y$$

   Write the dynamical system generated by the reaction network above in the following form: $\frac{dc}{dt} = YA_k\Psi$, where Y, A_k, Ψ denote the usual symbols used in the class and $c = (x, y, z)^T$ denotes the vector of concentrations corresponding to the species X, Y, Z.

3. (10 points) Consider a deficiency zero reaction network whose dynamics is given by $\frac{dx}{dt} = YA_k\Psi$, where Y, A_k, Ψ denote the usual symbols. Then show that $\ker(YA_k) \subseteq \ker(A_k)$.
4. Show that the following holds:
   (a) (5 points) $\mathrm{span}(\Delta) = \mathrm{span}(\Delta_{\rightarrow})$.
   (b) (5 points) For a reaction network having n complexes and ℓ linkage classes, we have $\dim(\mathrm{span}(\Delta)) = \dim(\mathrm{span}(\Delta_{\rightarrow})) = n - \ell$.
5. (5 points) State the Shinar-Feinberg criterion for a dynamical system to exhibit absolute concentration robustness.
6. (10 points) Consider the following reaction network:

$$X + 2Y \xrightarrow{k_1} 3Y$$
$$Y \xrightarrow{k_2} X$$
$$X + Y \underset{k_4}{\overset{k_3}{\rightleftharpoons}} Z$$

   - Does the above network satisfy the conditions of the absolute concentration robustness theorem with respect to all three species? (5 marks)
   - Is there absolute concentration robustness in any species? Justify. (5 marks)

7. Consider the following reaction network:

$$A + M \rightleftharpoons X$$
$$B + N \rightleftharpoons Y$$
$$Y \rightarrow 2A + N$$
$$B + X \rightleftharpoons Z$$
$$Z \rightarrow R + M$$

- Draw the species-reaction graph corresponding to the reaction network above. (10)
- Does the fully open extension of the network above have the capacity for multiple equilibria? (5)

8. Define the following terms:

(a) (3 points) Dynamically equivalent systems

(b) (3 points) Persistence

(c) (4 points) Permanence

9. State True or False with justification.

(a) (2 points) Union of siphons is a siphon.

(b) (2 points) Intersection of siphons is a siphon.

(c) (3 points) Union of critical sets is critical.

(d) (3 points) Any subset of a critical set is critical.

10. (15 points) Consider a reaction network. Let S denote the set of all species and let $T \subseteq S$ be a subset of species that is a minimal siphon (note that a set is minimal if it is the smallest set possessing that property. In this case, it means that there is no subset of T which is a siphon). A set A is said to be closed if and only if for every reaction $y \rightarrow y'$, if $\text{supp}(y) \subseteq A$, then $\text{supp}(y') \subseteq A$. For any set A, let $\text{Cl}(A)$ denote the smallest closed set containing A. Then for every species $i \in T$, prove that $\text{Cl}(\{i\} \cup (S - T)) = S$.