Real Analysis

Mid-Sem 2022

Full marks 50 (10 × 5)

1. a) If A and B are sets, then show that

$$(i)\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B), \quad ii) \quad \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B).$$

Here $\mathcal{P}$ denotes powerset.
b) Prove that a set and its powerset do not have the same cardinality.

2. Prove that for $p \in (1, \infty)$, we have $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$, with $(x, y) \in \mathbb{R}^+$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Set $s = supS$. Show that there exists a sequence $\{x_n\}$ in S with $n \in \mathbb{N}$, which converges to s.

4. Show that $\{x_n\}$ defined by

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdot \; \cdot \; \cdot \; \cdot \; \frac{1}{n} - log_e n,$$

is convergent.

5. Let $\{x_n\}$ be a sequence defined by

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{x_n^2 + \frac{1}{2^n}}.$$

Show that the sequence is convergent.