

Real Analysis  
End-Sem 2022  
Full marks 100 (10 × 10) Time - 3 hours

1. Prove that  $\sqrt{2}$  is not rational.

2. Consider the Fibonacci numbers  $\{F_n\}$  defined by  $F_1 = 1$ ,  $F_2 = 1$ , and  $F_{n+2} = F_{n+1} + F_n$ . Show that

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}, \quad n = 1, 2, 3, \dots$$

3. Show that the sequence  $\{x_n\}$  defined by  $x_n = \int_1^n \frac{\cos t}{t^2} dt$  is Cauchy.

4. Discuss the convergence or divergence of

$$x_n = \frac{[\alpha] + [2\alpha] + [3\alpha] + \dots + [n\alpha]}{n^2}, \quad n \in \mathbb{N},$$

where  $[x]$  represents the greatest integer less than or equal to the  $x$  and  $\alpha$  is an arbitrary real number.

5. Given  $x \geq 1$ , show that  $\lim_{n \rightarrow \infty} (2x^{1/n} - 1)^n = x^2$ .

6. Let  $f(x) = [x]$  and  $g(x) = x - [x]$ . Sketch the plots for  $f$  and  $g$ . Find the points at which they are continuous.

7. Show that any function continuous and periodic on  $\mathbb{R}$  must be uniformly continuous.

8. Show that there exists a continuous function  $F : [0, 1] \rightarrow \mathbb{R}$  whose derivative exists and equals zero almost everywhere but which is not constant.

9. Let  $f(x)$  is differentiable at  $a$ . Then find

$$\lim_{n \rightarrow \infty} \frac{a^n f(x) - x^n f(a)}{x - a}, \quad n \in \mathbb{N}.$$

10. Consider a function  $f(x)$ , whose second derivative  $f''(x)$  exists and continuous on  $(a, b)$  with  $c \in (a, b)$ . Show that

$$\lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Is the existence of the second derivative necessary to prove the existence of the above limit?