End Semester Examination

Discrete Structures
IIIT Hyderabad, Monsoon 2023

November 25, 2023

There are *ten* questions, 10 marks each. Calculators are allowed. Maximum Marks: 100.

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1. Fill in the following blanks: $10 \times 1 = 10$

   1. The coefficient of x^9y^3 in $(2x - 3y)^{12}$ is ________.
   2. The number of arrangements of the letters in MISSISSIPPI having no consecutive S's is ________.
   3. The number of positive integer solutions for $a + b + c + d = 10$ is ________.
   4. If two integers are selected at random without replacement from $\{1, 2, \ldots, 100\}$, the probability that the integers are consecutive is ________.
   5. If Pr(A) = 0.5, Pr(B) = 0.3, and $\Pr(A|B) + \Pr(B|A) = 0.8$, then $\Pr(A \cap B)$ is ________.
   6. State True or False: If eight people are in a room, at least two of them have birthdays that occur on the same day of the week: ________.
   7. State True or False: Let triangle ABC be equilateral with AB=1. If we select 10 points in the interiorof this triangle, there must be at least two whose distance apart is less than 1/3: ________.
   8. How many times must we roll a single die in order to get the same score at least thrice? ________.
   9. Solution to $a_{n+2} - 4a_{n+1} + 3a_n + 200 = 0$, with $a_0 = 2$, $a_1 = 104$ is ________.
   10. The chromatic number (minimum number of colors required to properly color a graph) of a connected bipartite graph is ________.

2. Give an example for each of the following: $2 + 3 + 5 = 10$

   1. A simple undirected graph G that has an Euler circuit (a circuit that has every edge once) and an example of a way to orient the edges (give directions to edges to obtain a directed graph) such that the resultant digraph does *not* have a Euler circuit and another way to orient the edges such that the digraph has a Euler circuit.
   2. A binary operation on graphs of n vertices such that the set of all graphs on n vertices forms a group (for that operation).
   3. Two binary operations on graphs of n vertices, say $+$ and $\star$, such that the set of all graphs on n vertices forms a ring (using $+, \star$).

3. Prove or disprove the following: $4 \times 2\frac{1}{2} = 10$

   1. If $\mathbb{F}$ is a finite field, the characteristic of $\mathbb{F}$ must be prime. However, the converse is not true.
   2. Any finite integral domain is a field.
   3. Any integral domain with finite characteristic must be of finite order.
   4. If U is an ideal of ring R and $1 \in U$, then $U = R$.

4. For any group G, let $A(G)$ denote the set of all automorphisms of G and let $F(G) = \{T_g \in A(G) \mid g \in G, T_g : G \to G \text{ where } \forall x \in G, T_g(x) = g^{-1}xg\}$. Prove the following: $1 + 2 + 3 + 4 = 10$

1. $A(G)$ is a group.
2. If $G = S_3$ (symmetric group of degree 3) then G is isomorphic to $F(G)$.
3. $F(G)$ is a normal subgroup of $A(G)$.
4. $F(G)$ is isomorphic to $G|Z$ where Z is the center of G.

5. Let G be a group in which , for some integer $n > 1$, $(ab)^n = a^n b^n$, for all $a, b \in G$. Prove the following: $4 \times 2\frac{1}{2} = 10$

   1. $G^{(n)} = \{x^n \mid x \in G\}$ is a normal subgroup of G.
   2. $G^{(n-1)} = \{x^{n-1} \mid x \in G\}$ is a normal subgroup of G.
   3. $a^{n-1} b^n = b^n a^{n-1}$ for all $a, b \in G$.
   4. $(aba^{-1}b^{-1})^{n(n-1)} = e$ for all $a, b \in G$.

6. Prove each of the following: (a) Lagrange's Theorem for finite groups (regarding order of a subgroup dividing the order of group), (b) If H and K are subgroups of group G then $(H \cap K)$ is a subgroup of G, and (c) any subgroup of a cyclic group is itself a cyclic group. $3 + 3 + 4 = 10$

7. Prove the following regarding simple planar graphs: $2 + 2 + 6 = 10$

   1. Theory of planar graphs is popular only for undirected graphs and not *directed* graphs. Why?
   2. Every planar graph is 6-colorable.
   3. Let p_n be the probability that a simple graph on n vertices, chosen uniformly at random from all the $2^{\binom{n}{2}}$ possible simple undirected graphs, is planar. What are the values of p_4, p_5 and p_6?

8. Given n distinct objects, prove that: $3 + 5 + 2 = 10$

   1. The number of *derangements* of n objects (arrangements where i^{th} object is not in i^{th} position, for all $1 \leq i \leq n$), is (approximately) $\frac{n!}{e}$.
   2. The number of times you need to pick an object uniformly at random (one at a time with replacement), such that the probability that you pick the same object more than once is at least 0.5, is $O(\sqrt{n})$.
   3. The number of ways in which the n objects can be permuted so that none of the following sequence of objects (assume objects are numbered as $1, \ldots, n$) occurs contiguously anywhere in the permutations: $1, 2, 3$ and $4, 5, 6, 7$, and $8, 9$ is ____________________. (Fill in the blanks and then prove it).

9. Given numbers a, b, c and d, suppose you have find $\gcd(a, b^{c^d})$ on your computer. How would you do it efficiently assuming that you know the prime factorization of a? (Note that machine may not be able to compute and store the value of b^{c^d}, even including all time/memory available in the Universe!)
   Hint: Use Euclid's algorithm, Chinese Remainder Theorem and Fermat's Little Theorem together).

10. Write in detail with proofs and applications about any *two* among of the following: $2 \times 5 = 10$

    1. Well-Ordering Principle
    2. Pigeonhole Principle
    3. Equivalence Relations and Partitions
    4. Principle of Inclusion and Exclusion
    5. Taxonomy of Recurrence Relations and Their Solutions
    6. Platonic Solids and Planar Graphs

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