Introduction to Quantum Information and Quantum Computation (CS312.9), Spring 2023, IIIT Hyderabad

Assignment – Solutions

February 26, 2023

Total Points: 25 (+5 Bonus points) Due date: 23rd February, 2023 (**Hard deadline**)

General Instructions: Submit handwritten or typed **PDFs**.

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1. [3 points] **(a)** Consider a trit with the initial probability distribution

$$\mathcal{P} = \left\{ \frac{1}{6}, \frac{2}{6}, \frac{3}{6} \right\}.$$

What is the minimum heat cost, determined by the Clausius inequality, due to resetting this trit?

(b) Show that the quantum states $(U \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ are mutually orthogonal. Here $U = \{I, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$.

Solution: (a) The initial entropy of the qutrit $S_i = \frac{1}{6}\ln\left(\frac{1}{6}\right) + \frac{2}{6}\ln\left(\frac{2}{6}\right) + \frac{3}{6}\ln\left(\frac{3}{6}\right) = 1.0114$. Now, resetting the qutrit takes its entropy to 0. So, this reduction in entropy of the system, ΔS, is accompanied by an increase in entropy of the system by at least ΔS in order to satisfy the second law of thermodynamics. Now, the Claussius inequality states that the heat lost by the system is atleast the change in entropy times the Boltzmann constant. Let us assume that the ambient temperature in 300K. So,

$$\begin{aligned}\Delta Q \geq k_B \Delta S &= 1.395 \times 10^{-23} \times 300 \\ &= 4.185 \times 10^{-21} J.\end{aligned}$$

This is the heat dissipated by the system to the environment.

(b) This is trivial.

2. [4 points] **Pauli rotations**

(a) Let A be matrix such that $A^2 = I$ and x be a real number. Prove that $e^{iAx} = \cos(x)I + i\sin(x)A$.

(b) Use the result of (a) to prove that

$$\exp(-i\theta\hat{\sigma}_x) = \cos(\theta)I - i\sin(\theta)\hat{\sigma}_x$$

(c) Consider the three component spin vector $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. If $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions, then prove that

$$\exp(-i\theta\hat{n} \cdot \vec{\sigma}) = \cos(\theta)I - i\sin(\theta)\hat{n} \cdot \vec{\sigma}$$

Solution: This is also quite trivial and can be found in any textbook on quantum computing.

3. [5 points] **Evolution of a quantum state and Quantum Zeno effect**

Suppose you are given the Hamiltonian $\hat{H} = \hbar\omega\hat{\sigma}_x$ and that we are working in units where $\hbar\omega = 1$. Assume that the quantum system is initialized in the state $|\psi(0)\rangle = |0\rangle$.

(a) Find the state of the system after time t.

(b) After what time T is the system in the state $|\psi(T)\rangle = |1\rangle$?

(c) Imagine that you make n projective measurements in the $\{|0\rangle, |1\rangle\}$ basis at time intervals $\delta T = T/n$. What is the probability of obtaining $|0\rangle$ in each of the n times when $n = 5$ and when $n = \infty$? Interpret the results.

Solution: **(a)** The evolution operator is $\hat{U}(t) = e^{-i\hat{\sigma}_x t} = \cos(t)\hat{I} + i\hat{\sigma}_x \sin(t)$. So, if $|\psi(0)\rangle = |0\rangle$, we have that the general dynamics would be

$$|\psi(t)\rangle = \cos(t)\,|0\rangle + i\sin(t)\,|1\rangle\,.$$

(b) After time $T = \pi/2$, the system would be in state $|1\rangle$.

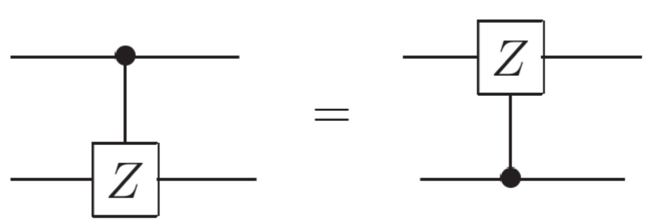
(c) Probability of observing $|0\rangle$ after n measurements in intervals of δT is $\cos^{2n}(\pi/2n)$, Which for large n is

$$p_{T,n} \approx \left(1 - \frac{\pi^2}{8n^2}\right)^{2n},$$

which goes to 1, as $n \to \infty$.

4. [5 points] **Quantum Gates**

(a) Show that



(b) Write the unitary corresponding to the controlled-Z quantum gate shown here.

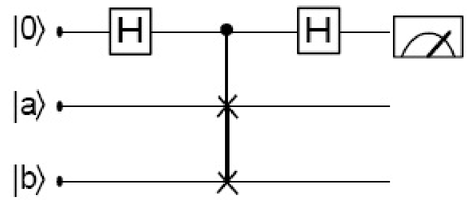
(c) What would be the output state of the given quantum circuit if the input state is $|+\rangle\,|0\rangle$?

(d) Construct a Controlled SWAP gate using only Toffoli gates.

Solution: Again, this is trivial to solve.

5. [4 points] **Swap test**

Consider the quantum circuit shown here. The two-qubit gate is a controlled-SWAP gate.



(a) What is the output state?

(b) What is the probability that the first qubit is in the state $|1\rangle$ after measurement?

Solution: The input to the circuit is the state $|0\rangle |a\rangle |b\rangle$ and applying Hadamard to the first (ancilla) qubit gives $(|0\rangle + |1\rangle)/\sqrt{2}$. The controlled swap operation swaps $|a\rangle$ and $|b\rangle$ if the ancilla qubit is $|1\rangle$. A second Hadamard on the ancilla results in the state

$$|\Psi\rangle = \frac{1}{2}|0\rangle (|a,b\rangle + |b,a\rangle) + \frac{1}{2}|1\rangle (|a,b\rangle - |b,a\rangle).$$

The probability of measuring the ancilla in the state $|0\rangle$ is

$$\begin{aligned} p(|0\rangle) &= \frac{1}{4}(\langle a,b| + \langle b,a|)(|a,b\rangle + |b,a\rangle) \\ &= \frac{1}{4}(2 + \langle a,b|b,a\rangle + \langle b,a|a,b\rangle) \\ &= \frac{1}{2} + \frac{1}{2}|\langle a|b\rangle|^2. \end{aligned}$$

On the other hand, the probability of measuring the ancilla in the state $|1\rangle$ is

$$\begin{aligned} p(|1\rangle) &= \frac{1}{4}(\langle a,b| - \langle b,a|)(|a,b\rangle - |b,a\rangle) \\ &= \frac{1}{4}(2 - \langle a,b|b,a\rangle + \langle b,a|a,b\rangle) \\ &= \frac{1}{2} - \frac{1}{2}|\langle a|b\rangle|^2. \end{aligned}$$

6. [4 points] **Randomized algorithm for the Deutsch-Jozsa problem**

In class, we have seen the Deutsch-Jozsa problem. We restate it here for convenience. Suppose we are given a black box for some Boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ with the promise that f is either *constant* or *balanced*. In order to determine which is the case with certainty, we have seen that a classical algorithm requires $2^{n-1}+1$ queries to the black box in the worst case, while a quantum query needs only one query. Now we impose some relaxation to the problem. Now, we want to determine whether f is *constant* or *balanced* with a success probability of $1-\varepsilon$, where $\varepsilon \in (0, 1/2)$. How many queries to the black box are needed by a classical algorithm and the quantum algorithm, respectively?

Solution: Let us consider that we make d queries to C_f with bit strings $S = \{x_1, x_2, \cdots, x_d\}$, where $x_i \in \{0,1\}^n$. Our strategy would be the following:

If, for all $x_i \in S$, $f(x_i)$ is constant, output *constant*, otherwise if there exists x_i, x_j such that $f(x_i) \neq f(x_j)$, output *balanced*. We will determine a value of d for which the error in the output is at most ε, by following this strategy.

First, notice that if our strategy outputs *balanced*, then the function was indeed *balanced*. So, when can we make an error? Well, when we output *constant* but the function was actually *balanced*. Let us compute the probability of error in this case.

When the function is *balanced*, for any $z \in \{0,1\}^n$, $Pr[f(z) = 0] = Pr[f(z) = 1] = 1/2$. So then, in our strategy, we make an error when the function is indeed *balanced* but for all the queries we made, $f(x_j)$ is the same $\forall x_j \in S$.

What is the probability that the outcome of all the d queries would yield the same value of the function? It is simply $p = \frac{1}{2^d} + \frac{1}{2^d} = \frac{1}{2^{d-1}}$. This is the probability of error. Now, we want $p < \varepsilon$. So,

$$\frac{1}{2^{d-1}} < \varepsilon$$
$$\implies d > \log(2/\varepsilon).$$

This means we only have to make $d = O(\log(1/\varepsilon))$ queries to ensure that the success probability is at least $1 - \varepsilon$. So, for instance, if we want the success probability to be $\geq 3/4$, we need to make at most 3 queries to C_f. Thus, the speedup for the Deustch-Jozsa algorithm is no longer exponential, when we allow the classical algorithm to make some small constant error.

7. [5 points] **Bonus Question: How does your wavefunction evolve?**

The time-independent Schrödinger equation for a free particle is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}.$$

(a) Verify that a typical solution to this equation is a wavefunction of the form $\psi(x,t) = Ae^{i(px-Et)/\hbar}$, where $E = E(p) = p^2/2m$.

(b) Now, consider a free particle localized in space at $t = 0$. To be more precise, assume that the wavefunction of the particle initially is a Gaussian wavepacket centered around the origin and is given by

$$\psi(x,0) = A \exp\left[-x^2/2d^2\right],$$

where d is the spread of the Gaussian wavepacket. Now, calculate how this wavepacket will evolve in time. Show that $|\psi(x,t)|^2 \propto \exp\left[-\frac{x^2}{d^2(1+\frac{\hbar^2 t^2}{m^2 d^4})}\right]$.

(c) What happens to the wavepacket with time? Does the probability of observing the particle in space spreads out or becomes more localized? How fast does it localize or delocalize?

(d) To the result obtained in (c), substitute (i) the mass of the particle by the mass of the electron ($\approx 10^{-27}$ g), the value of d for an electron is $\approx 10^{-8}$cm. (ii) the mass of the particle by your own mass in grams and assume

$d \approx 1$cm. For both cases, consider $\hbar \approx 10^{-27}$g cm$^2/s$. How fast does the wavefunction of the electron localize or delocalize? What happens to your wavefunction?

Solution: (a) Given that $\psi(x,t) = Ae^{i(px-Et)/\hbar}$. So, R.H.S.

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= (i\hbar)(\frac{-iAE}{\hbar})e^{i(px-Et)/\hbar} \\ &= EAe^{i(px-Et)/\hbar} \\ &= \frac{p^2}{2m}\psi(x,t) \end{aligned}$$

L.H.S:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{p^2}{2m}\psi(x,t).$$

(b)The initial wave function $\psi(x,0) = Ae^{-x^2/2d^2}$. In the momentum space,

$$\begin{aligned} \phi(p) &= A\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{2\pi\hbar}}e^{-x^2/2d^2}e^{-ipx/\hbar} \\ &= \frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}e^{(-x^2/2d^2-ipx/\hbar)}dx \\ &= \frac{A}{\sqrt{2\pi\hbar}}\sqrt{2\pi d^2}e^{-p^2d^2/2\hbar^2}. \end{aligned}$$

Now,

$$\psi(x,0) = \frac{Ad}{\sqrt{\hbar}}\int_{-\infty}^{+\infty}\frac{dp}{\sqrt{2\pi\hbar}}e^{-p^2d^2/2\hbar^2}e^{ipx/\hbar}.$$

Now, the Schrodinger equation gives us

$$\psi(x,t) = e^{-ip^2t/2m\hbar}\psi(x,0).$$

Thus,

$$\begin{aligned} \psi(x,t) &= \frac{Ad}{\sqrt{\hbar}}\int_{-\infty}^{+\infty}\frac{dp}{\sqrt{2\pi\hbar}}e^{-p^2d^2/2\hbar^2}e^{\frac{i}{\hbar}(px-p^2t/2m)} \\ &= \frac{Ad}{\sqrt{\hbar}}\int_{-\infty}^{+\infty}\frac{dp}{\sqrt{2\pi\hbar}}e^{-(\frac{d^2}{2\hbar^2}+\frac{it}{2m\hbar})p^2+(\frac{ix}{\hbar})p} \\ \psi(x,t) &\propto \exp\left[-\frac{x^2}{2d^2+2it\hbar/m}\right] \\ &\propto \exp\left[-\frac{x^2}{2d^2}\left(1+\frac{it\hbar}{md^2}\right)^{-1}\right] \end{aligned}$$

Thus,

$$\begin{aligned} |\psi(x,t)|^2 &\propto \exp\left[-\frac{x^2}{2d^2}\left\{\frac{1}{1-2it\hbar/md^2}+\frac{1}{1+2it\hbar/md^2}\right\}\right] \\ &\propto \exp\left[-\frac{x^2}{2d^2}\left\{\frac{2}{1+t^2\hbar^2/m^2d^4}\right\}\right] \\ &\propto \exp\left[-\frac{x^2}{d^2}\left\{\frac{1}{1+t^2\hbar^2/m^2d^4}\right\}\right]. \end{aligned}$$

Another, simpler way to look at the problem, is to observe the initial variance in the position of the wavepacket, $\langle x^2 \rangle \sim d^2$. By Heisenberg's uncertainty principle, the $\langle x^2 \rangle \langle p^2 \rangle \sim \hbar^2$. The velocity would thus be $\langle v^2 \rangle \sim \frac{\hbar^2}{m^2 d^2}$, and hence

$$\begin{aligned}
\langle x^2(t) \rangle &\sim d^2 + \frac{t^2 \hbar^2}{m^2 d^2} \\
&\sim d^2 \left(1 + \frac{t^2 \hbar^2}{m^2 d^4} \right) \\
&= d^2 (1 + t^2/\tau^2).
\end{aligned}$$

(c) Now choosing $\tau = md^2/\hbar$, we can write

$$|\psi(x,t)|^2 \propto \exp\left[-\frac{x^2}{d^2} \left\{ \frac{1}{1 + t^2/\tau^2} \right\} \right].$$

From this we see that the Gaussian wavepacket is unstable and spreads with time. This is manifested by an increase in the new variance of the Gaussian given by:

$$d^2(t) = d^2(1 + t^2/\tau^2).$$

The timescale of spreading is determined by $\tau = md^2/\hbar$.

(d) As $\tau = md^2/\hbar$, we have that for an electron, $\tau = 10^{-27} \times 10^{-16}/10^{-27} = 10^{-16} s$. So, the timescale of spreading of the wavefunction is approximately $10^{-16} s$. For myself, on the other hand, $\tau = 6000 \times 1/10^{-27} = 6 \times 10^{30} s$. This is several orders of magnitude higher than the lifetime of the universe. Clearly, this timescale is irrelevant; hence, quantum mechanics is irrelevant for classical objects.