(CS312.9), Spring 2023, IIIT Hyderabad

Final Exam

February 27, 2023

Total Marks: 30 **Time: 90 mins**

General Instructions: You are required to write your roll number on the question paper, attach it to the answer booklet and submit them together. Your copies will not be evaluated otherwise.

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1. [8 points] Consider Pauli matrices X, Y and Z. One can define single-qubit Pauli rotations $R_{\hat{n}}(\theta)$, where $\hat{n}$ is a three-dimensional unit vector in the Pauli basis. For example, when $\hat{n} = (1\ 0\ 0)^T$, we obtain $R_x(\theta) = \cos(\theta)I + i\sin(\theta)X$. Then, answer the following questions

   (a) Express the Hadamard gate as a product of R_x, R_z rotation and a phase.

   (b) Show that $XYX = -Y$ and prove that $XR_y(\theta)X = R_y(-\theta)$.

   (c) Prove the following identities: $HXH = Z$, $HYH = -Y$, $HZH = X$.

2. [6 points] Answer the following questions.

   (a) Construct a SWAP gate using only CNOT gates.

   (b) Prove that the following matrix is unitary

$$\hat{F}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \omega & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \cdots & \omega^{2(N-1)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{bmatrix},$$

   is unitary. Here, $\omega = e^{i2\pi/N}$ is the N^{th} root of unity.

   (c) What is $F_2 |0\rangle$?

3. [6 points] Let $|s\rangle$ be the n-qubit quantum state that is an equal superposition of all the $N = 2^n$ computational basis states and suppose,

$$|\psi\rangle = \sum_{k=1}^{N} c_k |k\rangle,$$

   is an arbitrary n-qubit quantum state, where $\sum_{k=1}^{N} |c_k|^2 = 1$. Furthermore, consider the circuit

$$D = H^{\otimes n}(2|0^n\rangle\langle 0^n| - I)H^{\otimes n}.$$

Show that

$$D|\psi\rangle = \sum_{j=1}^{N} (2\langle c\rangle - c_j)|j\rangle,$$

where $\langle c\rangle$ is the average of the amplitudes of $|\psi\rangle$, i.e. $\langle c\rangle = \sum_{j=1}^{N} c_j/N$.

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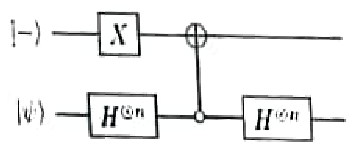
4. [5 points] Consider the unitary $\underline{U} = |0\rangle\langle 0| \otimes U_1 + |1\rangle\langle 1| \otimes U_2$. For any input state $|\psi\rangle$, if H is the Hadamard gate, what is the quantum state obtained after applying the circuit $(H \otimes I)\,U\,(H \otimes I)$ to the initial state $|0\rangle|\psi\rangle$? If a measurement is made in the first register and the state $|0\rangle$ is observed, what is the state of the second register?

5. [5 points] Consider the following quantum circuit where $|\psi\rangle$ is any n-qubit quantum state while $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Observe that for the controlled operation, the control qubits are denoted by an open circle instead of the usual solid circle (e.g: the conventional CNOT gate). This indicates that the controlled operation is conditioned on each control qubit set to 0.



(a) What is the output of this quantum circuit when

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{j\in\{0,1\}^n} |j\rangle?$$

(b) What is the matrix representation of the implemented unitary operation? You don't need to write the whole unitary matrix, the operator sum representation would suffice.