

Probability and Statistics

End Sem Solutions

Q1.

Using Law of Iterated Expectation

$$E[T] = E[E[T|N]]$$

$$E[T|N=n] = E\left[\sum_{i=1}^n x_i\right]$$

$$= nE[x] \quad (\because x_i \text{ s are iid})$$

$$\Rightarrow E[T|N] = NE[x]$$

$$\Rightarrow E[T] = E[E[T|N]] = E[NE[x]] = E[N]E[x] \quad \left(\because E[x] \text{ is a constant} \right)$$

Now if $N \sim \text{Geometric}(p)$

$$E[N] = \frac{1}{p}$$

if $x \sim \text{Exponential}(\lambda)$

$$E[x] = \frac{1}{\lambda}$$

$$\Rightarrow E[T] = E[N]E[x] = \frac{1}{p\lambda}$$

Q2.

$$2) f_X(x) = d_1 e^{-d_1 x}; f_Y(y) = d_2 e^{-d_2 y}.$$

$$Z = \min(X, Y)$$

$$F_Z(z) = P(Z \leq z) \quad \text{or } P(Z > z)$$

$$= 1 - P(\min(X, Y) > z)$$

$$= 1 - P(X > z \text{ and } Y > z)$$

$$= 1 - [1 - F_X(z)] [1 - F_Y(z)]$$

$$= 1 - e^{-(d_1 + d_2)z}, \quad \forall z \geq 0$$

$$F_W(w) = P(W \leq w) = P(\max(X, Y) \leq w)$$

$$= P(X \leq w, Y \leq w)$$

\therefore Independent, so:

$$= P(X \leq w) P(Y \leq w)$$

$$= (1 - e^{-d_1 w}) (1 - e^{-d_2 w})$$

$$F_W(w) = (1 - e^{-d_1 w}) (1 - e^{-d_2 w})$$

Q3.

$$3) F_Y(y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y})$$

$$\hookrightarrow (|X| \leq \sqrt{y})$$

or

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{d(F_Y(y))}{dy} = \frac{d(F_X(\sqrt{y}) - F_X(-\sqrt{y}))}{dy}$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} =$$

$$\Rightarrow \frac{1}{2\sqrt{y}} \left(\frac{e^{-y/2}}{\sqrt{2\pi}} + \frac{e^{-y/2}}{\sqrt{2\pi}} \right)$$

$$f_Y(y) = \frac{1}{\sqrt{y}} \frac{e^{-y/2}}{\sqrt{2\pi}} \rightarrow \begin{cases} 0, & y < 0 \\ \frac{e^{-y/2}}{\sqrt{2\pi y}}, & y \geq 0 \end{cases}$$

Q4.

Uniform Distribution

From the definition of the [continuous uniform distribution](#), X has probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

From the definition of a [moment generating function](#):

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

where $E(\cdot)$ denotes [expectation](#).

First, consider the case $t \neq 0$.

Then:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^a 0e^{tx} dx + \int_a^b \frac{e^{tx}}{b-a} dx + \int_b^{\infty} 0e^{tx} dx \\ &= \left[\frac{e^{tx}}{t(b-a)} \right]_a^b && \text{Primitive of } e^{ax}, \text{ Fundamental Theorem of Calculus} \\ &= \frac{e^{tb} - e^{ta}}{t(b-a)} \end{aligned}$$

In the case $t = 0$, we have $E(X^0) = E(1) = 1$.

Exponential Distribution

From the definition of the [Exponential distribution](#), X has probability density function:

$$f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$$

From the definition of a [moment generating function](#):

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f_X(x) dx$$

Then:

$$M_X(t) = \frac{1}{\beta} \int_0^{\infty} e^{x(-\frac{1}{\beta}+t)} dx \quad \text{Exponential of Sum}$$

$$= \frac{1}{\beta \left(-\frac{1}{\beta} + t\right)} \left[e^{x\left(-\frac{1}{\beta}+t\right)} \right]_0^{\infty} \quad \text{Primitive of Exponential Function}$$

Note that if $t > \frac{1}{\beta}$, then $e^{x\left(-\frac{1}{\beta}+t\right)} \rightarrow \infty$ as $x \rightarrow \infty$ by [Exponential Tends to Zero and Infinity](#), so the integral diverges in this case.

If $t = \frac{1}{\beta}$ then the integrand is identically 1, so the integral similarly diverges in this case.

If $t < \frac{1}{\beta}$, then $e^{x\left(-\frac{1}{\beta}+t\right)} \rightarrow 0$ as $x \rightarrow \infty$ from [Exponential Tends to Zero and Infinity](#), so the integral converges in this case.

Therefore, the function is only well defined for $t < \frac{1}{\beta}$.

Proceeding:

$$\begin{aligned} \frac{1}{\beta \left(-\frac{1}{\beta} + t\right)} \left[e^{x\left(-\frac{1}{\beta}+t\right)} \right]_0^{\infty} &= \frac{1}{\beta \left(-\frac{1}{\beta} + t\right)} (0 - 1) \quad \text{Exponential Tends to Zero and Infinity, Exponential of Zero} \\ &= \frac{1}{\beta \left(\frac{1}{\beta} - t\right)} \\ &= \frac{1}{1 - \beta t} \end{aligned}$$

Here, $\lambda = 1/\beta$.

Q5.

There is an alternative way to get the result by applying the the Law of Total Probability:

$$P[W] = \int_Z P[W | Z = z] f_Z(z) dz$$

As others have done, let $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$. What follows is the only slightly unintuitive step: instead of directly calculating the PDF of $Y - X$, first calculate the CDF: $P[X - Y \leq t]$ (we can then differentiate at the end).

$$P[Y - X \leq t] = P[Y \leq t + X]$$

This is where we'll apply total probability to get

$$\begin{aligned} &= \int_0^\infty P[Y \leq t + X | X = x] f_X(x) dx \\ &= \int_0^\infty P[Y \leq t + x] f_X(x) dx = \int_0^\infty F_Y(t + x) f_X(x) dx \end{aligned}$$

Note substituting the CDF here is only valid if $t \geq 0$,

$$\begin{aligned} &= \int_0^\infty (1 - e^{-\mu(t+x)}) \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-\lambda x} dx - \lambda e^{-\mu t} \int_0^\infty e^{-(\lambda+\mu)x} dx \\ &= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty - \lambda e^{-\mu t} \left[\frac{e^{-(\lambda+\mu)x}}{-(\lambda+\mu)} \right]_0^\infty = 1 - \frac{\lambda e^{-\mu t}}{\lambda + \mu} \end{aligned}$$

Differentiating this last expression gives us the PDF:

$$f_{Y-X}(t) = \frac{\lambda \mu e^{-\mu t}}{\lambda + \mu} \quad \text{for } t \geq 0$$

Here, $\lambda = \mu$, $Y = V$, $X = W$, $Y - X = V - W = Z$.

Q6.

If $X_i \sim \text{Exponential}(\theta)$, then

$$f_{X_i}(x; \theta) = \theta e^{-\theta x} u(x),$$

where $u(x)$ is the unit step function, i.e., $u(x) = 1$ for $x \geq 0$ and $u(x) = 0$ for $x < 0$. Thus, for $x_i \geq 0$, we can write

$$\begin{aligned} L(x_1, x_2, x_3, x_4; \theta) &= f_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4; \theta) \\ &= f_{X_1}(x_1; \theta) f_{X_2}(x_2; \theta) f_{X_3}(x_3; \theta) f_{X_4}(x_4; \theta) \\ &= \theta^4 e^{-(x_1 + x_2 + x_3 + x_4)\theta}. \end{aligned}$$

Since we have observed $(x_1, x_2, x_3, x_4) = (1.23, 3.32, 1.98, 2.12)$, we have

$$L(1.23, 3.32, 1.98, 2.12; \theta) = \theta^4 e^{-8.65\theta}.$$

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Here, it is easier to work with the log likelihood function, $\ln L(1.23, 3.32, 1.98, 2.12; \theta)$. Specifically,

$$\ln L(1.23, 3.32, 1.98, 2.12; \theta) = 4 \ln \theta - 8.65\theta.$$

By differentiating, we obtain

$$\frac{4}{\theta} - 8.65 = 0,$$

which results in

$$\hat{\theta}_{ML} = 0.46$$

Q7.

(a) Here we define the test statistic as

$$\begin{aligned} W &= \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \\ &= \frac{5.96 - 5}{1/\sqrt{5}} \\ &\approx 2.15. \end{aligned}$$

Here, $\alpha = .05$, so $z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$. Since $|W| > z_{\frac{\alpha}{2}}$, we reject H_0 and accept H_1 .

(b) The 95% CI is given by

$$\left(5.96 - 1.96 * \frac{1}{\sqrt{(5)}}, 5.96 + 1.96 * \frac{1}{\sqrt{(5)}} \right) = (5.09, 6.84).$$

Since μ_0 is not included in the interval, we are able to reject the null hypothesis and conclude that μ is not 5.

Q8.

Q8: let X be a random variable showing the no. of observed heads. $X = 700$ (given)

let n be the total no. of times the coin is tossed.
 $n = 1000$

Assume H_0 is true, W is a standard normal random variable $N(0,1)$

$$\begin{aligned} W &= \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}} = \frac{X - 1000 \times \frac{1}{2}}{\sqrt{1000 \times \frac{1}{2} \times \frac{1}{2}}} \\ &= \frac{(X - 500)}{5\sqrt{10}} \end{aligned}$$

If H_0 is true, then expect X to be close to 500, while if H_1 is true we expect X to be larger. Thus we can suggest the following test:

We choose a threshold c . If $W \leq c$, we accept H_0 ; otherwise we accept H_1 .

1. If we require significance level $\alpha = 0.05$, then

$$c = z_{0.05} = 1.645$$

$$W = \frac{700 - 500}{5\sqrt{10}} = \underline{12.65}$$

$W > c \rightarrow$ we reject H_0 and accept H_1 .

2. If significance level $\alpha = 0.01$, then

$$c = z_{0.01} = 2.33$$

$$W = 12.65$$

$W > c$, we reject H_0 and accept H_1 .

3. P-value

Since here $W = 12.65$, we will reject H_0 iff $c < 12.65$. Let $z_\alpha = c$, then

$$\alpha = 1 - \phi(c)$$

$$\alpha = 1 - \phi(12.65) \quad \boxed{c = 12.65}$$

From above eqⁿ, we will get α .

\therefore We reject H_0 for $\alpha > (1 - \phi(12.65))$

$$\therefore \text{P-value} = 1 - \phi(12.65)$$

Q9.

Solution: For $0 \leq x \leq 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy \\ &= \int_0^1 \left(x + \frac{3}{2}y^2 \right) dy \\ &= \left[xy + \frac{1}{2}y^3 \right]_0^1 \\ &= x + \frac{1}{2}. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $0 \leq y \leq 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dx \\ &= \int_0^1 \left(x + \frac{3}{2}y^2 \right) dx \\ &= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2x \right]_0^1 \\ &= \frac{3}{2}y^2 + \frac{1}{2}. \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The MAP estimate of X , given $Y = y$, is the value of x that maximizes

$$f_{X|Y}(x|y) = \frac{x + \frac{3}{2}y^2}{\frac{3}{2}y^2 + \frac{1}{2}}, \quad \text{for } 0 \leq x, y \leq 1.$$

For any $y \in [0, 1]$, the above function is maximized at $x = 1$. Thus, we obtain the MAP estimate of x as

$$\hat{x}_{MAP} = 1.$$

The ML estimate of X , given $Y = y$, is the value of x that maximizes

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{x + \frac{3}{2}y^2}{x + \frac{1}{2}} \\ &= 1 + \frac{\frac{3}{2}y^2 - \frac{1}{2}}{x + \frac{1}{2}}, \quad \text{for } 0 \leq x, y \leq 1. \end{aligned}$$

Therefore, we conclude

$$\hat{x}_{ML} = \begin{cases} 1 & 0 \leq y \leq \frac{1}{\sqrt{3}} \\ 0 & \text{otherwise} \end{cases}$$

Q10.

Solution: Since X and W are independent and normal, Y is also normal. The variance is

$$\begin{aligned}\text{Cov}(X, Y) &= \text{Cov}(X, X + W) \\ &= \text{Cov}(X) + \text{Cov}(X, W) \\ &= \text{Var}(X) = \sigma_X^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\rho(X, Y) &= \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \\ &= \frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_W^2}}.\end{aligned}$$

(a) The MMSE estimator of X given Y is

$$\begin{aligned}\hat{X}_M &= E[X|Y] \\ &= \mu_X + \rho\sigma_X \frac{Y - \mu_Y}{\sigma_Y} \\ &= \frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y.\end{aligned}$$

(b) The MSE of this estimator is given by

$$\begin{aligned}E[(X - \hat{X}_M)^2] &= E[\tilde{X}^2] \\ &= E[X^2] - E[\hat{X}_M^2] \\ &= \sigma_X^2 - \left(\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2}\right)^2 (\sigma_X^2 + \sigma_W^2) \\ &= \frac{\sigma_X^2 \sigma_W^2}{\sigma_X^2 + \sigma_W^2}.\end{aligned}$$