$P[|\frac{X}{n} - P(E)| > 0]$
$\leq P[\frac{X}{n} - P(E) > 0]$
$= P(\frac{X}{n} > P(E))$

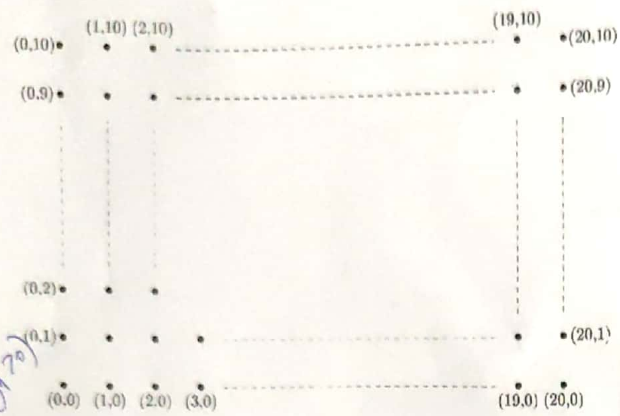
Probability & Statistics

Instructor: Girish Varma • Course Code: MA6.101 • IIIT Hyderabad

End Semester Exam

7 problems • 5 marks each

1 Counting Paths



A wireless sensor grid consists of $21 \times 11 = 231$ sensor nodes that are located at points (i,j) in the plane such that $i \in \{0,1,\cdots,20\}$ and $j \in \{0,1,2,\cdots,10\}$ as shown in Figure. The sensor node located at point $(0,0)$ needs to send a message to a node located at $(20,10)$. The messages are sent to the destination by going from each sensor to a neighboring sensor located above $(i+1,j)$ or to the right $(i,j+1)$. Assume all paths from $(0,0)$ to $(20,10)$ are equally likely.

a.) What is the probability that the sensor located at point $(10,5)$ receives the message? **(2)**

b.) Conditioned on the event that $(10,5)$ receives the message, what is the probability that $(15,8)$ also receives the message? **(1.5)**

c.) Now consider a different distribution of paths, in which each sensor which has a choice (that is not in the top row or the last column), sends the message to the above sensor with probability p_a and will send the message to the sensor to the right with probability $p_r = 1 - p_a$. Find the probability that $(10,5)$ receives the message. **(1.5)**

2 Random Variables

A router monitors incoming messages from a client and collects statistics. There are k different types of messages the client can send, each of which are equally likely. The client is sending a sequence of messages.

a.) Let X be the random variable corresponding the the number of messages the router should receive, for it to see t distinct messages. Find $\mathbb{E}X$. **(2)**

b.) Let Z_{ij} for $i \neq j$, $i,j \in \{1,\cdots,n\}$ be a binary valued random variable, which is 1 only when the ith and jth messages are different. Show that they are not 3-wise independent. That is there is some collection of 3 random variables which is not independent. **(1)**

c.) Let Y be the number of duplicates on receiving a sequence of length n. That is the number of tuples (i,j) such that $i \neq j$, $i,j \in \{1,\cdots,n\}$ and the ith message is same as the jth message. Find the $\mathbb{E}Y$ and $\text{Var}(Y)$. **(2)**

3 Tail Bounds

You are doing a quantum mechanics experiment, for which the outcome is uniformly random among k possibilities. Suppose you want to find out the probability of an unknown event $E \subseteq \{1,\cdots,k\}$ exactly, by repeating the experiment many times independently. After each experiment, you will only know if E happened or not, without knowing the outcome among k possibilities.

a.) Describe a possible method to find $P(E)$, such that it finds $P(E)$ in expectation (need to show that it indeed finds $P(E)$ in expectation). **(2)**

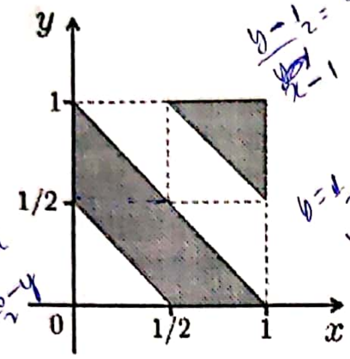
b.) How many times must you repeat the experiment to find out $P(E)$ exactly with confidence 99%? **(3)**

$\mathbb{E}[X] = \mathbb{E}[x_n]$

4 Continuous Random Variables

A pair of jointly continuous random variables, X and Y, have a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{in the shaded region of figure} \\ 0, & \text{otherwise} \end{cases}$$



a.) Find c. **(1)**

b.) Find marginal PDF of X and Y. **(1.5)**

c.) Find $\mathbb{E}(X|Y = 1/4)$ and $\text{Var}(X|Y = 1/4)$. **(1.5)**

d.) Find the conditional PDF for X given that $Y = 3/4$. **(1)**

5 Processes

Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson process with rate $\lambda_1 = 0.02$ goals per minute, and the number of goals scored by Team B is modeled by a Poisson process with rate $\lambda_2 = 0.03$ goals per minute. The two processes are assumed to be independent. The game lasts for 90 minutes.

a.) Find the probability that the game ends with the score A:1, B:2. **(1)**

b.) Find the probability that at least two goals are scored in the game. **(2)**

c.) Find the probability that Team B scores the first goal. **(2)**

6 Markov Chains

Consider the kings's tour on a chess board (8×8) without the diagonal moves: A kings selects one of the next positions except the diagonal ones at random independently of the past.

a.) What is the state space. Show that this process a Markov chain. **(1)**

b.) Is it irreducible (has a single recurrent class)? Is it aperiodic ? (need to show) **(2)**

c.) Find the stationary distribution. **(2)**

7 Statistics

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} 2x^2 + \frac{1}{3} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We also know that

$$f_{Y|X}(y|x) = \begin{cases} xy - \frac{x}{2} + 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a.) Find the posterior distribution. **(2)**

b.) Find the ML estimate for the observation $Y = y$. **(1.5)**

c.) Find the MAP estimate for the observation $Y = y$. **(1.5)**