INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY HYDERABAD

QUANTUM MECHANICS

Monsoon 2022 – CND Core – Credit 4

Instructor: Subhadip Mitra

Date: SEPTEMBER 19, 2022

Time: 1 H 30 M (08:30 - 10:00)

Mid-Semester Examination

Total Marks: 50

Instructions:

- **Class notes or books are not permitted. But you may bring one A4 sheet of handwritten material (not photo-copy/printed).**
- **Calculators are allowed.**
- **Do not write anything** (except roll number, seat no. etc.) **on the first page of the answer book.**
- You may skip 'trivial' steps. However, unless the logic is clear, you will not get any credit for a problem.
- Illegible answers will not be graded.
- No 'benefit of doubt' because of bad notation/illegible hand-writing etc.

Q 1. Show that

(a) For any two observables represented by two operators, A and B,

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$$

(where σ denotes the standard deviation) and that for $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$, the above equation reduces to the Heisenberg's uncertainty principle.

(b) If $\hat{A}$ and $\hat{B}$ have a complete set of common eigenstates (which then can form a basis), then $[\hat{A},\hat{B}]\,|\psi\rangle = 0$ for any $|\psi\rangle$ in the Hilbert space.

(c) Eigenvalues of Hermitian operators are real, and the eigenstates corresponding to different eigenvalues of a Hermitian operators are orthogonal.

[5+2+3 = 10 CO: 1,2,5]

Q 2. For a simple harmonic oscillator, the ladder operators are given by

$$\hat{a}_\pm = \frac{1}{\sqrt{2m\omega\hbar}}(\mp i\hat{p}+m\omega\hat{x}).$$

(a) Show that the Hamiltonian operator can be written as

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right).$$

(b) Obtain the normalized ground-state wave function. What is its energy?

(c) Let $\psi_n(x)$ be for the normalized (steady state) wavefunction of the n^{th} energy state. Find how $\psi_n(x)$ is related to $\psi_0(x)$.

[2+(3+1)+4 = 10 CO: 3]

Q 3. Let $|x\rangle$ denote the state (wave-function) at x. We can define an infinitesimal translation operator $\hat{\mathcal{T}}(dx)$ such that

$$\hat{\mathcal{T}}(dx')\,|x\rangle = |x+dx'\rangle.$$

(a) What properties should such an operator satisfy? In particular, argue for

(i) $\hat{\mathcal{T}}^\dagger(dx')$,

(ii) $\hat{\mathcal{T}}^{-1}(dx')$,

(iii) $\hat{\mathcal{T}}(dx')\cdot\hat{\mathcal{T}}(dx'')$ and

(iv) $\lim_{dx'\to 0}\hat{\mathcal{T}}(dx')$.

(b) Show that $\hat{\mathcal{T}}(dx') = 1 - i\,\hat{K}\,d\hat{x}'$ satisfies all the above properties if we ignore terms of second order or higher in dx'.

(c) Show that

$$[\hat{x},\hat{\mathcal{T}}(dx')]\,|x'\rangle = d\hat{x}'\,|x'+d\hat{x}'\rangle \approx d\hat{x}'\,|x'\rangle$$

and obtain $[\hat{x},\hat{K}]$.

[4+2+(3+1) = 10 CO: 2,4]

Q 4. (a) Show that the time evolution because of the Schrödinger equation does not affect the normalization of a wave function.

(b) However, if we assume that a particle is in a potential with an imaginary part, i.e.,

$$V = V_0 - i\Gamma$$

(where V_0 is the true potential and Γ is a positive real constant), show that the probability of finding the particle at any point $\rho(x,t)$ decreases with time, i.e., the particle decays. What is the lifetime of this particle?

(c) If the potential is real, the probability is conserved and hence, in 3D, it satisfies the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

where $\vec{J}$ is the probability current. Write the expression for $\vec{J}$. [4+4+2 = 10 CO: 3,4]

Q 5. (a) For the general spinor $\chi = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ find the probability of getting $\pm\hbar/2$ if one measures $\hat{S}_x$. Also find $\langle S_x \rangle$.

(b) Obtain the operator to measure the component of spin of an electron in the direction making 45^o with the x axis in the x-z plane?

(c) Argue that the eigenvalues of the operator $\hat{L}^2 - \hat{L}_x^2$ are always positive.

(d) Construct the $\hat{S}_z$ and $\hat{S}^2$ matrices and for a spin-1 particle. [2+2+2+(2+2) = 10 CO: 1,3,4]