SC3.316: Mathematical Methods in Biology
Midterm 1 solutions

1. (10 points) Solve the differential equation $\frac{dy}{dx} = 6x(y-1)^{\frac{2}{3}}$.

   Solution:
   Note that $y = 1$ is a singular solution. (2 marks).

   This is a separable equation. If $y \neq 1$, separating the variables, we get

   $$\frac{dy}{(y-1)^{\frac{2}{3}}} = 6xdx \ (3\,marks) \tag{1}$$

   Integrating this yields

   $$3(y-1)^{\frac{2}{3}} = 3x^2 + C \ (3\,marks) \tag{2}$$

   This implies that $y(x) = 1 + (x^2 + c)^3$ (2 marks)

2. (15 points) Consider the initial value problem

   $$\frac{dy}{dx} = e^x - e^{-x} + y \quad \text{with} \quad y(0) = 3/2.$$

   Solve the initial value problem and evaluate $y(2)$.

   Solution:
   We have $\frac{dy}{dx} - y = e^x - e^{-x}$. This is a first order linear equation. (2 marks)

   The integrating factor is e^{-x}. (3 marks)

   Multiplying by the integrating factor on both sides yields

   $$d(ye^{-x}) = (1 - e^{-2x})dx \ \text{(3 marks)}.$$

   This gives $ye^{-x} = \int(1 - e^{-2x})dx = x + \frac{e^{2x}}{2} + C$. Therefore $y(x) = e^x \left(x + \frac{e^{-2x}}{2} + C\right)$. (2 marks)

   Since $y(0) = 3/2$, we get that $C = 1$. (2 marks)

   Therefore the solution to the initial value problem is $y(x) = e^x \left(x + \frac{e^{-2x}}{2} + 1\right)$. (2 marks)

   At $x = 2$, we have $y(2) = e^2 \left(2 + \frac{e^{-4}}{2} + 1\right) = 3e^2 + \frac{1}{2}e^{-2}$. (1 mark)

3. (15 points) Consider a tank that has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 L of water. How much salt is in the tank after 30 minutes?

   Solution:

   Let $S(t)$ be the amount of salt at time t. (2 marks)

   Note that the inflow rate = outflow rate = 10 L/min. This implies that the volume of the solution in the tank does not change. (2 marks)

   The concentration of salt is $\frac{S}{100}$. Since contents flow out at 10 L/min, we have the rate at which salt leaves is $\frac{S}{10}$. (2 marks)

   However salt is added to the tank at the rate of 0.1 kg/min. Therefore the rate of change of concentration of salt is given by

   $$\frac{dS}{dt} = -\frac{S}{10} + 0.1 \text{ (3 marks)} \tag{3}$$

   Integrating this equation gives

   $-10 \log |-0.1S + 0.1| = t + C$ which implies that $S = 1 + Ce^{-0.1t}$ (2 marks).

   At $t = 0$, we have 10 kg of salt. This gives $C = 9$. (2 marks)

   After 30 mins, the amount of salt left is $1 + 9e^{-0.1t}$. (2 marks)

4. (15 points) Show that eigenvectors corresponding to distinct eigenvalues are linearly independent.

   Solution:

   Let the eigenvalues be given by $\lambda_1, \lambda_2, \cdots, \lambda_k$ and the eignvectors by $v_1, v_2, \cdots, v_k$. (2 marks)

   Let j be the maximal index so that that $v_1, \ldots, v_j$ are independent. (3 marks)

   This implies that there exists constants d_i such that $\sum_{i=1}^{j} d_i v_i = v_{j+1}$. (2 marks)

   Applying A on the above equation yields $A \sum_{i=1}^{j} d_i v_i = Av_{j+1} = \lambda_{j+1} v_{j+1}$. (1 mark)

   Further we know that $A \sum_{i=1}^{j} d_i v_i = \sum_{i=1}^{j} d_i \lambda_i v_i$. (2 marks)

   This yields $\sum_{i=1}^{j} d_i \lambda_i v_i = \lambda_{j+1} v_{j+1} = \lambda_{j+1} (\sum_{i=1}^{j} d_i v_i)$. (2 marks)

   Therefore, we have $\sum_{i=1}^{l} (\lambda_i - \lambda_{j+1}) d_i v_i = 0$, which is a contradiction since $\lambda_i \neq \lambda_{j+1}$. (3 marks)

5. (15 points) For each of (1)–(5), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

   1. Every real number is a fixed point.
   2. Every integer is a fixed point, and there are no others.

3. There are precisely three fixed points, and all of them are stable.
4. There are no fixed points.
5. There are precisely 2024 fixed points.

Solution:

1. $f(x) = 0$ for all x. (3 marks)
2. $f(x) = \sin n\pi$. (3 marks)
3. A stable or unstable fixed point implies changing the sign of the function values locally. Between any two fixed point of the same type (stable, unstable) must be a fixed point of the other type, because of the mean value theorem at a smooth function. Thus, this property cannot be fulfilled. (3 marks)
4. $f(x) = c$ for any constant c. (3 marks)
5. $f(x) = (x-1)(x-2)...(x-2024)$. (3 marks)

6. (15 points) Construct a differential equation of the form $y'' + p(x)y' + q(x)y = 0$, where both p and q are continuous everywhere and $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are its solutions.

Solution:

We claim that there is no such differential equation.
(2 marks).

For contradiction, assume that there exists such a differential equation with linearly independent solutions $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ (2 marks)

We calculate the Wronskian corresponding to the functions.

$W(x) = \begin{vmatrix} \sin(x^2) & \cos(x^2) \\ 2x\cos(x^2) & -2x\sin(x^2) \end{vmatrix}$. (4 marks).

The Wronskian vanishes at $x = 0$. (3 marks).

Therefore, there cannot exist such a differential equation, since the Wronskian of a linearly independent solutions of a differential equation is always non-zero on the interval if p and q are continuous everywhere. (4 marks)

7. (15 points) Suppose that A and B are $n \times n$ matrices satisfying $AB = BA$ and suppose that B has n distinct eigenvalues. Then AB is diagonalizable.

Solution:

Suppose v is an eigenvector of B with eigenvalue λ. Note that $(BA)v = (AB)v = \lambda Av$. So either $Av = 0$ or Av is also an eigenvector of B with eigenvalue λ. (3 marks)

Since B has n distinct eigenvalues, they all have multiplicity 1 which means that all of the eigenspaces of B are one-dimensional. Since v and Av both lie in the one dimensional eigenspace of B corresponding to the eigenvalue λ, v and Av must be linearly dependent. Since $v \neq 0$, this means that $Av = \mu v$ for some scalar μ. Therefore, v is an eigenvector of A corresponding to the eigenvalue μ. (6 marks)

Since B has n distinct eigenvalues, B is diagonalizable. Therefore B has n linearly independent eigenvectors $v_1, ..., v_n$. This implies that the vectors $v_1, .., v_n$ are also linearly independent eigenvectors of As, and hence A is diagonalizable. (3 marks)

This implies that $A = PD_1P^{-1}$ and $B = PD_2P^{-1}$ for diagonal matrices D_1 and D_2. This implies that $AB = PD_1D_2P^{-1}$ for the diagonal matrix D_1D_2. Hence AB is diagonalizable. (3 marks)