

## CS7.302: Computer Graphics

Final Exam on Feb 26, 2024. Total: 100 points  
(Answer any 5 out of 6 questions)

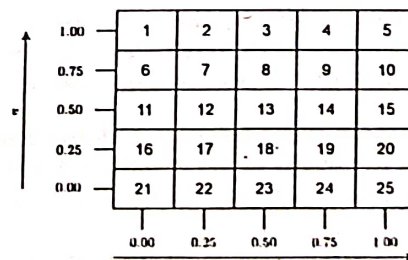
1. [20 points] Given  $p(\omega) = k \cdot \cos \theta \cdot e^\phi$ , which is a PDF to sample a direction vector on the upper hemisphere. Recall also that  $p(\omega) = \sin \theta \cdot p(\theta, \phi)$ . Solve the following sub-questions in order:
  - (a) Derive the normalization constant  $k$  of the PDF  $p(\omega)$ . [5 points]
  - (b) Write expression for  $p(\theta)$  &  $p(\phi)$ . [5 points]
  - (c) Write expression for the CDFs  $P(\theta)$  &  $P(\phi)$ . [5 points]
  - (d) Given two random numbers  $\xi_1 \in [0, 1]$  &  $\xi_2 \in [0, 1]$ , give the expressions that sample  $\theta$  and  $\phi$  proportional to  $p(\theta, \phi)$ . [5 points]

Hint:  $\sin 2x = 2 \sin x \cos x$

2. [20 points] Given an integral  $I = \int_D f(x) dx$  over a domain  $D$ , where  $|D| = 2\pi$ , solve the following:
  - (a) Write the Monte Carlo Estimator  $\langle I \rangle$ , assuming  $X_i$ 's are sampled uniformly over the domain. [5 points]
  - (b) Write the Monte Carlo Estimator  $\langle I \rangle$ , assuming  $X_i$ 's are sampled according to some PDF  $p(X_i)$ . [5 points]
  - (c) Prove that the Monte Carlo Estimator from the previous question computes the right answer on average. [10 points]

Hint: For a continuous random variable  $X$  sampled with probability  $p(X)$ ,  $E[f(X)] = \int_D f(x)p(x)dx$ .

3. [20 points] Given this  $5 \times 5$  monochromatic image, answer the following questions:



- (a) What will be the value at  $(u, v) = (0.4, 0.24)$  using nearest neighbour interpolation. [5 points]
- (b) What will be the value at  $(u, v) = (0.4, 0.24)$  using bi-linear interpolation. [5 points]
- (c) An object is using spherical mapping. Assume the following spherical mapping:

$$x = r \cos u \sin v$$

$$y = r \sin u \sin v$$

$$z = r \cos v$$

What will be the value at the point on the surface  $x = (1, 1, \sqrt{2})$  assuming bi-linear interpolation? [10 points]

4. [20 points] You are given the following function and an integral:

$$f(x) = (x + 2)^2, \quad I = \int_0^2 f(x) dx.$$

You are also given four uniform random numbers in  $[0, 1]$ :

$$\xi_1 = 0.58 \quad \xi_2 = 0.99 \quad \xi_3 = 0.27 \quad \xi_4 = 0.63.$$

- (a) Analytically integrate to find  $I$ . [2 points]
- (b) Use Monte Carlo (MC) integration, and sample  $X_i$ 's using the random numbers as:

$$X_i = 2 \cdot \xi_i$$

The PDF to be used in MC is  $p(X_i) = \frac{1}{2}$ .

Show the steps for each of the four Monte Carlo samples and write the final answer. [7 points]

- (c) Use Monte Carlo integration and sample  $X_i$ 's using the random numbers as:

$$X_i = \sqrt[3]{56 \cdot \xi_i + 8} - 2$$

The PDF to be used in MC is  $p(X_i) = \frac{3}{56}(x + 2)^2$ .

Show the steps for each of the four Monte Carlo samples and write the final answer. [7 points]

- (d) Plot a rough graph of number of MC samples on the x-axis, and for each sample plot the value of the MC estimate on the y-axis for (b) and (c). Also draw a line of  $y = A$  where  $A$  is the analytic answer from (a). Which converges faster, (b) or (c)? [4 points]
5. [20 points] Derive the expression to determine if a ray intersects with a sphere and the location of the intersection. A ray can be defined as  $\vec{o} + t\vec{d}$ , where  $\vec{o}$  is the start point of the ray and  $\vec{d}$  is the unit vector in the direction of the ray. A point  $\vec{p}$  lying on the sphere with center  $\vec{c}$  and radius  $r$  satisfies the property  $|\vec{p} - \vec{c}| = r$ . Find the expression for  $t$  in terms of  $\vec{o}$ ,  $\vec{p}$ ,  $\vec{c}$  and  $r$ , which would give the point of intersection with the sphere and the condition when the ray would intersect the sphere.
- Hint: The roots of a quadratic equation  $ax^2 + bx + c$  are given by  $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

6. [20 points] The rendering equation is as follows:

$$L_o(x, \omega_o) = \int_{\Omega} f(x, \omega_o, \omega_i) L_i(x, \omega_i) \cos \theta d\omega_i, \quad (1)$$

where  $\Omega$  is the upper hemisphere. The scene also contains an area light  $A$  in the scene, which is located at  $p$  with normal vector  $n_l$ . Answer the following questions:

- (a) Derive  $d\omega$  in terms of  $dA$ , where  $dA$  is a differential area on  $A$  and  $d\omega$  is a differential solid angle subtended by  $dA$  on  $\Omega$ . [8 points]
- (b) Given the previous derivation, write the rendering equation over  $A$  instead of over  $\Omega$ . [4 points]
- (c) Write a Monte Carlo Estimator for the modified rendering equation from the previous question using a uniform sampling PDF over the area light  $A$ . [8 points]