I QIC

1) $\vec{v}$ - 3D unit vector, θ - real no.

PT. $\exp(i\theta\, \vec{v}\cdot\vec{\sigma}) = \cos(\theta)\,\mathbb{1} + i\sin(\theta)\,\vec{v}\cdot\vec{\sigma}$

$$\vec{v}\cdot\vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z$$

$$= v_x\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + v_y\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + v_z\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\vec{v}\cdot\vec{\sigma} = \begin{bmatrix}v_z & v_x - iv_y\\ v_x+iv_y & -v_z\end{bmatrix}, \quad (\vec{v}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) = \begin{bmatrix}v_z & v_x-iv_y\\ v_x+iv_y & -v_z\end{bmatrix} = \begin{bmatrix}v_x^2+v_z^2+v_y^2 & 0\\ 0 & v_x^2+v_y^2+v_z^2\end{bmatrix}$$

$$\cancel{\exp(i\theta\,\vec{v}\cdot\vec{\sigma}) = \cos(\theta(\vec{v}\cdot\vec{\sigma})) + i\sin(\theta(\vec{v}\cdot\vec{\sigma}))} \qquad = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \mathbb{1}$$

$$\exp(i\theta(\vec{v}\cdot\vec{\sigma})) = \mathbb{1} + i\theta(\vec{v}\cdot\vec{\sigma}) + \frac{(i\theta(\vec{v}\cdot\vec{\sigma}))^2}{2!} + \frac{(i\theta(\vec{v}\cdot\vec{\sigma}))^3}{3!} + \frac{(i\theta\,\vec{v}\cdot\vec{\sigma})^4}{4!}$$

$$+ \frac{(i\theta\,\vec{v}\cdot\vec{\sigma})^5}{5!} + \frac{(i\theta\,\vec{v}\cdot\vec{\sigma})^6}{6!} + \cdots$$

$$= \mathbb{1} + i\theta\begin{bmatrix}v_z & v_x-iv_y\\ v_x+iv_y & -v_z\end{bmatrix} - \frac{\theta^2}{2!}\mathbb{1} - \frac{i\theta^3}{3!}\begin{bmatrix}v_z & v_x-iv_y\\ v_x+iv_y & -v_z\end{bmatrix}$$

$$+\frac{\theta^4}{4!}\mathbb{1} + \frac{i\theta^5}{5!}\begin{bmatrix}v_z & v_x-iv_y\\ v_x+iv_y & -v_z\end{bmatrix} - \frac{\theta^6}{6!}\mathbb{1} + \cdots$$

$$= \mathbb{1} - \frac{\theta^2}{2!}\mathbb{1} + \frac{\theta^4}{4!}\mathbb{1} - \frac{\theta^6}{6!}\mathbb{1} + \cdots$$

$$+ i(\vec{v}\cdot\vec{\sigma})\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right]$$

$$= \mathbb{1}\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right] + i(\vec{v}\cdot\vec{\sigma})\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right]$$

$$= \mathbb{1}\cos\theta + i(\vec{v}\cdot\vec{\sigma})\sin\theta = \cos(\theta)\,\mathbb{1} + i\sin(\theta)\,\vec{v}\cdot\vec{\sigma}$$

3.5/

2. $H = \frac{\gamma B \sigma_z}{2}$, $\rho_0 = \frac{1}{2}\left(\mathbb{1} + \frac{1}{2}\sigma_y + \frac{1}{2}\sigma_z\right)$

$$\rho_0 = \frac{1}{2}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right]$$

$$\rho_0 = \frac{1}{2}\begin{bmatrix} \frac{3}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -i \\ i & 1 \end{bmatrix}$$

$$\rho = e^{-\frac{iHt}{\hbar}} \rho_0 e^{iHt}$$

$$= \exp\left(-it\frac{\gamma B \sigma_z}{2}\right) \rho_0 \exp\left(it\frac{\gamma B \sigma_z}{2}\right)$$

$$= \exp\left(-\frac{it\gamma B}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \rho_0 \exp\left(\frac{it\gamma B}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$= \exp\begin{pmatrix} -\frac{it\gamma B}{2} & 0 \\ 0 & \frac{it\gamma B}{2} \end{pmatrix} \rho_0 \exp\begin{pmatrix} \frac{it\gamma B}{2} & 0 \\ 0 & -\frac{it\gamma B}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{it\gamma B}{2}} & 0 \\ 0 & e^{\frac{it\gamma B}{2}} \end{pmatrix} \frac{1}{4}\begin{pmatrix} 3 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{it\gamma B}{2}} & 0 \\ 0 & e^{-\frac{it\gamma B}{2}} \end{pmatrix}$$

$$\rho = \frac{1}{4}\begin{pmatrix} 3e^{-\frac{it\gamma B}{2}} & -ie^{-\frac{it\gamma B}{2}} \\ ie^{\frac{it\gamma B}{2}} & e^{\frac{it\gamma B}{2}} \end{pmatrix}\begin{pmatrix} e^{\frac{it\gamma B}{2}} & 0 \\ 0 & e^{-\frac{it\gamma B}{2}} \end{pmatrix}$$

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & -ie^{-it\gamma B} \\ ie^{it\gamma B} & 1 \end{pmatrix}$$

5)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

measurements $\rightarrow \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

~~P(a=0, b=0)~~

$$\rho = |\psi\rangle\langle\psi|_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\left(\frac{1}{\sqrt{2}}\left(\langle 00| + \langle 11|\right)\right)$$

$$\rho = |\psi\rangle\langle\psi|_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$P(a=0, b=0) = \cancel{\text{root}}\ Tr(M_0^\dagger M_0 \rho)$$

where $\quad M_0 = |0\rangle\langle 0| \otimes |0\rangle\langle 0| = |00\rangle\langle 00|$

$$M_0^\dagger M_0 = |00\rangle\langle 00|, \quad M_0^\dagger = M_0$$

$$\therefore P(a=0, b=0) = Tr(M_0\rho)$$

$$= Tr\left((|00\rangle\langle 00|)\left(\frac{1}{2}\right)\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)\right)$$

$$= \frac{1}{2} Tr\left(|00\rangle\langle 00| + |00\rangle\langle 11|\right)$$

Since the given state is pure, trace is 1.

$$\therefore P(a=0, b=0) = \frac{1}{2} \times 1 = \frac{1}{2}$$

3)

$\cancel{P(a=0, b=1)}$

$\cancel{M_1 = |01\rangle\langle 01|}$

$$P(a=1, b=1) = Tr(M_2 \rho)$$

$$= Tr\left((|11\rangle\langle 11|) \tfrac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)\right)$$

$$= Tr\left(\tfrac{1}{2}(|11\rangle\langle 00| + |11\rangle\langle 11|)\right) = \frac{1}{2}$$

$$P(b=0) = Tr(M^{+} M \rho)$$

$$= Tr\left((|00\rangle\langle 00| + |10\rangle\langle 10|) \tfrac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)\right)$$

$$= Tr\left(\tfrac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11|)\right) = \frac{1}{2}$$

$P(a=0, b=1)$, $M = |01\rangle\langle 01|$

$$P(a=0, b=1) = Tr\left(|01\rangle\langle 01| \tfrac{1}{2}|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$P(a=0, b=1) = \underline{\underline{0}}$$

3) $$|\phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d-1} |i\rangle_A \otimes |i\rangle_B$$

$$|A| = |B| = d$$

M - $d \times d$ matrix

M^T - transpose of M w.r.t. $\{|i\rangle_B\}_i$

we have to prove:

$$(M_A \otimes \mathbb{1})|\phi\rangle_{AB} = (\mathbb{1} \otimes M_B^T)|\Phi\rangle_{AB}$$

$$LHS = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} (M_A|i\rangle_A) \otimes |i\rangle_B = |u\rangle_{AB}$$

~~Let~~ $|m\rangle_A \otimes |n\rangle_B$ form an orthonormal basis in $H_A \otimes H_B$

Each component of the $d \times 1$ matrix can be written as:

$$\langle m|_A \otimes \langle n|_B |u\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle m|_A M|i\rangle_A \cdot \langle n|i\rangle_B$$

$$= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle m|_A M|i\rangle_A \, \delta_{ni}$$

$$= \frac{1}{\sqrt{d}} \langle m|M|n\rangle$$

Now, taking RHS,

$$RHS = (\mathbb{1} \otimes M_B^T)|\Phi\rangle_{AB}$$

$$= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes M_B^T|i\rangle_B = |v\rangle_{AB}$$

The arbitrary component of this $d \times 1$ matrix:

$$\langle m|_A \otimes \langle n|_B |v\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle m|i\rangle_A \langle n|M_B^T|i\rangle_B$$

$$\langle m|_A \otimes \langle n|_B |v\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \delta_{mi} \langle i|M|n\rangle$$

$\because \langle n|M^T|i\rangle = \langle i|M|n\rangle$

$$= \frac{1}{\sqrt{d}} \langle m|M|n\rangle$$

Hence proved

4) $\mathcal{E}_{A\to B}: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$

$$\mathcal{E}_{A\to B}(\rho) = p\rho + (1-p)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$$

Let $\rho = \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix}$

then, the transformation is:

$$\mathcal{E}_{A\to B}(\rho) = p\begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix} + (1-p)(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$$

$$\sigma_x \rho \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \rho_4 & \rho_3 \\ \rho_2 & \rho_1 \end{bmatrix}$$

$$\sigma_y \rho \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i\rho_3 & -i\rho_4 \\ i\rho_1 & i\rho_2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_y \rho \sigma_y = \begin{bmatrix} \rho_4 & -\rho_3 \\ -\rho_2 & \rho_1 \end{bmatrix}$$

$$\sigma_z \rho \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_z \rho \sigma_z = \begin{bmatrix} \rho_1 & \rho_2 \\ -\rho_3 & -\rho_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \rho_1 & -\rho_2 \\ -\rho_3 & \rho_4 \end{bmatrix}$$

$$\mathcal{E}_{A\to \cancel{B}}(\rho) = \begin{bmatrix} p\rho_1 & p\rho_2 \\ p\rho_3 & p\rho_4 \end{bmatrix} + (1-p)\begin{bmatrix} 2\rho_4 + \rho_1 & -\rho_2 \\ -\rho_3 & 2\rho_1 + \rho_4 \end{bmatrix}$$

$$\mathcal{E}_{A\to B}(\rho) = \begin{bmatrix} p\rho_1 + (1-p)(2\rho_4+\rho_1) & p\rho_2 - \rho_2(1-p) \\ p\rho_3 - \rho_3(1-p) & p\rho_4 + (1-p)(2\rho_1+\rho_4) \end{bmatrix}$$

$$= \begin{bmatrix} 2\rho_4 + \rho_1 - 2p\rho_4 & (2p-1)\rho_2 \\ (2p-1)\rho_3 & 2\rho_1 + \rho_4 - 2p\rho_1 \end{bmatrix}$$

$$\mathcal{E}_{A\to B}(\rho) = \begin{bmatrix} \rho_1 + 2(1-p)\rho_4 & (2p-1)\rho_2 \\ (2p-1)\rho_3 & \rho_4 + 2(1-p)\rho_1 \end{bmatrix}$$

The transformation of the channel is as shown.

Applying this on part A of maximally entangled state

$$|\phi\rangle\langle\phi| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Applying the transformation on each 2x2 block on the above state:

$$\begin{bmatrix} \Lambda\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ \Lambda\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} & \Lambda\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & (p-\frac{1}{2}) \\ 0 & (1-p) & 0 & 0 \\ 0 & 0 & (1-p) & 0 \\ (p-\frac{1}{2}) & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

This is the Choi state.