Probability and Statistics
Replacement Exam Solution

November 8, 2022

SECTION 1: 6 marks each

Question 1

Find the stationary distribution π for Markov Chain with the following transition probability matrix (4 marks). State if π is unique (1 mark). Is the chain irreducible? Give reasons (1 mark).

$$P = \begin{bmatrix} 0.1 & 0.9 & 0 \\ 0 & 0.9 & 0.1 \\ 0.1 & 0.9 & 0 \end{bmatrix}$$

(4 marks)

Since, $\pi \mathrm{P} = \pi$

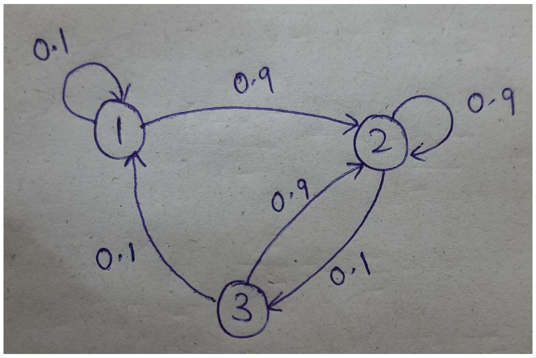
$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.9 & 0 \\ 0 & 0.9 & 0.1 \\ 0.1 & 0.9 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

$$\begin{pmatrix} 0.1\pi_1 + 0.1\pi_3 & 0.9\pi_1 + 0.9\pi_2 + 0.9\pi_3 & 0.1\pi_2 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

The stationary distribution π is given by the solution of the below equations:-

1. $0.1\pi_1 + 0.1\pi_3 = \pi_1$
2. $0.9\pi_1 + 0.9\pi_2 + 0.9\pi_3 = \pi_2$
3. $0.1\pi_2 = \pi_3$

And we also know that $\pi_1 + \pi_2 + \pi_3 = 1$. Hence, on solving we get $\pi_1 = 0.01$, $\pi_2 = 0.9$ and $\pi_3 = 0.09$. Clearly, π is unique. **(1 mark)**
The chain is irreducible as we can go to any node from all other nodes either directly or indirectly. **(1 mark)**



i.e. $1 \to 1, 2 \to 2, 3 \to 3, 1 \to 2, 1 \to 3, 2 \to 1, 2 \to 3, 3 \to 1, 3 \to 2$

Question 2

If A and B are exponential random variables with parameters a and b respectively, then prove $P(A < B) = E[e^{-bA}]$. Further show that this is equal to a/(a+b).

(4 marks)

$$\begin{aligned}
P(A < B) &= \int_0^\infty P(B > A | A = x) f_A(x) dx \\
&= \int_0^\infty e^{-bx} f_A(x) dx \\
&= \int_0^\infty g(x) f_A(x) dx \\
&= E[g(A)] \\
&= E[e^{-bA}]
\end{aligned}$$

(2 marks)

$$\begin{aligned}
E[e^{-bA}] &= \int_0^\infty e^{-bx} a e^{-ax} dx \\
&= a \int_0^\infty e^{-(b+a)x} dx \\
&= a/(a+b)
\end{aligned}$$

Question 3

Derive the expression for the Moment generating function of the following random variables

1. Standard Normal with mean 0 and variance 1 (3 marks)
2. Poisson random variable with parameter λ (3 marks)

Solution:
Moment Generating function is given by $M_X(t) = E[e^{tX}]$.

For Standard Normal X with mean 0 and variance 1:

$$
\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2} \qquad \forall x \in \mathbb{R} \\
\implies E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(x^2-2tx)} dx \\
&= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(x-t)^2} dx \\
&= e^{\frac{1}{2}t^2} I
\end{aligned}
$$

Now the above integral I is simply the integration of the pdf of a normal distribution with mean t and variance 1 over its complete support $\mathbb{R}$. Thus it will integrate to 1.

$$
\begin{aligned}
I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(x-t)^2} dx = 1 \\
\implies E[e^{tX}] = M_X(t) = e^{\frac{1}{2}t^2}
\end{aligned}
$$

For Poisson random variable Z with parameter λ:

$$
\begin{aligned}
Pr(Z = z) &= \frac{\lambda^z e^{-\lambda}}{z!} \qquad \forall z \in \{0, 1, 2, 3, 4...\} \\
\implies E[e^{tZ}] &= \sum_{z=0}^{\infty} e^{tz} \frac{\lambda^z e^{-\lambda}}{z!} \\
&= e^{-\lambda} \sum_{z=0}^{\infty} \frac{(\lambda e^t)^z}{z!}
\end{aligned}
$$

Now using the taylor series expansion of the exponential function:

$$\begin{aligned}
e^x &= \sum_{i=0}^{\infty} \frac{x^i}{i!} \\
\implies \sum_{z=0}^{\infty} \frac{(\lambda e^t)^z}{z!} &= e^{\lambda e^t} \\
\implies E[e^{tZ}] = M_Z(t) &= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}
\end{aligned}$$

Marking Scheme

For part 1 - 1 mark for writing the full equation of $E[e^{tX}]$, 2 marks for using the fact that the integral will evaluate to 1 and using it to get to the final answer. For part 2 - 1 mark for writing the full equation of $E[e^{tZ}]$, 2 marks for using the taylor series to get to the final answer.

Question 4

Let X be an exponential random variable with parameter 1. Find

1. Conditional PDF and CDF given $X > 1$
2. $E[X|X > 1]$

Solution

X is an exponential random variable with parameter 1. PDF of X is given by:

$$f_X(x) = \begin{cases} e^{-x}, x > 0 \\ 0, \text{Otherwise} \end{cases}$$

Let A be the event that $X > 1$.

$$\begin{aligned}
P(A) &= \int_1^{\infty} e^{-x} dx \\
&= \frac{1}{e} \quad \text{(1 Mark)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{X|X>1}(x) &= \begin{cases} \frac{f_X(x)}{P(A)}, & x > 1 \\ 0, & \text{Otherwise} \end{cases} \\
&= \begin{cases} e^{-x+1}, & x > 1 \\ 0, & \text{Otherwise} \end{cases} \quad \text{(1 Mark)}
\end{aligned}$$

$$
\begin{aligned}
F_{X|X>1}(x) &= \frac{F_X(x) - F_X(1)}{P(A)} \\
&= 1 - e^{-x+1} \quad \text{(1 Mark)}
\end{aligned}
$$

$$
\begin{aligned}
E[X|X>1] &= \int_1^\infty x f_{X|X>1}(x)dx \\
&= \int_1^\infty x e^{-x+1} dx \quad \text{(1 Mark)} \\
&= e\int_1^\infty x e^{-x} dx \\
&= e\Bigg[- e^{-x} - x e^{-x} \Bigg]_0^\infty \\
&= e\frac{2}{e} \\
&= 2 \quad \text{(2 Marks)}
\end{aligned}
$$

Question 5

Let X and Y be independent and identically distributed discrete random variables taking values on positive integers. Their pmf is $p(x) = C2^{-x}$ for $x \geq 1$. Find

1. The value of C that makes it a valid pmf
2. $P(\min\{\text{X,Y}\} \leq x)$
3. $P(X \text{ divides } Y)$

Solution

Given,

$$\text{PMF of } X = C2^{-x}, x \geq 1$$

Sum of probabilities of all possible values must be 1.

$$C\sum_{n=1}^{\infty} 2^{-x} = 1$$

Using sum of infinite GP,

$$C = 1 \quad \text{(2 marks)}$$

$$
\begin{aligned}
P(\min\{X,Y\} \le z) &= P(X \le z \cup Y \le z) \\
&= P(X \le z) + P(Y \le z) - P(X \le z \cap Y \le z) \\
&= 2P(X \le z) - P(X \le z)P(Y \le z) \\
&= 2F_X(z) - F_X(z)^2 \\
&= 2(1 - \frac{1}{2^z}) - (1 - \frac{1}{2^z})^2 \quad \text{(2 marks)}
\end{aligned}
$$

Calculating $F_X(z)$

$$
\begin{aligned}
F_X(z) &= \sum_{n=1}^{z} 2^{-x} \\
&= \frac{1}{2}\frac{1 - \frac{1}{2^z}}{1 - \frac{1}{2}} \\
&= 1 - \frac{1}{2^z}
\end{aligned}
$$

$$
\begin{aligned}
P(X \text{ divides } Y) &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P(X = n \cap Y = kn) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n}(\sum_{k=1}^{\infty} \frac{1}{2^{kn}}) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n}(\frac{\frac{1}{2^n}}{1 - \frac{1}{2^n}}) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{2^n - 1} \\
&= \sum_{n=1}^{\infty} \frac{2^n - (2^n - 1)}{2^n.(2^n - 1)} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n - 1} - \frac{1}{2^n} \\
&= 1 + \sum_{n=2}^{\infty} \frac{1}{2^n - 1} - 1 \\
&= \sum_{n=2}^{\infty} \frac{1}{2^n - 1} \quad \text{(2 marks)}
\end{aligned}
$$

SECTION 2: 5 marks each

Question 1

Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2. Let $Z = X + Y$. Then find the pmf of Z.

Solution:

We know X and Y are independent. We have for $k \geq 0$:

$$
\begin{aligned}
P(Z = X + Y = k) &= \sum_{i=0}^{k} P(X + Y = k, X = i) && (1)\\
&= \sum_{i=0}^{k} P(Y = k - i, X = i) && (2)\\
&= \sum_{i=0}^{k} P(Y = k - i)P(X = i) \qquad (\because \text{X, Y are independent}) && (3)\\
&= \sum_{i=0}^{k} e^{-\lambda_2} \frac{\lambda_2^{k-i}}{(k-i)!} e^{-\lambda_1} \frac{\lambda_1^{i}}{i!} && (4)\\
&= e^{-(\lambda_2+\lambda_1)} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \lambda_2^{k-i}\lambda_1^{i} && (5)\\
&= e^{-(\lambda_2+\lambda_1)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_2^{k-i}\lambda_1^{i} && (6)\\
&= \frac{(\lambda_2+\lambda_1)^k}{k!} \cdot e^{-(\lambda_2+\lambda_1)} && (7)
\end{aligned}
$$

$\implies Z \sim \mathcal{P}(\lambda_2 + \lambda_1)$.
$\implies$ <u>*H.P.*</u>

Marking Scheme

2 marks: Correctly using formula for PMF of Poisson random variable, and getting upto step (3)
3 marks: Correct calculations after (3)

Question 2

Let X_1, X_2, ..., be a sequence of i.i.d uniform $U[0, 1]$ random variables.
Let $Y_n = min(X_1, X_2, ..., X_n$. Prove the following independently.

- $Y_n \to 0$ in distribution (2.5 marks)

- $Y_n \to 0$ in probability (2.5 marks)

Solution:

1. $Y_n \xrightarrow{d} 0$: Note that

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Also, note that $R_{Y_n} = [0, 1]$. For $0 \le y \le 1$, we can write

$$\begin{aligned} F_{Y_n}(y) &= P(Y_n \le y) \\ &= 1 - P(Y_n > y) \\ &= 1 - P(X_1 > y, X_2 > y, \cdots, X_n > y) \\ &= 1 - P(X_1 > y)P(X_2 > y)\cdots P(X_n > y) \qquad \text{(since } X_i\text{'s are independent)} \\ &= 1 - (1 - F_{X_1}(y))(1 - F_{X_2}(y))\cdots(1 - F_{X_n}(y)) \\ &= 1 - (1 - y)^n. \end{aligned}$$

So, we get

$$F_{Y_n}(y) = \begin{cases} 0 & y < 0 \\ 1 - (1 - y)^n & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

From this, we can conclude

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & y \le 0 \\ 1 & y > 0 \end{cases}$$

Now, for the R.V. Z=0, we have

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ 1 & z \ge 0 \end{cases}$$

(Here, note that y=0 is a point of discontinuity in convergence of $Y_n \to 0$.

$$\text{So, } Y_n \to 0 \quad \text{if } y \neq 0$$

But for convergence in distribution we need to only look at those values of y for which F_{Y_n} is continuous. Therefore,

$$Y_n \xrightarrow{d} 0$$

2. $Y_n \xrightarrow{p} 0$: Note that as we found in part 1.

$$F_{Y_n}(y) = \begin{cases} 0 & y < 0 \\ 1 - (1-y)^n & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

In particular, note that Y_n is a continuous random variable. To show $Y_n \xrightarrow{p} 0$, we need to show that

$$\lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0. \tag{8}$$

Since $Y_n \geq 0$, it suffices to show that

$$\lim_{n\to\infty} P\big(Y_n \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.$$

(a) For $\epsilon \in (0, 1]$, we have

$$\begin{aligned} P\big(Y_n \geq \epsilon\big) &= 1 - P(Y_n < \epsilon) \\ &= 1 - P(Y_n \leq \epsilon) \qquad \text{(since } Y_n \text{ is a continuous random variable)} \\ &= 1 - F_{Y_n}(\epsilon) \\ &= (1-\epsilon)^n \end{aligned}$$

So,

$$\begin{aligned} \lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) &= \lim_{n\to\infty} (1-\epsilon)^n \\ &= 0, \qquad \text{for all } \epsilon \in (0, 1]. \end{aligned}$$

(b) For $\epsilon > 1$, we have

$$P\big(Y_n \geq \epsilon\big) = 1 - F_{Y_n}(\epsilon) = (1-1)^n = 0$$

So,

$$\lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) = 0$$

Therefore,

$$\lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0$$

.

Marking Scheme

Part 1: 1.5 marks: correct steps for finding out $F_{Y_n}(y)$
0.5 marks: getting correct distribution of Y_n
0.5 marks: point of discontinuity

Part 2: 0.5 marks: writing condition for $Y_n \xrightarrow{p} 0$
1.5 marks: correct steps for solving for $\epsilon \in (0, 1]$
0.5 marks: correct steps for solving for $\epsilon > 1$

Question 3

Let $u_1, u_2, u_3 ... u_n$ denote n samples drawn from a $U[0,1]$ random variable. Describe a procedure to use them to obtain one sample from a $Binomial(n,p)$ random variable.

Solution:
A $Binomial(n,p)$ distribution models the number of successes in a sequence of n independent trials/experiments, each asking a yes–no question, and each with its own Boolean-valued outcome: success (with probability p) or failure (with probability $q = 1 - p$). Each trial/experiment is thus a $Bernoulli(p)$ random variable.
Thus, if $X_1, X_2, ... X_n$ are n independent random variables with $X_i \sim Bernoulli(p)$ and $Z = X_1 + X_2 + X_3 ... X_n$, then $Z \sim Binomial(n,p)$.
We first convert each sample to a sample of a Bernoulli random variable using stochastic simulation for discrete random variables. If $X \sim Bernoulli(p)$

$$p_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

If u is a sample drawn from $U[0,1]$, then we can generate a sample from X as follows:

$$X = \begin{cases} 1 & \text{if } 0 \leq u < p \\ 0 & \text{if } p \leq u \leq 1 \end{cases}$$

Using above, we generate x_i from the corresponding u_i and then $z = \sum_i x_i$ is the sample from binomial distribution.
Marking Scheme
2.5 marks for the idea of using each sample from uniform distribution to create samples from Bernoulli distribution and then adding them up to get a sample from Binomial distribution.
2.5 marks for the procedure to convert a sample from uniform distribution to that from Bernoulli distribution.
If you have not used all the samples meaningfully and have just used one of them to create a sample from Binomial distribution you will be graded out of 2.5 marks.

Question 4

Let X have a Poisson distribution with parameter Λ, where Λ is an exponential random variable with parameter μ. Show that X has a geometric distribution.

Solution

Given PDF of Λ,

$$f_\Lambda(\lambda) = \mu e^{-\mu\lambda} \quad \text{where } \lambda \geq 0$$

and PMF of X (given $\Lambda = \lambda$),

$$P(X = x \mid \Lambda = \lambda) = e^{-\lambda}\frac{\lambda^x}{x!} \quad \text{where } x \in \mathbb{N} \cup \{0\}$$

$$\begin{aligned}
\therefore P(X = x) &= \int P(X = x \mid \Lambda = \lambda)\ f_\Lambda(\lambda)\ d\lambda \\
&= \int_0^\infty \left(e^{-\lambda}\frac{\lambda^x}{x!}\right)\left(\mu e^{-\mu\lambda}\right)\ d\lambda \\
&= \frac{\mu}{x!}\int_0^\infty e^{-(1+\mu)\lambda}\lambda^x d\lambda
\end{aligned}$$

Let $(1+\mu)\lambda = t \implies (1+\mu)d\lambda = dt$.

$$\begin{aligned}
\therefore P(X = x) &= \frac{\mu}{x!}\int_0^\infty e^{-(1+\mu)\lambda}\lambda^x d\lambda \\
&= \frac{\mu}{x!}\int_0^\infty e^{-t}\left(\frac{t}{1+\mu}\right)^x \frac{dt}{1+\mu} \\
&= \frac{\mu}{x!\ (1+\mu)^{x+1}}\int_0^\infty e^{-t}t^x dt
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty e^{-t}t^x dt &= \left[-t^x e^{-t}\right]_0^\infty + x\int_0^\infty e^{-t}t^{x-1}dt \\
&= 0 + x\int_0^\infty e^{-t}t^{x-1}dt \\
&= x\int_0^\infty e^{-t}t^{x-1}dt
\end{aligned}$$

Similarly,

$$\begin{aligned}
&= x(x-1)\int_0^\infty e^{-t}t^{x-2}dt \\
&= x(x-1)(x-2)\cdots(x-(x-1))\int_0^\infty e^{-t}t dt \\
&= x(x-1)(x-2)\cdots(x-(x-1))\left\{[-te^{-t}]_0^\infty + [-e^{-t}]_0^\infty\right\} \\
&= x(x-1)(x-2)\cdots 3\cdot 2\cdot 1 \\
&= x!
\end{aligned}$$

$$
\begin{aligned}
\therefore P(X=x) &= \frac{\mu}{x!\,(1+\mu)^{x+1}} \times x! \\
&= \frac{\mu}{(1+\mu)^{x+1}} \\
&= \frac{1}{(1+\mu)^{x}}\left(\frac{\mu}{1+\mu}\right) \\
&= \left(\frac{1}{1+\mu}\right)^{x}\left(1-\frac{1}{1+\mu}\right) \\
\implies X &\sim \mathit{Geometric}\left(1-\frac{1}{1+\mu}\right)
\end{aligned}
$$

Marking Scheme

- 0.5 mark for correctly identifying PDF of Λ.
- 0.5 mark for correctly identifying $P(X = x \mid \Lambda = \lambda)$.
- 1 mark for the formula of deriving $P(X = x)$ from $P(X = x \mid \Lambda = \lambda)$ and $f_\Lambda(\lambda)$.
- 2 marks for the integration.
- 1 mark for correctly identifying how X is Geometric.