Science-2 (Mid Sem Exam)
(2024)

Q1a. Consider the Compartmental models (Susceptible-Recovery-Infection) of epidemiology. If at any time t, the density of the susceptible, infected, and recovered population is captured by $s, i,$ and r respectively, construct the associated differential equation of each compartment. Explain each term in one or two sentences. Here the rate of infection is β, rate of recovery is γ, and take $s(t) + i(t) + r(t) = 1$.

3

Q1b. At initial phase of the disease spread, what should be the relation between β and γ such that the disease starts to propagate (exponentially)?

3

Q2. For a certain class of distributions, it is possible to create pseudo-random numbers from uniformly distributed random numbers by finding a mathematical transformation (inverse transform method). If z (drawn from uniform distribution) is the random element chosen from 0 to 1, and the target distribution is $f = k\,e^{-yk}$, then find the relation between y and z. k is constant.

3

Q3a. In 19th century, Robert Brown, a Scottish botanist, observed that pollen grains suspended in water, instead of remaining stationary or falling downwards, would trace out a random zig-zagging pattern. Using Langevin's approach, write down (with explanation) the equation of motion of one single pollen grain.

4

Q3b. Specify which principles/axioms (of statistical thermodynamics) are required for solving the equation of motion of a large number of pollen grains.
Using these axioms, calculate the average square displacement ($\langle x^2 \rangle$) of a large number pollen grains.

2+3=5

Q4a. The Predator-prey system consists of two kinds of animals. One of which preys on the other. If X symbolize prey, Y the predator, and A the food of the prey, we can write (this is also called as Lotka-Volterra model) following rate equations:

$$X+A \rightarrow 2X \quad ......(1)$$
$$X + Y \rightarrow 2Y \quad ......(2)$$
$$Y \rightarrow B \quad ......(3).$$

Construct the associated master equation for the model. k_1, k_2, k_3 are the rate constants of equations 1,2, and 3 respectively.

3

Q4b. Write down the related ODE models for the abovementioned rate reactions (assuming A is constant). Explain each term in one to two sentences.

3

Q5: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(1,2)$ and $f_y(2,1)$. Show that $u(x,y) = e^x \sin y$ is a solution of Laplace Equation $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$. **1+1+2=4**

Q6: If $f''(x_i) = \frac{f_{i+1} + f_{i-1} - 2f_i}{\Delta x^2}$, and if the heat conduction equation is $\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$, then show that $T_i^{j+1} = T_i^j + (T_{i+1}^j + T_{i-1}^j - 2T_i^j)\alpha \frac{\Delta t}{\Delta x^2}$. **3**