Quiz Solutions

Q1

$$P(X_3 < X_1 \text{ and } X_3 < X_2)$$

$$= P(X_1 > X_3 \text{ and } X_2 > X_3)$$

1 mark

$$= \int_0^\infty \underbrace{P(X_1 > x \text{ and } X_2 > x)}_{} f_{X_3}(x)\,dx$$

($\because$ Independent)

1 mark

$$= \int_0^\infty P(X_1 > x)\; P(X_2 > x)\; (\lambda_3 e^{-\lambda_3 x})\,dx$$

3 marks

$$= \lambda_3 \int_0^\infty e^{-\lambda_1 x} e^{-\lambda_2 x} e^{-\lambda_3 x}\,dx$$

$$= \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \quad \underline{\text{Ans}}$$

$\Rightarrow$ 2 for correct substitution of PDF & CDF

1 for correct working & final answer

Marking Scheme

(Total scaled by 2)

Q2 $N(t)$: Poisson Process with rate 'λ'

Check Markov Property 2 marks

$P(N(t)=k \mid N(t_1)=k_1, N(t_2)=k_2 \ldots\ldots N(t_m)=k_m)$

$= P(N(t-t_m) = k-k_m)$

$= P(N(t)=k \mid N(t_m)=k_m)$

$\therefore$ TPM $P_{ij}(t) = P(N(t)=j \mid N(0)=i)$

4 marks

$$= \begin{cases} 0 & \text{if } j<i \\ P(N(t)=j-i) & \text{otherwise} \end{cases}$$

$\rightarrow e^{-\lambda t}\dfrac{(\lambda t)^{j-i}}{(j-i)!}$ (By Definition)

2 marks

$$\therefore P = \begin{array}{c|cccc} & 0 & 1 & 2 & \cdots \\ \hline 0 & e^{-\lambda t} & e^{-\lambda t}(\lambda t) & e^{-\lambda t}\frac{(\lambda t)^2}{2!} & \\ 1 & 0 & e^{-\lambda t} & e^{-\lambda t}(\lambda t) & \\ 2 & 0 & 0 & e^{-\lambda t} & \\ \vdots & & & & \ddots \end{array}$$

Dimensions $\Rightarrow |\{0,1,2\ldots\}| \times |\{0,1,2\ldots\}|$

2 marks

Q3. Binomial Process $\{S_n, n \geq 1\}$

we know $S_n = \sum_{i=1}^{n} X_i$ where X_i are iid Bernoulli r.v.. Let $X_i \sim \text{Bernoulli}(p)$

So, 2 marks
$$\left\{\begin{aligned} &P(S_{n+1} = j \mid S_n = i) \\ = &P(X_{n+1} = j - i) \end{aligned}\right. \quad \text{(By Bernoulli r.v. PMF)}$$

$$= \begin{cases} (1-p) & \text{if } j = i \\ p & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{3 marks}$$

$$\therefore \quad P = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & & \\ \hline 1 & 1-p & p & 0 & 0 & \cdots & \\ 2 & 0 & 1-p & p & 0 & \cdots & \\ 3 & 0 & 0 & 1-p & p & 0 & \cdots \\ 4 & 0 & 0 & 0 & 1-p & p & \cdots \\ & & & & \vdots & & \end{array}$$

Marking Scheme

Total Scaled by 2

Q4

Stationary Distribution

1 mark $\Rightarrow \pi Q = 0$

$$[\pi_1 \quad \pi_2] \begin{bmatrix} -\lambda_H & \lambda_H \\ \lambda_T & -\lambda_T \end{bmatrix} = 0$$

2 marks $\Rightarrow \lambda_H \pi_1 = \lambda_T \pi_2$ or $\pi_1 = \dfrac{\lambda_T}{\lambda_H} \pi_2$

1 mark and $\pi_1 + \pi_2 = 1$

$\Rightarrow$

$(1/2 + 1/2)$ $\left\{ \pi_1 = \dfrac{\lambda_T}{\lambda_H + \lambda_T}, \quad \pi_2 = \dfrac{\lambda_H}{\lambda_H + \lambda_T} \right\}$

For Embedded DTMC, we only consider jumps from one state to a different state — 2 marks

So $P_{11} = P_{22} = 0$ — 1 mark

and $P_{ij} = \dfrac{q_{ij}}{|q_{ii}|}$ — 1 mark

$\therefore$ TPM $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ — 1 mark