

1. Statement 1: If A is a skew-Hermitian Matrix, then iA and -iA are Hermitian
 Statement 2: If A is any square matrix, the $A - A^*$ is a skew-Hermitian Matrix

Which of the above statement(s) is/are correct? **(1 mark)**

- a. Statement 1
 - b. Statement 2
 - c. Both Statement 1 and 2**
 - d. None of the above
2. If A is symmetric and positive definite, then LU factorization can be arranged so that $A=LL^T$
 Where L is ? **(1 mark)**
- a. Lower Triangular with positive diagonal entries**
 - b. Upper Triangular with positive diagonal entries
 - c. Lower Triangular with real diagonal entries
 - d. upper Triangular with real diagonal entries

3.
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ 9 \\ 9 \end{bmatrix}$$

Find the values of x_1 , x_2 , and x_3 by using forward and back substitution. **(1 mark)**

- a. 2,3,3**
 - b. 3,2,3
 - c. 3,3,2
 - d. 3,3,3
4. A is a 3X3 invertible matrix. $\text{Tr}(A)=11$, $\text{Tr}(A^2)=53$, and $\text{Tr}(A^{-1}) = 1/7$. Find $\det(A)$. **(2 marks)**
- a. 245
 - b. 236
 - c. 238**
 - d. 224
5. A matrix P is an orthogonal projector if it is? **(1 mark)**
- a. Idempotent
 - b. Symmetric
 - c. Square
 - d. Both A and B**

6. Which of the following is true regarding invertible matrices A and B with eigenvalues λ_1 and λ_2 respectively? **(1 mark)**

- a. $\text{tr}(A)=\text{tr}(B)$ when $B = A^{-1}$
- b. $\lambda_1\lambda_2=1$ when $B = A^{-1}$**
- c. $\lambda_1\lambda_2=1$ when $B = A^T$
- d. $\det(A)\det(B)=1$ when $B = A^T$

7. Eigen values of a real symmetric matrix are always: **(1 mark)**

- a. Positive
- b. Negative
- c. Real**
- d. Complex

8. How does the Householder transformation preserve orthogonality of a matrix in QR decomposition? **(1 mark)**

- a. By multiplying the matrix with the identity matrix
- b. By transforming the matrix into an upper triangular matrix
- c. By transforming the matrix into a lower triangular matrix
- d. By preserving the dot product of the columns of the matrix**

9. Let x_0 be a least squares solution to $Ax = b$. Which of the following statement is true in general about the residual $r = Ax_0 - b$? **(1 mark)**

- a. r is the projection of b onto the null space of A^T
- b. r is the projection of b onto the column space of A**
- c. r is perpendicular to b
- d. r lies in the null space of A

10. For the model below:

$$\dot{N} = rN \left(1 - \frac{N}{K} \right)$$

- a) Find all the fixed points of this model. **(3 marks)**
- b) Which of these fixed points are stable? **(2 marks)**
- c) Solve it analytically and check whether long term evolution of the solution converges to the stable fixed point. **(5 marks)**

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EXAMPLE 2.4.2:

Classify the fixed points of the logistic equation, using linear stability analysis, and find the characteristic time scale in each case.

Solution: Here $f(N) = rN \left(1 - \frac{N}{K} \right)$, with fixed points $N^* = 0$ and $N^* = K$. Then $f'(N) = r - \frac{2rN}{K}$ and so $f'(0) = r$ and $f'(K) = -r$. Hence $N^* = 0$ is unstable and $N^* = K$ is stable, as found earlier by graphical arguments. In either case, the characteristic time scale is $1/|f'(N^*)| = 1/r$. ■

11. Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$: **(8 marks)**

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2 = 4$ and the determinant $\lambda_1\lambda_2 = 3$. **(2 marks)**

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2 + 2$ agrees with $1 + 3$. The determinant 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors come separately by solving $(A - \lambda I)x = 0$ which is $Ax = \lambda x$:

$$\lambda = 1: \quad (A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the eigenvector } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the eigenvector } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A . Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \quad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \quad A + 4I \text{ has } \begin{matrix} 1 + 4 = 5 \\ 3 + 4 = 7 \end{matrix}$$