Drasil Geometric Algebra Extension

Overview of Requirements

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Overview

1. Einstein Summation Notation
2. Vectors
3. Matrices
4. Geometric Algebra
5. Geometric Algebraic Definition of Vectors and Matrices
6. Software Requirements

The purpose of this slide deck is to define the objects and operations that will be implemented in the project. We will begin with a discussion about vectors and matrices, then generalize with geometric algebra. Next, we will redefine the operations using geometric algebra. Finally, we will finish with a discussion of some high-level requirements for the project.

Revision History

Date	Version	Notes
January 23rd, 2025	1.0	First draft of document version for presentation
February 7th, 2025	1.1	First draft of slide version of document

Einstein Summation Notation

- Used to describe repeated summations and multiplications in a compact notation.
- They behave according to **four rules** Faculty of Khan (2023).
    1. Any twice-repeated index in a single term is summed over.
    2. A twice-repeated index is called a dummy index; a once-repeated index is called a free index.
    3. No index may occur 3 or more times in a given term.
    4. In an equation with Einstein notation, the free indices on both sides must match.

Einstein Summation Notation Rules

Here are the rules with some more explanations Faculty of Khan (2023):

1. *Any twice-repeated index in a single term is summed over.*
   For example, $a_{ij}b_i$ represents the term $\sum_{i=1}^{j} a_{ij}b_i$.
2. *A twice-repeated index is called a dummy index; a once-repeated index is called a free index.*
   In the example above, i is a dummy index — it can be renamed however you would like. However, j is a free index and has restrictions on naming.
3. *No index may occur 3 or more times in a given term.*
   For example, $a_{ii}b_i$ is not legal.

We will use this convention from now on. Note that this convention also allows us to drop any mention of "$\forall i. 1 \leq i \leq n$" since this is implied, so we will not write these to keep things clean.

Einstein Summation Notation Rules

Here are the rules with some more explanations Faculty of Khan (2023):

4. *In an equation with Einstein notation, the free indices on both sides must match.*
   Some examples of correctly-formed equations:
   - $x_i = a_{jj} b_j$ is valid because i is free on both the LHS and RHS
   - $a_i = A_{ki} B_{kj} x_j + C_{ik} u_k$ is valid because i is a free variable on the LHS, and in every term it is the free variable on the RHS.

   Some examples of incorrectly-formed equations:
   - $x_i = A_{ji}$ is invalid because i is the only free variable on the LHS, but i and j are both free on the RHS.
   - $x_j = Aik u_k$ is invalid because j is free on the LHS, but i is free on the RHS.
   - $x_i = A_{ik} u_k + c_j$ is invalid because i is free on the LHS, but on the RHS, one term has i free while the other term has j free.

We will use this convention from now on. Note that this convention also allows us to drop any mention of "$\forall i.1 \leq i \leq n$" since this is implied, so we will not write these to keep things clean.

Vectors

- A vector is a quantity having a magnitude and a direction Wikipedia contributors (2024b) (geometric interpretation).
- Used in mathematics and physics problems
- Vectors support two main operations: vector addition and scalar multiplication
- Notation
  - As a convention, we will denote vectors with lowercase boldface font, such as $\mathbf{v}$.
  - The type of a vector can be written as $\mathbb{R}^n$, meaning a vector of real numbers, with length n

Vector Operations

Vector Addition

Two vectors with the same size can be added together. Adding them adds their components together. That is,

$$(\mathbf{a} + \mathbf{b})_i = \mathbf{a}_i + \mathbf{b}_i$$

Scalar Multiplication

The scalar multiplication of a vector is equivalent to multiplying the magnitude by that number. Equivalently, it is the same as multiplying each component by the number. That is, a vector $\mathbf{a}$ scaled by the real number r is denoted $r\mathbf{a}$ and is defined as:

$$(r\mathbf{a})_i = r\mathbf{a}_i$$

Vector Subtraction

Vector subtraction can be defined in terms of addition and scalar multiplication:

$$(\mathbf{a} - \mathbf{b})_i = \mathbf{a}_i + (-1)\mathbf{b}_i$$

Vector Operations

Dot Product

The *dot product* of two vectors each of size n is represented by

$$(\mathbf{a} \cdot \mathbf{b})_i = \mathbf{a}_i \mathbf{b}_i$$

. Some notes:

- $a \cdot a = \|a\|^2$, where $\|a\|$ is the length of a
- $a \cdot b = 0$ if a and b are perpendicular

Notice that it takes two vectors and outputs a real number.

Cross Product

The *cross product* $\mathbf{c}$ of 3-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted as $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, and is defined by Einstein notation as McGinty (2012):

$$\mathbf{c}_i = \epsilon_{i,j,k} \mathbf{a}_j \mathbf{b}_k, \text{ where } \epsilon_{1,2,3} = \epsilon_{2,3,1} = \epsilon_{3,1,2} = 1 \text{ and } \epsilon_{3,2,1} = \epsilon_{2,1,3} = \epsilon_{1,3,2} = -1$$

Sidenote: Linear Transformations

- Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a function on vectors.
- T is said to be a linear transformation if, for all scalars k and p and vectors $\mathbf{a}$ and $\mathbf{b}$, the following linearity property holds:

$$T(k\mathbf{a} + p\mathbf{b}) = kT(\mathbf{a}) + pT(\mathbf{b})$$

  In other words, if we scale the input vectors and add them together before passing the result into the function, the result is the same as scaling and adding after passing the vectors into the function Kuttler (2008).

Basis Vectors

- **Basis vector**: A set B of vectors in a vector space V such that every vector in V can be written as a unique linear combination of the vectors in B Wikipedia contributors (2024a).
- A basis must be:
  - Linearly independent: for every finite subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of B, if $c_1\mathbf{v}_1 + \ldots c_n\mathbf{v}_n = \mathbf{0}$ then $c_1 = \ldots = c_n = 0$. In other words, no basis vector should be able to be written as a linear combination of other basis vectors.
  - A spanning set: for every vector $\mathbf{v}$ in V, one can write $\mathbf{v}$ as $\mathbf{v} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$. In other words, the basis must be able to construct all vectors in the set.
  - For a given vector space V, every basis set B will be of the same size. This is called the *dimension* of the vector space.

Matrices

- A *matrix* is a representation of a linear operator with respect to a basis.
- Matrices can be implemented as a rectangular array of numbers, symbols, or expressions with entries arranged in rows and columns Wikipedia contributors (2025a). For example, the following is a 2-by-3 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- Matrices are also widely used in mathematics and physics.
- Notation
  - By convention, we will denote matrices as boldface uppercase letters, e.g. $\mathbf{A}$
  - Sometimes, you will see the notation $\mathbb{R}^{m \times n}$ representing an $m \times n$ matrix of real numbers.

Matrix Operations

Addition
The addition of two $m \times n$ matrices is calculated as:

$$(\mathbf{A} + \mathbf{B})_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$$

Scalar Multiplication The scalar multiplication of a real number c and a matrix $\mathbf{A}$ is denoted as $c\mathbf{A}$ and is computed as:

$$(c\mathbf{A})_{ij} = c\mathbf{A}_{ij}$$

Subtraction
The subtraction of two $m \times n$ matrices is denoted as $\mathbf{A} - \mathbf{B}$, and is the addition of the scalar multiplication by -1:

$$(\mathbf{A} - \mathbf{B})_{ij} = \mathbf{A}_{ij} + (-1)\mathbf{B}_{ij}$$

Matrix Operations

Transposition
The *transpose* of an $m \times n$ matrix is the $n \times m$ matrix $\mathbf{A}^T$ formed by turning rows into columns and vice versa:

$$(\mathbf{A}^T)_{ij} = \mathbf{A}_{ji}$$

Matrix Multiplication
The *matrix multiplication* of matrices $\mathbf{A}$ and $\mathbf{B}$, is defined when matrix $\mathbf{A}$ is of size $m \times p$ and matrix $\mathbf{B}$ is of size $p \times n$. Then, the resulting matrix $\mathbf{AB}$ is a matrix of size $m \times n$, such that

$$(\mathbf{AB})_{ij} = a_{ir} b_{rj}$$

Matrix Row Operations

There are three kinds of row operations:

1. Row addition: adding one row to another
2. Row multiplication: multiplying all entries in a row by a non-zero constant
3. Row switching: interchanging two rows of a matrix

Matrix Row Operations

There are three kinds of row operations:

1. Row addition: adding one row to another This can be done by using a multiplying by a matrix that looks like an identity matrix, but with a non-zero entry in one of the non-diagonal spaces:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & b \end{bmatrix} = \begin{bmatrix} a & b \\ c + 2a & d + 2b \end{bmatrix}$$

Matrix Row Operations

There are three kinds of row operations:

2. Row multiplication: multiplying all entries in a row by a non-zero constant This can be done by an identity matrix with a non-one in the diagonal:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & b \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$$

Matrix Row Operations

There are three kinds of row operations:

3. Row switching: interchanging two rows of a matrix This can be done by swapping the positions of the ones in the identity matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & b \end{bmatrix} = \begin{bmatrix} d & c \\ a & b \end{bmatrix}$$

Submatrix

Taking a submatrix involves taking only a portion of the rows and/or columns in the matrix. To delete the rows or columns you want, you multiply by the identity matrix with that row or column deleted. For example, to extract the bottom-right 2×2 matrix, we must first select the rows, then the columns that we want Alger (2015):

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & f \\ h & i \end{bmatrix}$$

Geometric Algebra

- *Geometric algebra*, also known as *Clifford algebra*, is an abstraction of vectors and matrices.
- It allows us to think about more abstract objects, beyond just scalars and vectors, which are called *bivectors*, *trivectors*, etc.
- Clifford Algebra defines many rules and operations, many of which have geometric interpretations.
- We will first define these, then continue to redefine vector and matrix operations using them.
- Notation
  - We will use s to represent scalars, V to represent vectors, and B to represent bivectors.
  - A, B, and C will be used to represent any geometric algebra object (also known as clifs)
- Unless otherwise indicated, this content comes from Denker (2025). It is a great resource if you'd like to learn more.

Visalizing Scalars, Vectors, Bivectors, . . .

- Most people know about how to visualize scalars (a point in space) and vectors (an arrow in space), but how do we visualize higher-dimensional objects?
- The visualization is shown in Figure 1
- A bivector is a patch of flat surface
- A trivector is a piece of three-dimensional space, with a volume and orientation.

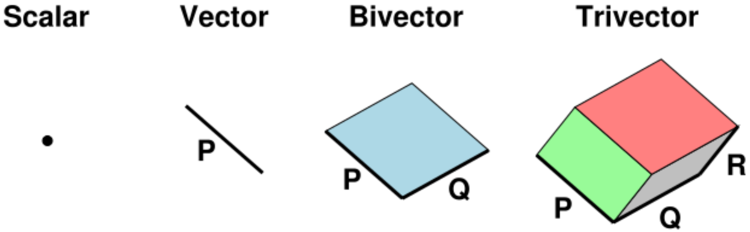


Figure 1: Visualization of geometric algebraic objects

Grades

- Each of these (scalar, vector, bivector, ...) has a *grade*, which corresponds to the number of dimensions involved. We have summarized these in Table 1.
- We call these objects (regardless of their grade) *clifs*.

Table 1: Grades of Geometric Objects

Object	**Visualized as**	**Geometric Extent**	**Grade**
Scalar	Point	No geometric extent	0
Vector	Line segment	Extent in 1 direction	1
Bivector	Patch of surface	Extent in 2 directions	2
Trivector	Piece of shape	Extent in 3 directions	3
Etc.			

Visualization of Scaling Objects

- For every vector $\mathbf{v}$, you can visualize $2\mathbf{v}$ as being twice as much length.
- For a bivector B, $2B$ has twice as much area.
- For scalars? Twice as much "heat"? Or other quantity.

Basic Arithmetic Properties

- Scalars are familiar real numbers.
  - The obey the laws of: addition, subtraction, multiplication, etc.
  - Addition is associative and commutative.
  - Multiplication of scalars is associative and commutative, and distributes over addition.
- Vectors
  - can be added to each other.
  - can c be multiplied by scalars in the usual way.
  - follow associativity and commutativity laws for addition.
  - follow distributivity laws in multiplication (will be introduced a bit later)

Addition of Clifs

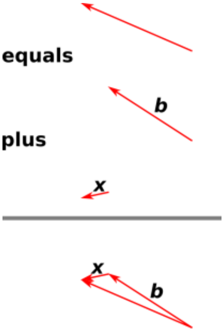
- Unique property: *any* clif, regardless of grade, can be added together. This includes adding scalars to vectors, vectors to bivectors, etc. For instance, if s is a scalar, V is a vector and B is a bivector, we can have

$$C = s + V + B$$

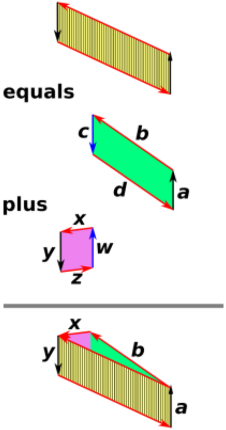
- Note: From one perspective, this can be seen as a formal sum; that is, adding things that are not the same type. However, another view is that these are all clifs of the same clif space of a given dimension, so adding them makes perfect sense.
- Note: addition is allowed (e.g. $s + V$), comparison is not (e.g. $s < V$).

Addition Visually

Bivectors are added edge-to-edge, analogously to vectors' tip-to-tail placement.



(a) Addition of vectors



(b) Addition of bivectors.

Figure 2: Addition of vectors and bivectors.

Grade Selection

- Since clifs can be additions of objects of many different grades, it is nice to have a way to extract only the component you would like.
- Thus, we define the following notation: $\langle C \rangle_N$ is the grade-N piece of C.
- For example, $\langle C \rangle_0$ extracts the scalar part of C.
- Similar to the $\mathcal{R}()$ and $\mathcal{I}()$ operators in complex numbers, but the types are unchanged. In complex numbers, the $\mathcal{I}()$ operator returns the complex part as a real number. In geometric algebra, the grade selection maintains the grade of what's being selected.

Multiplication: General Properties

- We write multiplication of two clifs A and B by juxtaposition AB.
- The geometric product AB follows the laws of:
  - associativity: $(AB)C = A(BC)$
  - distributivity: $A(B + C) = AB + AC$

  for all clifs A, B, and C.
- Not commutative in general, but the special case of scalar multiplication is.
- Multiplying vectors by scalars works as before, and is commutative: $sV = Vs$.
- In fact, multiplying any clif by a scalar follows similarly and is also commutative. That is, $sC = Cs$ for any scalar s and clif C.

Multiplication: Vectors by Vectors

- Given two vectors, P and Q, we know that the geometric product PQ exists.
- On vectors, we can define two new operations:
  - $P \cdot Q = \frac{PQ+QP}{2}$, where P and Q have grade $= 1$.
  - $P \wedge Q = \frac{PQ-QP}{2}$, where P and Q have grade ≤ 1.
- We call the first one the *dot product*, and the second one the *wedge product*.
- Rearranging, we get:

$$PQ = P \cdot Q + P \wedge Q$$

  where P and Q have grade $= 1$.
  - Note: It's important to note that this only works for vectors. It is not the general definition of the geometric product. We will define that later.

Properties of Dot Product

- The dot product produces a scalar.
- Rotation by 180 degrees does not change the dot product, since $P \cdot P = (-P) \cdot (-P)$. Rotation in perpendicular to P also does not change the dot product.
- Parallel vectors: if P and Q are parallel, symbolized $P \| Q$, then

$$PQ = QP = P \cdot Q$$

.

- Perpendicular vectors: If $P \cdot Q = 0$ then the vectors P and Q are perpendicular or *orthogonal*, symbolized $P \perp Q$, and

$$PQ = -QP = P \wedge Q$$

Properties of Wedge Product

- The wedge product is invariant to rotations of 180 degrees in the PQ plane.
- If we rotate 180 degrees in a plane perpendicular to the plane, that will flip the sign of the wedge product (analogous to the cross product).
- The idea of wedge generalized to more than two vectors:

$$P \wedge Q \wedge R = \frac{1}{6}(PQR + QRP + RPQ - RQP - QPR - PRQ)$$

- Even more generally,

$$q_1 \wedge q_2 \wedge q_3 \cdots \wedge q_r = \frac{1}{r} \sum_{\pi} \operatorname{sign}(\pi) q_{\pi(1)} q_{\pi(2)} q_{\pi(3)} \cdots q_{\pi(r)}$$

  where the sum runs over all permutations π and $\operatorname{sign}(\pi)$ is defined as $+1$ for even permutations and -1 for odd ones.
  - This will be an object of grade r if all vectors are linearly independent, otherwise it will be 0.

Some Terminology

- A *blade* is any scalar, vector, or the wedge product of any number of vectors.
- Any clif that has a definite grade (consists of one single grade) is called *homogeneous*. It is either a blade or a sum of blades, all of the same grade.
- Example: $s + V$ (where s is a scalar and V is a vector) is not homogenous. Does not have a definite grade, and is not a blade.

Other Wedge Products

- We can define the wedge product between a blade and another blade. By unpacking the wedge product, and getting rid of parentheses, we get:

$$P \wedge (Q \wedge R) = P \wedge Q \wedge R$$

$$(P \wedge Q) \wedge R = P \wedge Q \wedge R$$

- Then, the wedge product contains only the top-grade terms from a geometric product: If $P = \langle P \rangle_p$ and $Q = \langle Q \rangle_q$, then $P \wedge Q = \langle PQ \rangle_{p+q}$.
- Finally, we need the fact that the wedge product distributes over addition:

$$V \wedge (A + B) = V \wedge A + V \wedge B$$

- Now we have generalized the wedge product to clif wedge clif.

Other Dot Products

- Analogously, the dot product of two blades is the lowest-grade part: If $P = \langle P \rangle_p$ and $Q = \langle Q \rangle_q$, then $P \wedge Q = \langle PQ \rangle_{p-q}$.
- Symmetry does not hold in general.
- As before, given this definition, we can expand to all clifs knowing that

$$V \cdot (A + B) = V \cdot A + V \cdot B$$

Geometric Product

- At last, we can define the geometric product in general!
- To compute it we must:
  1. Pick a basis γ
  2. Project the factors onto that basis
  3. Multiply everything term-by-term
  4. Then simplify
- Example: $A = \gamma_1 + 2\gamma_1\gamma_2$ and $B = 3\gamma_3 + 5\gamma_1\gamma_2\gamma_3$. Table 2 shows the terms.

Table 2: Term-by-Term Multiplication

	γ_1	$2\gamma_1\gamma_2$
$3\gamma_3$	$3\gamma_3\gamma_1$	$6\gamma_3\gamma_1\gamma_2$
$5\gamma_1\gamma_2\gamma_3$	$5\gamma_1\gamma_2\gamma_3\gamma_1$	$10\gamma_1\gamma_2\gamma_3\gamma_3\gamma_1$

Geometric Product

- Next, we permute the orthonormal vectors, multiplying by -1 each time we do.

Table 3: Permuting the Basis Vectors

	γ_1	$2\gamma_1\gamma_2$
$3\gamma_3$	$-3\gamma_1\gamma_3$	$6\gamma_1\gamma_2\gamma_3$
$5\gamma_1\gamma_2\gamma_3$	$5\gamma_1\gamma_1\gamma_2\gamma_3$	$-10\gamma_1\gamma_1\gamma_2\gamma_2\gamma_3$

- Finally, we simplify, removing any vectors that are the same, since a multiplication of orthonormal vectors $\gamma_i\gamma_i = 1$

Table 4: Simplifying using Orthonormal Vector Properties

	γ_1	$2\gamma_1\gamma_2$
$3\gamma_3$	$-3\gamma_1\gamma_3$	$6\gamma_1\gamma_2\gamma_3$
$5\gamma_1\gamma_2\gamma_3$	$5\gamma_2\gamma_3$	$-10\gamma_3$

- The final result is $-3\gamma_1\gamma_3 + 6\gamma_1\gamma_2\gamma_3 + 5\gamma_2\gamma_3 + -10\gamma_3$.

Geometric Algebraic Definition of Vectors and Matrices

- Next, we will define the vector and matrix operations using geometric algebra.
- The vector operations we defined earlier can be implemented in geometric algebra using grade-1 objects (vectors). The operations follow closely.
- Matrix operations depend mostly on the notion of a matrix multiplication.

Vector Operations in Geometric Algebra

Vector Addition

Follows from the addition of clif (two vector clifs).

Scalar Multiplication

Follows from scalar multiplication of a vector clif.

Vector Subtraction

Follows from scalar multiplication and addition of two vector clifs and the scalar -1.

Dot Product

Follows from dot product of two vector clifs.

Cross Product

The cross product can be defined using the wedge product Wikipedia contributors (2025b). If we have two vectors (using the standard basis $\mathbf{e}$) $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ and $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, then the result is

$$\mathbf{u} \wedge \mathbf{v} = (u_1v_2 - u_2v_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) + (u_3v_1 - u_1v_3)(\mathbf{e}_3 \wedge \mathbf{e}_1) + (u_2v_3 - u_3v_2)(\mathbf{e}_2 \wedge \mathbf{e}_3)$$

Here, the coefficients are the same as before, but the result is a bivector instead of a vector.

Matrix Representation in Geometric Algebra

- In geometric algebra, a matrix can be represented as a linear transformation on a vector Mathoma (2023).
- That is, it is a function $\mathbf{A} : \mathbb{R}^m \to \mathbb{R}^n$, such that $\mathbf{A}(k\mathbf{a} + p\mathbf{b}) = k\mathbf{A}(\mathbf{a}) + p\mathbf{A}(\mathbf{b})$.
- We'll continue to use the boldface notation to indicate that the function is supposed to be a matrix.
- We can use this definition to redefine the idea of a matrix in geometric algebra.

Matrix Operations in Geometric Algebra

Addition
The addition of two matrices is calculated as the addition of the two functions representing those matrices, since they are linear functions. That is,

$$\mathbf{C}(\mathbf{v}) = \mathbf{A}(\mathbf{v}) + \mathbf{B}(\mathbf{v})$$

Scalar Multiplication The scalar multiplication of a real number s and a matrix represented by $\mathbf{A}(v)$ is denoted as $s\mathbf{A}$ and is computed using the scalar product with the resulting vector:

$$s\mathbf{A}(v)$$

Subtraction
The subtraction of two matrices is denoted as $\mathbf{A} - \mathbf{B}$, and is the addition of the scalar multiplication by -1, as before.

$$(\mathbf{A} - \mathbf{B})(v) = \mathbf{A}(v) + (-1)\mathbf{B}(v)$$

Matrix Operations in Geometric Algebra

Transposition
The *transpose* of a matrix $\mathbf{A}(\mathbf{v}) : \mathbb{R}^m \to \mathbb{R}^n$ is the matrix $\mathbf{A}^T(\mathbf{v}) : \mathbb{R}^n \to \mathbb{R}^m$ such that the matrices behave like the original definition:

$$(\mathbf{A}^T)_{i,j} = \mathbf{A}_{j,i}$$

Matrix Multiplication
The *matrix multiplication* of matrices $\mathbf{A}(\mathbf{v}) : \mathbb{R}^m \to \mathbb{R}^p$ and $\mathbf{A}(\mathbf{v}) : \mathbb{R}^p \to \mathbb{R}^n$, is the function composition of the two matrices:

$$(\mathbf{AB})(\mathbf{v}) = (\mathbf{A} \cdot \mathbf{B})(\mathbf{v})$$

Other Operations
Other described operations, like row operations and submatrix operations, depend on matrix multiplication and thus follow from their descriptions earlier in this document. Their sizes will be encoded in the function type.

Functional Requirements

1. The system shall contain an internal representation of geometric algebra.
2. The system shall contain a smart constructor for vectors, represented using geometric algebra internally.
3. The system shall contain a smart constructor for matrices, represented using geometric algebra internally.
4. The system shall allow the specification of vector, and matrix operations with fixed or variable sizes at specification-time.
5. The system shall support at least the vector, matrix operations defined in this document.
6. The system shall allow the generation of documentation for vectors, matrices, and generalized clifs.
7. The system shall allow the generation of code to perform the operations for vectors, matrices, and generalized clifs.
8. The system shall ensure or check the validity of operations at specification time, generation-time, or runtime, as appropriate.

Functional Requirements

9. For a given problem specified using this extension, generated documents shall logically match the generated code for performing the given operations.
10. The addition of these new features to Drasil shall not break any existing examples.

References

Nick Alger. How to define the dot product of vectors in the complex plane., 2015. URL https://math.stackexchange.com/a/1322946. Online; accessed 2025-02-07.

John S. Denker. Clifford algebra: A visual introduction, 2025. URL https://www.av8n.com/physics/clifford-intro.htm#sec-basis. Online; accessed 2025-02-07.

Faculty of Khan. Einstein Summation Convention: an Introduction. https://www.youtube.com/watch?v=CLrTj7D2fLM, 2023. Accessed: 2025-01-23.

Kenneth Kuttler. *A First Course in Linear Algebra*. LibreTexts, 2008.

Mathoma. Geometric Algebra - The Matrix Representation of a Linear Transformation. https://www.youtube.com/watch?v=d0gSuPVlMoI, 2023. Accessed: 2025-02-07.

Bob McGinty. Tensor notation basics. https://www.continuummechanics.org/tensornotationbasic.html, 2012. Accessed: 2025-01-23.

Wikipedia contributors. Basis (linear algebra), October 2024a. URL https://en.wikipedia.org/wiki/Basis_(linear_algebra).

Wikipedia contributors. Vector (mathematics and physics), October 2024b. URL https://en.wikipedia.org/wiki/Vector_(mathematics_and_physics).

Wikipedia contributors. Matrix (mathematics), January 2025a. URL https://en.wikipedia.org/wiki/Matrix_(mathematics).

Wikipedia contributors. Exterior algebra — Wikipedia, the free encyclopedia, 2025b. URL https://en.wikipedia.org/w/index.php?title=Exterior_algebra&oldid=1272926547. [Online; accessed 8-February-2025].