Generating Functions

Let $s = (s_0, s_1, s_2, s_3, \dots)$ be a sequence
we define the generating function $f_s(x)$ for s by

$$f_s(x) = s_0 + s_1 x + s_2 x^2 + s_3 x^3 + \dots = \sum_0^\infty s_n x^n$$

One way to find a formula for s_n is to use properties of the sequence to derive properties of f_s and then use these properties to calculate the coefficients s_n.

① Lets look at the sequence $s_0 = (1, 1, 1, 1, \dots)$ of all 1's. then

$f_s(x) = 1 + x + x^2 + x^3 + \dots$

Observe what happens when we multiply $f_s(x)$ by $1-x$

$$f_s(x) \cdot (1-x) = f_s(x) - x f_s(x)$$
$$= 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$
$$- \quad x - x^2 - x^3 - x^4 - x^5 - \dots$$
$$= 1 + 0 + 0 + 0 + \dots$$
$$= 1$$

So $f_s(x)(1-x) = 1 \Rightarrow f_s(x) = \dfrac{1}{1-x}$

② Lets look at $s_2 = (1, 2, 4, 8, 16, 32, \dots)$

so $f_{s_2}(x) = 1 + 2x + 4x^2 + 8x^3 + \dots$

and lets multiply them by $1-2x$

$$f_{s_2}(x)(1-2x) = f_{s_2}(x) - 2x f_{s_2}(x)$$

$$= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \dots$$
$$- 2x - 4x^2 - 8x^3 - 16x^4 - 32x^5 - \dots$$
$$= 1$$

So $f_{s_2}(x) = \dfrac{1}{1-2x}$

③ Let's look at $(1-ax-bx^2) f_s(x)$

$$= f_s(x) - ax f_s(x) - bx^2 f_s(x)$$

$$= s_0 + s_1 x + s_2 x^2 + s_3 x^3 + s_4 x^4 + \dots$$
$$- a s_0 x - a s_1 x^2 - a s_2 x^3 - a s_3 x^4 - \dots$$
$$- b s_0 x^2 - b s_1 x^3 - b s_2 x^4 - \dots$$

$$= s_0 + (s_1 - a s_0) x$$
$$+ (s_2 - a s_1 - b s_0) x^2$$
$$+ (s_3 - a s_2 - b s_1) x^3$$
$$+ \dots$$
$$+ (s_n - a s_{n-1} - b s_{n-2}) x^n$$
$$+ \dots$$

So if s satisfies

$$s_n = a s_{n-1} + b s_{n-2}$$

then $(1 - ax - bx^2) \cdot f_s(x) = s_0 + (s_1 - a s_0) x$

So $f_s(x) = \dfrac{s_1 - a s_0 x}{1 - ax - bx^2}$

If we factor the denominator it will be one of two forms.

ⓐ $(1-\beta x)^2$

ⓑ $(1-\beta_1 x)(1-\beta_2 x) \quad \beta_1 \neq \beta_2$

In case ⓐ

$$\frac{1}{(1-\beta x)^2} = \left(1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \cdots\right)^2$$

$$= 1 + 2\beta x + 3\beta^2 x^2 + 4\beta^3 x^3 + \cdots + (n+1)\beta^n x^n + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)\beta^n x^n$$

so $\dfrac{s+tx}{(1-\beta x)^2} = \begin{array}{l} s + 2s\beta x + 3s\beta^2 x^2 + 4s\beta^3 x^3 + \cdots \\ \quad + tx + 2t\beta x^2 + 3t\beta^2 x^3 + \cdots \end{array}$

$$= s + (2s\beta + t)x + (3s\beta + 2t)\beta x^2 + \cdots + ((n+1)s\beta + nt)\beta^{n-1} x^n$$

so $s_n = ((s\beta + t)n + s\beta)\beta^{n-1}$

$= ((s + t/\beta)n + s)\beta^n$

has the form $(an+b)\beta^n$

in case (b) $\dfrac{1}{(1-\beta_1 x)(1-\beta_2 x)}$ $\beta_1 \neq \beta_2$

$$= \frac{a_1}{(1-\beta_1 x)} + \frac{a_2}{(1-\beta_2 x)} = \frac{a_1 - a_1\beta_2 x + a_2 - a_2\beta_1 x}{(1-\beta_1 x)(1-\beta_2 x)}$$

so $a_1 + a_2 = 1$ and $a_1\beta_2 = a_2\beta_1$

i.e. $a_1\beta_2 = (1-a_1)\beta_1$ $a_1(\beta_2+\beta_1) = \beta_1$

$$a_1 = \frac{\beta_1}{\beta_1+\beta_2} \qquad a_2 = \frac{\beta_2}{\beta_1+\beta_2}$$

so in case 2

$$\frac{1}{(1-x\beta_1)(1-x\beta_2)} = \frac{\gamma_1}{1-\beta_1 x} + \frac{\gamma_2}{1-\beta_2 x}$$

$$= \sum_0^\infty (\gamma_1\beta_1^n + \gamma_2\beta_2^n)x^n$$

so $s_n = \gamma_1\beta_1^n + \gamma_2\beta_2^n$ for some γ_1 & γ_2

We've shown how to use recurrence equations to find a closed form for the generating function $f_S(x)$ and then to use that formula to find the coefficients of f_S which gives a closed form for the sequence elements s_n