Χειμερινό Εξάμηνο
Ακαδημαϊκό Έτος 2009-2010

Πανεπιστήμιο Κρήτης
Τμήμα Επιστήμης Υπολογιστών
ΗΥ-110 Απειροστικός Λογισμός Ι
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Άσκηση 1η

a)

$$\lim_{x \to \infty} \frac{2x+3}{5x+7} = \lim_{x \to \infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$$

b)

$$\lim_{x \to \infty} \frac{x+1}{x^2+3} = \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$$

c)
Όπως οι παραπάνω, αποτέλεσμα -12

d)

$$\lim_{x \to \infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x \to \infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} = 7$$

e)

$$\lim_{x \to -\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x \to -\infty} \frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}} = \lim_{x \to -\infty} \frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)} = 1$$

f)

$$\lim_{x \to \infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} = \lim_{x \to \infty} \frac{x+\frac{1}{x^2}}{1-\frac{1}{x}} = \infty$$

g)

$$\lim_{x \to \infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}} = \lim_{x \to \infty} \frac{2x^{1/15}-\frac{1}{x^{19/15}}+\frac{7}{x^{8/5}}}{1+\frac{3}{x^{3/5}}+\frac{1}{x^{11/10}}} = \infty$$

h)

$$\lim_{x \to -\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4} = \lim_{x \to -\infty} \frac{\frac{1}{x^{2/3}}-5+\frac{3}{x}}{2+\frac{1}{x^{1/3}}-\frac{4}{x}} = -\frac{5}{2}$$

i)

$$\lim_{x \to \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0,$$

and function continuous at $x = \pi$.

j)

$$\lim_{t \to 0} \sin\left(\tfrac{\pi}{2}\cos(\tan t)\right) = \sin\left(\tfrac{\pi}{2}\cos(\tan(0))\right) = \sin\left(\tfrac{\pi}{2}\cos(0)\right) = \sin\left(\tfrac{\pi}{2}\right) = 1,$$

, and function continuous at $t = 0$.

k)

$$\lim_{y \to 1} \sec(y\sec^2 y - \tan^2 y - 1) = \lim_{y \to 1} \sec(y\sec^2 y - \sec^2 y) =$$

$$\lim_{y \to 1} \sec((y-1)\sec^2 y) = \sec((1-1)\sec^2 1)$$

$= \sec 0 = 1$, , and function continuous at $y = 1$.

l)

$$\lim_{x \to 0} \tan\left[\tfrac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right] = \tan\left[\tfrac{\pi}{4}\cos(\sin(0))\right] = \tan\left(\tfrac{\pi}{4}\cos(0)\right) = \tan\left(\tfrac{\pi}{4}\right) = 1,$$

and function continuous at $x = 0$.

Άσκηση 2η

(a) $\lim_{t \to t_0} (3f(t)) = 3\lim_{t \to t_0} f(t) = 3(-7) = -21$

(b) $\lim_{t \to t_0} (f(t))^2 = \left(\lim_{t \to t_0} f(t)\right)^2 = (-7)^2 = 49$

(c) $\lim_{t \to t_0} (f(t) \cdot g(t)) = \lim_{t \to t_0} f(t) \cdot \lim_{t \to t_0} g(t) = (-7)(0) = 0$

(d) $\lim_{t \to t_0} \frac{f(t)}{g(t)-7} = \frac{\lim_{t \to t_0} f(t)}{\lim_{t \to t_0} (g(t)-7)} = \frac{\lim_{t \to t_0} f(t)}{\lim_{t \to t_0} g(t) - \lim_{t \to t_0} 7} = \frac{-7}{0-7} = 1$

(e) $\lim_{t \to t_0} \cos(g(t)) = \cos\left(\lim_{t \to t_0} g(t)\right) = \cos 0 = 1$

(f) $\lim_{t \to t_0} |f(t)| = \left|\lim_{t \to t_0} f(t)\right| = |-7| = 7$

(g) $\lim_{t \to t_0} (f(t) + g(t)) = \lim_{t \to t_0} f(t) + \lim_{t \to t_0} g(t) = -7 + 0 = -7$

(h) $\lim_{t \to t_0} \left(\frac{1}{f(t)}\right) = \frac{1}{\lim_{t \to t_0} f(t)} = \frac{1}{-7} = -\frac{1}{7}$

Άσκηση 3η

(a) Suppose x_0 is rational $\Rightarrow f(x_0) = 1$. Choose $\epsilon = \frac{1}{2}$. For any $\delta > 0$ there is an irrational number x (actually infinitely many) in the interval $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 0$. Then $0 < |x - x_0| < \delta$ but $|f(x) - f(x_0)| = 1 > \frac{1}{2} = \epsilon$, so $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 rational.

On the other hand, x_0 irrational $\Rightarrow f(x_0) = 0$ and there is a rational number x in $(x_0 - \delta, x_0 + \delta) \Rightarrow f(x) = 1$. Again $\lim_{x \to x_0} f(x)$ fails to exist $\Rightarrow$ f is discontinuous at x_0 irrational. That is, f is discontinuous at every point.

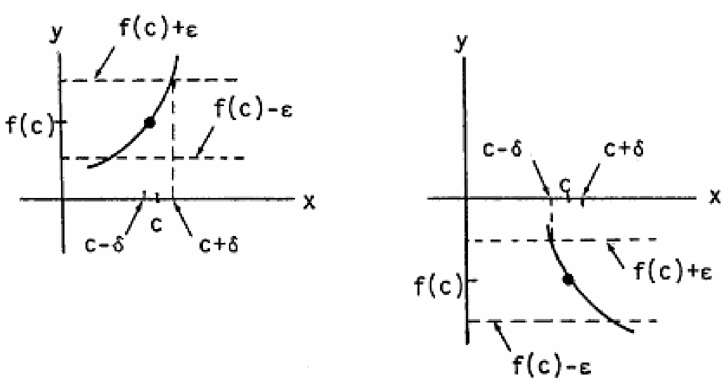
(b) f is neither right-continuous nor left-continuous at any point x_0 because in every interval $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ there exist both rational and irrational real numbers. Thus neither limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist by the same arguments used in part (a).

Άσκηση 4η

Let $\epsilon = \frac{|f(c)|}{2} > 0$. Since f is continuous at $x = c$ there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ $\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$.

If $f(c) > 0$, then $\epsilon = \frac{1}{2} f(c) \Rightarrow \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c) \Rightarrow f(x) > 0$ on the interval $(c - \delta, c + \delta)$.

If $f(c) < 0$, then $\epsilon = -\frac{1}{2} f(c) \Rightarrow \frac{3}{2} f(c) < f(x) < \frac{1}{2} f(c) \Rightarrow f(x) < 0$ on the interval $(c - \delta, c + \delta)$.



Άσκηση 5η

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(100 - 4.9(2+h)^2) - (100 - 4.9(2)^2)}{h} = \lim_{h \to 0} \frac{-4.9(4 + 4h + h^2) + 4.9(4)}{h}$$

$= \lim_{h \to 0} (-19.6 - 4.9h) = -19.6$. The minus sign indicates the object is falling downward at a speed of 19.6 m/sec.

Άσκηση 6η

α)

Yes. Let R be the radius of the equator (earth) and suppose at a fixed instant of time we label noon as the zero point, 0, on the equator $\Rightarrow$ $0 + \pi R$ represents the midnight point (at the same exact time). Suppose x_1 is a point on the equator "just after" noon $\Rightarrow$ $x_1 + \pi R$ is simultaneously "just after" midnight. It seems reasonable that the temperature T at a point just after noon is hotter than it would be at the diametrically opposite point just after midnight: That is, $T(x_1) - T(x_1 + \pi R) > 0$. At exactly the same moment in time pick x_2 to be a point just before midnight $\Rightarrow$ $x_2 + \pi R$ is just before noon. Then $T(x_2) - T(x_2 + \pi R) < 0$. Assuming the temperature function T is continuous along the equator (which is reasonable), the Intermediate Value Theorem says there is a point c between 0 (noon) and πR (simultaneously midnight) such that $T(c) - T(c + \pi R) = 0$; i.e., there is always a pair of antipodal points on the earth's equator where the temperatures are the same.

b)

At most 1 horizontal asymptote: If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$, then the ratio of the polynomials' leading coefficients is L, so $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = L$ as well.

c)

$\cos x = x \Rightarrow (\cos x) - x = 0$. If $x = -\frac{\pi}{2}$, $\cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0$. If $x = \frac{\pi}{2}$, $\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} < 0$. Thus $\cos x - x = 0$ for some x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ according to the Intermediate Value Theorem.

Άσκηση 7η

$|A - 9| \le 0.01 \Rightarrow -0.01 \le \pi\left(\frac{x}{2}\right)^2 - 9 \le 0.01 \Rightarrow 8.99 \le \frac{\pi x^2}{4} \le 9.01 \Rightarrow \frac{4}{\pi}(8.99) \le x^2 \le \frac{4}{\pi}(9.01)$ $\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \le x \le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \le x \le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.