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Πανεπιστήμιο Κρήτης
Τμήμα Επιστήμης Υπολογιστών
ΗΥ-110 Απειροστικός Λογισμός Ι
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Έκτη Σειρά Ασκήσεων και Λύσεις

Ποιες από τις ακολουθίες των οποίων οι n-στοί όροι δίνονται στις Ασκήσεις 1-18 συγκλίνουν, και ποιες αποκλίνουν; Βρείτε το όριο κάθε συγκλίνουσας ακολουθίας.

1. $a_n = 1 + \frac{(-1)^n}{n}$

2. $a_n = \frac{1-(-1)^n}{\sqrt{n}}$

3. $a_n = \frac{1-2^n}{2^n}$

4. $a_n = 1 + (0.9)^n$

5. $a_n = \sin\frac{n\pi}{2}$

6. $a_n = \sin n\pi$

7. $a_n = \frac{\ln(n^2)}{n}$

8. $a_n = \frac{\ln(2n+1)}{n}$

9. $a_n = \frac{n+\ln n}{n}$

10. $a_n = \frac{\ln(2n^3+1)}{n}$

11. $a_n = \left(\frac{n-5}{n}\right)^n$

12. $a_n = \left(1+\frac{1}{n}\right)^{-n}$

13. $a_n = \sqrt[n]{\frac{3^n}{n}}$

14. $a_n = \left(\frac{3}{n}\right)^{1/n}$

15. $a_n = n(2^{1/n}-1)$

16. $a_n = \sqrt[n]{2n+1}$

17. $a_n = \frac{(n+1)!}{n!}$

18. $a_n = \frac{(-4)^n}{n!}$

Λύσεις:

1. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right) = 1$

2. converges to 0, since $0 \le a_n \le \frac{2}{\sqrt{n}}$, $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0$ using the Sandwich Theorem for Sequences

3. converges to -1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1-2^n}{2^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2^n}-1\right) = -1$

4. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}[1+(0.9)^n] = 1+0 = 1$

5. diverges, since $\left\{\sin\frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, 1, \dots\}$

6. converges to 0, since $\{\sin n\pi\} = \{0, 0, 0, \dots\}$

7. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\ln n^2}{n} = 2\lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$

8. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\ln(2n+1)}{n} = \lim\limits_{n\to\infty} \frac{\left(\frac{2}{2n+1}\right)}{1} = 0$

9. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n+\ln n}{n}\right) = \lim\limits_{n\to\infty} \frac{1+\left(\frac{1}{n}\right)}{1} = 1$

10. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\ln(2n^3+1)}{n} = \lim\limits_{n\to\infty} \frac{\left(\frac{6n^2}{2n^3+1}\right)}{1} = \lim\limits_{n\to\infty} \frac{12n}{6n^2} = \lim\limits_{n\to\infty} \frac{2}{n} = 0$

11. converges to e^{-5}, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n-5}{n}\right)^n = \lim\limits_{n\to\infty} \left(1 + \frac{(-5)}{n}\right)^n = e^{-5}$ by Theorem 5

12. converges to $\frac{1}{e}$, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ by Theorem 5

13. converges to 3, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{3^n}{n}\right)^{1/n} = \lim\limits_{n\to\infty} \frac{3}{n^{1/n}} = \frac{3}{1} = 3$ by Theorem 5

14. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \lim\limits_{n\to\infty} \frac{3^{1/n}}{n^{1/n}} = \frac{1}{1} = 1$ by Theorem 5

15. converges to $\ln 2$, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n\left(2^{1/n}-1\right) = \lim\limits_{n\to\infty} \frac{2^{1/n}-1}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \frac{\left[\frac{\left(-2^{1/n}\ln 2\right)}{n^2}\right]}{\left(\frac{-1}{n^2}\right)} = \lim\limits_{n\to\infty} 2^{1/n}\ln 2$

$= 2^0 \cdot \ln 2 = \ln 2$

16. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt[n]{2n+1} = \lim\limits_{n\to\infty} \exp\left(\frac{\ln(2n+1)}{n}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\frac{2}{2n+1}}{1}\right) = e^0 = 1$

17. diverges, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty$

18. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(-4)^n}{n!} = 0$ by Theorem 5

Βρείτε τα αθροίσματα των σειρών στις Ασκήσεις 19-24

19. $\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$ **20.** $\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$

21. $\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)}$ **22.** $\sum_{n=3}^{\infty} \frac{-8}{(4n-3)(4n+1)}$

23. $\sum_{n=0}^{\infty} e^{-n}$ **24.** $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

Λύσεις:

19. $\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Rightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \ldots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1}$
$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$

20. $\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1} \Rightarrow s_n = \left(\frac{-2}{2} + \frac{2}{3}\right) + \left(\frac{-2}{3} + \frac{2}{4}\right) + \ldots + \left(\frac{-2}{n} + \frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1} \Rightarrow \lim_{n\to\infty} s_n$
$= \lim_{n\to\infty} \left(-1 + \frac{2}{n+1}\right) = -1$

21. $\frac{9}{(3n-1)(3n+2)} = \frac{3}{3n-1} - \frac{3}{3n+2} \Rightarrow s_n = \left(\frac{3}{2} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{8}\right) + \left(\frac{3}{8} - \frac{3}{11}\right) + \ldots + \left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$
$= \frac{3}{2} - \frac{3}{3n+2} \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{3}{2} - \frac{3}{3n+2}\right) = \frac{3}{2}$

22. $\frac{-8}{(4n-3)(4n+1)} = \frac{-2}{4n-3} + \frac{2}{4n+1} \Rightarrow s_n = \left(\frac{-2}{9} + \frac{2}{13}\right) + \left(\frac{-2}{13} + \frac{2}{17}\right) + \left(\frac{-2}{17} + \frac{2}{21}\right) + \ldots + \left(\frac{-2}{4n-3} + \frac{2}{4n+1}\right)$
$= -\frac{2}{9} + \frac{2}{4n+1} \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(-\frac{2}{9} + \frac{2}{4n+1}\right) = -\frac{2}{9}$

23. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \frac{1}{e}$ and $a = 1 \Rightarrow$ the sum is $\frac{1}{1-\left(\frac{1}{e}\right)} = \frac{e}{e-1}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)\left(\frac{-1}{4}\right)^n$ a convergent geometric series with $r = -\frac{1}{4}$ and $a = \frac{-3}{4} \Rightarrow$ the sum is
$\frac{\left(-\frac{3}{4}\right)}{1-\left(\frac{-1}{4}\right)} = -\frac{3}{5}$

Ποιες από τις παρακάτω σειρές των Ασκήσεων 25-40 συγκλίνουν απολύτως, ποιες από συνθήκη, και ποιες αποκλίνουν; Αιτιολογήστε τις απαντήσεις σας.

25. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

26. $\sum_{n=1}^{\infty} \frac{-5}{n}$

27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

28. $\sum_{n=1}^{\infty} \frac{1}{2n^3}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

30. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

31. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

32. $\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$

35. $\sum_{n=1}^{\infty} \frac{n+1}{n!}$

36. $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$

37. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

38. $\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

39. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

40. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Λύσεις:

25. diverges, a p-series with $p = \frac{1}{2}$

26. $\sum\limits_{n=1}^{\infty} \frac{-5}{n} = -5 \sum\limits_{n=1}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series

27. Since $f(x) = \frac{1}{x^{1/2}} \Rightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{\sqrt{n}} = 0$, the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.

28. converges absolutely by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \geq 1$, which is the nth term of a convergent p-series

29. The given series does not converge absolutely by the Direct Comparison Test since $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$, which is the nth term of a divergent series. Since $f(x) = \frac{1}{\ln(x+1)} \Rightarrow f'(x) = -\frac{1}{(\ln(x+1))^2(x+1)} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{1}{\ln(n+1)} = 0$, the given series converges conditionally by the Alternating Series Test.

30. $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim\limits_{b \to \infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim\limits_{b \to \infty} \left[-(\ln x)^{-1}\right]_2^b = -\lim\limits_{b \to \infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ the series converges absolutely by the Integral Test

31. converges absolutely by the Direct Comparison Test since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$, the nth term of a convergent p-series

32. diverges by the Direct Comparison Test for $e^{n^n} > n \Rightarrow \ln\left(e^{n^n}\right) > \ln n \Rightarrow n^n > \ln n \Rightarrow \ln n^n > \ln(\ln n)$ $\Rightarrow n \ln n > \ln(\ln n) \Rightarrow \frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$, the nth term of the divergent harmonic series

33. $\lim\limits_{n \to \infty} \frac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim\limits_{n \to \infty} \frac{n^2}{n^2+1}} = \sqrt{1} = 1 \Rightarrow$ converges absolutely by the Limit Comparison Test

34. Since $f(x) = \frac{3x^2}{x^3+1} \Rightarrow f'(x) = \frac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2 \Rightarrow a_{n+1} < a_n$ for $n \geq 2$ and $\lim\limits_{n \to \infty} \frac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit Comparison Test, $\lim\limits_{n \to \infty} \frac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim\limits_{n \to \infty} \frac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.

35. converges absolutely by the Ratio Test since $\lim\limits_{n \to \infty} \left[\frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1}\right] = \lim\limits_{n \to \infty} \frac{n+2}{(n+1)^2} = 0 < 1$

36. diverges since $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist

37. converges absolutely by the Ratio Test since $\lim\limits_{n \to \infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right] = \lim\limits_{n \to \infty} \frac{3}{n+1} = 0 < 1$

38. converges absolutely by the Root Test since $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n 3^n}{n^n}} = \lim_{n\to\infty} \frac{6}{n} = 0 < 1$

39. converges absolutely by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n(n+1)(n+2)}}\right)} = \sqrt{\lim_{n\to\infty} \frac{n(n+1)(n+2)}{n^3}} = 1$

40. converges absolutely by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n\sqrt{n^2-1}}\right)} = \sqrt{\lim_{n\to\infty} \frac{n^2(n^2-1)}{n^4}} = 1$

Στις Ασκήσεις 41-50, (α) βρείτε την ακτίνα και το διάστημα σύγκλισης κάθε σειράς. Έπειτα εντοπίστε τις τιμές x για τις οποίες η σειρά συγκλίνει (β) απολύτως και (γ) υπό συνθήκη.

41. $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$

42. $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$

44. $\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$

45. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

46. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

47. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

48. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{2n+1}$

49. $\sum_{n=1}^{\infty} (\operatorname{csch} n)x^n$

50. $\sum_{n=1}^{\infty} (\coth n)x^n$

Λύσεις:

41. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+4)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x+4)^n}\right| < 1 \Rightarrow \frac{|x+4|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+4|}{3} < 1$

$\Rightarrow |x+4| < 3 \Rightarrow -3 < x+4 < 3 \Rightarrow -7 < x < -1$; at $x = -7$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have $\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \le x < -1$

(b) the interval of absolute convergence is $-7 < x < -1$

(c) the series converges conditionally at $x = -7$

42. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}}\right| < 1 \Rightarrow (x-1)^2 \lim_{n\to\infty} \frac{1}{(2n)(2n+1)} = 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

43. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n}\right| < 1 \Rightarrow |3x-1| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} < 1 \Rightarrow |3x-1| < 1$

$\Rightarrow -1 < 3x-1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n^2}$

$= -\sum_{n=1}^{\infty} \frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series, which is absolutely convergent; at $x = \frac{2}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, which converges absolutely

(a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \le x \le \frac{2}{3}$

(b) the interval of absolute convergence is $0 \le x \le \frac{2}{3}$

(c) there are no values for which the series converges conditionally

44. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{(2x+1)^{n+1}}{2^{n+1}} \cdot \frac{2n+1}{n+1} \cdot \frac{2^n}{(2x+1)^n}\right| < 1 \Rightarrow \frac{|2x+1|}{2} \lim_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{2n+1}{n+1}\right| < 1$

$\Rightarrow \frac{|2x+1|}{2}(1) < 1 \Rightarrow |2x+1| < 2 \Rightarrow -2 < 2x+1 < 2 \Rightarrow -3 < 2x < 1 \Rightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$ which diverges by the nth-Term Test for Divergence since $\lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \frac{1}{2} \ne 0$; at $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{2^n}{2^n} = \sum_{n=1}^{\infty} \frac{n+1}{2n+1}$, which diverges by the nth-Term Test

(a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(c) there are no values for which the series converges conditionally

45. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\right| < 1 \Rightarrow \frac{|x|}{e} \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ $\Rightarrow \frac{|x|}{e} \cdot 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

46. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} < 1 \Rightarrow |x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

47. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+2)x^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| < 1 \Rightarrow \frac{x^2}{3} \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; the series $\sum_{n=1}^{\infty} -\frac{n+1}{\sqrt{3}}$ and $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{3}}$, obtained with $x = \pm\sqrt{3}$, both diverge

(a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$

(b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$

(c) there are no values for which the series converges conditionally

48. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n+3}}{2n+3} \cdot \frac{2n+1}{(x-1)^{2n+1}}\right| < 1 \Rightarrow (x-1)^2 \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right) < 1 \Rightarrow (x-1)^2(1) < 1$ $\Rightarrow (x-1)^2 < 1 \Rightarrow |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1}$ $= \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ which converges conditionally by the Alternating Series Test and the fact that $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges; at $x = 2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$, which also converges conditionally

(a) the radius is 1; the interval of convergence is $0 \le x \le 2$

(b) the interval of absolute convergence is $0 < x < 2$

(c) the series converges conditionally at $x = 0$ and $x = 2$

49. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\operatorname{csch}(n+1)x^{n+1}}{\operatorname{csch}(n)x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{\left(\frac{2}{e^{n+1}-e^{-n-1}}\right)}{\left(\frac{2}{e^n - e^{-n}}\right)}\right| < 1$ $\Rightarrow |x| \lim_{n\to\infty} \left|\frac{e^{-1}-e^{-2n-1}}{1-e^{-2n-2}}\right| < 1 \Rightarrow \frac{|x|}{e} < 1 \Rightarrow -e < x < e$; the series $\sum_{n=1}^{\infty} (\pm e)^n \operatorname{csch} n$, obtained with $x = \pm e$, both diverge since $\lim_{n\to\infty} (\pm e)^n \operatorname{csch} n \ne 0$

(a) the radius is e; the interval of convergence is $-e < x < e$

(b) the interval of absolute convergence is $-e < x < e$

(c) there are no values for which the series converges conditionally

50. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}\coth(n+1)}{x^n \coth(n)}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{1+e^{-2n-2}}{1-e^{-2n-2}} \cdot \frac{1-e^{-2n}}{1+e^{-2n}}\right| < 1 \Rightarrow |x| < 1$ $\Rightarrow -1 < x < 1$; the series $\sum_{n=1}^{\infty} (\pm 1)^n \coth n$, obtained with $x = \pm 1$, both diverge since $\lim_{n\to\infty} (\pm 1)^n \coth n \ne 0$

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

Σε καθεμιά από τις ασκήσεις 51-56 δίνεται η τιμή μιας σειράς Maclaurin της συνάρτησης f(x) σε κάποιο σημείο. Ποια είναι η συνάρτηση και ποιο το σημείο; Με τι ισούται το άθροισμα της σειράς;

51. $1 - \frac{1}{4} + \frac{1}{16} - \cdots + (-1)^n \frac{1}{4^n} + \cdots$

52. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1} \frac{2^n}{n3^n} + \cdots$

53. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$

54. $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$

55. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$

56. $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots + (-1)^{n-1} \frac{1}{(2n-1)(\sqrt{3})^{2n-1}} + \cdots$

Λύσεις:

51. The given series has the form $1 - x + x^2 - x^3 + \ldots + (-x)^n + \ldots = \frac{1}{1+x}$, where $x = \frac{1}{4}$; the sum is $\frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

52. The given series has the form $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots = \ln(1+x)$, where $x = \frac{2}{3}$; the sum is $\ln\left(\frac{5}{3}\right) \approx 0.510825624$

53. The given series has the form $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sin x$, where $x = \pi$; the sum is $\sin \pi = 0$

54. The given series has the form $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \cos x$, where $x = \frac{\pi}{3}$; the sum is $\cos \frac{\pi}{3} = \frac{1}{2}$

55. The given series has the form $1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^n}{n!} + \ldots = e^x$, where $x = \ln 2$; the sum is $e^{\ln(2)} = 2$

56. The given series has the form $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n-1}}{(2n-1)} + \ldots = \tan^{-1} x$, where $x = \frac{1}{\sqrt{3}}$; the sum is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Βρείτε τις σειρές Maclaurin για τις συναρτήσεις των Ασκήσεων 57-64

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57. $\dfrac{1}{1-2x}$

58. $\dfrac{1}{1+x^3}$

59. $\sin \pi x$

60. $\sin \dfrac{2x}{3}$

61. $\cos (x^{5/2})$

62. $\cos \sqrt{5x}$

63. $e^{(\pi x/2)}$

64. e^{-x^2}

Λύση:

57. Consider $\frac{1}{1-2x}$ as the sum of a convergent geometric series with $a = 1$ and $r = 2x \Rightarrow \frac{1}{1-2x}$ $= 1 + (2x) + (2x)^2 + (2x)^3 + \ldots = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ where $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

58. Consider $\frac{1}{1+x^3}$ as the sum of a convergent geometric series with $a = 1$ and $r = -x^3 \Rightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$ $= 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{3n}$ where $|-x^3| < 1 \Rightarrow |x^3| < 1 \Rightarrow |x| < 1$

59. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$

60. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1}(2n+1)!}$

61. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(x^{5/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{5/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{(2n)!}$

62. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos \sqrt{5x} = \cos\left((5x)^{1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left((5x)^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{(2n)!}$

63. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{(\pi x/2)} = \sum_{n=0}^{\infty} \frac{\left(\frac{\pi x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n x^n}{2^n n!}$

64. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

Στις Ασκήσεις 65-68 βρείτε τους πρώτους τέσσερις μη μηδενικούς όρους της σειράς Taylor την οποία παράγει η f στο x=α.

65. $f(x) = \sqrt{3 + x^2}$ at $x = -1$

66. $f(x) = 1/(1 - x)$ at $x = 2$

67. $f(x) = 1/(x + 1)$ at $x = 3$

68. $f(x) = 1/x$ at $x = a > 0$

Λύσεις:

65. $f(x) = \sqrt{3+x^2} = (3+x^2)^{1/2} \Rightarrow f'(x) = x\,(3+x^2)^{-1/2} \Rightarrow f''(x) = -x^2\,(3+x^2)^{-3/2} + (3+x^2)^{-1/2}$ $\Rightarrow f'''(x) = 3x^3\,(3+x^2)^{-5/2} - 3x\,(3+x^2)^{-3/2}$; $f(-1) = 2$, $f'(-1) = -\frac{1}{2}$, $f''(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$, $f'''(-1) = -\frac{3}{32} + \frac{3}{8} = \frac{9}{32} \Rightarrow \sqrt{3+x^2} = 2 - \frac{(x+1)}{2\cdot 1!} + \frac{3(x+1)^2}{2^3\cdot 2!} + \frac{9(x+1)^3}{2^5\cdot 3!} + \dots$

66. $f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$; $f(2) = -1$, $f'(2) = 1$, $f''(2) = -2$, $f'''(2) = 6 \Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \dots$

67. $f(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(x) = -(x+1)^{-2} \Rightarrow f''(x) = 2(x+1)^{-3} \Rightarrow f'''(x) = -6(x+1)^{-4}$; $f(3) = \frac{1}{4}$, $f'(3) = -\frac{1}{4^2}$, $f''(3) = \frac{2}{4^3}$, $f'''(2) = \frac{-6}{4^4} \Rightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \dots$

68. $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$; $f(a) = \frac{1}{a}$, $f'(a) = -\frac{1}{a^2}$, $f''(a) = \frac{2}{a^3}$, $f'''(a) = \frac{-6}{a^4} \Rightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \dots$