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ΗΥ-110 Απειροστικός Λογισμός Ι
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Άσκηση 1^η

(a) Let the point located at $(\cosh u, 0)$ be called T. Then $A(u) =$ area of the triangle ΔOTP minus the area under the curve $y = \sqrt{x^2-1}$ from A to T $\Rightarrow A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2-1}\,dx$.

(b) $A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2-1}\,dx \Rightarrow A'(u) = \frac{1}{2}(\cosh^2 u + \sinh^2 u) - \left(\sqrt{\cosh^2 u - 1}\right)(\sinh u)$
$= \frac{1}{2}\cosh^2 u + \frac{1}{2}\sinh^2 u - \sinh^2 u = \frac{1}{2}(\cosh^2 u - \sinh^2 u) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$

(c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C$, and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$

Άσκηση 2^η

1. $u = (\sin^{-1} x)^2$, $du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}}$; $dv = dx$, $v = x$;
$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}}$;
$u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = -\frac{2x\,dx}{\sqrt{1-x^2}}$, $v = 2\sqrt{1-x^2}$;
$-\int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = 2(\sin^{-1} x)\sqrt{1-x^2} - \int 2\,dx = 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C$; therefore
$\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C$

2. $\frac{1}{x} = \frac{1}{x}$,
$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$,
$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$,
$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,
$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)} \Rightarrow$ the following pattern:
$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k| \right] + C$$

3. $u = \sin^{-1} x,\ du = \frac{dx}{\sqrt{1-x^2}};\ dv = x\,dx,\ v = \frac{x^2}{2};$

$$\int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2\,dx}{2\sqrt{1-x^2}};\ \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} \to \int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta\cos\theta\,d\theta}{2\cos\theta}$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$$

4. $\int \sin^{-1}\sqrt{y}\,dy;\ \begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \to \int 2z \sin^{-1} z\,dz$; from Exercise 3, $\int z \sin^{-1} z\,dz$

$$= \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y \sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$$

$$= y \sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$$

5. $\int \frac{d\theta}{1-\tan^2\theta} = \int \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}\,d\theta = \int \frac{1+\cos 2\theta}{2\cos 2\theta}\,d\theta = \frac{1}{2}\int (\sec 2\theta + 1)\,d\theta = \frac{\ln|\sec 2\theta + \tan 2\theta| + 2\theta}{4} + C$

6. $u = \ln\left(\sqrt{x} + \sqrt{1+x}\right),\ du = \left(\frac{dx}{\sqrt{x}+\sqrt{1+x}}\right)\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1+x}}\right) = \frac{dx}{2\sqrt{x}\sqrt{1+x}};\ dv = dx,\ v = x;$

$$\int \ln\left(\sqrt{x} + \sqrt{1+x}\right)dx = x\ln\left(\sqrt{x} + \sqrt{1+x}\right) - \frac{1}{2}\int \frac{x\,dx}{\sqrt{x}\sqrt{1+x}};\ \frac{1}{2}\int \frac{x\,dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}};$$

$$\begin{bmatrix} x + \frac{1}{2} = \frac{1}{2}\sec\theta \\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta \end{bmatrix} \to \frac{1}{4}\int \frac{(\sec\theta - 1)\cdot\sec\theta\tan\theta\,d\theta}{\left(\frac{1}{2}\tan\theta\right)} = \frac{1}{2}\int (\sec^2\theta - \sec\theta)\,d\theta$$

$$= \frac{\tan\theta - \ln|\sec\theta + \tan\theta|}{2} + C = \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{2} + C$$

$$\Rightarrow \int \ln\left(\sqrt{x} + \sqrt{1+x}\right)dx = x\ln\left(\sqrt{x} + \sqrt{1+x}\right) - \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{4} + C$$

7. $\int \frac{dt}{t - \sqrt{1-t^2}};\ \begin{bmatrix} t = \sin\theta \\ dt = \cos\theta\,d\theta \end{bmatrix} \to \int \frac{\cos\theta\,d\theta}{\sin\theta - \cos\theta} = \int \frac{d\theta}{\tan\theta - 1};\ \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \\ d\theta = \frac{du}{u^2+1} \end{bmatrix} \to \int \frac{du}{(u-1)(u^2+1)}$

$$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\,du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\ln\left|\frac{\tan\theta - 1}{\sec\theta}\right| - \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1} t + C$$

8. $\int \frac{(2e^{2x}-e^x)\,dx}{\sqrt{3e^{2x}-6e^x-1}}$; $\left[\begin{array}{c} u = e^x \\ du = e^x\,dx \end{array}\right] \to \int \frac{(2u-1)\,du}{\sqrt{3u^2-6u-1}} = \frac{1}{\sqrt{3}} \int \frac{(2u-1)\,du}{\sqrt{(u-1)^2-\frac{4}{3}}}$;

$$\left[\begin{array}{c} u-1 = \frac{2}{\sqrt{3}}\sec\theta \\ du = \frac{2}{\sqrt{3}}\sec\theta\tan\theta\,d\theta \end{array}\right] \to \frac{1}{\sqrt{3}}\int\left(\frac{4}{\sqrt{3}}\sec\theta + 1\right)(\sec\theta)\,d\theta = \frac{4}{3}\int \sec^2\theta\,d\theta + \frac{1}{\sqrt{3}}\int \sec\theta\,d\theta$$

$$= \frac{4}{3}\tan\theta + \frac{1}{\sqrt{3}}\ln|\sec\theta+\tan\theta| + C_1 = \frac{4}{3}\cdot\sqrt{\frac{3}{4}(u-1)^2-1} + \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}}{2}(u-1)+\sqrt{\frac{3}{4}(u-1)^2-1}\right| + C_1$$

$$= \frac{2}{3}\sqrt{3u^2-6u-1} + \frac{1}{\sqrt{3}}\ln\left|u-1+\sqrt{(u-1)^2-\frac{4}{3}}\right| + \left(C_1 + \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{\sqrt{3}}\left[2\sqrt{e^{2x}-2e^x-\frac{1}{3}} + \ln\left|e^x-1+\sqrt{e^{2x}-2e^x-\frac{1}{3}}\right|\right] + C$$

9. $\int \frac{1}{x^4+4}\,dx = \int \frac{1}{(x^2+2)^2-4x^2}\,dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\,dx$

$$= \frac{1}{16}\int\left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right]dx$$

$$= \frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] + C$$

10. $\int \frac{1}{x^6-1}\,dx = \frac{1}{6}\int\left(\frac{1}{x-1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1}\right)dx$

$$= \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\int\left[\frac{2x-1}{x^2-x+1} - \frac{3}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{2x+1}{x^2+x+1} - \frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right]dx$$

$$= \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\left[\ln\left|\frac{x^2-x+1}{x^2+x+1}\right| - 2\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - 2\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right] + C$$

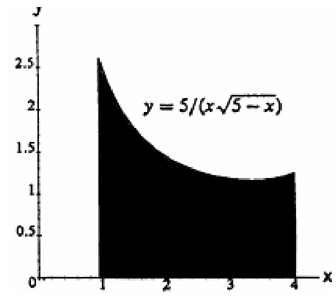
Άσκηση 3^η

$$V = \int_a^b \pi y^2\,dx = \pi\int_1^4 \frac{25\,dx}{x^2(5-x)}$$

$$= \pi\int_1^4\left(\frac{dx}{x} + \frac{5\,dx}{x^2} + \frac{dx}{5-x}\right)$$

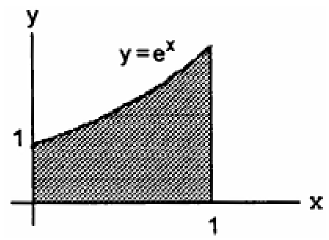
$$= \pi\left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x}\right]_1^4 = \pi\left(\ln 4 - \frac{5}{4}\right) - \pi\left(\ln\frac{1}{4} - 5\right)$$

$$= \frac{15\pi}{4} + 2\pi\ln 4$$



$$V = \int_a^b 2\pi\left(\begin{array}{c}\text{shell}\\\text{radius}\end{array}\right)\left(\begin{array}{c}\text{shell}\\\text{height}\end{array}\right)dx = \int_0^1 2\pi x e^x\,dx$$

$$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$$



$$V = \int_a^b 2\pi\left(\begin{array}{c}\text{shell}\\\text{radius}\end{array}\right)\left(\begin{array}{c}\text{shell}\\\text{height}\end{array}\right)dx = \int_0^1 2\pi xy\,dx$$

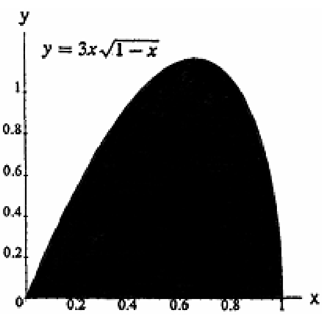
$$= 6\pi\int_0^1 x^2\sqrt{1-x}\,dx;\ \left[\begin{array}{c} u = 1-x \\ du = -dx \\ x^2 = (1-u)^2 \end{array}\right]$$

$$\to\ -6\pi\int_1^0 (1-u)^2\sqrt{u}\,du$$

$$= -6\pi\int_1^0\left(u^{1/2} - 2u^{3/2} + u^{5/2}\right)du$$

$$= -6\pi\left[\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_1^0 = 6\pi\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right)$$

$$= 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$$



Άσκηση $4^{η}$

The area of the shaded region is $\int_0^1 \sin^{-1} x\, dx = \int_0^1 \sin^{-1} y\, dy$, which is the same as the area of the region to the left of the curve $y = \sin x$ (and part of the rectangle formed by the coordinate axes and dashed lines $y = 1$, $x = \frac{\pi}{2}$). The area of the rectangle is $\frac{\pi}{2} = \int_0^1 \sin^{-1} y\, dy + \int_0^{\pi/2} \sin x\, dx$, so we have

$\frac{\pi}{2} = \int_0^1 \sin^{-1} x\, dx + \int_0^{\pi/2} \sin x\, dx \Rightarrow \int_0^{\pi/2} \sin x\, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x\, dx.$

Άσκηση $5^{η}$

$$f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}\, g'(x), \text{ where } g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0 \left(\frac{2}{1+16}\right) = \frac{2}{17}$$

Άσκηση $6^{η}$

$$\lim_{b\to 1-} \int_0^b \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$$

$$\lim_{x\to 0+} \left(\cos\sqrt{x}\right)^{1/x} = \frac{1}{\sqrt{e}}$$

$$\lim_{x\to\infty} (x+e^x)^{2/x} = e^2$$

$$\lim_{x\to\infty} \int_{-x}^{x} \sin t\, dt = \lim_{x\to\infty} [-\cos t]_{-x}^{x} = \lim_{x\to\infty} [-\cos x + \cos(-x)] = \lim_{x\to\infty} (-\cos x + \cos x) = \lim_{x\to\infty} 0 = 0$$

$\lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\, dt$; $\lim_{t\to 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim_{t\to 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\, dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim_{x\to 0^+} x \int_x^1 \frac{\cos t}{t^2}\, dt$ is an indeterminate $0 \cdot \infty$ form and we apply l'Hôpital's rule:

$$\lim_{x\to 0^+} x \int_x^1 \frac{\cos t}{t^2}\, dt = \lim_{x\to 0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\, dt}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \cos x = 1$$