Χειμερινό Εξάμηνο
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Πανεπιστήμιο Κρήτης
Τμήμα Επιστήμης Υπολογιστών
ΗΥ-110 Απειροστικός Λογισμός Ι
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Άσκηση 1η

1. $\int (x^3 + 5x - 7)dx = \frac{x^4}{4} + 5\frac{x^2}{2} - 7x$
2. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt = 2t^{\frac{3}{2}} - \frac{4}{\mathrm{t}}$
3. $\int \frac{r\,dr}{(r^2+5)^2} = -\frac{1}{2(\mathrm{r}^2+5)}$
4. $\int 3\theta\sqrt{2-\theta^2}d\theta = -(2-\theta^2)^{3/2}$
5. $\int x^3(1+x^4)^{-1/4}dx = \frac{1}{3}(\mathrm{x}^4+1)^{3/4}$
6. $\int \sec^2\frac{s}{10}ds = 10\tan\frac{s}{10}$
7. $\int \csc\sqrt{2\theta}\ \cot\sqrt{2\theta}\ d\theta = -\sqrt{2\theta}\csc\sqrt{2\theta}\,[+log\left(2\sin\sqrt{\frac{\theta}{2}}\right) - log\left(2\cos\sqrt{\frac{\theta}{2}}\right)]$
8. $\int \sin^{-2}\frac{4}{x}dx$
9. $\int 2(\cos x)^{-1/2}\sin x\,dx = -4\sqrt{\cos x}$
10. $\int \left(t-\frac{2}{t}\right)\left(t+\frac{2}{t}\right)dt = \frac{t^3}{3} + \frac{4}{t}$

Άσκηση 2η

(a) $\int_{-2}^{2} f(x)\,dx = \frac{1}{3}\int_{-2}^{2} 3\,f(x)\,dx = \frac{1}{3}(12) = 4$

(b) $\int_{2}^{5} f(x)\,dx = \int_{-2}^{5} f(x)\,dx - \int_{-2}^{2} f(x)\,dx = 6 - 4 = 2$

(c) $\int_{5}^{-2} g(x)\,dx = -\int_{-2}^{5} g(x)\,dx = -2$

(d) $\int_{-2}^{5} (-\pi\,g(x))\,dx = -\pi\int_{-2}^{5} g(x)\,dx = -\pi(2) = -2\pi$

(e) $\int_{-2}^{5} \left(\frac{f(x)+g(x)}{5}\right) dx = \frac{1}{5}\int_{-2}^{5} f(x)\,dx + \frac{1}{5}\int_{-2}^{5} g(x)\,dx = \frac{1}{5}(6) + \frac{1}{5}(2) = \frac{8}{5}$

(a) $\int_{0}^{2} g(x)\,dx = \frac{1}{7}\int_{0}^{2} 7\,g(x)\,dx = \frac{1}{7}(7) = 1$

(b) $\int_{1}^{2} g(x)\,dx = \int_{0}^{2} g(x)\,dx - \int_{0}^{1} g(x)\,dx = 1 - 2 = -1$

(c) $\int_{2}^{0} f(x)\,dx = -\int_{0}^{2} f(x)\,dx = -\pi$

(d) $\int_{0}^{2} \sqrt{2}\,f(x)\,dx = \sqrt{2}\int_{0}^{2} f(x)\,dx = \sqrt{2}(\pi) = \pi\sqrt{2}$

(e) $\int_{0}^{2} [g(x) - 3\,f(x)]\,dx = \int_{0}^{2} g(x)\,dx - 3\int_{0}^{2} f(x)\,dx = 1 - 3\pi$

Άσκηση 3η

For the sketch given, $a = 0$, $b = \pi$; $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x = \frac{1 - \cos 2x}{2}$;

$$A = \int_0^{\pi} \frac{(1-\cos 2x)}{2}\, dx = \frac{1}{2}\int_0^{\pi}(1 - \cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi}{2}$$

For the sketch given, $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(t) - g(t) = \frac{1}{2}\sec^2 t - (-4\sin^2 t) = \frac{1}{2}\sec^2 t + 4\sin^2 t$;

$$A = \int_{-\pi/3}^{\pi/3}\left(\tfrac{1}{2}\sec^2 t + 4\sin^2 t\right) dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\, dt + 4\int_{-\pi/3}^{\pi/3}\sin^2 t\, dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\, dt + 4\int_{-\pi/3}^{\pi/3}\frac{(1-\cos 2t)}{2}\, dt$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\, dt + 2\int_{-\pi/3}^{\pi/3}(1 - \cos 2t)\, dt = \frac{1}{2}[\tan t]_{-\pi/3}^{\pi/3} + 2\left[t - \frac{\sin 2t}{2}\right]_{-\pi/3}^{\pi/3} = \sqrt{3} + 4\cdot\frac{\pi}{3} - \sqrt{3} = \frac{4\pi}{3}$$

For the sketch given, $a = -2$, $b = 2$; $f(x) - g(x) = 2x^2 - (x^4 - 2x^2) = 4x^2 - x^4$;

$$A = \int_{-2}^{2}(4x^2 - x^4)\, dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right] = \frac{64}{3} - \frac{64}{5} = \frac{320-192}{15} = \frac{128}{15}$$

For the sketch given, $a = -1$, $b = 1$; $f(x) - g(x) = x^2 - (-2x^4) = x^2 + 2x^4$;

$$A = \int_{-1}^{1}(x^2 + 2x^4)\, dx = \left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_{-1}^{1} = \left(\frac{1}{3} + \frac{2}{5}\right) - \left[-\frac{1}{3} + \left(-\frac{2}{5}\right)\right] = \frac{2}{3} + \frac{4}{5} = \frac{10+12}{15} = \frac{22}{15}$$

AREA = A1 + A2

A1: For the sketch given, $a = -3$ and we find b by solving the equations $y = x^2 - 4$ and $y = -x^2 - 2x$ simultaneously for x: $x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \Rightarrow 2(x+2)(x-1) \Rightarrow x = -2$ or $x = 1$ so $b = -2$: $f(x) - g(x) = (x^2 - 4) - (-x^2 - 2x) = 2x^2 + 2x - 4 \Rightarrow A1 = \int_{-3}^{-2}(2x^2 + 2x - 4)\, dx$

$$= \left[\frac{2x^3}{3} + \frac{2x^2}{2} - 4x\right]_{-3}^{-2} = \left(-\frac{16}{3} + 4 + 8\right) - (-18 + 9 + 12) = 9 - \frac{16}{3} = \frac{11}{3};$$

A2: For the sketch given, $a = -2$ and $b = 1$: $f(x) - g(x) = (-x^2 - 2x) - (x^2 - 4) = -2x^2 - 2x + 4$

$$\Rightarrow A2 = -\int_{-2}^{1}(2x^2 + 2x - 4)\, dx = -\left[\frac{2x^3}{3} + x^2 - 4x\right]_{-2}^{1} = -\left(\frac{2}{3} + 1 - 4\right) + \left(-\frac{16}{3} + 4 + 8\right)$$

$$= -\frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 = 9;$$

Therefore, AREA = A1 + A2 $= \frac{11}{3} + 9 = \frac{38}{3}$

AREA = A1 + A2 + A3

A1: For the sketch given, $a = -2$ and $b = -1$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A1 = \int_{-2}^{-1}(x^2 - x - 2)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right) = \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \frac{11}{6};$$

A2: For the sketch given, $a = -1$ and $b = 2$: $f(x) - g(x) = (4 - x^2) - (-x + 2) = -(x^2 - x - 2)$

$$\Rightarrow A2 = -\int_{-1}^{2}(x^2 - x - 2)\, dx = -\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} = -\left(\frac{8}{3} - \frac{4}{2} - 4\right) + \left(-\frac{1}{3} - \frac{1}{2} + 2\right) = -3 + 8 - \frac{1}{2} = \frac{9}{2};$$

A3: For the sketch given, $a = 2$ and $b = 3$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A3 = \int_{2}^{3}(x^2 - x - 2)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3} = \left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = 9 - \frac{9}{2} - \frac{8}{3};$$

Therefore, AREA = A1 + A2 + A3 $= \frac{11}{6} + \frac{9}{2} + \left(9 - \frac{9}{2} - \frac{8}{3}\right) = 9 - \frac{5}{6} = \frac{49}{6}$

Άσκηση 4η

$$y = x^2 + \int_1^x \frac{1}{t}\, dt \Rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} \Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{1}{x^2};\ y(1) = 1 + \int_1^1 \frac{1}{t}\, dt = 1 \text{ and } y'(1) = 2 + 1 = 3$$

$$y = \int_0^x\left(1 + 2\sqrt{\sec t}\right) dt \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec x} \Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{1}{2}\right)(\sec x)^{-1/2}(\sec x \tan x) = \sqrt{\sec x}\,(\tan x);$$

$$x = 0 \Rightarrow y = \int_0^0\left(1 + 2\sqrt{\sec t}\right) dt = 0 \text{ and } x = 0 \Rightarrow \frac{dy}{dx} = 1 + 2\sqrt{\sec 0} = 3$$

Άσκηση 5η

$$A(x) = \tfrac{\pi}{4}(\text{diameter})^2 = \tfrac{\pi}{4}\left(2\sqrt{x} - \tfrac{x^2}{4}\right)^2 = \tfrac{\pi}{4}\left(4x - x^{5/2} + \tfrac{x^4}{16}\right); \; a = 0,\ b = 4 \;\Rightarrow\; V = \int_a^b A(x)\,dx$$

$$= \tfrac{\pi}{4}\int_0^4 \left(4x - x^{5/2} + \tfrac{x^4}{16}\right) dx = \tfrac{\pi}{4}\left[2x^2 - \tfrac{2}{7}x^{7/2} + \tfrac{x^5}{5\cdot 16}\right]_0^4 = \tfrac{\pi}{4}\left(32 - 32\cdot\tfrac{8}{7} + \tfrac{2}{5}\cdot 32\right)$$

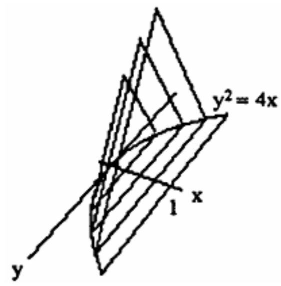
$$= \tfrac{32\pi}{4}\left(1 - \tfrac{8}{7} + \tfrac{2}{5}\right) = \tfrac{8\pi}{35}(35 - 40 + 14) = \tfrac{72\pi}{35}$$

$$A(x) = \tfrac{1}{2}(\text{edge})^2 \sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{4}\left[2\sqrt{x} - (-2\sqrt{x})\right]^2$$

$$= \tfrac{\sqrt{3}}{4}\left(4\sqrt{x}\right)^2 = 4\sqrt{3}\,x; \; a = 0,\ b = 1$$

$$\Rightarrow\; V = \int_a^b A(x)\,dx = \int_0^1 4\sqrt{3}\,x\,dx = \left[2\sqrt{3}\,x^2\right]_0^1$$

$$= 2\sqrt{3}$$



(a) *washer method*:

$$R(x) = \tfrac{4}{x^3},\ r(x) = \tfrac{1}{2} \;\Rightarrow\; V = \int_a^b \pi[R^2(x) - r^2(x)]\,dx = \int_1^2 \pi\left[\left(\tfrac{4}{x^3}\right)^2 - \left(\tfrac{1}{2}\right)^2\right] dx = \pi\left[-\tfrac{16}{5}x^{-5} - \tfrac{x}{4}\right]_1^2$$

$$= \pi\left[\left(\tfrac{-16}{5\cdot 32} - \tfrac{1}{2}\right) - \left(-\tfrac{16}{5} - \tfrac{1}{4}\right)\right] = \pi\left(-\tfrac{1}{10} - \tfrac{1}{2} + \tfrac{16}{5} + \tfrac{1}{4}\right) = \tfrac{\pi}{20}(-2 - 10 + 64 + 5) = \tfrac{57\pi}{20}$$

(b) *shell method*:

$$V = 2\pi\int_1^2 x\left(\tfrac{4}{x^3} - \tfrac{1}{2}\right) dx = 2\pi\left[-4x^{-1} - \tfrac{x^2}{4}\right]_1^2 = 2\pi\left[\left(-\tfrac{4}{2} - 1\right) - \left(-4 - \tfrac{1}{4}\right)\right] = 2\pi\left(\tfrac{5}{4}\right) = \tfrac{5\pi}{2}$$

(c) *shell method*:

$$V = 2\pi\int_a^b \binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}} dx = 2\pi\int_1^2 (2 - x)\left(\tfrac{4}{x^3} - \tfrac{1}{2}\right) dx = 2\pi\int_1^2 \left(\tfrac{8}{x^3} - \tfrac{4}{x^2} - 1 + \tfrac{x}{2}\right) dx$$

$$= 2\pi\left[-\tfrac{4}{x^2} + \tfrac{4}{x} - x + \tfrac{x^2}{4}\right]_1^2 = 2\pi\left[(-1 + 2 - 2 + 1) - \left(-4 + 4 - 1 + \tfrac{1}{4}\right)\right] = \tfrac{3\pi}{2}$$

(d) *washer method*:

$$V = \int_a^b \pi[R^2(x) - r^2(x)]\,dx$$

$$= \pi\int_1^2 \left[\left(\tfrac{7}{2}\right)^2 - \left(4 - \tfrac{4}{x^3}\right)^2\right] dx$$

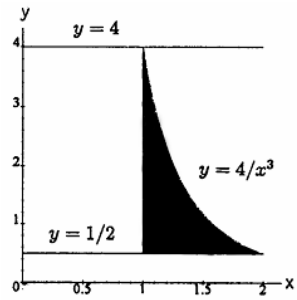
$$= \tfrac{49\pi}{4} - 16\pi\int_1^2 \left(1 - 2x^{-3} + x^{-6}\right) dx$$

$$= \tfrac{49\pi}{4} - 16\pi\left[x + x^{-2} - \tfrac{x^{-5}}{5}\right]_1^2$$

$$= \tfrac{49\pi}{4} - 16\pi\left[\left(2 + \tfrac{1}{4} - \tfrac{1}{5\cdot 32}\right) - \left(1 + 1 - \tfrac{1}{5}\right)\right]$$

$$= \tfrac{49\pi}{4} - 16\pi\left(\tfrac{1}{4} - \tfrac{1}{160} + \tfrac{1}{5}\right)$$

$$= \tfrac{49\pi}{4} - \tfrac{16\pi}{160}(40 - 1 + 32) = \tfrac{49\pi}{4} - \tfrac{71\pi}{10} = \tfrac{103\pi}{20}$$



Άσκηση 6^{η}

$\frac{dx}{dt} = -5\sin t + 5\sin 5t$ and $\frac{dy}{dt} = 5\cos t - 5\cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
$= \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}$
$= 5\sqrt{\sin^2 5t - 2\sin t\sin 5t + \sin^2 t + \cos^2 t - 2\cos t\cos 5t + \cos^2 5t} = 5\sqrt{2 - 2(\sin t\sin 5t + \cos t\cos 5t)}$
$= 5\sqrt{2(1-\cos 4t)} = 5\sqrt{4\left(\frac{1}{2}\right)(1-\cos 4t)} = 10\sqrt{\sin^2 2t} = 10|\sin 2t| = 10\sin 2t$ (since $0 \le t \le \frac{\pi}{2}$)
$\Rightarrow \text{Length} = \int_0^{\pi/2} 10\sin 2t\,dt = [-5\cos 2t]_0^{\pi/2} = (-5)(-1) - (-5)(1) = 10$

$\frac{dx}{dt} = 3t^2 - 12t$ and $\frac{dy}{dt} = 3t^2 + 12t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2-12t)^2 + (3t^2+12t)^2} = \sqrt{288t^2 + 18t^4}$
$= 3\sqrt{2}\,|t|\sqrt{16+t^2} \Rightarrow \text{Length} = \int_0^1 3\sqrt{2}\,|t|\sqrt{16+t^2}\,dt = 3\sqrt{2}\int_0^1 t\sqrt{16+t^2}\,dt$; $\big[u = 16+t^2 \Rightarrow du = 2t\,dt$
$\Rightarrow \frac{1}{2}du = t\,dt;\ t=0 \Rightarrow u=16;\ t=1 \Rightarrow u=17\big]$; $\frac{3\sqrt{2}}{2}\int_{16}^{17}\sqrt{u}\,du = \frac{3\sqrt{2}}{2}\left[\frac{2}{3}u^{3/2}\right]_{16}^{17} = \frac{3\sqrt{2}}{2}\left(\frac{2}{3}(17)^{3/2} - \frac{2}{3}(16)^{3/2}\right)$
$= \frac{3\sqrt{2}}{2}\cdot\frac{2}{3}\left((17)^{3/2} - 64\right) = \sqrt{2}\left((17)^{3/2} - 64\right) \approx 8.617.$

$\frac{dx}{d\theta} = -3\sin\theta$ and $\frac{dy}{d\theta} = 3\cos\theta \Rightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(-3\sin\theta)^2 + (3\cos\theta)^2} = \sqrt{3(\sin^2\theta + \cos^2\theta)} = 3$
$\Rightarrow \text{Length} = \int_0^{3\pi/2} 3\,d\theta = 3\int_0^{3\pi/2} d\theta = 3\left(\frac{3\pi}{2} - 0\right) = \frac{9\pi}{2}$

$x = t^2$ and $y = \frac{t^3}{3} - t,\ -\sqrt{3} \le t \le \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = t^2 - 1 \Rightarrow \text{Length} = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{(2t)^2 + (t^2-1)^2}\,dt$
$= \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{t^4 + 2t^2 + 1}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{t^4 + 2t^2 + 1}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{(t^2+1)^2}\,dt = \int_{-\sqrt{3}}^{\sqrt{3}}(t^2+1)\,dt = \left[\frac{t^3}{3} + t\right]_{-\sqrt{3}}^{\sqrt{3}}$
$= 4\sqrt{3}$