Χειμερινό Εξάμηνο 2009-2010

Πανεπιστήμιο Κρήτης
Τμήμα Επιστήμης Υπολογιστών
ΗΥ-110 Απειροστικός Ι
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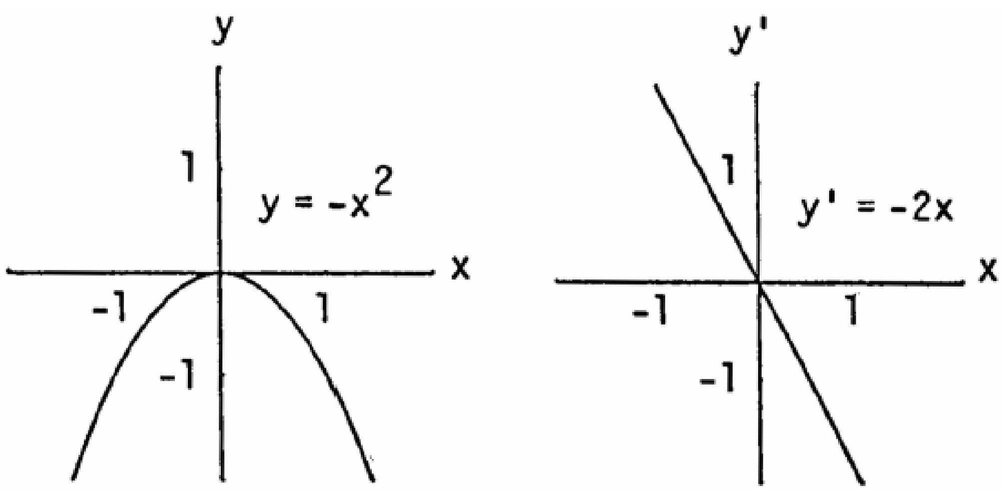
<u>Άσκηση 1η</u>

1) (α) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{hh}=\lim_{h\to 0}\frac{-(x+h)^2-(-x)^2}{}=\lim_{h\to 0}\frac{-x^2-2xh-h^2+x^2}{h}$$
$$=\lim_{h\to 0}(-2x-h)=-2x$$

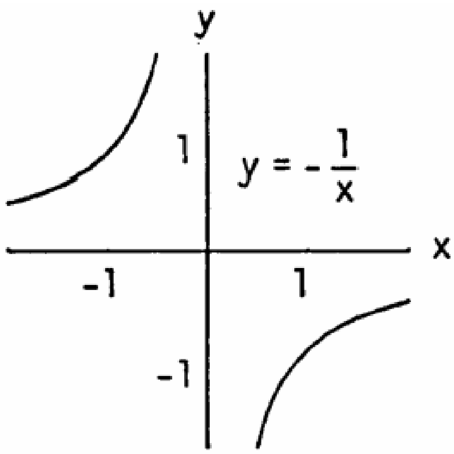
(β)

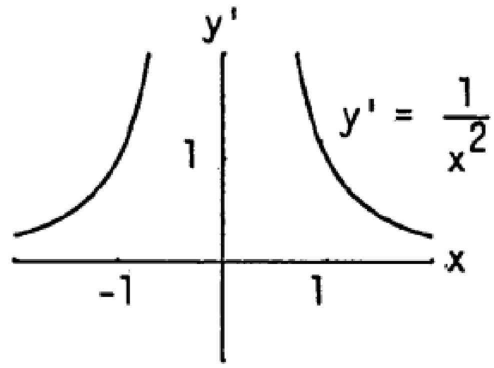


(γ) $y'=-2x$ είναι θετική για $x<0$, y’ ισούται με μηδέαν για x=0 και y’ είναι αρνητική για x>0.
(δ) $y=-x^2$ είναι αύξουσα για: $-\infty<x<0$ και φθίνουσα για $0<x<\infty$. Η συνάρτηση είναι αύξουσα στα διαστήματα όπου η παράγωγός της είναι θετική, και φθίνουσα όταν η παράγωγός της είναι αρνητική.

2)
(α)
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left(\frac{-1}{x+h}-\frac{-1}{x}\right)}{h}=\lim_{h\to 0}\frac{-x+(x+h)}{x(x+h)h}=\lim_{h\to 0}\frac{1}{x(x+h)}=\frac{1}{x^2}$$

(β)



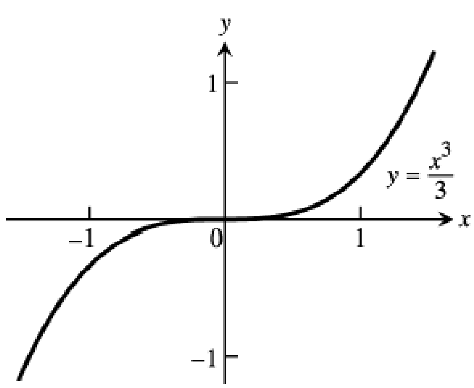


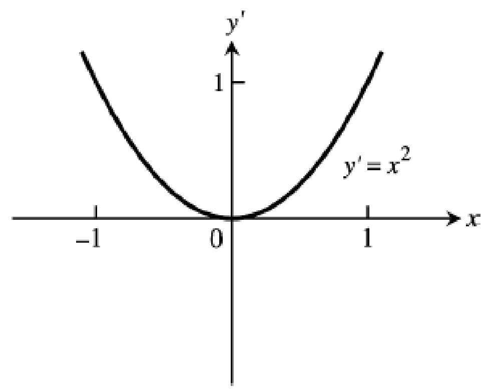
(γ) Η y’ είναι θετική για όλα τα $x \neq 0$, δεν μηδενίζεται ποτέ και δεν είναι ποτέ αρνητική.
(δ) Η y είναι αύξουσα για $-\infty < x < 0$ και $0 < x < \infty$

3) (α)

Using the alternate formula for calculating derivatives: $f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\left(\frac{z^3}{3} - \frac{x^3}{3}\right)}{z - x}$
$= \lim_{z \to x} \frac{z^3 - x^3}{3(z - x)} = \lim_{z \to x} \frac{(z - x)(z^2 + zx + x^2)}{3(z - x)} = \lim_{z \to x} \frac{z^2 + zx + x^2}{3} = x^2 \Rightarrow f'(x) = x^2$

(β)



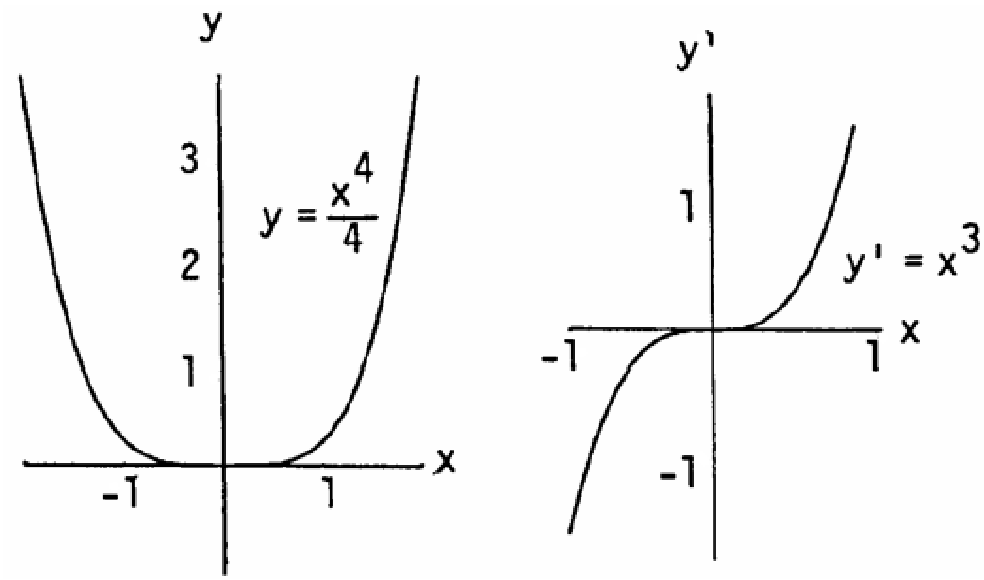


(γ) Η y’ είναι θετική για κάθε $x \neq 0$ και y’=0 όταν x=0. Η y’ δεν είναι ποτέ αρνητική.
(δ) Η y είναι αύξουσα για κάθε $x \neq 0$ επειδή είναι θετική η παράγωγός της σε αυτό το διάστημα.

4) (α)

Using the alternate form for calculating derivatives: $f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\left(\frac{z^4}{4} - \frac{x^4}{4}\right)}{z - x}$
$= \lim_{z \to x} \frac{z^4 - x^4}{4(z - x)} = \lim_{z \to x} \frac{(z - x)(z^3 + xz^2 + x^2z + x^3)}{4(z - x)} = \lim_{z \to x} \frac{z^3 + xz^2 + x^2z + x^3}{4} = x^3 \Rightarrow f'(x) = x^3$

(β)



(γ) Η y’ είναι θετική για x>0, μηδέν για x=0 και αρνητική για x<0.
(δ) Η y είναι αύξουσα για $0 < x < \infty$, και φθίνουσα για $-\infty < x < 0$.

Άσκηση 2

(α) $s = 55 - 4.9t^2 \Rightarrow v = -9.8t \Rightarrow speed = |v| = 9.8t, \frac{m}{\sec} \Rightarrow a = -9.8\frac{m}{\sec^2}$

(β) $s = 0 \Rightarrow 55 - 4.9t^2 = 0 \Rightarrow t = \sqrt{\frac{55}{4.9}} \approx 3.3$

(γ) Όταν $t = \sqrt{\frac{55}{4.9}} \approx 3.3$, $v = -9.8\sqrt{\frac{55}{4.9}} \approx -32.8329\frac{m}{\sec}$

Άσκηση 3

- $y = \frac{4x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4(-x^2+1)}{(x^2+1)^2}$. When $x = 0$, $y = 0$ and $y' = \frac{4(0+1)}{1}$ $= 4$, so the tangent to the curve at $(0,0)$ is the line $y = 4x$. When $x = 1$, $y = 2 \Rightarrow y' = 0$, so the tangent to the curve at $(1,2)$ is the line $y = 2$.

- $y = \frac{8}{x^2+4} \Rightarrow y' = \frac{(x^2+4)(0)-8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$. When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2+4)^2} = -\frac{1}{2}$, so the tangent line to the curve at $(2,1)$ has the equation $y - 1 = -\frac{1}{2}(x-2)$, or $y = -\frac{x}{2} + 2$.

Άσκηση 4

$y = ax^2 + bx + c$ passes through $(0,0) \Rightarrow 0 = a(0) + b(0) + c \Rightarrow c = 0$; $y = ax^2 + bx$ passes through $(1,2)$ $\Rightarrow 2 = a + b$; $y' = 2ax + b$ and since the curve is tangent to $y = x$ at the origin, its slope is 1 at $x = 0$ $\Rightarrow y' = 1$ when $x = 0 \Rightarrow 1 = 2a(0) + b \Rightarrow b = 1$. Then $a + b = 2 \Rightarrow a = 1$. In summary $a = b = 1$ and $c = 0$ so the curve is $y = x^2 + x$.

Άσκηση 5

$v = t^2 - 4t + 3 \Rightarrow a = 2t - 4$

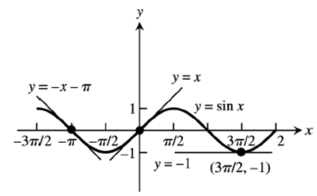
(a) $v = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $3 \Rightarrow a(1) = -2$ m/sec^2 and $a(3) = 2$ m/sec^2

(b) $v > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow 0 \le t < 1$ or $t > 3$ and the body is moving forward; $v < 0 \Rightarrow (t-3)(t-1) < 0$
$\Rightarrow 1 < t < 3$ and the body is moving backward

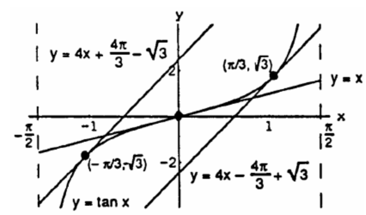
(c) velocity increasing $\Rightarrow a > 0 \Rightarrow 2t - 4 > 0 \Rightarrow t > 2$; velocity decreasing $\Rightarrow a < 0 \Rightarrow 2t - 4 < 0 \Rightarrow 0 \le t < 2$

Άσκηση 6

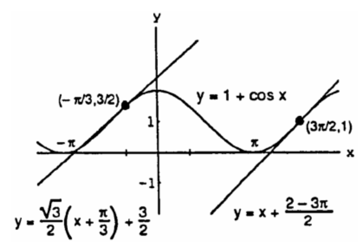
$y = \sin x \Rightarrow y' = \cos x \Rightarrow$ slope of tangent at $x = -\pi$ is $y'(-\pi) = \cos(-\pi) = -1$; slope of tangent at $x = 0$ is $y'(0) = \cos(0) = 1$; and slope of tangent at $x = \frac{3\pi}{2}$ is $y'\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$. The tangent at $(-\pi, 0)$ is $y - 0 = -1(x + \pi)$, or $y = -x - \pi$; the tangent at $(0, 0)$ is $y - 0 = 1(x - 0)$, or $y = x$; and the tangent at $\left(\frac{3\pi}{2}, -1\right)$ is $y = -1$.



- $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec^2\left(-\frac{\pi}{3}\right) = 4$; slope of tangent at $x = 0$ is $\sec^2(0) = 1$; and slope of tangent at $x = \frac{\pi}{3}$ is $\sec^2\left(\frac{\pi}{3}\right) = 4$. The tangent at $\left(-\frac{\pi}{3}, \tan\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, -\sqrt{3}\right)$ is $y + \sqrt{3} = 4\left(x + \frac{\pi}{3}\right)$; the tangent at $(0, 0)$ is $y = x$; and the tangent at $\left(\frac{\pi}{3}, \tan\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$ is $y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$.



- $y = 1 + \cos x \Rightarrow y' = -\sin x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $-\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$; slope of tangent at $x = \frac{3\pi}{2}$ is $-\sin\left(\frac{3\pi}{2}\right) = 1$. The tangent at the point $\left(-\frac{\pi}{3}, 1 + \cos\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, \frac{3}{2}\right)$ is $y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{3\pi}{2}, 1 + \cos\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, 1\right)$ is $y - 1 = x - \frac{3\pi}{2}$



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Άσκηση 7

Step 1: The formula holds for $n = 2$ (a single product) since $y = u_1 u_2 \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx} u_2 + u_1 \frac{du_2}{dx}$.

Step 2: Assume the formula holds for $n = k$:

$$y = u_1 u_2 \cdots u_k \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \ldots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx}.$$

If $y = u_1 u_2 \cdots u_k u_{k+1} = (u_1 u_2 \cdots u_k) u_{k+1}$, then $\frac{dy}{dx} = \frac{d(u_1 u_2 \cdots u_k)}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$

$= \left(\frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx}\right) u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$

$= \frac{du_1}{dx} u_2 u_3 \cdots u_{k+1} + u_1 \frac{du_2}{dx} u_3 \cdots u_{k+1} + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$.

Thus the original formula holds for $n = (k+1)$ whenever it holds for $n = k$.

Άσκηση 8

$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \frac{dT}{dL} = 2\pi \cdot \frac{1}{2\sqrt{\frac{L}{g}}} \cdot \frac{1}{g} = \frac{\pi}{g\sqrt{\frac{L}{g}}} = \frac{\pi}{\sqrt{gL}}$. Therefore, $\frac{dT}{du} = \frac{dT}{dL} \cdot \frac{dL}{du} = \frac{\pi}{\sqrt{gL}} \cdot kL = \frac{\pi k\sqrt{L}}{\sqrt{g}} = \frac{1}{2} \cdot 2\pi k\sqrt{\frac{L}{g}}$
$= \frac{kT}{2}$, as required.

Άσκηση 9

$s^2 = y^2 + x^2 \Rightarrow 2s\,\frac{ds}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{\sqrt{169}}\,[5(-442) + 12(-481)]$
$= -614$ knots

Άσκηση 10

(a) For all a, b and for all $x \neq 2$, f is differentiable at x. Next, f differentiable at $x = 2 \Rightarrow$ f continuous at $x = 2$ $\Rightarrow \lim_{x \to 2^-} f(x) = f(2) \Rightarrow 2a = 4a - 2b + 3 \Rightarrow 2a - 2b + 3 = 0$. Also, f differentiable at $x \neq 2$ $\Rightarrow f'(x) = \begin{cases} a, & x < 2 \\ 2ax - b, & x > 2 \end{cases}$. In order that $f'(2)$ exist we must have $a = 2a(2) - b \Rightarrow a = 4a - b \Rightarrow 3a = b$. Then $2a - 2b + 3 = 0$ and $3a = b \Rightarrow a = \frac{3}{4}$ and $b = \frac{9}{4}$.

(b) For $x < 2$, the graph of f is a straight line having a slope of $\frac{3}{4}$ and passing through the origin; for $x \geq 2$, the graph of f is a parabola. At $x = 2$, the value of the y-coordinate on the parabola is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x = 2$. In addition, the slope of the parabola at the match up point is $\frac{3}{4}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.