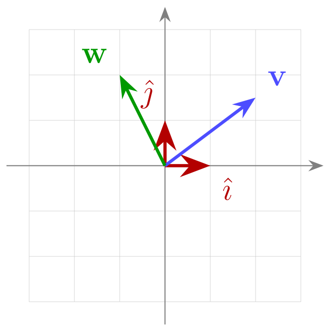
Linear Algebra

Essence of Linear Algebra

Comprehensive Lecture Notes

with emphasis on geometric intuition and practical applications



Based on the video series

Essence of Linear Algebra

by Grant Sanderson

YouTube Channel: 3Blue1Brown

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Sepehr Talaei

Based on the “Essence of Linear Algebra” video series by 3Blue1Brown

Sharif University of Technology Physics Scientific Association

Contents

Preface

These notes are based on the famous "Essence of Linear Algebra" video series by Grant Sanderson from the 3Blue1Brown YouTube channel. The main goal of this series is to provide an intuitive and geometric understanding of linear algebra concepts.

Features of These Notes

- **Geometric Intuition:** Emphasis on visual understanding of concepts, not just calculations
- **Rigorous Definitions:** Standard mathematical formulations
- **Practical Applications:** Applications in physics, computer science, and engineering
- **Exercises:** Diverse problems to reinforce learning

Prerequisites

- Familiarity with basic mathematics (high school algebra)
- Understanding of Cartesian coordinate systems
- A desire for deep understanding of mathematics!

How to Use These Notes

Each chapter contains the following sections:

Definitions: Key concepts with mathematical notation

Geometric Intuition: Visual explanations (in blue boxes)

Practical Applications: Real-world examples (in green boxes)

Exercises: Problems for practice

Best wishes for learning linear algebra

Chapter 1

Introduction to Vectors

Vectors are the fundamental building block of linear algebra. In this chapter, we explore three different perspectives on vectors: the physics view, the computer science view, and the mathematics view. We also learn the basic operations of vector addition and scalar multiplication.

1.1 Three Perspectives on Vectors

The concept of a vector has different meanings depending on your field of study. Understanding these three perspectives helps us form a complete picture of vectors.

1.1.1 The Physics Perspective

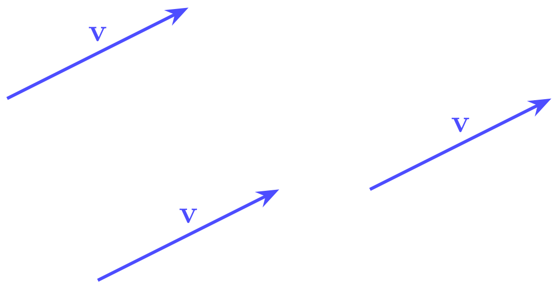
Definition 1.1 (Vector from Physics Perspective)**.** A vector is an **arrow** in space defined by two properties:

1. **Length** (magnitude)
2. **Direction**

As long as these two properties remain the same, a vector can be moved anywhere in space and still be considered the same vector.

Geometric Intuition

Draw an arrow on paper. Now move it to another location without rotating it or changing its size. From a physicist's perspective, this is still the same vector!
For example, the gravitational force on an apple always points downward with a constant magnitude—regardless of where the apple is in the room.



All of these are the same vector

1.1.2 The Computer Science Perspective

Definition 1.2 (Vector from Computer Science Perspective)**.** A vector is an **ordered list of numbers**. For example:

$$\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

In this view, "vector" is almost synonymous with "list."

Practical Application

Example: Housing Price Model

Suppose you want to model houses based on two features:

- Square footage (in square meters)
- Price (in thousands of dollars)

Each house is a two-dimensional vector:

$$\text{House}_1 = \begin{bmatrix} 85 \\ 250 \end{bmatrix}, \quad \text{House}_2 = \begin{bmatrix} 120 \\ 420 \end{bmatrix}, \quad \text{House}_3 = \begin{bmatrix} 65 \\ 180 \end{bmatrix}$$

Important: The order of numbers matters! $\begin{bmatrix} 85 \\ 250 \end{bmatrix}$ is different from $\begin{bmatrix} 250 \\ 85 \end{bmatrix}$.

1.1.3 The Mathematics Perspective

Definition 1.3 (Vector from Mathematics Perspective)**.** Mathematicians define vectors abstractly: a vector is anything on which you can perform these two operations:

1. **Add two vectors**
2. **Multiply a vector by a number (scalar)**

We will explore the details of this definition in the final chapter (Abstract Vector Spaces).

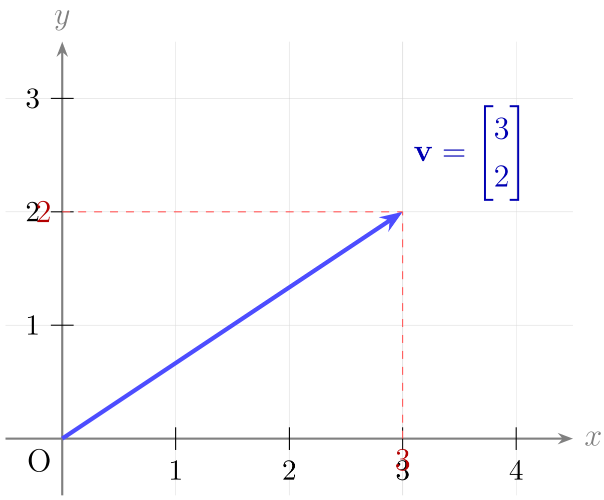
Important

Throughout this course, we visualize a vector as an **arrow in a coordinate system** with its **tail at the origin**. This geometric perspective helps us understand concepts more deeply.

1.2 Coordinate Systems and Vector Representation

Definition 1.4 (Two-Dimensional Cartesian Coordinate System)**.** A coordinate system consists of:

- Horizontal axis: ***x*-axis**
- Vertical axis: ***y*-axis**
- Point of intersection: **origin**



Definition 1.5 (Coordinates of a Vector)**.** The **coordinates** of a vector are a pair of numbers describing how to get from the origin to the tip of the vector:

- First number: how far to move along the x-axis (right is positive, left is negative)
- Second number: how far to move along the y-axis (up is positive, down is negative)

Geometric Intuition

The coordinates $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ mean:

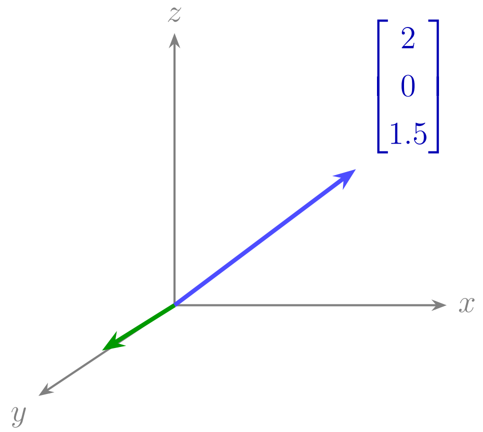
1. From the origin, move 3 units to the right
2. Then move 2 units up

3. The tip of the vector is right here!

1.3 Three-Dimensional Space

Definition 1.6 (Three-Dimensional Vector)**.** In three-dimensional space, a third axis called the ***z*-axis** is added, perpendicular to both the x and y axes. Each vector is specified by three numbers:

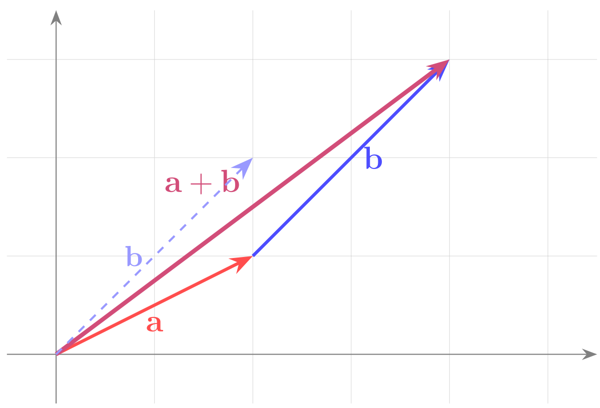
$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



1.4 Vector Addition

Definition 1.7 (Adding Two Vectors - Geometric Method)**.** To add two vectors **a** and **b**:

1. Draw vector **a**
2. Move vector **b** so that its tail is at the tip of **a**
3. The sum vector is drawn from the origin (tail of **a**) to the tip of **b**



Definition 1.8 (Adding Two Vectors - Algebraic Method)**.** If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$:

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

We add corresponding components together.

Example 1.1.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 2+(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Geometric Intuition

Vector addition can be interpreted as **walking a path**:

- First walk along vector $\mathbf{a}$
- Then walk along vector $\mathbf{b}$
- The net effect is the same as walking directly along $\mathbf{a} + \mathbf{b}$

Like walking: if you take 2 steps right, then 5 steps right, it's the same as taking 7 steps right.

Practical Application

Example: Airplane Speed in Wind

An airplane flies at $\begin{bmatrix} 500 \\ 0 \end{bmatrix}$ km/h toward the east (positive x direction). Wind blows at $\begin{bmatrix} 0 \\ 50 \end{bmatrix}$ km/h toward the north (positive y direction).
Actual speed of the airplane relative to ground:

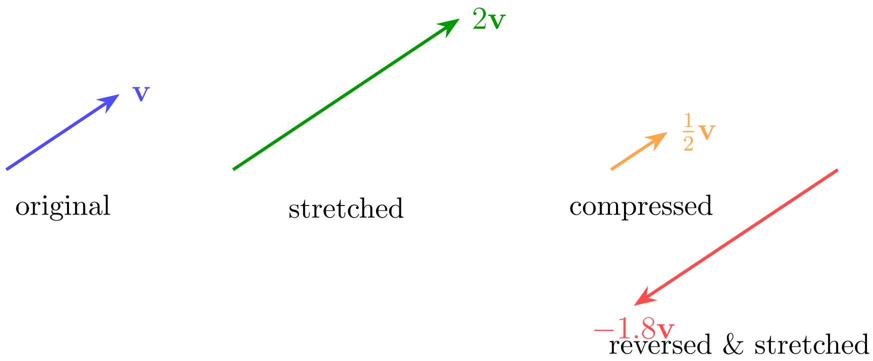
$$\mathbf{v}_{\text{actual}} = \begin{bmatrix} 500 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 50 \end{bmatrix} = \begin{bmatrix} 500 \\ 50 \end{bmatrix}$$

The airplane moves both east and slightly north.

1.5 Scalar Multiplication

Definition 1.9 (Scalar Times Vector)**.** Multiplying a number (scalar) c by a vector $\mathbf{v}$ **scales** the vector by factor c:

$$c \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \end{bmatrix}$$



Important

Effects of scalar multiplication:

- $c > 1$: vector is **stretched**
- $0 < c < 1$: vector is **compressed**
- $c < 0$: vector is **reversed** and then scaled
- $c = 0$: vector becomes zero
- $c = 1$: vector is unchanged

Warning

If $c < 0$, the vector not only changes in magnitude but also **reverses direction**!

Example 1.2. If $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$:

$$2\mathbf{v} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \qquad \text{(doubled)}$$

$$-\mathbf{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \qquad \text{(reversed)}$$

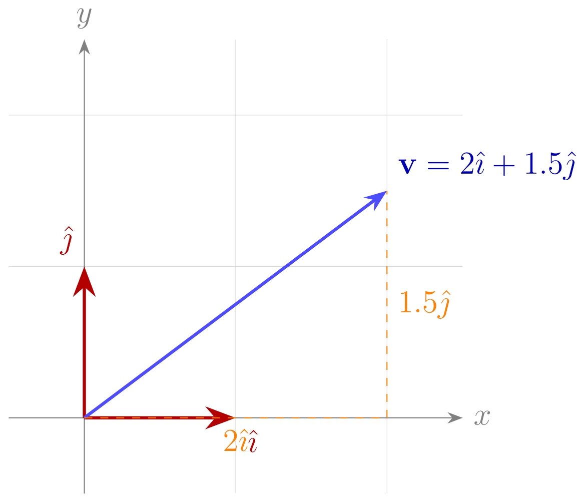
$$\frac{1}{2}\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \text{(halved)}$$

Definition 1.10 (Scalar)**.** In linear algebra, numbers that multiply vectors are called **scalars**. This name comes from the verb "to scale." The word "scalar" is essentially synonymous with "number."

1.6 Basis Vectors

Definition 1.11 (Standard Basis Vectors in $\mathbb{R}^2$)**.** The two standard basis vectors are:

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Theorem 1.1. *Any vector in $\mathbb{R}^2$ can be written as a linear combination of basis vectors:*

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\hat{\imath} + b\hat{\jmath} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Geometric Intuition

When we write $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, we're really saying:

"Take 3 copies of $\hat{\imath}$ and 2 copies of $\hat{\jmath}$, then add them together"

That is:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\hat{\imath} + 2\hat{\jmath} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.7 Connecting the Perspectives

Summary

The power of linear algebra lies in translating between different perspectives:

- **Data analyst:** can visualize long lists of numbers as vectors in space
- **Physicist:** can describe motion and forces using numbers
- **Graphics programmer:** can implement geometric transformations using matrices

1.8 Exercises

Exercise 1.1. Draw the following vectors in a coordinate system:

$$\mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Exercise 1.2. Calculate the sum and difference of the following vectors:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(a) $\mathbf{a} + \mathbf{b}$

(b) $\mathbf{a} - \mathbf{b}$

(c) $2\mathbf{a} + 3\mathbf{b}$

Exercise 1.3. If $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, calculate and sketch the following vectors:

(a) $2\mathbf{v}$

(b) $-\mathbf{v}$

(c) $\frac{1}{2}\mathbf{v}$

(d) $-2.5\mathbf{v}$

Exercise 1.4 (Applied)**.** A ship moves at 30 km/h toward the north. The water current flows at 10 km/h toward the east. What is the ship's actual velocity relative to the shore?

Exercise 1.5. Show that for any vector $\mathbf{v}$:

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

where $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the zero vector.

Exercise 1.6. Prove that vector addition is commutative:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

Exercise 1.7 (Challenge)**.** Three points $A(1, 2)$, $B(4, 6)$, and $C(7, 2)$ are given. Show that these three points form an isosceles triangle.

Hint: The length of vector $\begin{bmatrix} a \\ b \end{bmatrix}$ *equals* $\sqrt{a^2 + b^2}$*.*

Problem 1.1. In three-dimensional space, vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is given. Calculate:

(a) $3\mathbf{v}$

(b) $\mathbf{v} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

(c) $-\frac{1}{2}\mathbf{v}$

Chapter 2

Linear Combinations, Span, and Basis

In this chapter, we explore the key concepts of "linear combination," "span," and "basis." These concepts form the foundation for a deep understanding of linear algebra and allow us to understand how vectors "fill" space.

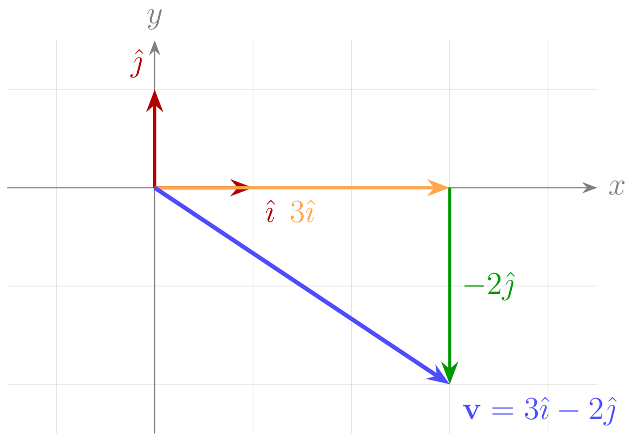
2.1 A New Look at Coordinates

In the previous chapter, we introduced vector coordinates as "directions for movement." But there's another perspective that is very important.

Geometric Intuition

When we write $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, instead of thinking "3 units right, 2 units down," think of it this way:

- The number 3 is a **scalar** that scales vector $\hat{\imath}$
- The number -2 is a **scalar** that scales vector $\hat{\jmath}$
- The final vector is the **sum** of these two scaled vectors



2.2 Linear Combination

Definition 2.1 (Linear Combination)**.** A **linear combination** of two vectors $\mathbf{v}$ and $\mathbf{w}$ is:

$$a\mathbf{v} + b\mathbf{w}$$

where a and b are arbitrary scalars.

More generally, a linear combination of n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i$$

Geometric Intuition

Where does the name "linear" come from? If you fix one of the scalars and vary the other, the tip of the resulting vector moves along a **straight line**.
For example, if $b = 1$ is fixed and a varies, the vector $a\mathbf{v} + \mathbf{w}$ moves along a line parallel to $\mathbf{v}$.

Example 2.1. Suppose $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Some linear combinations:

$$2\mathbf{v} + 1\mathbf{w} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$-1\mathbf{v} + 3\mathbf{w} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

$$0\mathbf{v} + 0\mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

2.3 Span

Definition 2.2 (Span)**.** The **span** of a set of vectors is the set of all possible linear combinations of those vectors:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_i \in \mathbb{R}\}$$

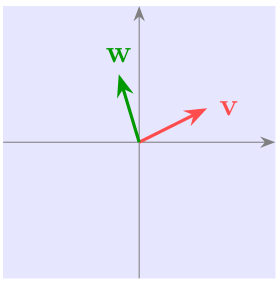
Geometric Intuition

The key question: Given some vectors and using only the two basic operations (addition and scalar multiplication), what vectors can we reach?
The answer: The **span** of those vectors.

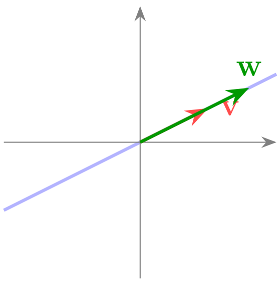
2.3.1 Span in Two Dimensions

Theorem 2.1. *For two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^2$*, there are three possible cases:*

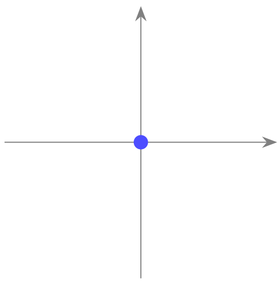
1. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are not collinear:* $\text{span}\{\mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$ *(the entire plane)*
2. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are collinear (but non-zero):* $\text{span}\{\mathbf{v}, \mathbf{w}\}$ *is a line*
3. *If both are zero:* $\text{span}\{\mathbf{v}, \mathbf{w}\} = \{\mathbf{0}\}$ *(only the origin)*



Case 1: Entire plane



Case 2: A line


Case 3: Only origin

2.3.2 Span in Three Dimensions

Theorem 2.2. *For vectors in* $\mathbb{R}^3$*:*

- ***One non-zero vector:*** *The span is a* ***line***
- ***Two non-collinear vectors:*** *The span is a* ***plane***
- ***Three vectors not in one plane:*** *The span is all of* $\mathbb{R}^3$

Geometric Intuition

Imagine two vectors in three-dimensional space. Their linear combinations form a flat plane through the origin. Now if you add a third vector that's not on this plane, it's like "sweeping" the plane through space, covering all of 3D space.

2.4 Linear Independence and Dependence

Definition 2.3 (Linear Dependence)**.** A set of vectors is **linearly dependent** if one of them can be removed without changing the span. In other words, at least one vector is "redundant."

Mathematically: vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent if and only if there exist scalars $c_1, \ldots, c_n$, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

Definition 2.4 (Linear Independence)**.** A set of vectors is **linearly independent** if none of them is redundant—that is, each vector adds a new dimension to the span.

Mathematically: vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if and only if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \implies c_1 = c_2 = \cdots = c_n = 0$$

Example 2.2. Consider vectors $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

These vectors are **linearly dependent** because $\mathbf{w} = 2\mathbf{v}$. We can write:

$$2\mathbf{v} - \mathbf{w} = \mathbf{0} \quad \text{i.e.,} \quad 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 2.3. The vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent.

If $a\mathbf{v} + b\mathbf{w} = \mathbf{0}$:

$$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $a = 0$ and $b = 0$. The only way to get the zero vector is if all coefficients are zero.

Warning

In n-dimensional space, at most n vectors can be linearly independent. If you have more than n vectors, they must be linearly dependent.

2.5 Basis

Definition 2.5 (Basis)**.** A basis of a vector space is a set of vectors that:

1. Are **linearly independent**
2. **Span** the entire space

Theorem 2.3. *The standard basis of $\mathbb{R}^2$ is:*

$$\left\{ \hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

And the standard basis of $\mathbb{R}^3$ is:

$$\left\{ \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Geometric Intuition

A basis is like a "language" for describing vectors. When we say $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, we're really saying "3 of the first basis vector and 2 of the second."
If we choose a different basis, the same vector will have different coordinates—like translating a sentence to another language.

2.5.1 Non-Standard Bases

Example 2.4. The following set is also a basis for $\mathbb{R}^2$:

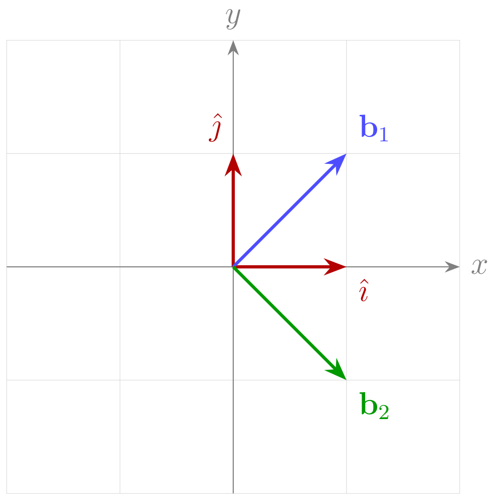
$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Proof of linear independence: If $a\mathbf{b}_1 + b\mathbf{b}_2 = \mathbf{0}$:

$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $a + b = 0$ and $a - b = 0$, which gives $a = b = 0$.

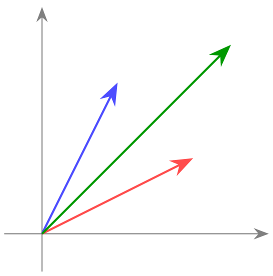
Spans the entire plane: Since we have two linearly independent vectors in $\mathbb{R}^2$, they span the entire plane.



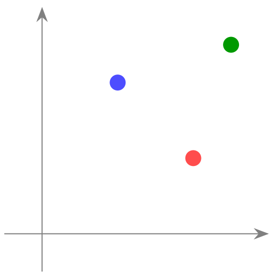
2.6 Vectors as Points

Remark 2.1. Sometimes instead of thinking of a vector as an arrow, it's easier to think of it as a **point**—the point where the tip of the vector lands.

- When thinking about a specific vector: visualize it as an **arrow**
- When thinking about a collection of vectors: visualize them as **points**



Vectors as arrows



Vectors as points

2.7 Summary of Key Concepts

Summary

Linear combination: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ with arbitrary scalars

Span: Set of all possible linear combinations

Linear independence: No vector is redundant

Linear dependence: At least one vector can be removed

Basis: Linearly independent set that spans the entire space

2.8 Exercises

Exercise 2.1. Write vector $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ as a linear combination of $\hat{\imath}$ and $\hat{\jmath}$.

Exercise 2.2. Are the following vectors linearly independent? Explain.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Exercise 2.3. Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hint: In $\mathbb{R}^2$, what is the maximum number of linearly independent vectors?

Exercise 2.4. Describe the span of the following vectors:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 2.5 (Challenge)**.** Show that the following set is a basis for $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

Then find the coordinates of vector $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ in this new basis.

Exercise 2.6. Three vectors in $\mathbb{R}^3$ are given:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Are these vectors linearly independent? What is their span?

Problem 2.1. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$, then $\{2\mathbf{v}_1, 3\mathbf{v}_2\}$ is also a basis.

Problem 2.2. Is it possible to find three vectors in $\mathbb{R}^2$ that are linearly independent? Why or why not?

Chapter 3

Linear Transformations and Matrices

Linear transformations are the heart of linear algebra. In this chapter, we learn how any linear transformation can be represented by a matrix, and how matrix multiplication relates to composing transformations.

3.1 What is a Transformation?

Definition 3.1 (Transformation)**.** A **transformation** is a function that takes vectors to other vectors:

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

Each input vector is mapped to an output vector.

Geometric Intuition

We use the word "transformation" instead of "function" because we want to think about **movement**. Imagine every vector in space being "moved" to a new location. Picture the grid lines of a coordinate system. A transformation stretches, compresses, rotates, or otherwise changes this grid.

3.2 Linear Transformation

Definition 3.2 (Linear Transformation)**.** A transformation T is a **linear transformation** if and only if these two conditions hold:

1. **Additivity:** $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$
2. **Homogeneity:** $T(c\mathbf{v}) = c \cdot T(\mathbf{v})$

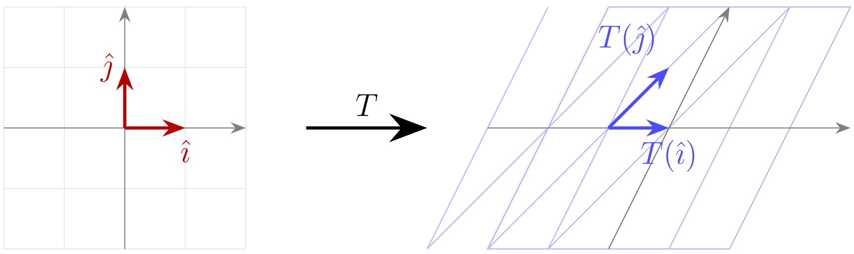
for any vectors $\mathbf{v}, \mathbf{w}$ and any scalar c.

Geometric Intuition

A linear transformation can be recognized by two geometric properties:

1. **Straight lines remain straight** (they don't curve)
2. **The origin stays fixed**

If grid lines remain parallel and evenly spaced after the transformation, it's linear.



Before transformation   After transformation

3.3 The Matrix of a Linear Transformation

Theorem 3.1 (Fundamental Theorem)**.** *Every linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is completely determined by knowing where T sends the basis vectors $\hat{\imath}$ and $\hat{\jmath}$.*

Proof. Any vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as: $\mathbf{v} = x\hat{\imath} + y\hat{\jmath}$

Using linearity:

$$T(\mathbf{v}) = T(x\hat{\imath} + y\hat{\jmath}) = xT(\hat{\imath}) + yT(\hat{\jmath})$$

So if we know $T(\hat{\imath})$ and $T(\hat{\jmath})$, we can compute $T(\mathbf{v})$ for any $\mathbf{v}$. □

Definition 3.3 (Transformation Matrix)**.** If $T(\hat{\imath}) = \begin{bmatrix} a \\ c \end{bmatrix}$ and $T(\hat{\jmath}) = \begin{bmatrix} b \\ d \end{bmatrix}$, the **transformation matrix** of T is:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The first column is where $\hat{\imath}$ lands, and the second column is where $\hat{\jmath}$ lands.

Important

Golden Rule: The columns of a matrix = where the basis vectors land

3.4 Matrix-Vector Multiplication

Definition 3.4 (Matrix Times Vector)**.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Geometric Intuition

Multiplying a matrix by a vector means:

1. Multiply the x component of the input by the first column
2. Multiply the y component of the input by the second column
3. Add the results

This is exactly the linear combination of the new basis vectors!

Example 3.1. Rotation by 90 counterclockwise:

$$T(\hat{\imath}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T(\hat{\jmath}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So the rotation matrix is:

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Check: $\mathbf{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

3.5 Examples of Important Transformations

3.5.1 Rotation

Definition 3.5 (Rotation Matrix)**.** Rotation by angle θ counterclockwise:

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example 3.2. Rotation by 45:

$$\mathbf{R}_{45} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

3.5.2 Scaling

Definition 3.6 (Scaling Matrix)**.** Scaling by factors s_x along x and s_y along y:

$$\mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Practical Application

Application in Computer Graphics: To enlarge or shrink an image:

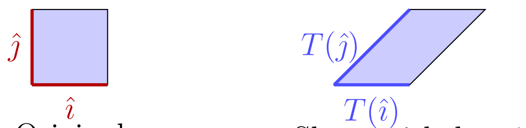
- $s_x = s_y = 2$: Image doubles in size
- $s_x = 1, s_y = 0.5$: Image is compressed vertically

3.5.3 Shear

Definition 3.7 (Horizontal Shear Matrix)**.**

$$\mathbf{H} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

where k is the shear amount.



Original   Shear with $k = 1$

3.5.4 Reflection

Example 3.3. Reflection across the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection across the line $y = x$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3.6 Matrix Multiplication = Composing Transformations

Theorem 3.2 (Composition of Transformations)**.** *If* $\mathbf{A}$ *is the matrix of transformation* T_1 *and* $\mathbf{B}$ *is the matrix of transformation* T_2*, then:*

$$\mathbf{B} \cdot \mathbf{A} = \textit{matrix of } (T_2 \circ T_1)$$

meaning first apply T_1*, then apply* T_2*.*

Warning

Order matters! $\mathbf{AB} \neq \mathbf{BA}$ in general.
Rotation then shear $\neq$ Shear then rotation

Definition 3.8 (Matrix Multiplication)**.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Geometric Intuition

To compute the columns of $\mathbf{BA}$:

- First column: $\mathbf{B}$ times the first column of $\mathbf{A}$ = where $\hat{\imath}$ lands after both transformations
- Second column: $\mathbf{B}$ times the second column of $\mathbf{A}$ = where $\hat{\jmath}$ lands after both transformations

Example 3.4. Rotation by 90 followed by shear:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Check: $\hat{\imath} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

3.7 Practical Applications

Practical Application

Computer Graphics and Video Games

Every object in a game is described by a set of vectors (vertices). To move, rotate, or resize an object, we just multiply all vertices by the appropriate matrix.

An animation = a sequence of matrix multiplications

Practical Application

Image Processing

Image filters like blur, sharpen, and edge detection are implemented using matrix multiplication.

3.8 Exercises

Exercise 3.1. Write the matrix of the transformation that takes $\hat{\imath}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\hat{\jmath}$ to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Exercise 3.2. Compute the following product:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Exercise 3.3. Write the matrix for rotation by 180 and show that it equals $-\mathbf{I}$.

Exercise 3.4. Multiply the following two matrices (in both orders) and show that $\mathbf{AB} \neq \mathbf{BA}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Exercise 3.5 (Challenge)**.** Find the matrix for reflection across the line $y = 2x$.

Hint: This line makes an angle of $\arctan(2)$ *with the x-axis.*

Problem 3.1. Show that the composition of two rotations by angles α and β equals a rotation by angle $\alpha + \beta$.

Problem 3.2. If $\mathbf{A}^2 = \mathbf{A}$ (i.e., $\mathbf{A}$ is a projection matrix), what is the geometric interpretation?

Chapter 4

3D Transformations and Determinants

The determinant is a number that measures how much a linear transformation "stretches" or "compresses" space. In this chapter, we learn the geometric meaning of determinants and how to compute them.

4.1 Transformations in Three-Dimensional Space

Definition 4.1 (3D Transformation Matrix)**.** A linear transformation in $\mathbb{R}^3$ is represented by a 3×3 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The columns are where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ land.

Geometric Intuition

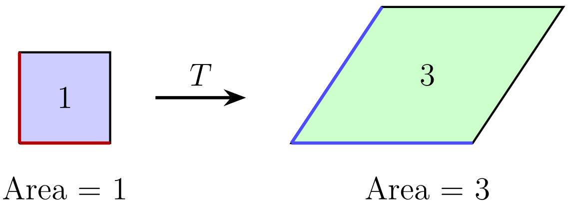
The same logic from 2D applies in 3D. Imagine a three-dimensional grid being stretched, compressed, rotated, or sheared. Lines still remain straight and the origin stays fixed.

4.2 Determinant: Geometric Meaning

Definition 4.2 (Determinant - Geometric Definition)**.** The **determinant** of a matrix is the factor by which the corresponding transformation scales area (in 2D) or volume (in 3D).

If $\mathbf{A}$ is the matrix of transformation T:

$$\det(\mathbf{A}) = \frac{\text{area/volume after transformation}}{\text{area/volume before transformation}}$$



Important

Determinant = scaling factor for area/volume

If $\det(\mathbf{A}) = 3$, every shape becomes 3 times larger after the transformation.

4.3 The Sign of the Determinant

Theorem 4.1 (Meaning of the Sign)**.**

- $\det(\mathbf{A}) > 0$: *Orientation of space is preserved*
- $\det(\mathbf{A}) < 0$: *Orientation of space is reversed (like a mirror)*
- $\det(\mathbf{A}) = 0$: *Space is collapsed to a lower dimension*

Geometric Intuition

In 2D, if $\hat{\jmath}$ is to the left of $\hat{\imath}$, the orientation is "positive." If a transformation reverses this relationship (like a reflection), the determinant becomes negative.

In 3D, use the right-hand rule: if curling fingers from $\hat{\imath}$ to $\hat{\jmath}$ makes the thumb point toward $\hat{k}$, the orientation is positive.

4.4 Computing the Determinant

4.4.1 Determinant of a 2×2 Matrix

Definition 4.3.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example 4.1.

$$\det \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = 3 \times 2 - 1 \times 0 = 6$$

Every shape's area is multiplied by 6.

4.4.2 Determinant of a 3×3 Matrix

Definition 4.4 (Expansion Formula)**.**

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Geometric Intuition

The 3×3 determinant equals the volume of the parallelepiped formed by the three column vectors of the matrix (with sign).

Example 4.2.

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{aligned} &= 1(5 \times 9 - 6 \times 8) - 2(4 \times 9 - 6 \times 7) + 3(4 \times 8 - 5 \times 7) \\ &= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\ &= 1(-3) - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 = 0 \end{aligned}$$

A zero determinant means the three columns lie in the same plane!

4.5 Properties of the Determinant

Theorem 4.2 (Main Properties)**.** *1.* $\det(\mathbf{I}) = 1$

2. $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$

3. $\det(\mathbf{A}^\mathsf{T}) = \det(\mathbf{A})$

4. $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$ *for an $n \times n$ matrix*

5. If a row/column is all zeros: $\det(\mathbf{A}) = 0$

6. If two rows/columns are equal: $\det(\mathbf{A}) = 0$

Geometric Intuition

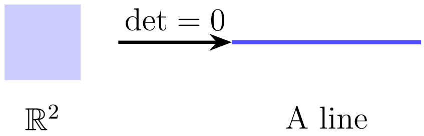
The property $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ is very important:
If $\mathbf{A}$ scales area by 3 and $\mathbf{B}$ scales area by 2, their composition scales area by

$3 \times 2 = 6$.

4.6 Zero Determinant: Collapsing Space

Theorem 4.3. $\det(\mathbf{A}) = 0$ *if and only if:*

- *The columns of* $\mathbf{A}$ *are linearly dependent*
- *The corresponding transformation collapses space to a lower dimension*



Practical Application

Testing Linear Independence:
Want to know if three vectors in space are linearly independent? Put them as columns of a matrix. If $\det \neq 0$, they're independent.

4.7 Practical Examples

Practical Application

Computing Triangle Area from Vertices
If a triangle has vertices (x_1, y_1), (x_2, y_2), (x_3, y_3):

$$\text{Area} = \frac{1}{2}\left|\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|$$

Practical Application

Physics: Torque and Force
Determinants are used in computing cross products, which appear in calculations of torque, angular momentum, and electromagnetic fields.

4.8 Exercises

Exercise 4.1. Compute the determinant of the following matrices:

(a) $\begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

Exercise 4.2. If $\det(\mathbf{A}) = 5$ and $\det(\mathbf{B}) = -2$, what is $\det(\mathbf{AB})$?

Exercise 4.3. Compute the area of the parallelogram with vertices $(0, 0)$, $(3, 1)$, $(1, 4)$, $(4, 5)$.

Exercise 4.4. Compute the determinant of the rotation matrix for angle θ and interpret the result.

Exercise 4.5 (Challenge)**.** Prove that for any matrix $\mathbf{A}$:

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

(Assume $\mathbf{A}$ is invertible)

Problem 4.1. Compute the volume of the parallelepiped formed by the following vectors:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Chapter 5

Inverse Matrices, Column Space, and Null Space

In this chapter, we learn about inverse matrices, solving systems of linear equations, and important spaces associated with matrices (column space, null space, and rank).

5.1 Systems of Linear Equations

Definition 5.1 (System of Equations in Matrix Form)**.** A system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

can be written as $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Geometric Intuition

Interpret the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ as:

"What vector $\mathbf{x}$ lands on $\mathbf{b}$ after applying transformation $\mathbf{A}$?"

In other words: "reverse" the transformation $\mathbf{A}$ to get from $\mathbf{b}$ to $\mathbf{x}$.

5.2 Inverse Matrix

Definition 5.2 (Inverse Matrix)**.** The matrix $\mathbf{A}^{-1}$ is the **inverse** of matrix $\mathbf{A}$ if:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix.

Geometric Intuition

If $\mathbf{A}$ is a transformation, $\mathbf{A}^{-1}$ is the transformation that undoes the effect of $\mathbf{A}$:

- If $\mathbf{A}$ is rotation by 90, then $\mathbf{A}^{-1}$ is rotation by -90
- If $\mathbf{A}$ scales by 2, then $\mathbf{A}^{-1}$ scales by $\frac{1}{2}$

Theorem 5.1 (Solving Systems with the Inverse)**.** *If* $\mathbf{A}$ *is invertible, the unique solution to* $\mathbf{Ax} = \mathbf{b}$ *is:*

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

5.2.1 Formula for 2×2 Inverse

Theorem 5.2. *If* $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\det(\mathbf{A}) \neq 0$*:*

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 5.1.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \quad \det(\mathbf{A}) = 12 - 2 = 10$$

$$\mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

5.3 When Does the Inverse Exist?

Theorem 5.3. *Matrix* $\mathbf{A}$ *is invertible if and only if* $\det(\mathbf{A}) \neq 0$.

Geometric Intuition

If $\det(\mathbf{A}) = 0$, the transformation $\mathbf{A}$ collapses space (e.g., plane to line). Such a transformation is not reversible—information is lost.
It's like adding several numbers together: you can't recover the original numbers from just the sum.

5.4 Column Space

Definition 5.3 (Column Space)**.** The **column space** of matrix $\mathbf{A}$, denoted $\text{Col}(\mathbf{A})$, is the span of the columns of $\mathbf{A}$:

$$\text{Col}(\mathbf{A}) = \text{span}\{\text{columns of } \mathbf{A}\}$$

Geometric Intuition

Column space = set of all possible outputs of transformation $\mathbf{A}$
Question: "Does $\mathbf{Ax} = \mathbf{b}$ have a solution?" is equivalent to "Is $\mathbf{b}$ in the column space of $\mathbf{A}$?"

Example 5.2.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

The second column = 3 times the first column, so:

$$\text{Col}(\mathbf{A}) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \text{a line through the origin}$$

5.5 Null Space

Definition 5.4 (Null Space)**.** The **null space** (or kernel) of matrix $\mathbf{A}$ is the set of all vectors that $\mathbf{A}$ sends to zero:

$$\text{Null}(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\}$$

Geometric Intuition

Null space = vectors that transformation $\mathbf{A}$ "squishes" to zero
If $\det(\mathbf{A}) \neq 0$: only the zero vector gets squished, so $\text{Null}(\mathbf{A}) = \{\mathbf{0}\}$
If $\det(\mathbf{A}) = 0$: an entire line or plane gets compressed to the origin

Example 5.3. For matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$:

Solve $\mathbf{Ax} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equation: $x + 2y = 0$, so $x = -2y$

Null space: $\text{Null}(\mathbf{A}) = \left\{ t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$ (a line)

5.6 Rank

Definition 5.5 (Rank)**.** The **rank** of matrix $\mathbf{A}$ equals:

- The dimension of the column space
- The number of linearly independent columns
- The number of linearly independent rows

$$\text{rank}(\mathbf{A}) = \dim(\text{Col}(\mathbf{A}))$$

Theorem 5.4 (Rank-Nullity Theorem)**.** *For an $m \times n$ matrix:*

$$\text{rank}(\mathbf{A}) + \dim(\text{Null}(\mathbf{A})) = n$$

(the number of columns)

Geometric Intuition

Rank = dimensions that the transformation preserves
$\dim(\text{Null}(\mathbf{A}))$ = dimensions that are lost
Their sum = dimension of the input space

5.7 Non-Square Matrices

Definition 5.6 (Transformations Between Dimensions)**.** An $m \times n$ matrix represents a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$:

- $m > n$: transformation from lower to higher dimension
- $m < n$: transformation from higher to lower dimension

Example 5.4. A 2×3 matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} 1 - 1$$

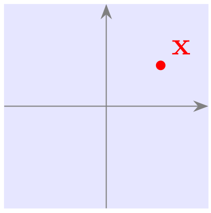
is a transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. It "projects" 3D space onto a plane.

5.8 Different Cases for Systems of Equations

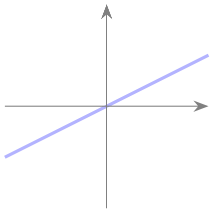
Theorem 5.5 (Analysis of $\mathbf{Ax} = \mathbf{b}$)**.** *1.* ***Unique solution:*** *If* $\det(\mathbf{A}) \neq 0$

2. ***Infinitely many solutions:*** *If* $\det(\mathbf{A}) = 0$ *and* $\mathbf{b} \in \text{Col}(\mathbf{A})$

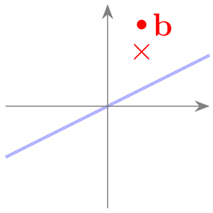
3. ***No solution:*** *If* $\mathbf{b} \notin \mathrm{Col}(\mathbf{A})$



Unique solution
$\det \neq 0$



Infinite solutions
$\det = 0$, $\mathbf{b} \in \mathrm{Col}$



No solution
$\mathbf{b} \notin \mathrm{Col}$

5.9 Exercises

Exercise 5.1. Compute the inverse of:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

Exercise 5.2. Solve the following system using the inverse matrix:

$$\begin{cases} 2x + 3y = 7 \\ x + 2y = 4 \end{cases}$$

Exercise 5.3. Find the null space of:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Exercise 5.4. Determine the rank of:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 5.5 (Challenge)**.** Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (if the inverses exist).

Problem 5.1. For what values of k is the following matrix not invertible?

$$\mathbf{A} = \begin{bmatrix} k & 2 \\ 3 & k \end{bmatrix}$$

Chapter 6

Dot Products and Duality

The dot product is one of the fundamental operations in linear algebra. In this chapter, we learn its definition, properties, and the deep concept of duality that connects vectors to linear functions.

6.1 Definition of the Dot Product

Definition 6.1 (Dot Product)**.** The **dot product** of two vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

In n-dimensional space:

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

Important

The dot product of two vectors is a **number** (scalar), not a vector!

Example 6.1.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 3 \times 1 + 2 \times 4 = 3 + 8 = 11$$

6.2 Geometric Interpretation

Theorem 6.1 (Geometric Formula for Dot Product)**.**

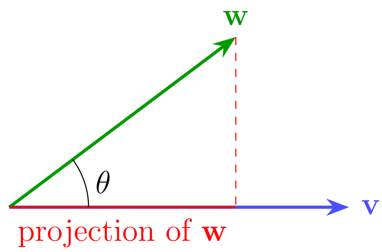
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$$

where θ is the angle between the two vectors.

Geometric Intuition

The dot product can be interpreted two ways:

1. Length of $\mathbf{v}$ times the length of the projection of $\mathbf{w}$ onto $\mathbf{v}$
2. Length of $\mathbf{w}$ times the length of the projection of $\mathbf{v}$ onto $\mathbf{w}$



6.3 Properties of the Dot Product

Theorem 6.2 (Main Properties)**.** *1.* ***Commutative:*** $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

2. ***Distributive:*** $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$

3. ***Scalar multiplication:*** $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$

4. ***Positive definiteness:*** $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$

Definition 6.2 (Length of a Vector)**.**

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

6.4 Orthogonality

Theorem 6.3 (Condition for Orthogonality)**.** *Two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are* ***orthogonal*** *(perpendicular) if and only if:*

$$\mathbf{v} \cdot \mathbf{w} = 0$$

Geometric Intuition

If $\mathbf{v} \cdot \mathbf{w} = 0$, then $\cos\theta = 0$, so $\theta = 90$.
The projection of any vector onto a perpendicular vector is zero.

Example 6.2. Vectors $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are orthogonal because:

$$\mathbf{v} \cdot \mathbf{w} = 3 \times 2 + 2 \times (-3) = 6 - 6 = 0$$

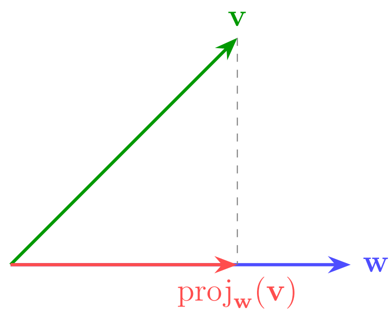
6.5 Vector Projection

Definition 6.3 (Projection of a Vector)**.** The projection of vector $\mathbf{v}$ onto vector $\mathbf{w}$:

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$$

Geometric Intuition

The projection of $\mathbf{v}$ onto $\mathbf{w}$ is a vector in the direction of $\mathbf{w}$ that represents the "shadow" of $\mathbf{v}$ on the line of $\mathbf{w}$.



6.6 Duality

Geometric Intuition

A deep insight: every $1 \times n$ row vector can be thought of as a **linear function** that takes n-dimensional vectors to numbers.
For example, $\begin{bmatrix} 2 & 1 \end{bmatrix}$ is a linear function:

$$\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x + y$$

Theorem 6.4 (Duality)**.** *Every linear function* $f : \mathbb{R}^n \to \mathbb{R}$ *can be written as a dot product with a fixed vector:*

$$f(\mathbf{x}) = \mathbf{v}\cdot\mathbf{x}$$

for a unique vector $\mathbf{v}$*.*

Definition 6.4 (Dual Vector)**.** The vector $\mathbf{v}$ that represents a linear function f is called the **dual vector** of that function.

Practical Application

Application in Machine Learning:
In neural networks, each neuron computes a linear function of its inputs:

$$\text{output} = w_1x_1 + w_2x_2 + \cdots + w_nx_n = \mathbf{w} \cdot \mathbf{x}$$

The weights $\mathbf{w}$ are the dual vector of that neuron.

6.7 Practical Applications

Practical Application

Physics: Mechanical Work
Work done by force $\vec{F}$ over displacement $\vec{d}$:

$$W = \vec{F} \cdot \vec{d} = \left\|\vec{F}\right\| \left\|\vec{d}\right\| \cos\theta$$

If the force is perpendicular to the direction of motion, no work is done!

Practical Application

Computer Graphics: Lighting Calculations
The intensity of light reflected from a surface:

$$I = \max(0, \vec{n} \cdot \vec{l})$$

where $\vec{n}$ is the surface normal and $\vec{l}$ is the light direction.

Practical Application

Text Similarity
To compare two text documents, convert each to a vector (e.g., TF-IDF) and compute cosine similarity:

$$\text{similarity} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \cos\theta$$

6.8 Exercises

Exercise 6.1. Compute the dot product of:

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

Exercise 6.2. Are vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ orthogonal?

Exercise 6.3. Find the projection of $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ onto $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Exercise 6.4. Calculate the angle between vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Exercise 6.5 (Challenge)**.** Show that for any two vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2$$

Problem 6.1. A force of magnitude 10 Newtons acts at an angle of 60 to the horizontal on an object. If the object moves 5 meters horizontally, how much work is done?

Chapter 7

Cross Products and Applications

The cross product (vector product) is an operation that creates a vector perpendicular to two vectors in three-dimensional space. In this chapter, we study its definition, properties, and deep connection to determinants.

7.1 Cross Product in Two Dimensions

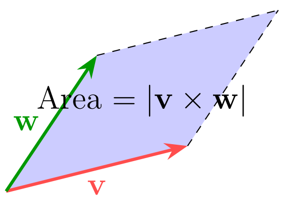
Definition 7.1 (2D Cross Product)**.** For two vectors in $\mathbb{R}^2$:

$$\mathbf{v} \times \mathbf{w} = v_1 w_2 - v_2 w_1 = \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$$

The result is a **number** (not a vector).

Geometric Intuition

2D cross product = signed area of the parallelogram formed by the two vectors
Positive sign: $\mathbf{w}$ is to the left of $\mathbf{v}$
Negative sign: $\mathbf{w}$ is to the right of $\mathbf{v}$



7.2 Cross Product in Three Dimensions

Definition 7.2 (3D Cross Product)**.** For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$:

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

Theorem 7.1 (Determinant Formula)**.**

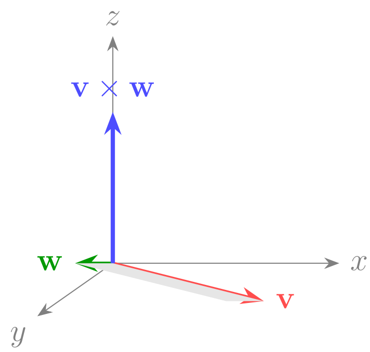
$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

(symbolic expansion along the first row)

Geometric Intuition

The cross product $\mathbf{v} \times \mathbf{w}$:

- **Direction:** Perpendicular to both vectors (right-hand rule)
- **Magnitude:** Area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$



7.3 Properties of the Cross Product

Theorem 7.2 (Main Properties)**.**

1. ***Anti-commutativity:*** $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$
2. ***Distributivity:*** $\mathbf{v} \times (\mathbf{w} + \mathbf{u}) = \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{u}$
3. ***Scalar multiplication:*** $(c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w})$
4. ***Orthogonality:*** $\mathbf{v} \times \mathbf{w} \perp \mathbf{v}$ *and* $\mathbf{v} \times \mathbf{w} \perp \mathbf{w}$

5. ***Self-cross is zero:*** $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Warning

The cross product is **not commutative**!

$$\mathbf{v} \times \mathbf{w} \neq \mathbf{w} \times \mathbf{v}$$

In fact, they are opposite to each other.

Theorem 7.3 (Magnitude Formula)**.**

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta$$

where θ is the angle between the two vectors.

7.4 Cross Product from the Linear Transformation Viewpoint

Geometric Intuition

A deeper view: the cross product can be defined through duality.
The function $f(\mathbf{x}) = \det[\mathbf{v} \,|\, \mathbf{w} \,|\, \mathbf{x}]$ is a linear function of $\mathbf{x}$. By duality, there must exist a vector such that:

$$f(\mathbf{x}) = \cdot\mathbf{x}$$

This is exactly $\mathbf{v} \times \mathbf{w}$!

Theorem 7.4.

$$\det[\mathbf{v} \,|\, \mathbf{w} \,|\, \mathbf{x}] = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{x}$$

7.5 Basis Vector Products

Theorem 7.5 (Cross Products of Basis Vectors)**.**

$$\begin{aligned} \hat{\imath} \times \hat{\jmath} &= \hat{k} & \qquad \hat{\jmath} \times \hat{\imath} &= -\hat{k} \\ \hat{\jmath} \times \hat{k} &= \hat{\imath} & \qquad \hat{k} \times \hat{\jmath} &= -\hat{\imath} \\ \hat{k} \times \hat{\imath} &= \hat{\jmath} & \qquad \hat{\imath} \times \hat{k} &= -\hat{\jmath} \end{aligned}$$

Example 7.1.

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

$$\begin{aligned}
&= (3 \times 7 - 4 \times 6)\hat{\imath} - (2 \times 7 - 4 \times 5)\hat{\jmath} + (2 \times 6 - 3 \times 5)\hat{k} \\
&= (21 - 24)\hat{\imath} - (14 - 20)\hat{\jmath} + (12 - 15)\hat{k} \\
&= -3\hat{\imath} + 6\hat{\jmath} - 3\hat{k} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}
\end{aligned}$$

7.6 Applications

Practical Application

Physics: Torque

Torque of force $\vec{F}$ about a point at distance $\vec{r}$:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The direction of torque is perpendicular to the plane containing the force and the lever arm.

Practical Application

Physics: Lorentz Force

Force on a charged particle moving in a magnetic field:

$$\vec{F} = q\vec{v} \times \vec{B}$$

Practical Application

Computer Graphics: Surface Normal

To find the normal vector to a triangular surface with vertices A, B, C:

$$\vec{n} = (\vec{B} - \vec{A}) \times (\vec{C} - \vec{A})$$

Practical Application

Computing Triangle Area

Area of triangle with vertices A, B, C:

$$\text{Area} = \frac{1}{2}\left\|(\vec{B}-\vec{A})\times(\vec{C}-\vec{A})\right\|$$

7.7 Triple Products

Definition 7.3 (Scalar Triple Product)**.**

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det[\mathbf{a}\,|\,\mathbf{b}\,|\,\mathbf{c}]$$

Result = signed volume of the parallelepiped formed by the three vectors

Theorem 7.6 (Cyclic Property)**.**

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

7.8 Exercises

Exercise 7.1. Compute the cross product:

$$\mathbf{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$$

Exercise 7.2. Find the area of the parallelogram with sides $\mathbf{a} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

Exercise 7.3. Show that $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Exercise 7.4. Find the normal vector to the plane passing through points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Exercise 7.5 (Challenge)**.** Show that:

$$\|\mathbf{a}\times\mathbf{b}\|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2$$

(This is called Lagrange's identity)

Problem 7.1. Calculate the volume of the parallelepiped with edges:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Chapter 8

Cramer's Rule

Cramer's rule is an elegant method for solving systems of linear equations using determinants. In this chapter, we learn the geometric interpretation of this rule and its conditions for use.

8.1 Statement of Cramer's Rule

Theorem 8.1 (Cramer's Rule)**.** *For the system* $\mathbf{Ax} = \mathbf{b}$ *with an* $n \times n$ *matrix and* $\det(\mathbf{A}) \neq 0$*:*

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

where $\mathbf{A}_i$ *is the matrix with column* i *of* $\mathbf{A}$ *replaced by vector* $\mathbf{b}$*.*

Example 8.1. For a 2×2 system:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

Solution:

$$x = \frac{\det \begin{bmatrix} e & b \\ f & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{ed - bf}{ad - bc}$$

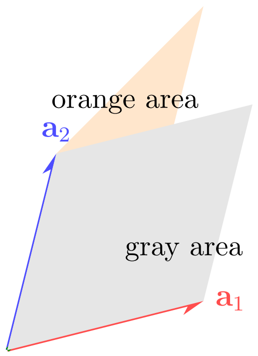
$$y = \frac{\det \begin{bmatrix} a & e \\ c & f \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{af - ec}{ad - bc}$$

8.2 Geometric Interpretation

Geometric Intuition

In two dimensions, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ asks:
"What linear combination of the columns of $\mathbf{A}$ equals $\mathbf{b}$?"
If $\mathbf{A} = [\mathbf{a}_1 \mid \mathbf{a}_2]$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$:

$$x\mathbf{a}_1 + y\mathbf{a}_2 = \mathbf{b}$$



Theorem 8.2 (Area Interpretation)**.**

$$x = \frac{\textit{Area of parallelogram}(\mathbf{b}, \mathbf{a}_2)}{\textit{Area of parallelogram}(\mathbf{a}_1, \mathbf{a}_2)}$$

$$y = \frac{\textit{Area of parallelogram}(\mathbf{a}_1, \mathbf{b})}{\textit{Area of parallelogram}(\mathbf{a}_1, \mathbf{a}_2)}$$

Geometric Intuition

Why does this work?
Consider the parallelogram $(\mathbf{b}, \mathbf{a}_2)$. Since $\mathbf{b} = x\mathbf{a}_1 + y\mathbf{a}_2$:

$$\text{Area}(\mathbf{b}, \mathbf{a}_2) = \text{Area}(x\mathbf{a}_1 + y\mathbf{a}_2, \mathbf{a}_2)$$

Since $\mathbf{a}_2 \times \mathbf{a}_2 = 0$:

$$= \text{Area}(x\mathbf{a}_1, \mathbf{a}_2) = x \cdot \text{Area}(\mathbf{a}_1, \mathbf{a}_2)$$

Therefore:

$$x = \frac{\text{Area}(\mathbf{b}, \mathbf{a}_2)}{\text{Area}(\mathbf{a}_1, \mathbf{a}_2)}$$

8.3 Computational Example

Example 8.2. Solve the system:

$$\begin{cases} 3x + 2y = 7 \\ x + 4y = 9 \end{cases}$$

Step 1: Compute $\det(\mathbf{A})$:

$$\det \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = 12 - 2 = 10$$

Step 2: Compute x:

$$x = \frac{\det \begin{bmatrix} 7 & 2 \\ 9 & 4 \end{bmatrix}}{\det(\mathbf{A})} = \frac{28 - 18}{10} = \frac{10}{10} = 1$$

Step 3: Compute y:

$$y = \frac{\det \begin{bmatrix} 3 & 7 \\ 1 & 9 \end{bmatrix}}{\det(\mathbf{A})} = \frac{27 - 7}{10} = \frac{20}{10} = 2$$

Check: $3(1) + 2(2) = 7$ ✓ and $1(1) + 4(2) = 9$ ✓

8.4 Three Dimensions and Higher

Example 8.3. A 3×3 system:

$$\begin{cases} 2x + y - z = 3 \\ x - y + 2z = 1 \\ 3x + 2y + z = 4 \end{cases}$$

Coefficient matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$x = \frac{\det \begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 4 & 2 & 1 \end{bmatrix}}{\det(\mathbf{A})}, \quad y = \frac{\det \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix}}{\det(\mathbf{A})}, \quad z = \frac{\det \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 2 & 4 \end{bmatrix}}{\det(\mathbf{A})}$$

Geometric Intuition

In three dimensions, instead of ratios of areas, we have ratios of **volumes**.
x = ratio of volume of parallelepiped $(\mathbf{b}, \mathbf{a}_2, \mathbf{a}_3)$ to volume of $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$

8.5 Limitations and Applications

Warning

Cramer's rule:

- Only works for square systems (n equations, n unknowns)
- Only when $\det(\mathbf{A}) \neq 0$ (unique solution)
- For large n, computationally **very expensive**

Remark 8.1. In practice, for large systems, methods like Gaussian elimination are used which are more efficient. However, Cramer's rule:

- Provides deeper theoretical understanding
- Is useful for analytical formulas
- Is used in theorem proofs

Practical Application

Application: Finding Line Intersections
Two lines $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ intersect at:

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

This formula comes directly from Cramer's rule.

8.6 Exercises

Exercise 8.1. Solve using Cramer's rule:

$$\begin{cases} 2x + 3y = 8 \\ 4x - y = 2 \end{cases}$$

Exercise 8.2. Solve using Cramer's rule:

$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - z = 2 \end{cases}$$

Exercise 8.3. Find the intersection point of lines $2x + 3y = 7$ and $x - y = 1$ using Cramer's rule.

Exercise 8.4. Explain the geometric interpretation of Cramer's rule when $\det(\mathbf{A}) = 0$.

Exercise 8.5 (Challenge)**.** Show that the Cramer formula is consistent with computing $\mathbf{A}^{-1}\mathbf{b}$.

Problem 8.1. A triangle with vertices $A(1,1)$, $B(4,2)$, $C(2,5)$ is given. Using Cramer's rule, find the coordinates of the centroid.

Chapter 9

Change of Basis

The same vector has different coordinates in different bases. In this chapter, we learn how to translate between different coordinate systems and how this concept relates to matrices.

9.1 Coordinates Relative to Different Bases

Geometric Intuition

The coordinates of a vector depend on the chosen basis. If you speak a different language, you express the same concept differently.

Example: A vector $\mathbf{v}$ that is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in the standard basis might be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in a different basis!

Definition 9.1 (Coordinates in a Basis)**.** If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis, the coordinates of vector $\mathbf{v}$ in this basis are the coefficients c_1, c_2 such that:

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$$

Notation: $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Example 9.1. Basis $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Express vector $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in this basis.

We need to find c_1, c_2 such that:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{cases} 2c_1 - c_2 = 3 \\ c_1 + c_2 = 2 \end{cases}$$

Solution: $c_1 = \frac{5}{3}$, $c_2 = \frac{1}{3}$

So $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$

9.2 Change of Basis Matrix

Definition 9.2 (Change of Basis Matrix)**.** The **change of basis matrix** from basis $\mathcal{B}$ to the standard basis is a matrix whose columns are the basis vectors of $\mathcal{B}$:

$$\mathbf{P} = [\mathbf{b}_1 \,|\, \mathbf{b}_2 \,|\, \cdots \,|\, \mathbf{b}_n]$$

Theorem 9.1.

$$\mathbf{v} = \mathbf{P}\,[\mathbf{v}]_{\mathcal{B}}$$

That is: coordinates in basis $\mathcal{B}$ $\times$ change of basis matrix = vector in standard basis

Theorem 9.2.

$$[\mathbf{v}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{v}$$

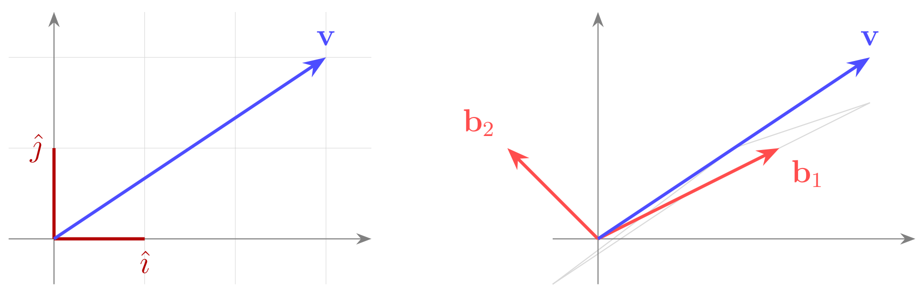
That is: to convert from standard basis to basis $\mathcal{B}$, we use the inverse.

9.3 Change of Basis Transformation

Geometric Intuition

Matrix $\mathbf{P}$ is an identity transformation that only changes the language:

- $\mathbf{P}$: translates from language $\mathcal{B}$ to standard language
- $\mathbf{P}^{-1}$: translates from standard language to language $\mathcal{B}$



Standard basis
$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Basis $\mathcal{B}$
$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} ? \\ ? \end{bmatrix}$

9.4 Representing Transformations in Different Bases

Theorem 9.3 (Transformation in New Basis)**.** *If* $\mathbf{A}$ *is the matrix of transformation* T *in the standard basis, the matrix of the same transformation in basis* $\mathcal{B}$ *is:*

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Geometric Intuition

This formula has three steps:

1. $\mathbf{P}$: translate from basis $\mathcal{B}$ to standard basis
2. $\mathbf{A}$: apply the transformation in standard basis
3. $\mathbf{P}^{-1}$: translate the result back to basis $\mathcal{B}$

$$\begin{tikzcd} {[\mathbf{v}]_{\mathcal{B}}} \arrow[r, "\mathbf{P}"] \arrow[rrr, bend right, "\mathbf{P}^{-1}\mathbf{A}\mathbf{P}"'] & \mathbf{v} \arrow[r, "\mathbf{A}"] & T(\mathbf{v}) \arrow[r, "\mathbf{P}^{-1}"] & {[T(\mathbf{v})]_{\mathcal{B}}} \end{tikzcd}$$

9.5 Importance of Change of Basis

Geometric Intuition

Why is change of basis important?
Some transformations look **simpler** in certain bases. For example:

- Rotation in the standard basis is complicated
- But in a basis where one axis lies on the axis of rotation, it becomes simpler

The best basis for a transformation? The **eigenbasis** - next chapter!

Practical Application

Application: Simplifying Computations
Suppose you want to compute $\mathbf{A}^{100}$. If in a suitable basis, $\mathbf{A}$ becomes diagonal:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} \quad \text{(diagonal)}$$

Then:

$$\mathbf{A}^{100} = \mathbf{P}\mathbf{D}^{100}\mathbf{P}^{-1}$$

And $\mathbf{D}^{100}$ is very easy to compute!

9.6 Exercises

Exercise 9.1. Given basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, find the coordinates of vector $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in this basis.

Exercise 9.2. Write the change of basis matrix from $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ to the standard basis.

Exercise 9.3. If $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ in the standard basis, find the matrix of this transformation in basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Exercise 9.4 (Challenge)**.** Show that $\det(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \det(\mathbf{A})$.

Problem 9.1. Given two bases $\mathcal{B}_1$ and $\mathcal{B}_2$. How do you compute the change of basis matrix directly from $\mathcal{B}_1$ to $\mathcal{B}_2$ (without going through the standard basis)?

Chapter 10

Eigenvectors and Eigenvalues

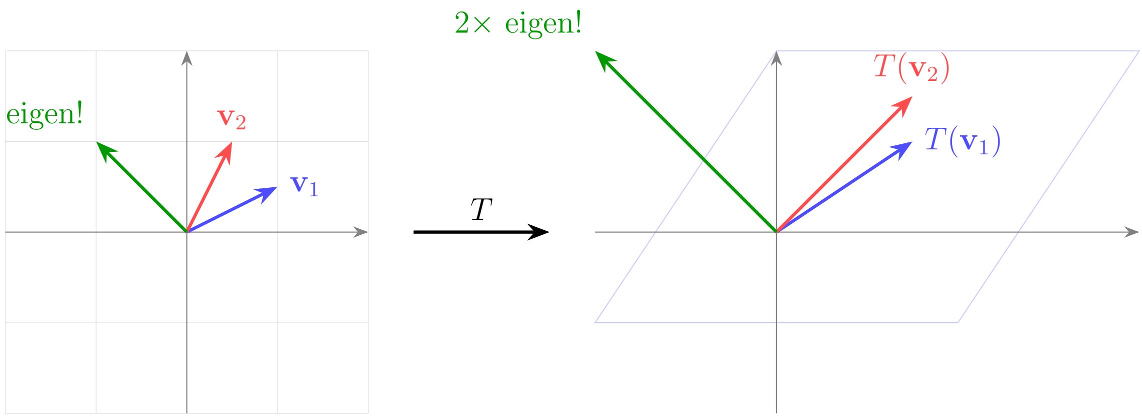
Eigenvectors are special directions that a linear transformation only scales without changing their direction. This concept is one of the most important ideas in linear algebra with broad applications in physics, engineering, and data science.

10.1 Motivation: Special Vectors

Geometric Intuition

When you apply a linear transformation, most vectors get knocked off their original direction. But some special vectors only get stretched or compressed and **stay on the same line**.

These special vectors are called **eigenvectors**.



Before transformation

After transformation

10.2 Formal Definition

Definition 10.1 (Eigenvector and Eigenvalue)**.** A nonzero vector $\mathbf{v}$ is an **eigenvector** of matrix $\mathbf{A}$ if:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ. This scalar is called the corresponding **eigenvalue**.

Geometric Intuition

$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ means:

"Transformation $\mathbf{A}$ acts on vector $\mathbf{v}$ as just a scalar multiplication"

Vector $\mathbf{v}$ stays on the same line, just scaled by λ.

Example 10.1. For matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$:

Vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\mathbf{v}$$

Corresponding eigenvalue: $\lambda = 3$

10.3 Finding Eigenvalues

Theorem 10.1 (Characteristic Equation)**.** *λ is an eigenvalue of $\mathbf{A}$ if and only if:*

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Proof. $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ can be written as:

$$\mathbf{A}\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \implies (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

This equation has a nonzero solution if and only if $\mathbf{A} - \lambda\mathbf{I}$ is not invertible, i.e., $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. □

Definition 10.2 (Characteristic Polynomial)**.** $\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial in λ called the **characteristic polynomial**. Its roots are the eigenvalues.

Example 10.2. For $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) - 0 = 0$$

Roots: $\lambda_1 = 3$, $\lambda_2 = 2$

10.4 Finding Eigenvectors

Theorem 10.2. *For each eigenvalue λ, the corresponding eigenvectors are found by solving the homogeneous system:*

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

Example 10.3. Continuing the previous example with $\lambda = 2$:

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}$$

Equation: $v_1 + v_2 = 0$, so $v_1 = -v_2$

Eigenvector: $\mathbf{v} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for any $t \neq 0$

10.5 Eigenspace

Definition 10.3 (Eigenspace)**.** The **eigenspace** corresponding to eigenvalue λ:

$$E_\lambda = \text{Null}(\mathbf{A} - \lambda\mathbf{I}) = \{\mathbf{v} \mid \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\}$$

Geometric Intuition

The eigenspace contains all vectors that transformation $\mathbf{A}$ only scales by factor λ. This space is always a vector subspace.

10.6 Geometric Interpretation

Practical Application

3D Rotation

For a rotation in $\mathbb{R}^3$, the eigenvector with $\lambda = 1$ is the **axis of rotation**! This vector stays fixed.

Describing rotation with axis and angle is much simpler than a 3×3 matrix.

Example 10.4. Shear matrix: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenvalue: $\lambda = 1$ (repeated)

Eigenvector: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (only one eigen-direction)

Interpretation: Shear keeps the x-axis fixed.

10.7 Special Cases

Theorem 10.3 (2D Rotation)**.** *The rotation matrix* $\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *for* $\theta \neq 0, \pi$*:*

Eigenvalues: $\lambda = \cos\theta \pm i\sin\theta$ *(complex!)*

No real vector stays in place - all vectors rotate.

Warning

Eigenvalues can be **complex** even for real matrices! This happens in rotations.

10.8 Applications

Practical Application

Google PageRank

Consider web pages as vectors and links as a matrix. The dominant eigenvector (with the largest eigenvalue) shows the relative importance of pages.

Practical Application

Principal Component Analysis (PCA)

In machine learning, eigenvectors of the covariance matrix show the principal directions of variation in the data.

Practical Application

Quantum Mechanics

Eigenvalues of operators = possible measurement outcomes

Eigenvectors = stable states of the system

10.9 Exercises

Exercise 10.1. Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Exercise 10.2. Show that the eigenvalues of a diagonal matrix are its diagonal entries.

Exercise 10.3. Find the eigenvalues of the 90 rotation matrix.

Exercise 10.4. If λ is an eigenvalue of $\mathbf{A}$, show that λ^2 is an eigenvalue of $\mathbf{A}^2$.

Exercise 10.5 (Challenge)**.** Prove that the trace of a matrix (sum of diagonal entries) equals the sum of eigenvalues.

Problem 10.1. Population matrix:

$$= \begin{bmatrix} 0 & 4 \\ 0.5 & 0 \end{bmatrix}$$

Find the dominant eigenvalue and interpret it.

Chapter 11

Computing Eigenvalues and Eigenbasis

In this chapter, we learn faster methods for computing eigenvalues of 2×2 matrices and become familiar with the concept of eigenbasis and diagonalization.

11.1 Quick Trick for 2×2 Matrices

Theorem 11.1 (Quick Formula). *For matrix* $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*:*

Sum of eigenvalues:

$$\lambda_1 + \lambda_2 = a + d = \mathrm{tr}(\mathbf{A})$$

Product of eigenvalues:

$$\lambda_1 \cdot \lambda_2 = ad - bc = \det(\mathbf{A})$$

Geometric Intuition

Knowing $m = \lambda_1 + \lambda_2$ and $p = \lambda_1\lambda_2$, you can find $\lambda_{1,2}$:

$$\lambda = \frac{m}{2} \pm \sqrt{\left(\frac{m}{2}\right)^2 - p}$$

Or: find two numbers whose sum is m and whose product is p.

Example 11.1. $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

Trace: $m = 3 + 1 = 4$

Determinant: $p = 3 - 4 = -1$

Eigenvalues: two numbers with $x + y = 4$ and $xy = -1$:

$$\lambda = 2 \pm \sqrt{4 - (-1)} = 2 \pm \sqrt{5}$$

11.2 Eigenbasis

Definition 11.1 (Eigenbasis)**.** If the eigenvectors of a matrix can form a **basis** (i.e., we have n linearly independent eigenvectors), this basis is called an **eigenbasis**.

Theorem 11.2. *In an eigenbasis, the transformation matrix becomes* ***diagonal****:*

$$[\mathbf{A}]_{eigenbasis} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

Geometric Intuition

Why diagonal?
In an eigenbasis, each basis vector only gets scaled:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \to \text{first column: } \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \to \text{second column: } \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix}$$

11.3 Diagonalization

Definition 11.2 (Diagonalization)**.** Matrix $\mathbf{A}$ is **diagonalizable** if:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

where $\mathbf{D}$ is diagonal and $\mathbf{P}$ is the matrix of eigenvectors.

Theorem 11.3 (Diagonalizability Condition)**.** *An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.*

Example 11.2. $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$

Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Verification: $\mathbf{PDP}^{-1} = \mathbf{A}$ ✓

11.4 Matrix Powers via Diagonalization

Theorem 11.4. *If* $\mathbf{A} = \mathbf{PDP}^{-1}$*:*

$$\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$$

and $\mathbf{D}^n$ *is very simple:*

$$\mathbf{D}^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

Example 11.3. Computing $\mathbf{A}^{100}$ for the previous example:

$$\mathbf{A}^{100} = \mathbf{P} \begin{bmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \mathbf{P}^{-1}$$

Without diagonalization, we would need 100 matrix multiplications!

Practical Application

Application: Fibonacci Numbers

Fibonacci sequence: $F_{n+1} = F_n + F_{n-1}$

Matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

With diagonalization, we find a closed-form formula:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

11.5 Non-Diagonalizable Matrices

Example 11.4. Shear matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$:

Characteristic polynomial: $(1 - \lambda)^2 = 0$

Eigenvalue: $\lambda = 1$ (repeated)

Eigenvector: only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (one-dimensional)

This matrix is not diagonalizable!

Warning

A repeated eigenvalue is not necessarily problematic. The problem arises when the number of independent eigenvectors is less than the multiplicity of the eigenvalue.

11.6 Symmetric Matrices

Theorem 11.5 (Spectral Theorem)**.** *A symmetric matrix ($\mathbf{A} = \mathbf{A}^\top$):*

1. *Has real eigenvalues*
2. *Eigenvectors corresponding to distinct eigenvalues are orthogonal*
3. *Is always diagonalizable (with an orthogonal basis)*

Geometric Intuition

Symmetric matrices are "well-behaved." They always become diagonal and their eigenvectors are perpendicular - like the principal axes of an ellipse.

11.7 Exercises

Exercise 11.1. Using the quick trick, find the eigenvalues of:

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

Exercise 11.2. Diagonalize matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Exercise 11.3. Using diagonalization, compute $\mathbf{A}^{10}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Exercise 11.4. Is the following matrix diagonalizable?

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Exercise 11.5 (Challenge)**.** Show that $\mathbf{A}$ and $\mathbf{A}^\top$ have the same eigenvalues.

Problem 11.1. The population of rabbits and foxes is modeled by:

$$\begin{bmatrix} R_{n+1} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1.1 & -0.4 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} R_n \\ F_n \end{bmatrix}$$

Analyze the long-term behavior of the population.

Chapter 12

Abstract Vector Spaces

In this final chapter, we extend the concept of vectors beyond geometric arrows. We will see that functions, polynomials, and even music can be "vectors" - as long as they follow the rules of linear algebra.

12.1 Motivation: What is a Vector?

Geometric Intuition

So far we have known vectors as arrows or lists of numbers. But mathematicians ask a deeper question:
"What things behave like vectors?"
Answer: Anything that can be added and multiplied by scalars!

Example 12.1. Functions can be added: $(f + g)(x) = f(x) + g(x)$
Functions can be multiplied by numbers: $(cf)(x) = c \cdot f(x)$
So functions can be "vectors"!

12.2 Formal Definition of Vector Space

Definition 12.1 (Vector Space)**.** A **vector space** over the field $\mathbb{R}$ is a set V with two operations:

- **Addition:** $+ : V \times V \to V$
- **Scalar multiplication:** $\cdot : \mathbb{R} \times V \to V$

that satisfy the following axioms.

Theorem 12.1 (Vector Space Axioms)**.** *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *and* $a, b \in \mathbb{R}$*:*
Addition axioms:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *(commutativity)*
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *(associativity)*
3. *Existence of identity:* $\exists \mathbf{0} : \mathbf{v} + \mathbf{0} = \mathbf{v}$
4. *Existence of inverse:* $\forall \mathbf{v}, \exists (-\mathbf{v}) : \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

Scalar multiplication axioms:

5. $a(b\mathbf{v}) = (ab)\mathbf{v}$
6. $1 \cdot \mathbf{v} = \mathbf{v}$

Distributive axioms:

7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
8. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

12.3 Examples of Vector Spaces

12.3.1 $\mathbb{R}^n$ - The Standard Space

Example 12.2. $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ with usual addition and multiplication.
This is the same space we have been working with until now.

12.3.2 The Space of Polynomials

Example 12.3. $\mathcal{P}_n$ = the set of polynomials of degree at most n:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Addition: $(p + q)(x) = p(x) + q(x)$
Scalar multiplication: $(cp)(x) = c \cdot p(x)$
Zero vector: $p(x) = 0$
Basis: $\{1, x, x^2, \ldots, x^n\}$ - dimension of space: $n + 1$

12.3.3 The Space of Functions

Example 12.4. $C[a, b]$ = the set of continuous functions on interval $[a, b]$
This space is **infinite-dimensional**!

12.3.4 The Space of Matrices

Example 12.5. $M_{m\times n}$ = the set of $m \times n$ matrices

Addition: element-wise addition

Scalar multiplication: multiply all entries by the scalar

Dimension: $m \times n$

12.4 Abstract Linear Transformations

Definition 12.2 (Linear Transformation Between Spaces)**.** A function $T : V \to W$ is a **linear transformation** if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{v}) = c \cdot T(\mathbf{v})$

Example 12.6. Differentiation: $D : \mathcal{P}_n \to \mathcal{P}_{n-1}$ with $D(p) = p'$

This is linear because $(f + g)' = f' + g'$ and $(cf)' = cf'$.

Example 12.7. Definite integral: $I : C[0, 1] \to \mathbb{R}$ with $I(f) = \int_0^1 f(x)dx$

This is linear because $\int(f + g) = \int f + \int g$ and $\int cf = c\int f$.

12.5 Functions as Infinite-Dimensional Vectors

Geometric Intuition

A function $f : [0, 2\pi] \to \mathbb{R}$ can be thought of as an infinite-dimensional vector:

- Each point x is a "component"
- The value $f(x)$ is the value of that component

Inner product of functions:

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx$$

Practical Application

Fourier Series

Sine and cosine functions form a "basis" for periodic functions:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The coefficients a_n, b_n are like "coordinates" of the function in this basis!

12.6 Why Does Abstraction Matter?

Summary

The Power of Abstraction:

When something is a vector space, **all the tools of linear algebra** are applicable:

- Linear independence and dependence
- Basis and dimension
- Linear transformations and matrices
- Eigenvalues and eigenvectors
- Image and null space

One theorem in linear algebra = a theorem for all these spaces!

Practical Application

Quantum Mechanics

Quantum states form a vector space (Hilbert space). Physical operators are linear transformations. Measurement values = eigenvalues!

Practical Application

Signal Processing

Audio signals are vectors in function space. Fourier transform is a change of basis. Filters are linear transformations.

Practical Application

Machine Learning

Data are vectors in feature space. Linear models are linear transformations. Dimensionality reduction = finding a better basis.

12.7 Looking Ahead

Remark 12.1. This course was an introduction to linear algebra. More advanced topics include:

- **Quadratic forms** and classification of conic sections
- **Singular Value Decomposition (SVD)** - powerful tool in data science
- **Numerical linear algebra** - efficient algorithms
- **Hilbert spaces** - infinite-dimensional linear algebra
- **Representation theory** - groups and linear algebra

12.8 Exercises

Exercise 12.1. Show that the set of 2×2 symmetric matrices is a vector space. What is its dimension?

Exercise 12.2. Is the set of polynomials with $p(0) = 1$ a vector space? Why or why not?

Exercise 12.3. Show that differentiation $D : \mathcal{P}_3 \to \mathcal{P}_2$ is a linear transformation. Write its matrix in the standard basis.

Exercise 12.4. Prove that in any vector space: $0 \cdot \mathbf{v} = \mathbf{0}$

Exercise 12.5 (Challenge)**.** Consider the solution space of the differential equation $y'' + y = 0$. Show that this is a vector space and find a basis for it.

Problem 12.1. For the linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ with $T(p) = p + p'$:

(a) Write the matrix of T in the basis $\{1, x, x^2\}$

(b) Find the eigenvalues

(c) Find the eigenpolynomials (eigenvectors)

12.9 Course Summary

Summary

The Essence of Linear Algebra
We started from vectors and matrices and arrived at abstract spaces. Key ideas:

1. **Vector:** Something that has addition and scalar multiplication

2. **Linear transformation:** A function that preserves lines
3. **Matrix:** The numerical representation of a linear transformation
4. **Determinant:** The volume scaling factor
5. **Eigenvalues:** Special directions that only get scaled
6. **Abstraction:** All these concepts work beyond arrows

Linear algebra is the common language of mathematics, physics, engineering, and computer science.

Appendix A

Glossary

Term	**Symbol/Notation**
Vector	$\mathbf{v}, \mathbf{w}, \mathbf{u}$
Matrix	$\mathbf{A}, \mathbf{B}, \mathbf{C}$
Linear Transformation	$T : \mathbb{R}^2 \to \mathbb{R}^2$
Determinant	$\det(\mathbf{A})$
Eigenvalue	λ
Eigenvector	$\mathbf{v}$
Basis	$\{\hat{\imath}, \hat{\jmath}\}$
Span	$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$
Null Space	$\text{Null}(\mathbf{A})$
Column Space	$\text{Col}(\mathbf{A})$
Row Space	$\text{Row}(\mathbf{A})$
Rank	$\text{rank}(\mathbf{A})$
Linear Independence	vectors with no redundancy
Linear Dependence	at least one redundant vector
Dot Product	$\mathbf{v} \cdot \mathbf{w}$
Cross Product	$\mathbf{v} \times \mathbf{w}$
Identity Matrix	$\mathbf{I}$
Inverse Matrix	$\mathbf{A}^{-1}$
Transpose	$\mathbf{A}^{\mathsf{T}}$
Change of Basis	$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$
Eigenbasis	basis of eigenvectors
Diagonal Matrix	non-zero only on diagonal
Scalar	a number (real or complex)
Linear Combination	$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots$
Vector Space	set closed under $+$ and scalar $\times$