

$$\vec{r} \cdot \vec{r} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (r^\alpha x, r^\alpha y, r^\alpha z) \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r^\alpha x}{\partial x} = \frac{\partial}{\partial x} \left(x^2 + y^2 + z^2 \right)^{\frac{\alpha}{2}} = (x^2 + y^2 + z^2)^{\frac{\alpha}{2}} + x \frac{\partial (x^2 + y^2 + z^2)^{\frac{\alpha}{2}}}{\partial x} = r^\alpha + x \frac{\alpha}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} 2x = r^\alpha + \alpha x^2 \frac{r^\alpha}{r^2}$$

$$\vec{r} \cdot \vec{f} = 3r^\alpha + \alpha (x^2 + y^2 + z^2) \frac{r^\alpha}{r^2} = r^\alpha (3 + \alpha)$$

$$\nabla^2 \equiv \nabla \cdot \nabla \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$$

$$f(x, y) = f(x(r, \theta), y(r, \theta)) \quad \frac{\partial r}{\partial x} = \cos \theta \quad \frac{\partial r}{\partial y} = \sin \theta \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r} \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x} = \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta}{r} \frac{\partial f}{\partial \theta} \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial y} = \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) \frac{\partial \theta}{\partial x} = \cos^2 \theta \frac{\partial^2 f}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \sin^2 \theta \frac{\partial^2 f}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial f}{\partial \theta} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 f}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\cos^2 \theta}{r} \frac{\partial^2 f}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$\Rightarrow \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$