Computer Animation & Physical Simulation

Lecture 12: Soft-Body Simulation – Deformable Solids

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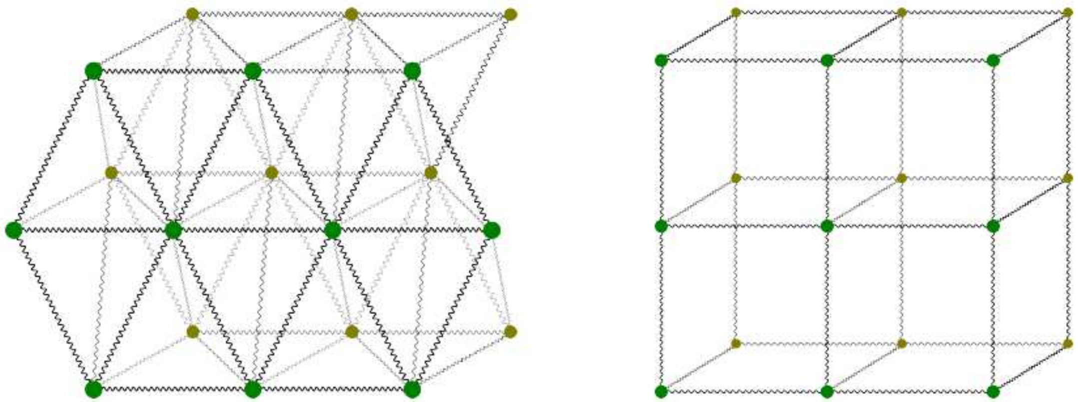
Mass-Spring Simulation for Deformable Objects

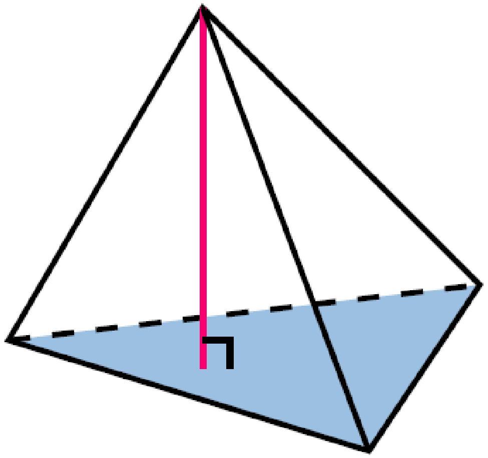
- **Volumetric meshing**
  - Tetrahedral mesh
  - Hexahedral mesh

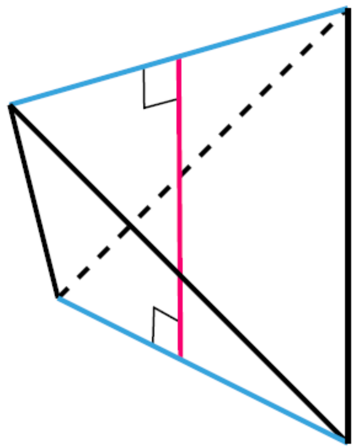


(a) 3D tetrahedral; (b) 3D hexahedral.

Mass-Spring Simulation for Deformable Objects

- **Altitude spring**
  - Point-face altitude spring
  - Edge-edge altitude spring





I. Basics for Deformable Solids

Notation

- **Partial differentiation**

$$\mathbf{x}_{,i} \equiv \partial \mathbf{x} / \partial \theta_i, \; \mathbf{u}_{,ik} \equiv \frac{\partial^2 \mathbf{u}}{\partial \theta_i \partial \theta_k}$$

- **Operators**
  - Vector dot product: dot $(\cdot)$
  - Vector cross product: cross $(\times)$
  - Tensor double contraction: colon $(:)$

I.a Linear Elasticity

Linear Elasticity

- **Commonly used in computer graphics**
  - Relatively simple formulation
  - Resulting efficient simulations

- **Three essential parts**
  - Geometry: study of deformation a body can undergo
  - Internal and external forces: how they affect an object's equilibrium or dynamics
  - Constitutive relation: how deforming geometry relates to internal forces

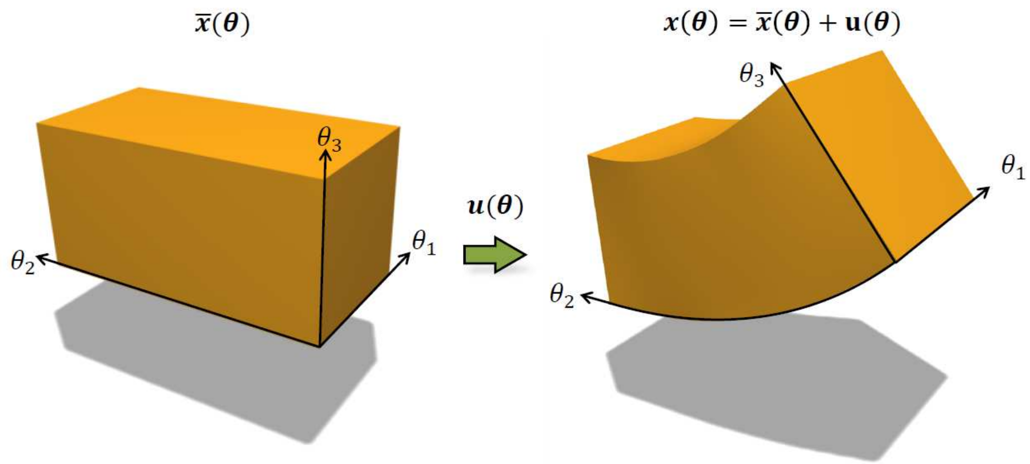
Linear Elasticity

- **Geometry**
  - Restrict to Lagrangian description
    - Undeformed positions

$$\bar{\mathbf{x}}(\boldsymbol{\theta}) \qquad \boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$$

    - Undergo a deformation

$$\mathbf{x}(\boldsymbol{\theta}) = \bar{\mathbf{x}}(\boldsymbol{\theta}) + \mathbf{u}(\boldsymbol{\theta})$$



Linear Elasticity

- **Geometry**
  - Cauchy strain
    - Assuming only small displacements

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,j}\right)$$

    - The main diagonal of the tensor
      - The amount of stretch in the three (normal) spatial directions
    - Off-diagonal values
      - Amount of shear in the according planes
    - Linearization of the more general strain (non-linear)

Linear Elasticity

- **Forces, equilibrium and dynamics**
  - Cauchy stress
    - Introduce a virtual cut plane with normal **n**
    - Force distribution at a point can compactly be described by the product of the Cauchy stress tensor with the plane normal

$$\sigma \cdot \mathbf{n} = \mathbf{f_n}$$

    - Main diagonal
      - Normal stress
    - Off-diagonal
      - Shear stress

Linear Elasticity

- **Forces, equilibrium and dynamics**
  - Total force
    - Summing up all traction forces on the side of the corresponding infinitesimal cube
    - Using the divergence (Gauss) theorem

$$\sigma \cdot \mathbf{n} = \mathbf{f_n}$$

$$\Big\downarrow \text{Volume integral}$$

$$\nabla \cdot \sigma + \mathbf{f} = 0$$

Linear Elasticity

- **Forces, equilibrium and dynamics**
  - Energy
    - Internal forces in an elastic body are conservative
    - Related to an underlying scalar energy potential characterizing the amount of work for deformation

$$W_{int}(\mathbf{u}) = \int_{\Omega} \epsilon(\mathbf{u}) : \sigma(\mathbf{u})\, d\Omega$$

    - Conservative forces

$$\mathbf{f}_{int} = -\frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}}$$

Linear Elasticity

- **Forces, equilibrium and dynamics**
  - Static equilibrium
    - All internal and external forces need to cancel each other

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}_{ext}$$

    - Making use of the elastic potential

$$-\frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

      - Better suited for setting up the corresponding discrete problems

Linear Elasticity

- **Forces, equilibrium and dynamics**
  - Equations of motion
    - If an object is not in static equilibrium: difference between internal and external forces results in net forces
    - Acceleration of the material according to Newton's second law

$$\rho\,\ddot{\mathbf{u}} + \mathbf{f}_d(\dot{\mathbf{u}}) + \nabla\cdot\sigma \;=\; \mathbf{f}_{ext}$$

      - $\mathbf{f}_d(\dot{\mathbf{u}})$: a damping force
    - Making use of the elastic energy potential

$$\rho\,\ddot{\mathbf{u}} + \mathbf{f}_d(\dot{\mathbf{u}}) - \frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}} \;=\; \mathbf{f}_{ext}$$

Linear Elasticity

- **Constitutive relation**
  - Establish the relation between internal deformation and force
  - The simplest is the linear relation
    - a Hookean material

$$\sigma = \mathbf{C} : \epsilon$$

    - Young modulus: material's stiffness
    - Poisson's ratio: how much (linearized) change in volume (compression) is penalized during deformation

Linear Elasticity

- **Strong form**
  - Derivation results in a point-wise description

$$\sigma \cdot \mathbf{n} = \mathbf{f}_{\mathbf{n}} \qquad \nabla \cdot \sigma = \mathbf{f}_{ext}$$

$$\rho\,\ddot{\mathbf{u}} + \mathbf{f}_d(\dot{\mathbf{u}}) - \frac{\partial W_{int}(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

    - Taking variational viewpoint on the problem
    - Other forms are better for discrimination and for handling discontinuous problems

Linear Elasticity

- **Energy formulation**
  - From the point of view of variational calculus $\nabla \cdot \sigma = \mathbf{f}_{ext}$
  - Euler-Lagrange equation of a corresponding energy functional
    - Necessary condition for being at the minimum of the potential energy

$$W_{tot}(\mathbf{u}) = \frac{1}{2}\int_{\Omega}(\epsilon(\mathbf{u}) : \sigma(\mathbf{u}) - \mathbf{f}_{ext} \cdot \mathbf{u})\, d\Omega$$

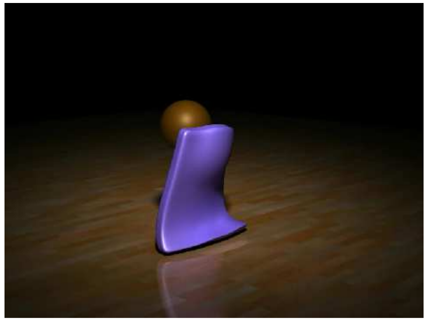
$$\Big\Downarrow \quad \sigma = \mathbf{C} : \epsilon$$

$$W_{tot}(\mathbf{u}) = \frac{1}{2}\int_{\Omega}(\epsilon(\mathbf{u}) : \mathbf{C} : \epsilon(\mathbf{u}) - \mathbf{f}_{ext} \cdot \mathbf{u})\, d\Omega$$

I.b Nonlinear Elasticity

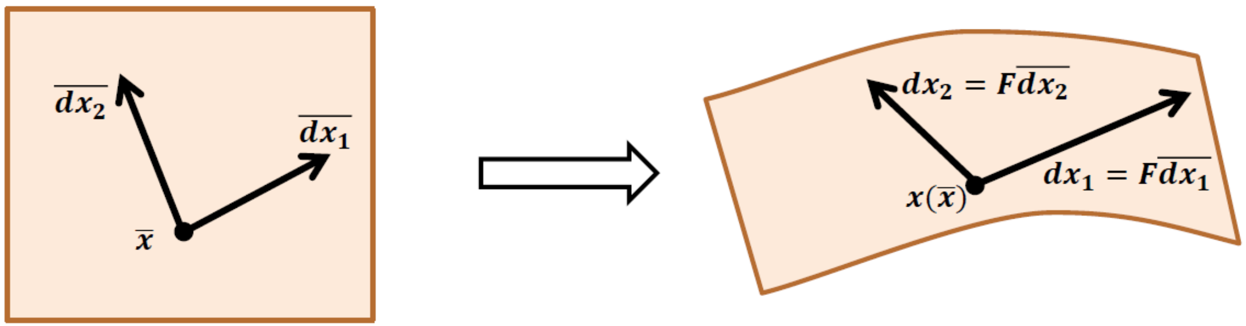
Nonlinear Elasticity

- **Handle large deformations**
  - More common in computer graphics
  - More complex but also allow for a much more realistic description of a wide variety of scenarios
  - Geometric measures and constitutive relations are more versatile



Nonlinear Elasticity

- **Geometry**
  - Describe the deformed geometry directly by
    - The mapping: $\mathbf{x}(\bar{\mathbf{x}})$
    - Deforming domain points: $\bar{\mathbf{x}} \in \Omega \subset \mathbb{R}^3$
    - Do not refer to a displacement field



Nonlinear Elasticity

- **Geometry**
  - Deformation gradient tensor

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}}$$

    - How neighboring material particles are deformed relative to each other
      - For a given particle at position $\bar{\mathbf{x}}$

$$\bar{\mathbf{x}} + d\bar{\mathbf{x}} \quad \Longrightarrow \quad \mathbf{x}(\bar{\mathbf{x}} + d\bar{\mathbf{x}})$$

$$\mathbf{x}(\bar{\mathbf{x}} + d\bar{\mathbf{x}}) \approx \mathbf{x}(\bar{\mathbf{x}}) + \mathbf{F}d\bar{\mathbf{x}} = \mathbf{x}(\bar{\mathbf{x}}) + d\mathbf{x}$$

$$\Downarrow$$

$$d\mathbf{x} = \mathbf{F}\bar{d}\mathbf{x}$$

Nonlinear Elasticity

- **Geometry**
  - Green strain
    - Consider the scalar products of two arbitrary direction vectors $d\bar{\mathbf{x}}_1$ and $d\bar{\mathbf{x}}_2$ after deformation

$$d\mathbf{x}_1 \cdot d\mathbf{x}_2 = d\bar{\mathbf{x}}_1 \cdot \mathbf{F}^T \mathbf{F} d\bar{\mathbf{x}}_2 = d\bar{\mathbf{x}}_1 \cdot \mathbf{C} d\bar{\mathbf{x}}_2$$

    - Right Cauchy-Green deformation tensor

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

    - Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$

Nonlinear Elasticity

- **Constitutive relation**
  - Saint-Venant Kirchhoff
    - Simplest nonlinear material models
    - Extend the linear stress-strain relation of linear elasticity

$$\Psi_{StVK} = \frac{\lambda}{2} tr(\mathbf{E})^2 + \mu tr(\mathbf{E}^2)$$

    - Lamé constants: λ and μ
    - λ : isotropic stiffness
    - μ : (linearized) volume preservation properties of the modeled material

Nonlinear Elasticity

- **Constitutive relation**
  - Neo-Hookean
    - Compressible Neo-Hookean model
    - Share characteristics and material parameters that are known from linear elasticity

$$\Psi_{NH} = \frac{\mu}{2}(tr(\mathbf{C}) - 3) - \mu \, lnJ + \frac{\lambda}{2}(lnJ)^2$$

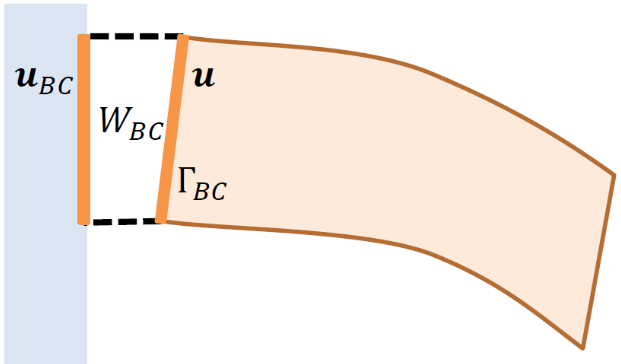
- $J = det\mathbf{F}$ measures the change in volume

I.c Handling Boundary and Collisions

Boundary Conditions and Collisions

- **Boundary conditions**

$$\mathbf{u}(\boldsymbol{\theta}) = \mathbf{u}_{BC}(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \Gamma_{BC} \subset \Gamma = \partial\Omega$$

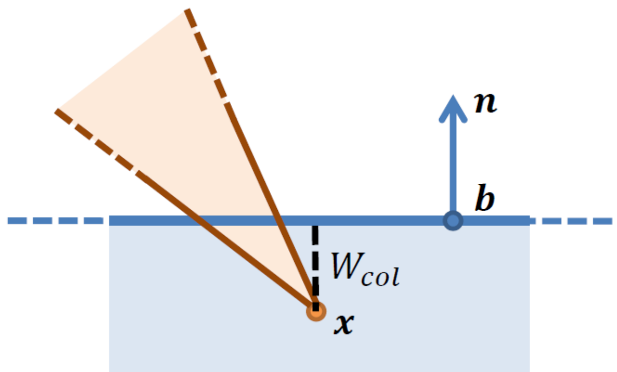


  - Modeled as an additional elastic energy $W_{BC}(\mathbf{u}) = \frac{\beta}{2}\int_{\Gamma_{BC}} |\mathbf{u} - \mathbf{u}_{BC}|^2$

Boundary Conditions and Collisions

- **Collisions**
  - Define the simple quadratic penalty potential

$$W_{plane} = \frac{\gamma}{2}((\mathbf{x} - \mathbf{b}) \cdot \mathbf{n})^2$$



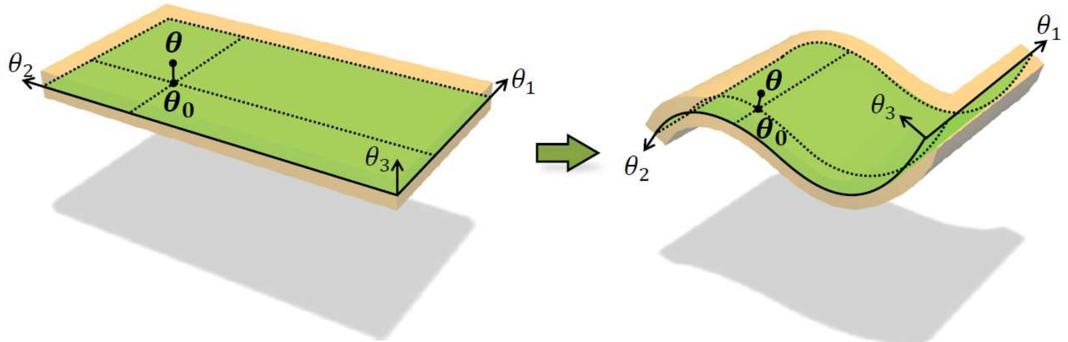
II. Resultant-Based Formulations

Resultant-based Formulations

- **Thin geometries**
  - Specialized variants of the theory
  - More efficient and numerically better suited
- **Resultant-based models**
  - Material has only small extent in certain spatial directions
  - Simplified by making certain assumption on how the material can deform in these directions
  - Thin shell theory: reduction along one direction
  - Rod theory: reduction along two directions

Resultant-based Formulations

- **Shells**
  - Consider a volumetric surface-like solid
  - Extent along tangent directions is much greater than along normal direction



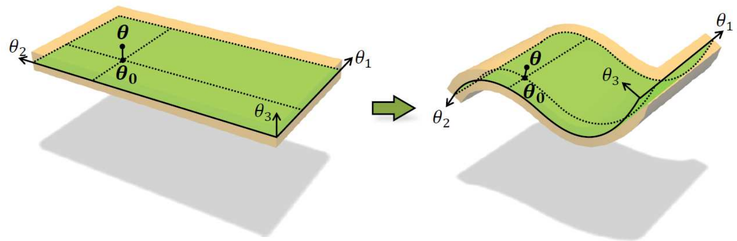
Resultant-based Formulations

- **Shells**
  - Strain about middle surface
    - Middle surface parameterized by the material-domain surface

$$\boldsymbol{\theta}_0 = (\theta_1, \theta_2, 0)$$

    - Assuming the shell to be sufficiently thin in normal direction

$$\begin{aligned}\bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \theta_3 \bar{\mathbf{x}}_{,3}(\boldsymbol{\theta}_0) \\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \theta_3 \mathbf{u}_{,3}(\boldsymbol{\theta}_0)\end{aligned}$$



Resultant-based Formulations

- **Shells**
  - In the view of the linear elasticity theory
    - Substitute into the linear Cauchy strain

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,j}\right)$$

$$\begin{aligned} \bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \theta_3\, \bar{\mathbf{x}}_{,3}(\boldsymbol{\theta}_0) \\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \theta_3\, \mathbf{u}_{,3}(\boldsymbol{\theta}_0) \end{aligned}$$

$$\Downarrow$$

$$\boldsymbol{\epsilon}(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\, \boldsymbol{\beta}^3(\boldsymbol{\theta}_0)$$

- Membrane strain $\alpha_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,j}\right)$
- Bending strain $\beta^k_{ij} = \frac{1}{2}\left(\mathbf{u}_{,ik} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,jk} + \mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,jk} + \bar{\mathbf{x}}_{,ik} \cdot \mathbf{u}_{,j}\right)$

Resultant-based Formulations

- **Shells**
  - Energy integration
    - Elastic energy of the volumetric shell model

$$W_{tot}(\mathbf{u}) = \frac{1}{2}\int_{\Omega}(\epsilon(\mathbf{u}):\mathbf{C}:\epsilon(\mathbf{u}) - \mathbf{f}_{ext}\cdot\mathbf{u})\,d\Omega \qquad \epsilon(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\,\boldsymbol{\beta}^3(\boldsymbol{\theta}_0)$$

$$\Downarrow$$

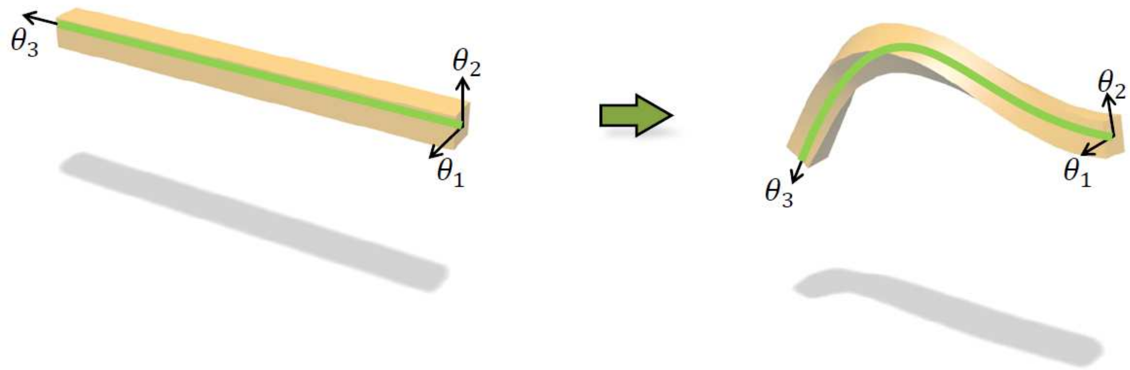
$$W = \frac{1}{2}\int_{\Omega}\Big(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\,\boldsymbol{\beta}^3(\boldsymbol{\theta}_0)\Big):\mathbf{C}:\Big(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_3\,\boldsymbol{\beta}^3(\boldsymbol{\theta}_0)\Big)\mathrm{d}\Omega$$

    - Integration in normal direction can be performed analytically

$$W = \frac{h_3}{2}\int_{\mathcal{S}}\boldsymbol{\alpha}:\mathbf{C}:\boldsymbol{\alpha} + \frac{h_3^2}{12}\,\boldsymbol{\beta}^3:\mathbf{C}:\boldsymbol{\beta}^3\,\mathrm{d}\mathcal{S}$$

Resultant-based Formulations

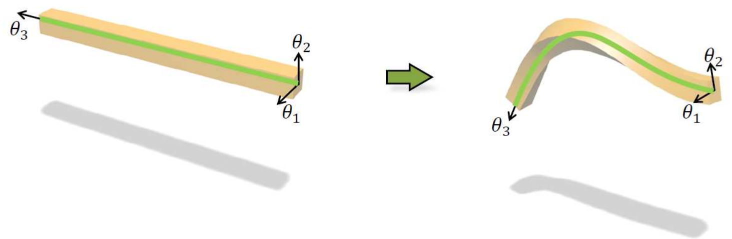
- **Rods**
  - A volumetric curve-like solid
  - Extent along tangent direction is much greater than along normal directions



Resultant-based Formulations

- **Rods**
  - Strain about centerline
    - Let the centerline curve Γ be parameterized by

$$\boldsymbol{\theta}_0 = (\theta_1, 0, 0)$$



    - For small extents along both normals

$$\begin{aligned}\bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \theta_2 \bar{\mathbf{x}}_{,2}(\boldsymbol{\theta}_0) + \theta_3 \bar{\mathbf{x}}_{,3}(\boldsymbol{\theta}_0) \\ \mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \theta_2 \mathbf{u}_{,2}(\boldsymbol{\theta}_0) + \theta_3 \mathbf{u}_{,3}(\boldsymbol{\theta}_0)\end{aligned}$$

    - The same steps for the derivation of the small strain

$$\boldsymbol{\epsilon}(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \theta_2 \boldsymbol{\beta}^2(\boldsymbol{\theta}_0) + \theta_3 \boldsymbol{\beta}^3(\boldsymbol{\theta}_0)$$

Resultant-based Formulations

- **Rods**
  - Energy integration
    - Analytic integration in the normal directions yields the one-dimensional integral of axial energy density over the rod's centerline

$$W = \frac{h_2 h_3}{2} \int_{\Gamma} \boldsymbol{\alpha} : \mathbf{C} : \boldsymbol{\alpha} + \frac{h_2^2}{12} \boldsymbol{\beta}^2 : \mathbf{C} : \boldsymbol{\beta}^2 + \frac{h_3^2}{12} \boldsymbol{\beta}^3 : \mathbf{C} : \boldsymbol{\beta}^3 \, \mathrm{d}\Gamma$$

III. Spatial and Temporal Discretization

Space Discretization for Linear Formulations

- **Discrete formulation**
  - Representation of the finite dimensional space
    - Using a basis of shape functions

$$\mathbf{u}_N(\bar{\mathbf{x}}) = \sum_i^N \mathbf{u}_i N_i(\bar{\mathbf{x}}) \in V_N$$

    - Basis functions must also fulfill the completeness property
      - Constant reproduction: In order to represent arbitrary translations of the body
        - Partition of unity $\sum_i N_i(\bar{\mathbf{x}}) = 1$
      - Linear reproduction: necessary for a basis to represent constant strain fields as well as arbitrary rigid body motions

Space Discretization for Linear Formulations

- **Energy-based approach**
  - Inserting solution representation

$$
\begin{aligned}
W_N(\mathbf{u}_N) &= \frac{1}{2}a(\sum_i \mathbf{u}_i N_i, \sum_j \mathbf{u}_j N_j) - f(\sum_i \mathbf{u}_i N_i) \\
&= \sum_{ij} \mathbf{u}_i \mathbf{u}_j a(N_i, N_j) - \sum_i \mathbf{u}_i f(N_i) \\
&= \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{f}_{ext},
\end{aligned}
$$

$$
\mathbf{K}_{ij} = a(N_i, N_j),\ \mathbf{f}_{ext,i} = f(N_i)
$$

Space Discretization for Linear Formulations

- **Energy-based approach**
  - Taking gradient to yield equation to solve
    - Static case

$$\mathbf{K}\mathbf{u} = \mathbf{f}_{ext}$$

    - Dynamic case

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}_{ext}$$

$$\mathbf{M}_{ij} = \mathbf{I} \cdot \int_{\Omega} \rho N_i N_j \, d\Omega$$

Space Discretization for Linear Formulations

- **Finite-element method (FEM)**
  - A specific instantiation of the Galerkin methodology
  - The object is partitioned into finite elements

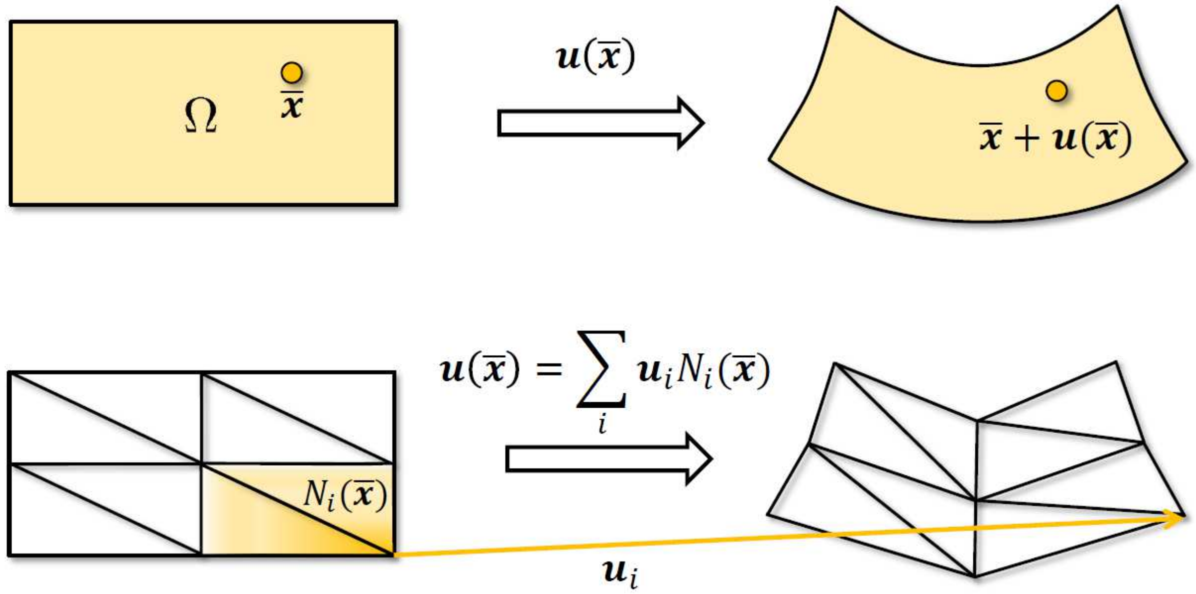
$$\bigcup e = \Omega$$

  - $\mathbf{u}(\bar{\mathbf{x}})$ is approximated by interpolating the displacements of the nodes within elements

$$\mathbf{u}(\bar{\mathbf{x}})|_e \approx \mathbf{u}^e(\bar{\mathbf{x}}) := \sum_{i=1}^{k} \mathbf{u}_i N_i^e(\bar{\mathbf{x}})$$

Space Discretization for Linear Formulations

- **Finite-Element Method (FEM)**



Polyhedral Elements

- **Traditional FEM simulations in computer graphics**
  - Rely on strictly tetrahedral or hexahedral meshes
  - Require complex remeshing in case of topological changes
  - Support more general convex polyhedral elements in finite element simulations
  - Using harmonic coordinates

Harmonic Basis Functions

- **A generalization of**
  - Linear barycentric basis functions to general polyhedral elements
  - A shape function is harmonic if its Laplacian vanishes in e

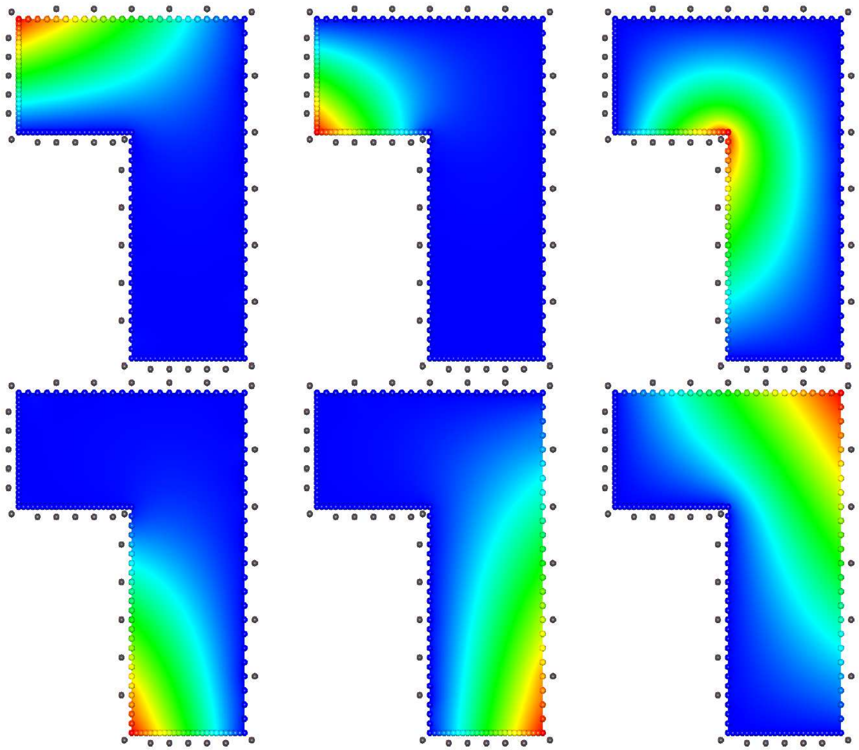
$$\Delta N_i^e(\bar{\mathbf{x}}) = 0, \qquad \text{for} \quad \bar{\mathbf{x}} \in e,$$
$$N_i^e(\bar{\mathbf{x}}) = b_i(\bar{\mathbf{x}}), \qquad \text{for} \quad \bar{\mathbf{x}} \in \partial e.$$

$$N_i^e(\bar{\mathbf{x}}_j) = \delta_{ij} \quad \forall\, i,j = 1,\ldots,k$$

$$N_i^{e_1}(\bar{\mathbf{x}}) = N_i^{e_2}(\bar{\mathbf{x}}) \quad \text{for} \quad \bar{\mathbf{x}} \in e_1 \cap e_2$$

Harmonic Basis Functions

- **Illustration**
  - Different element boundary conditions



Numerical Approximation

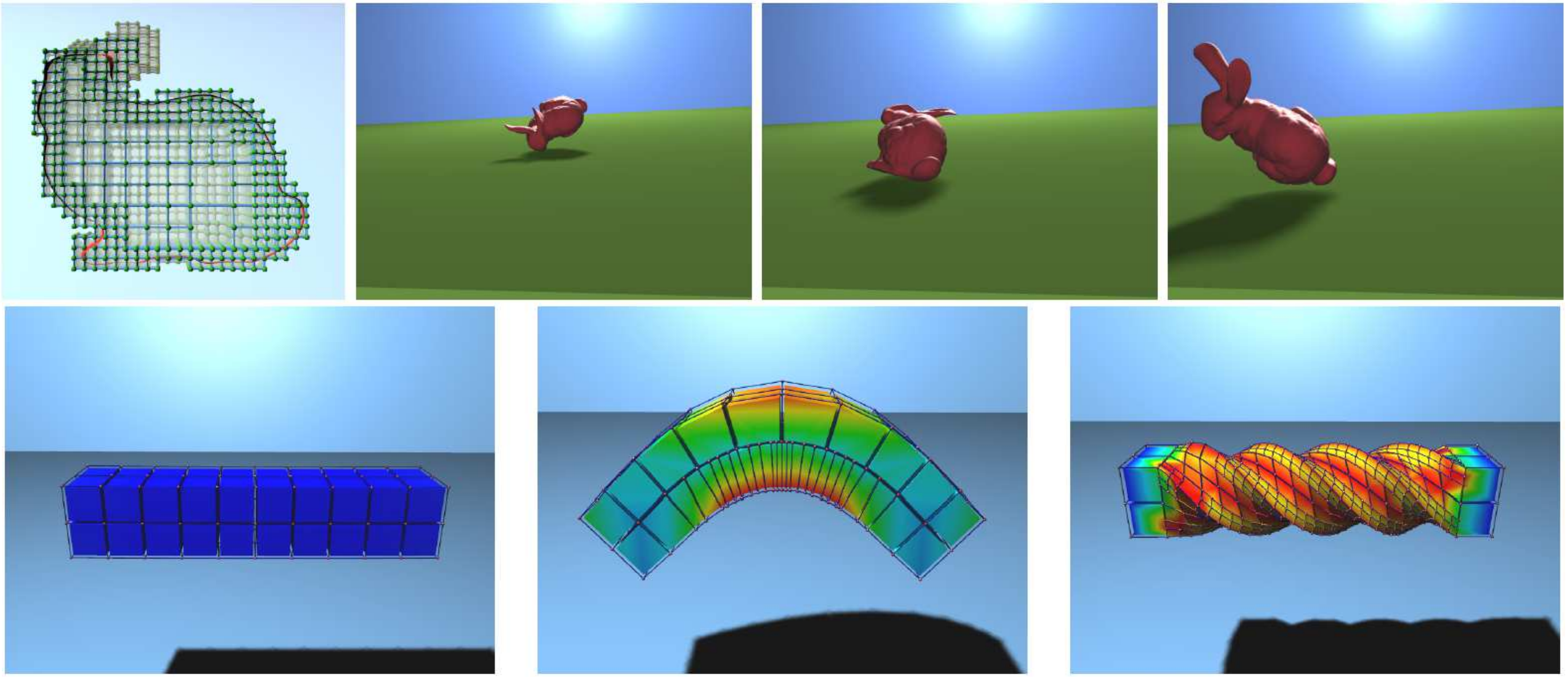
- **Closed form expressions for harmonic basis functions**
  - Exist for simple element shapes only
- **For more general elements**
  - Have to be computed numerically
  - Method of Fundamental Solutions

$$N(\bar{\mathbf{x}}) = \sum_{j=1}^{n} w_j \cdot \psi(\|\bar{\mathbf{x}} - \mathbf{k}_j\|) + \mathbf{a}_1^T \bar{\mathbf{x}} + a_0$$

    - The kernel function ψ is chosen as fundamental solution of the Laplace PDE

$$\psi(r) = \log r \text{ in 2D and } \psi(r) = 1/r \text{ in 3D}$$

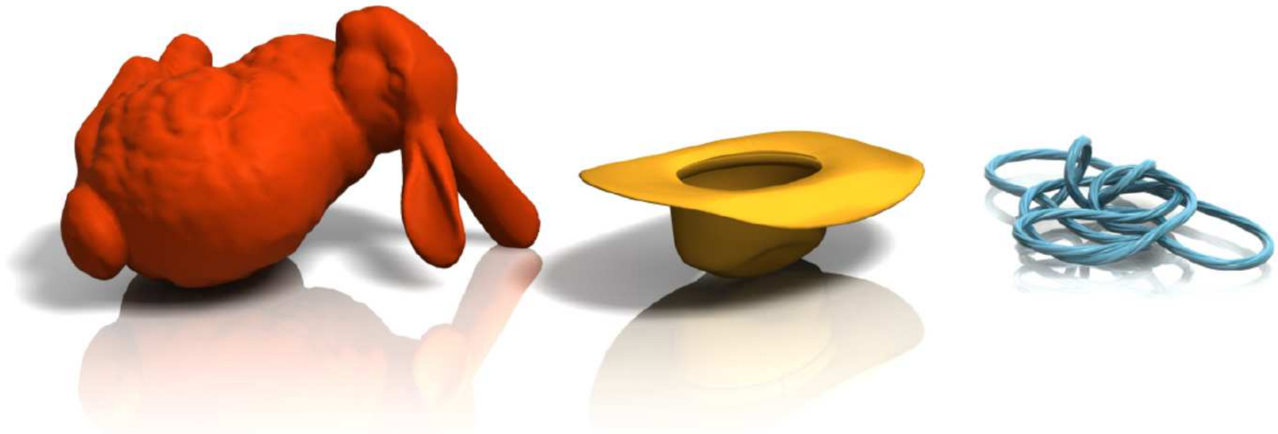
Results



IV. Unifying Resultant-based Models

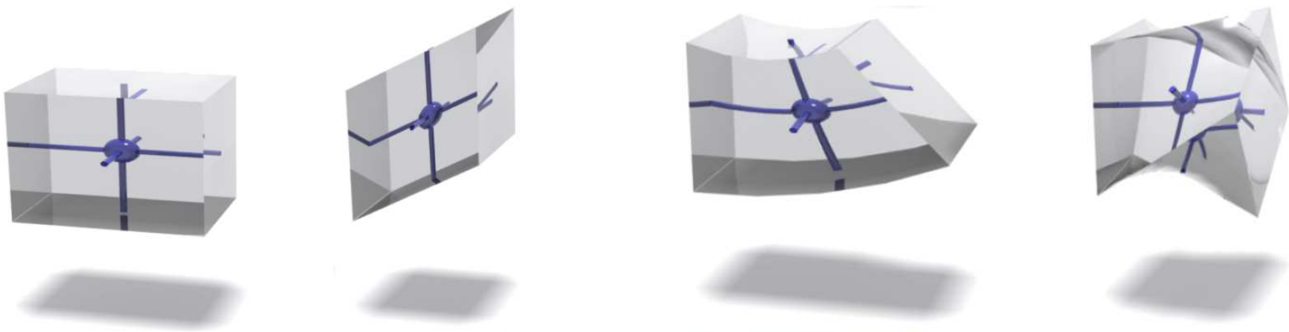
Unifying Resultant-based Models

- **Resultant-based models**
  - Only valid for single types of geometry (thin shell or rod)
  - Handle all three types in an unified manner?



Elastons

- **Consider a volumetric point-like solid**
  - Extent along all three directions is small
  - Strain in the vicinity of this point will measure
    - Linear deformations: stretch and shear at the center
    - Quadratic deformations: bending and twist along all three “normal” directions



Elastons

- **Linearizing strain**
  - Employ curvilinear coordinates describe an elaston centered at

$$\boldsymbol{\theta}_0 = (0,0,0)$$

  - First-order Taylor approximation of positions and displacements
    - In all three normal directions

$$\begin{aligned}
\bar{\mathbf{x}}(\boldsymbol{\theta}) &\approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \, \bar{\mathbf{x}}_{,k}(\boldsymbol{\theta}_0) \\
\mathbf{u}(\boldsymbol{\theta}) &\approx \mathbf{u}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \, \mathbf{u}_{,k}(\boldsymbol{\theta}_0)
\end{aligned}$$

Elastons

- **Linearizing strain**
  - Strain centered about the elaston

$$\epsilon_{ij} = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \bar{\mathbf{x}}_{,j} + \bar{\mathbf{x}}_{,i} \cdot \mathbf{u}_{,j}\right) \quad +$$

$$\bar{\mathbf{x}}(\boldsymbol{\theta}) \approx \bar{\mathbf{x}}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \bar{\mathbf{x}}_{,k}(\boldsymbol{\theta}_0)$$

$$\mathbf{u}(\boldsymbol{\theta}) \approx \mathbf{u}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \mathbf{u}_{,k}(\boldsymbol{\theta}_0)$$

$$\Downarrow$$

$$\epsilon(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0)$$

  - Naturally generalizes its shell and rod analogues
  - Capture stretching, shearing, bending, and twisting along all three axes

Elastons

- **Energy integration**
  - Integral over the elaston's volume

$$\epsilon(\boldsymbol{\theta}) \approx \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \qquad \sigma = \mathbf{C} : \epsilon \qquad W_{int}(\mathbf{u}) = \int_\Omega \epsilon(\mathbf{u}) : \sigma(\mathbf{u})\, d\Omega$$

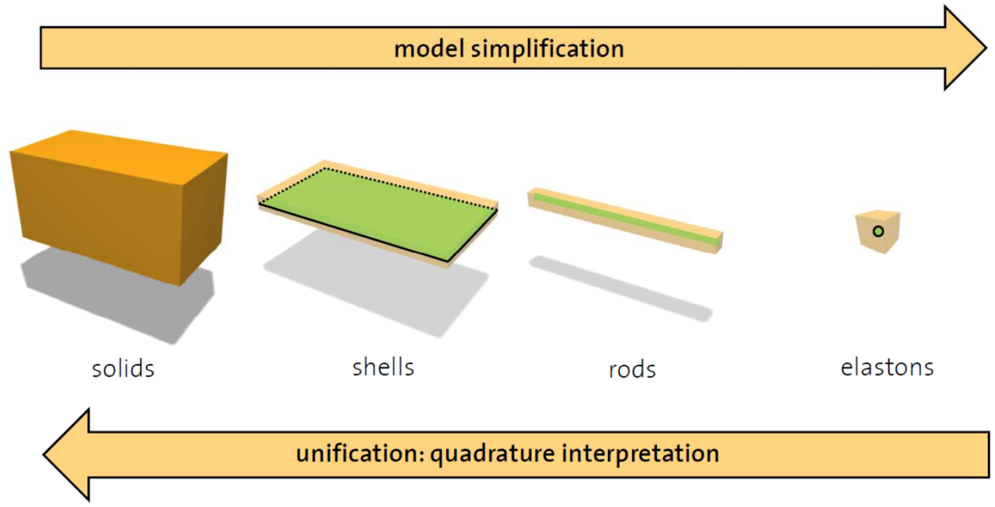
$$W = \frac{1}{2} \int_{\Omega_e} \left(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \right) : \mathbf{C} : \left(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \theta_k \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \right) \mathrm{d}\Omega$$

  - All three directions have thin extent
    - Analytically integrate

$$W = \frac{V}{2} \left(\boldsymbol{\alpha}(\boldsymbol{\theta}_0) : \mathbf{C} : \boldsymbol{\alpha}(\boldsymbol{\theta}_0) + \sum_{k=1}^{3} \frac{h_k^2}{12} \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) : \mathbf{C} : \boldsymbol{\beta}^k(\boldsymbol{\theta}_0) \right) \qquad V = h_1 h_2 h_3$$

Elastons

- **Basic building blocks**
  - For assembling the elastic energy of any deformable object, independent of its form



Elastons

- **Summing up: a new integration rule**
  - The classical goal of resultant-based model
    - Reduce the dimensionality of the model
    - Simplify its numerical treatment
    - Energy integration can be performed analytically
  - Elastons offer the most general integration rule
    - Approximate the stored elastic energy of rods, shells, or solids, respectively

$$W_{int}(\mathbf{u}) = \int_{\Omega} \epsilon(\mathbf{u}) : \sigma(\mathbf{u})\, d\Omega \longrightarrow W = \sum_{e \in \mathcal{E}} \frac{V^e}{2} \left(\boldsymbol{\alpha}^e : \mathbf{C} : \boldsymbol{\alpha}^e + \sum_{k=1}^{3} \frac{(h_k^e)^2}{12} \boldsymbol{\beta}^{ke} : \mathbf{C} : \boldsymbol{\beta}^{ke} \right)$$

Elastons

- **Summing up: a new integration rule**
  - Further requirements for integration rule in simulations
    - The elastons have to sample the material sufficiently densely
      - All relevant deformations are measured
    - An admissible basis of the solution space
      - Basis functions must now be twice differentiable in order to be able to measure bending strains
      - Reproduce constant and linear functions for accurate preservation of linear and angular momenta

V. Mesh-Free Methods

Meshfree Method

- **Drawback of FEM mesh-based approach**
  - Not always the optimal choice
    - Undergoes large, possibly plastic, deformations
    - Distorted meshes need to be adapted in order to maintain quality and accuracy

- **Point-based discretization**
  - Allow for the simplest possible discretization structure (sampling)
  - Use of moving least squares (MLS) as scattered data interpolation procedure

Classical MLS Formulation

- **A method for scattered data interpolation**
  - Assume sample points $\bar{\mathbf{x}}_1, \ldots, \bar{\mathbf{x}}_n$
    - Each with an associated function value $\mathbf{u}_i \in \mathbb{R}^3$
  - Around a given point
    - Local polynomial approximation of displacement field

$$\mathbf{a}(\hat{\bar{\mathbf{x}}})^T \mathbf{p}(\bar{\mathbf{x}})$$

    - Using a weight function of the form $w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i)$ $w(\mathbf{d}) = (1 - \|\mathbf{d}\|^2)^3$

Classical MLS Formulation

- **A method for scattered data interpolation**
  - Define error function

$$J(\mathbf{a}(\hat{\mathbf{x}})) = \frac{1}{2}\sum_{i=1}^{n} w(\hat{\mathbf{x}} - \bar{\mathbf{x}}_i) \left\| \mathbf{a}^T \mathbf{p}(\bar{\mathbf{x}}_i) - \mathbf{u}_i \right\|^2$$

  - The derivative of the error function is

$$\frac{\partial J}{\partial \mathbf{a}(\hat{\mathbf{x}})} = \sum_{i=1}^{n} w(\hat{\mathbf{x}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)(\mathbf{a}(\hat{\mathbf{x}})^T \mathbf{p}(\bar{\mathbf{x}}_i) - \mathbf{u}_i)$$

  - Setting it to zero and solving the system for the coefficient vector yields

$$\mathbf{a}(\hat{\mathbf{x}}) = (\sum_{i=1}^{n} w(\hat{\mathbf{x}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)^T)^{-1} \sum_{i=1}^{n} w(\hat{\mathbf{x}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)\mathbf{u}_i$$

Classical MLS Formulation

- **A method for scattered data interpolation**
  - Insert into the polynomial to obtain the approximation

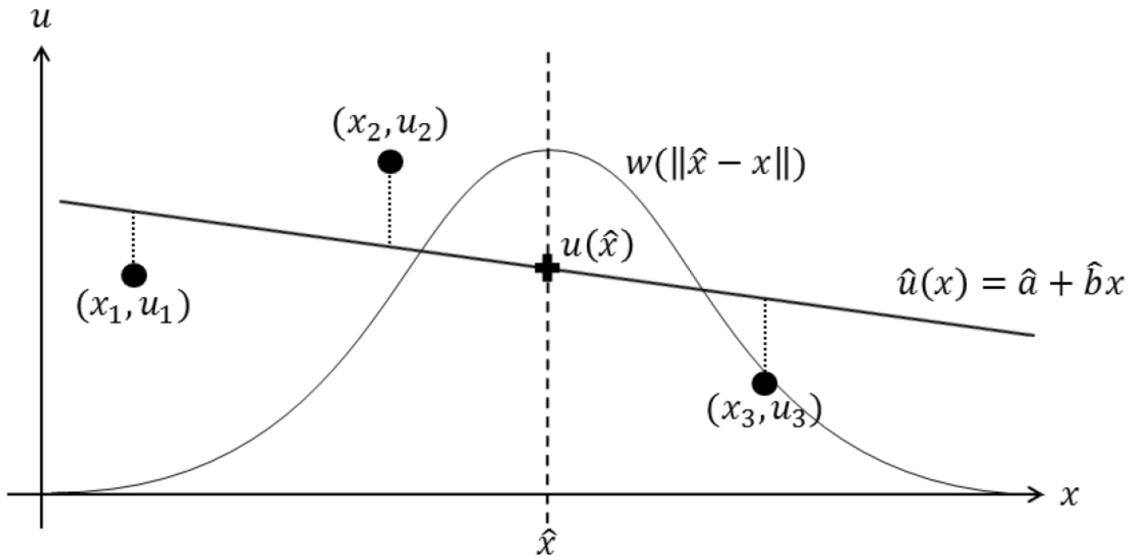
$$\mathbf{a}(\hat{\bar{\mathbf{x}}})^T\mathbf{p}(\bar{\mathbf{x}}) \Leftarrow \mathbf{a}(\hat{\bar{\mathbf{x}}}) = \left(\sum_{i=1}^{n} w(\hat{\bar{\mathbf{x}}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)^T\right)^{-1} \sum_{i=1}^{n} w(\hat{\bar{\mathbf{x}}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)\mathbf{u}_i$$

$$\Downarrow$$

$$\mathbf{u}(\bar{\mathbf{x}}) = \mathbf{p}(\bar{\mathbf{x}})^T\left(\sum_{i=1}^{n} w(\hat{\bar{\mathbf{x}}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)^T\right)^{-1} \sum_{i=1}^{n} w(\hat{\bar{\mathbf{x}}} - \bar{\mathbf{x}}_i)\mathbf{p}(\bar{\mathbf{x}}_i)\mathbf{u}_i$$

Classical MLS Formulation

- **A simple example of a linear 1D MLS fit**



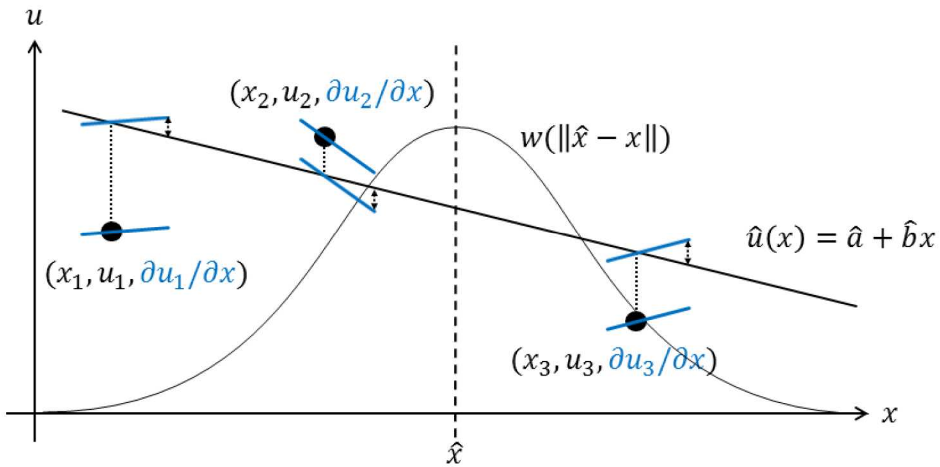
Generalized MLS

- **A point-based discretization**
  - Compatible with our thinking of elastons as “elastic points”
  - Take advantage of elastons to local form, topology, etc.

- **Limitation of the classical MLS**
  - To guarantee an invertible matrix $\mathbf{G}(\bar{\mathbf{x}})$
    - Sufficiently many samples
    - Must not be co-linear or co-planar

Generalized MLS

- **Motivation for GMLS**
  - Pursue a simpler approach
  - An extension of classical MLS



Generalized MLS

- **Linear GMLS**
  - Linear polynomial fitting
  - Fit also to the additional first derivative

$$J(\mathbf{a}(\hat{\bar{\mathbf{x}}})) = \frac{1}{2}\sum_{i=1}^{n} w(\hat{\bar{\mathbf{x}}} - \bar{\mathbf{x}}_i)\left\|\mathbf{a}^T\mathbf{p}(\bar{\mathbf{x}}_i) - \mathbf{u}_i\right\|^2 + \sum_{i=1}^{n}\sum_{j=1}^{3} w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i)\left\|\mathbf{a}^T\mathbf{p}_{,j}(\bar{\mathbf{x}}_i) - \mathbf{u}_{i,j}\right\|^2$$

$$\Downarrow$$

$$\mathbf{u}(\bar{\mathbf{x}}) = \sum_{i=1}^{n}\left[\mathbf{u}_i N_i(\bar{\mathbf{x}}) + \sum_{j=1}^{3}\mathbf{u}_{i,j} N_i^j(\bar{\mathbf{x}})\right]$$

$$N_i^j(\bar{\mathbf{x}}) = \mathbf{p}(\bar{\mathbf{x}})^T \mathbf{G}^{-1}(\bar{\mathbf{x}})\,\mathbf{p}_{,j}(\bar{\mathbf{x}}_i)\, w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i)$$

$$\mathbf{G}(\bar{\mathbf{x}}) = \sum_{i=1}^{n} w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i)\left[\mathbf{p}(\bar{\mathbf{x}}_i)\,\mathbf{p}(\bar{\mathbf{x}}_i)^T + \sum_{j=1}^{3}\mathbf{p}_{,j}(\bar{\mathbf{x}}_i)\,\mathbf{p}_{,j}(\bar{\mathbf{x}}_i)^T\right]$$

Generalized MLS

- **Quadratic GMLS**
  - For higher accuracy and faster convergence, consider second order derivative
  - Use a quadratic polynomial $\mathbf{p}(\bar{\mathbf{x}}) = (1, \bar{x}, \bar{y}, \bar{z}, \bar{x}\bar{x}, \bar{x}\bar{y}, \bar{x}\bar{z}, \bar{y}\bar{y}, \bar{y}\bar{z}, \bar{z}\bar{z})^T$
  - Add a third error term to the objective

$$\sum_{i=1}^{n} \sum_{j,k=1}^{3} w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i) \left\| \mathbf{a}^T \mathbf{p}_{,jk}(\bar{\mathbf{x}}_i) - \mathbf{u}_{i,jk} \right\|^2$$

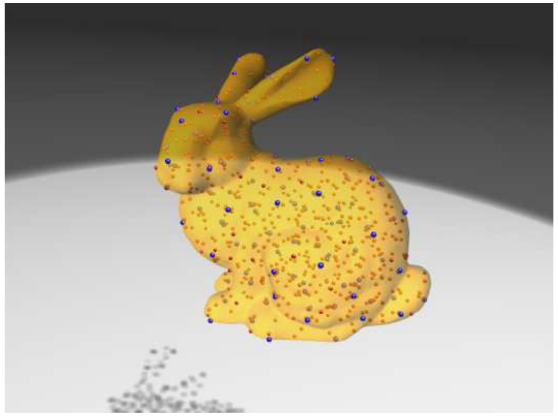
$$\mathbf{u}(\bar{\mathbf{x}}) = \sum_{i=1}^{n} \left[\mathbf{u}_i N_i(\bar{\mathbf{x}}) + \sum_{j=1}^{3} \mathbf{u}_{i,j} N_i^j(\bar{\mathbf{x}}) + \sum_{j,k=1}^{3} \mathbf{u}_{i,jk} N_i^{jk}(\bar{\mathbf{x}}) \right]$$

$$N_i^{jk}(\bar{\mathbf{x}}) = \mathbf{p}(\bar{\mathbf{x}})^T \mathbf{G}^{-1}(\bar{\mathbf{x}}) \mathbf{p}_{,jk}(\bar{\mathbf{x}}_i) w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i)$$

$$\sum_{i=1}^{n} \sum_{j,k=1}^{3} w(\bar{\mathbf{x}} - \bar{\mathbf{x}}_i) \mathbf{p}_{,jk}(\bar{\mathbf{x}}_i) \mathbf{p}_{,jk}(\bar{\mathbf{x}}_i)^T$$

Implementation

- **Sampling**
  - Given an input cloud
    - Generate the positions of GMLS sample points
$$\{\bar{\mathbf{x}}_1, \ldots \bar{\mathbf{x}}_n\}$$
    - Generate the positions of elaston centers
$$\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$$
    - Sub-sampling the dense material point set
    - The material is partitioned into
$$\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n$$
    - Associate each material point $\mathbf{m}_j \in \mathcal{M}$ to its closest sample
    - Iterate until convergence $\bar{\mathbf{x}}_i$



Extensions

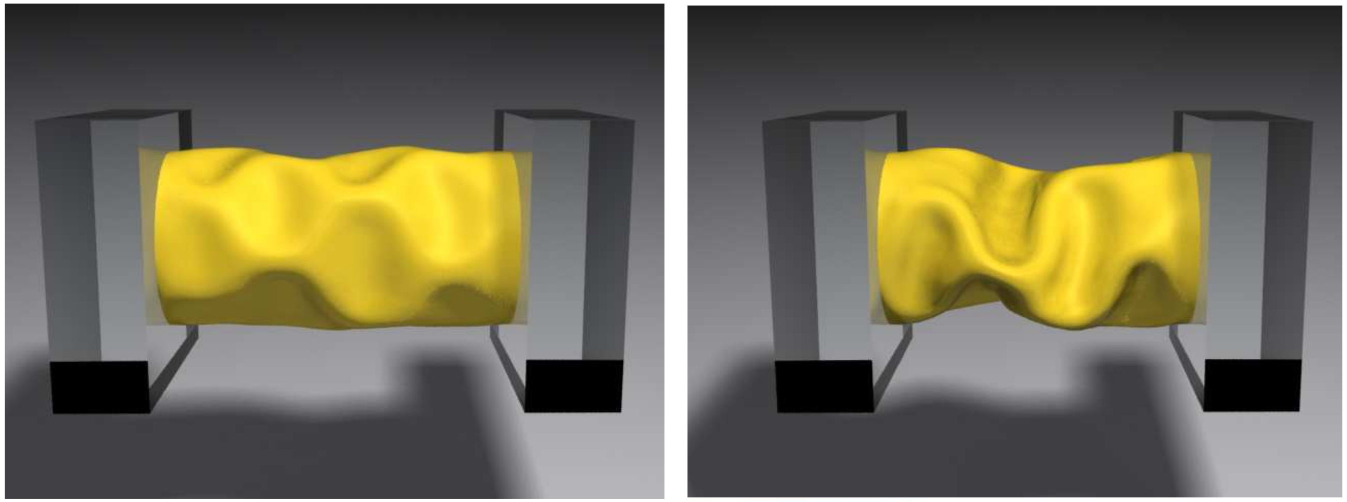
- **Plasticity**
  - Plasticity model
    - Define plastic membrane and bending strain variables for each elaston $\boldsymbol{\alpha}_p, \boldsymbol{\beta}_p^k$
    - Define the effective elastic strain
      - Taking the difference between the measured geometric strains and the stored plastic strains

$$\boldsymbol{\alpha}_e = \boldsymbol{\alpha}_g - \boldsymbol{\alpha}_p, \quad \boldsymbol{\beta}_e^k = \boldsymbol{\beta}_g^k - \boldsymbol{\beta}_p^k$$

    - New elastic energy stored in a single elaston $W_\mathrm{e} = \frac{V}{2}\left(\boldsymbol{\alpha}_e : \mathbf{C} : \boldsymbol{\alpha}_e + \sum_{k=1}^{3} \frac{(h_k^e)^2}{12} \boldsymbol{\beta}_e^k : \mathbf{C} : \boldsymbol{\beta}_e^k\right)$

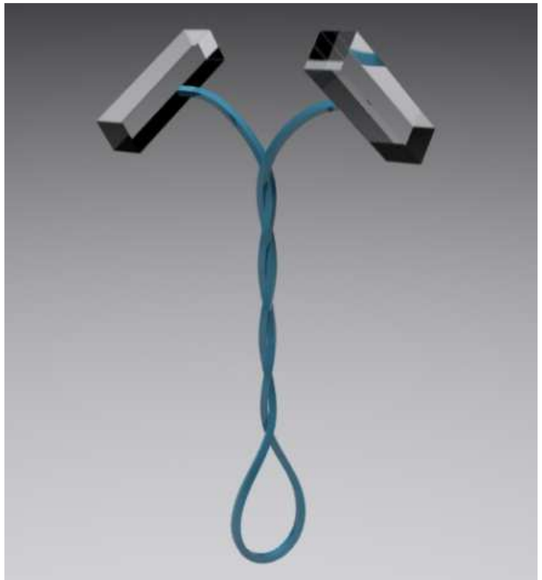
Results

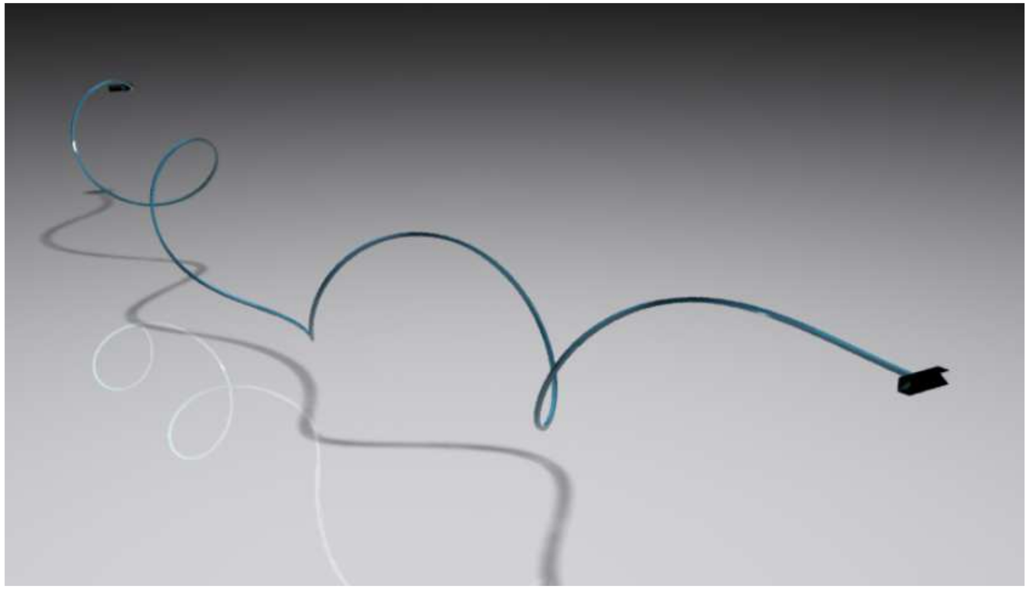
- **Cylinder compression**
  - A cylinder shows the typical buckling patterns as it is getting more and more compressed.



Results

- **Twisting a thin rod**



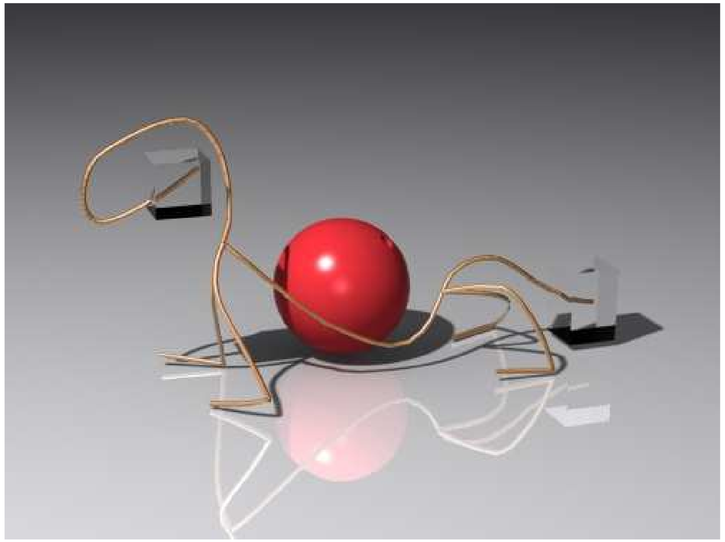


Results

- **Complex interaction between different types of geometry**

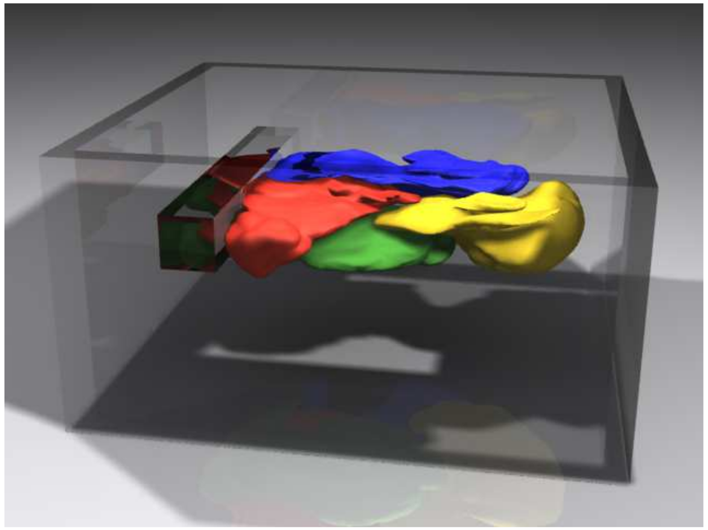
Results

- **Ball drop and elasto-plastic cuboid**

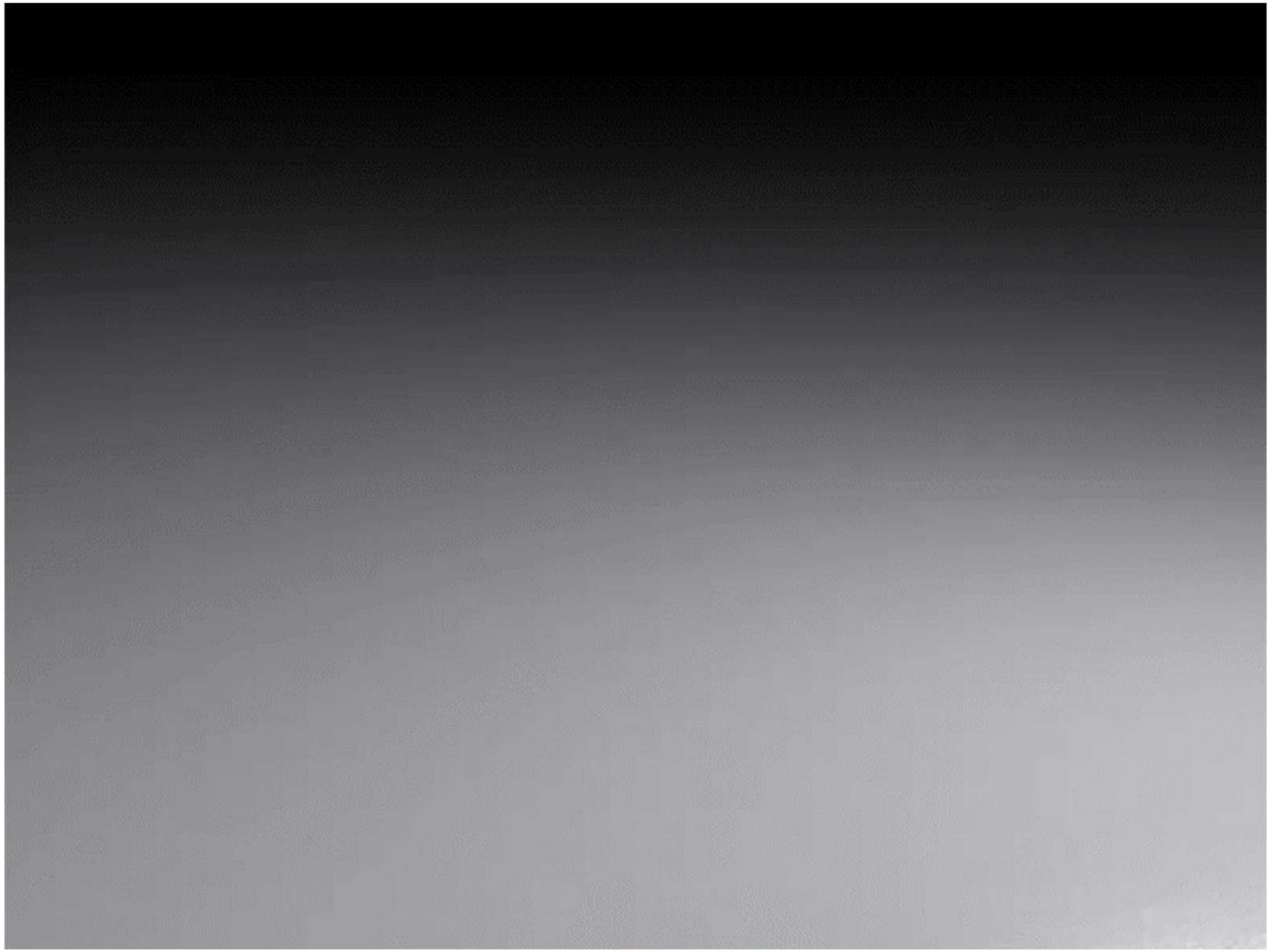


Results

- **Fish model and elasto-plastic bunnies**



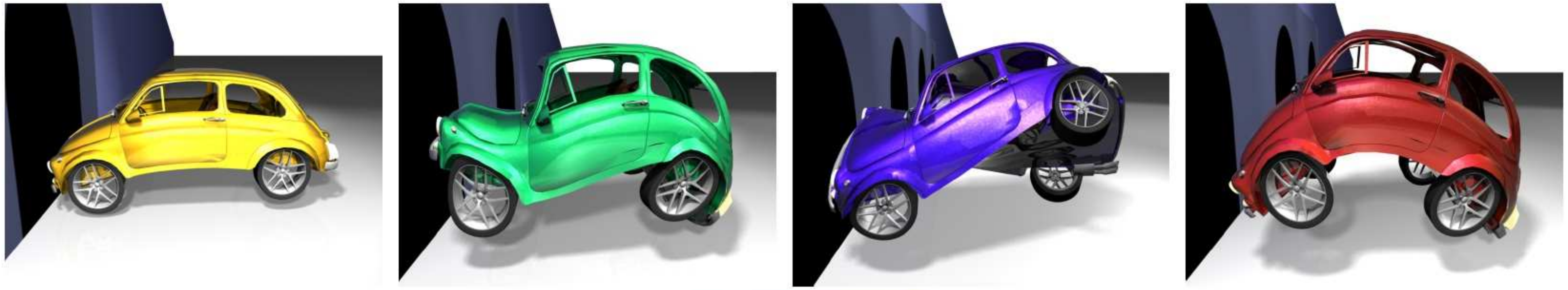
Results



VI. Art-Directable Elastic Potentials

Art-Directable Elastic Potentials

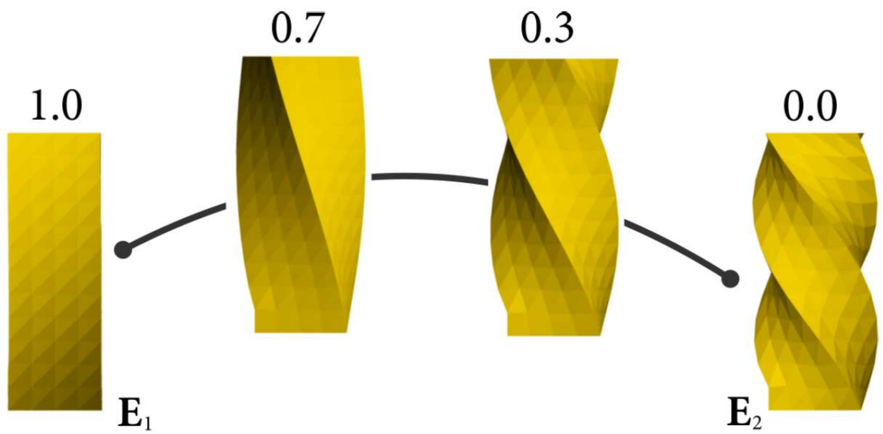
- **Control the deformation**
  - Classical deformation mainly driven by the chosen material model and its parameter values
  - Often, an artist starts with a vision on how an object should deform
  - Setting up a specialized elastic model to follow the preferred example poses



Example Manifold

- **Example manifold by example interpolation**
  - Interpolate between these examples by interpolating their descriptors

$$\mathbf{E}(w) = (1 - w)\mathbf{E}_1 + w\mathbf{E}_2$$



Example Manifold

- **Example manifold by example interpolation**
  - The interpolated descriptor is generally not realizable
    - Find the closest realizable strain $\mathbf{E}(\mathbf{x}_w) \in \mathcal{F}$ and corresponding configuration
    - Solve the least squares minimization

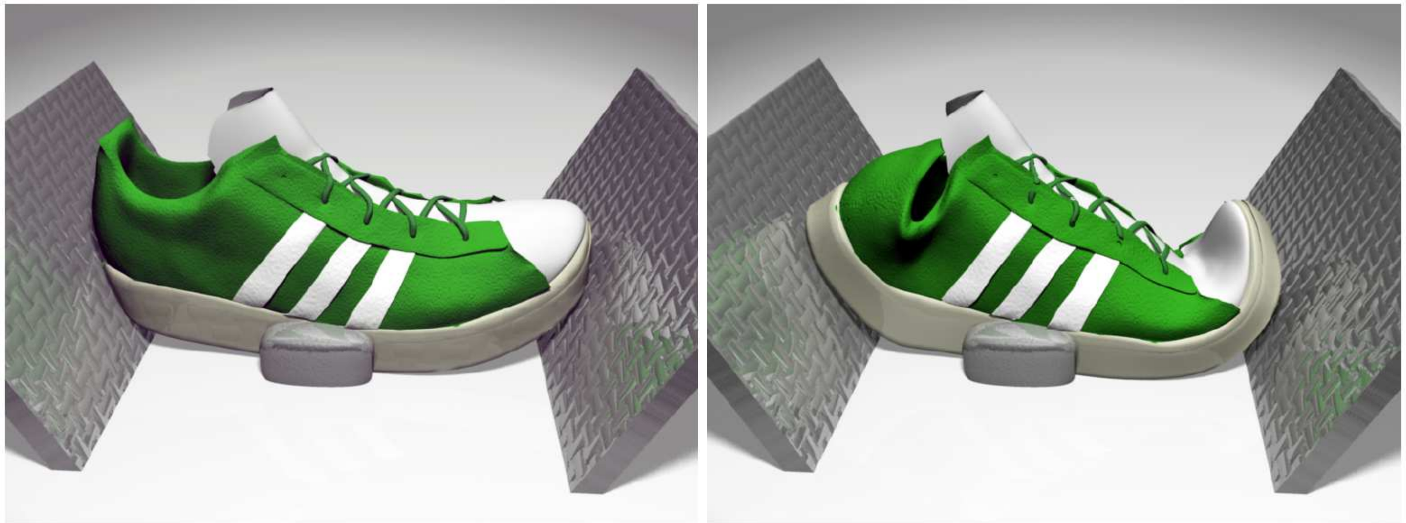
$$\min_{\mathbf{x}_w} W_I(\mathbf{x}_w, w) = \min_{\mathbf{x}_w} \frac{1}{2} |\mathbf{E}(\mathbf{x}_w) - \mathbf{E}(w)|_F^2$$

      - $|\cdot|_F$: Frobenius norms of the elemental strain tensors
    - Generalize to an arbitrary number of poses n

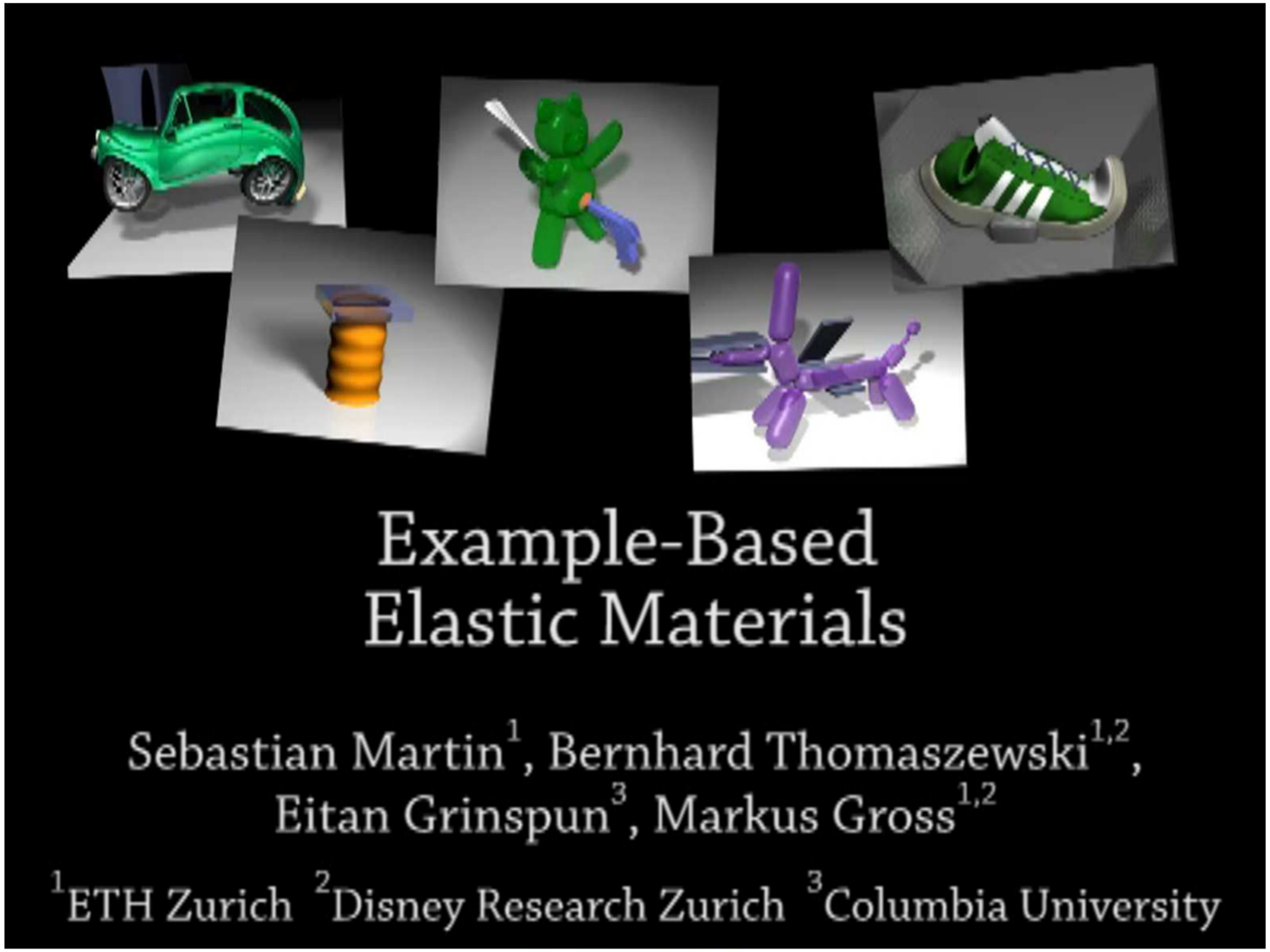
$$\mathbf{E}(\mathbf{w}) = \sum_i^n w_i \mathbf{E}_i$$

Results

- **Compressed sneaker simulated as a coarse solid**
  - Without examples (left) and augmented with two local examples (right)



Results


Example-Based
Elastic Materials
Sebastian Martin[1], Bernhard Thomaszewski[1,2],
Eitan Grinspun[3], Markus Gross[1,2]
[1]ETH Zurich [2]Disney Research Zurich [3]Columbia University

Next Lecture: Fluid Simulation I