Computer Animation & Physical Simulation

Lecture 9: Soft-Body Simulation – Hair II

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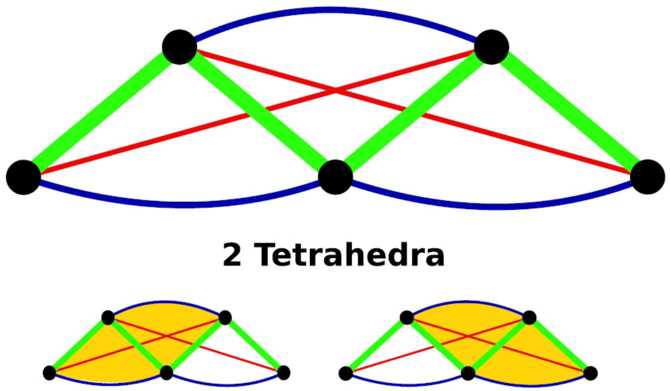
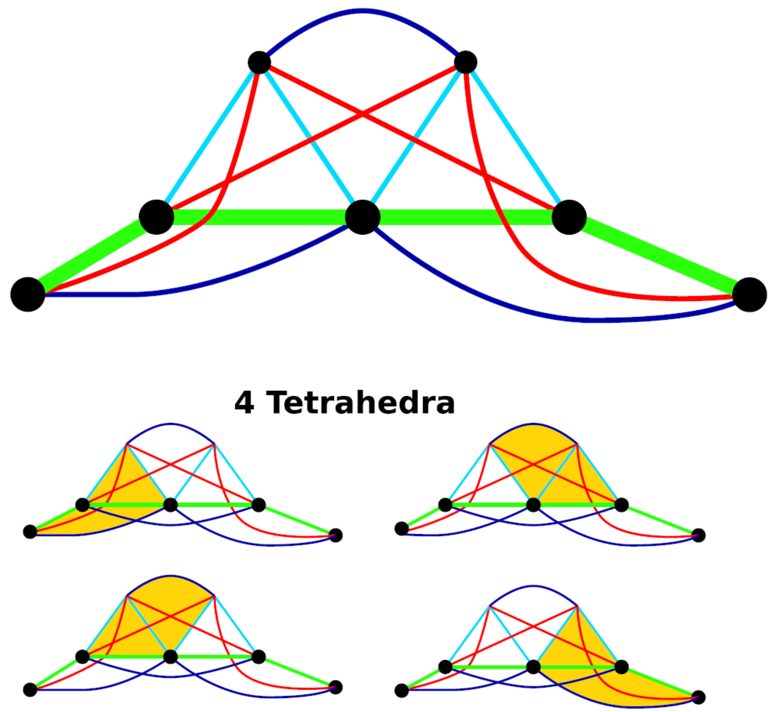
Simulation of Hair Dynamics

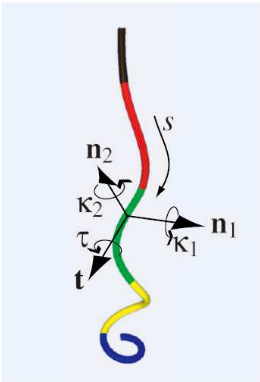
- **Mass-Spring Model**

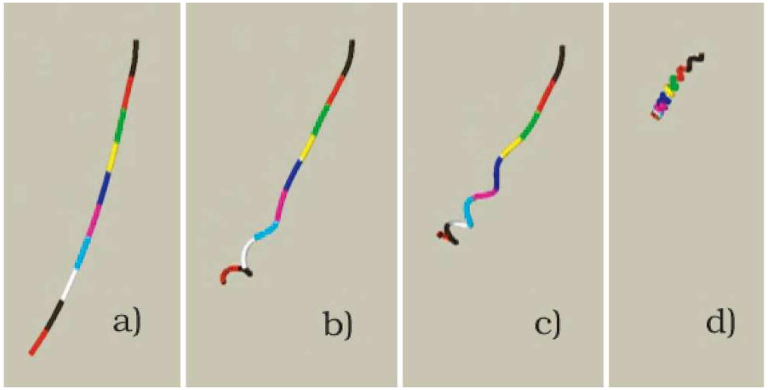




Simulation of Hair Dynamics

- **Super-Helix Model**
  - Built upon the Cosserat and Kirchhoff theories of rods





$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_{iQ}}\right) - \frac{\partial T}{\partial q_{iQ}} + \frac{\partial U}{\partial q_{iQ}} + \frac{\partial D}{\partial \dot{q}_{iQ}} = \int_0^L \mathbf{J}_{iQ}(s,\mathbf{q},t)\cdot\mathbf{F}(s,t)\,\mathrm{d}s$$

I. Hair Interactions

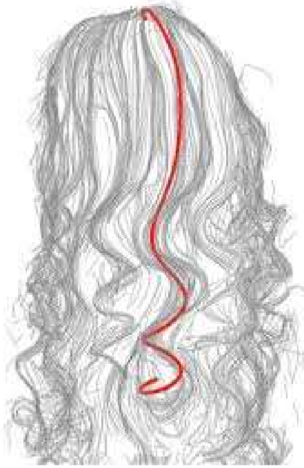
Hair Interaction

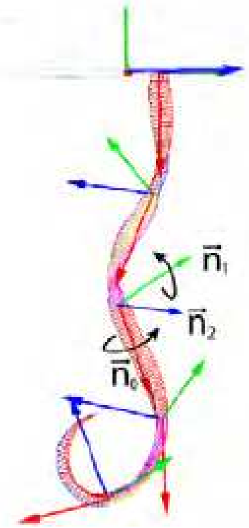
- **Hair-body interaction**
  - Collisions and contacts between hair strands

- **Hair-hair interaction**
  - Collisions and contacts between hair strands cause hair to occupy a pretty high volume
  - Largely due to the surface of individual hair strands (composed of scales)
  - Anisotropic friction inside hair

Wisp of Hair

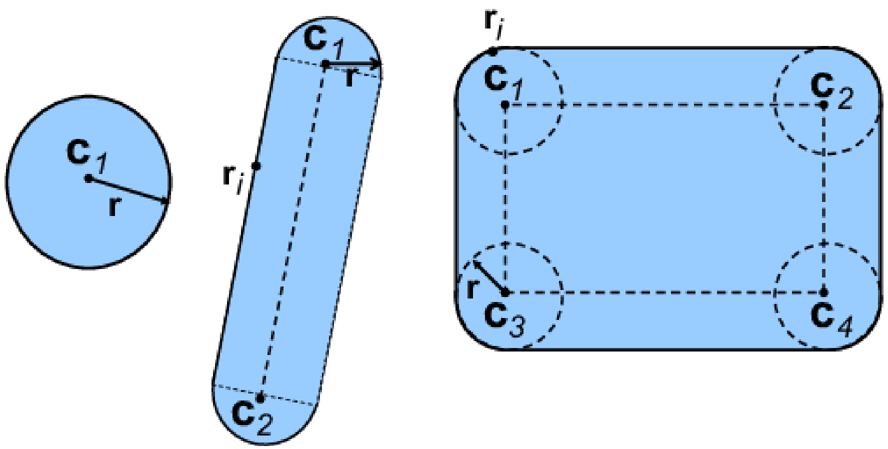
- **A cluster of hair around the hair strand**
  - Occupy certain volumes

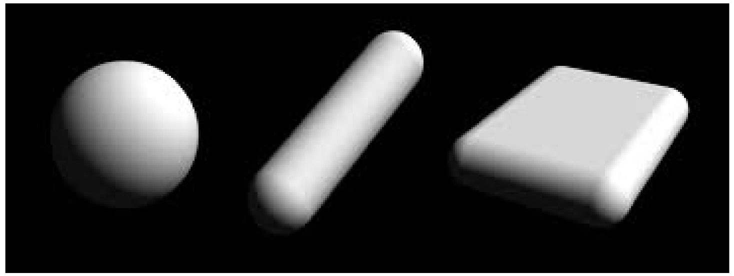





Representation of Wisp of Hair

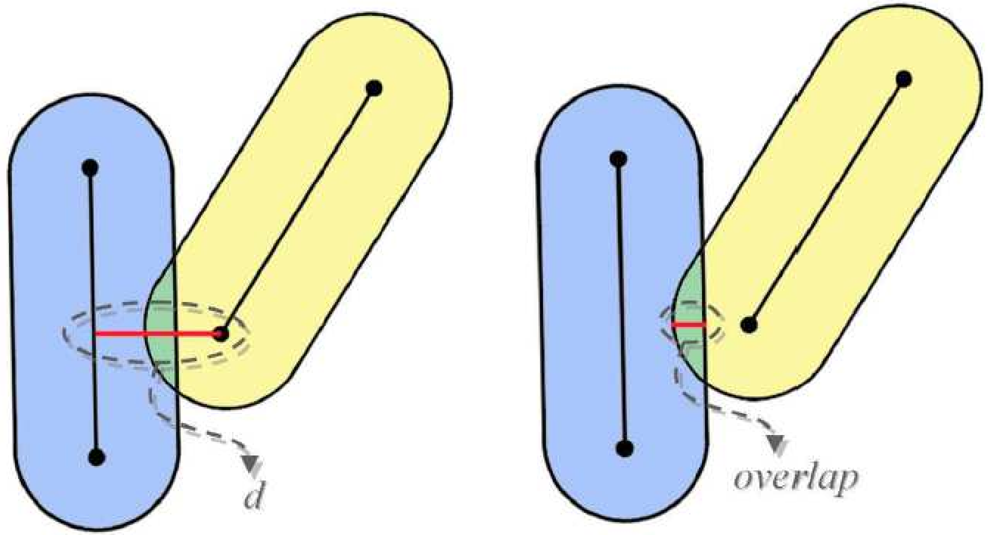
- **Swept sphere volumes (SSV)**
  - Construction





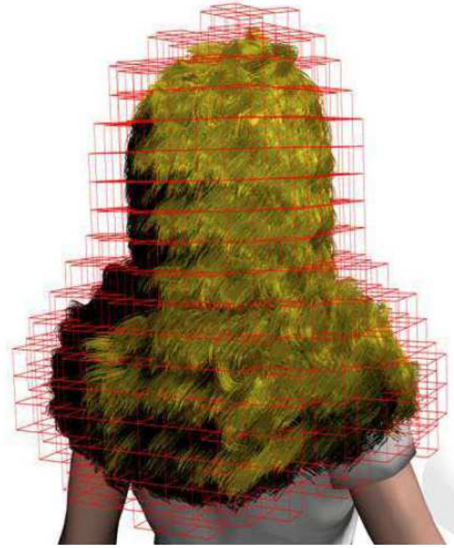
Detecting Guide-Strand Interactions

- **Swept sphere volumes (SSV)**
  - Collision detection



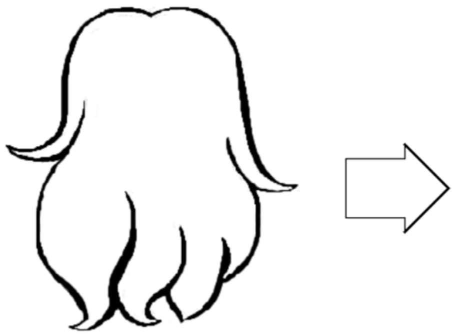
Detecting Guide-Strand Interactions

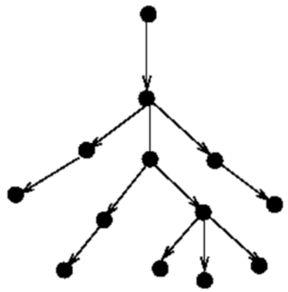
- **Naive implementation**
  - Directly compare the distance between hair strand segments
  - O(N^2) complexity
- **Regular 3D grid**
  - Get rid of most non-intersecting cases
  - Also be used between hair and the character model

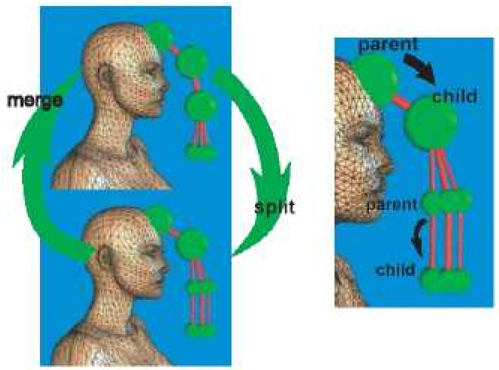


Adaptive Wisp Tree (AWT)

- **During hair motion, clusters form and split due to frictions**
  - Frictions, static charges
  - Features of the initial hairstyles
  - Clustering behaviors are observed moving from hair tips to roots



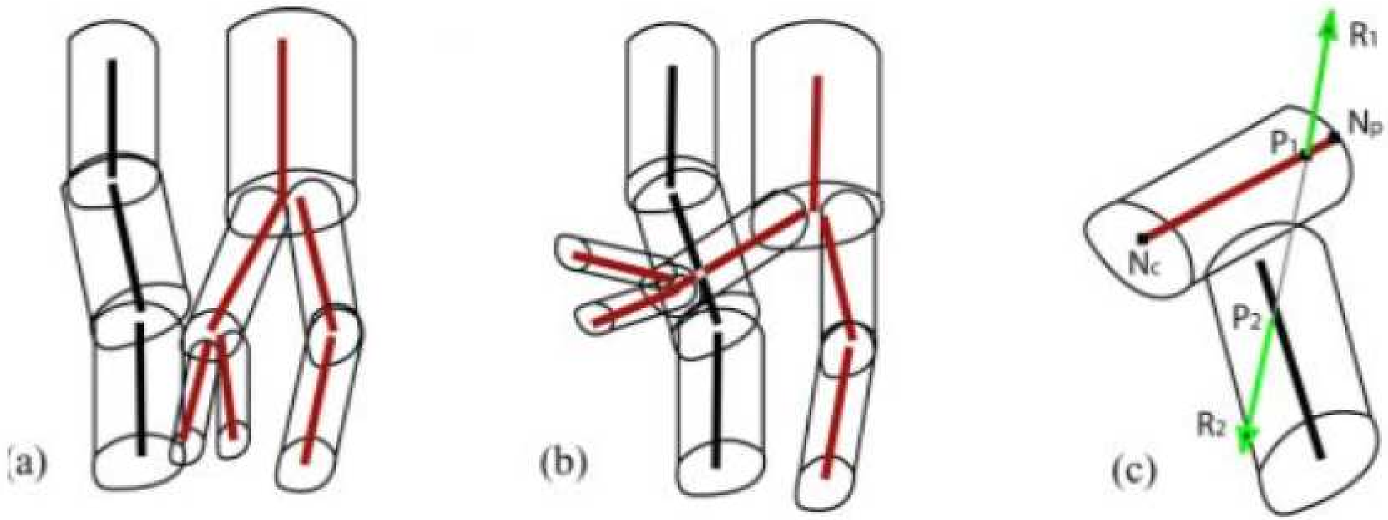




Adaptive Wisp Tree(AWT)

Adaptive Wisp Tree (AWT)

- **Hair interaction using AWT**
  - Use cylinders to approximate the hair segments



Response to guide-strand

- **Hair self-collisions should be very soft**
  - Frictional rather than bouncing behavior
  - Using soft penalty forces together with friction forces

- **Penalty and friction forces**

Direction of collision

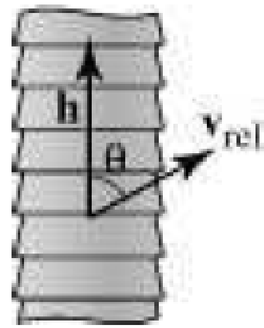
$$\begin{cases} \text{if } (gap \leq 0) & \boldsymbol{R_N} = \mathbf{0} \\ \text{if } (0 \leq gap \leq \delta_{reg}) & \boldsymbol{R_N} = \frac{k_c\, gap^2}{2\,\delta_{reg}} \boldsymbol{n_c} \\ \text{else} & \boldsymbol{R_N} = k_c\,(gap - \frac{\delta_{reg}}{2})\,\boldsymbol{n_c} \end{cases}$$

Penalty force

$$\mathbf{R_T} = -\nu\,(\mathbf{v_{rel}} - (\mathbf{v_{rel}}.\mathbf{n_c})\,\mathbf{n_c})$$

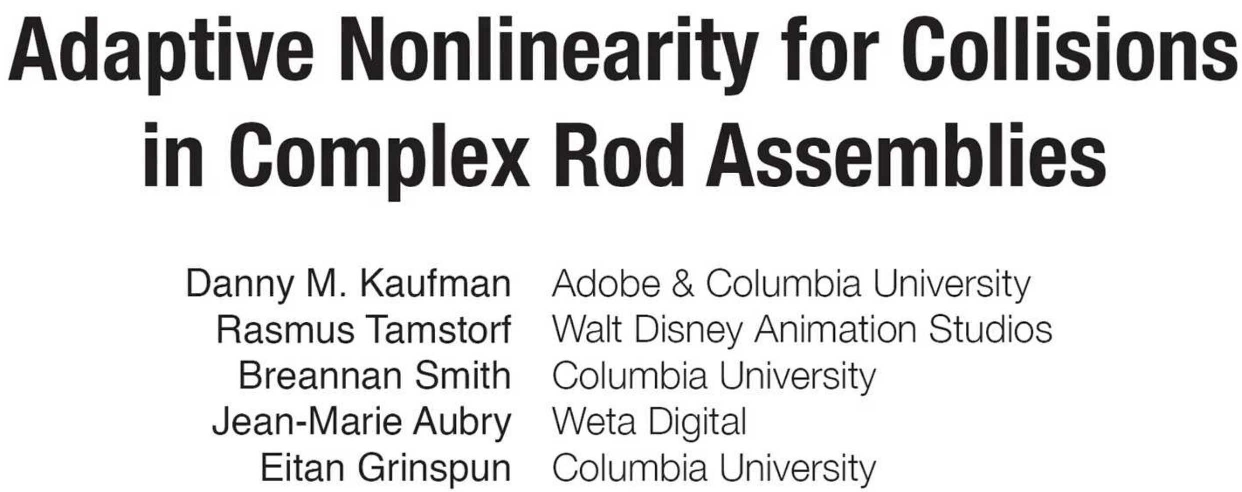
$$\nu = \nu_0\,(1 + \sin(\theta/2))$$

Friction force



Adaptive Nonlinearity for Collisions

- **A collision response algorithm**
  - Adapting the degree of nonlinearity


Adaptive Nonlinearity for Collisions in Complex Rod Assemblies

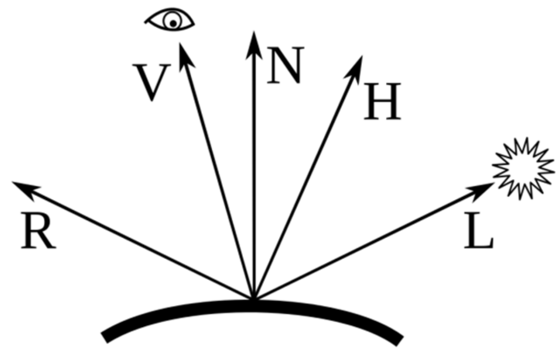
Danny M. Kaufman	Adobe & Columbia University
Rasmus Tamstorf	Walt Disney Animation Studios
Breannan Smith	Columbia University
Jean-Marie Aubry	Weta Digital
Eitan Grinspun	Columbia University

III. Hair Rendering

Traditional Rendering Model

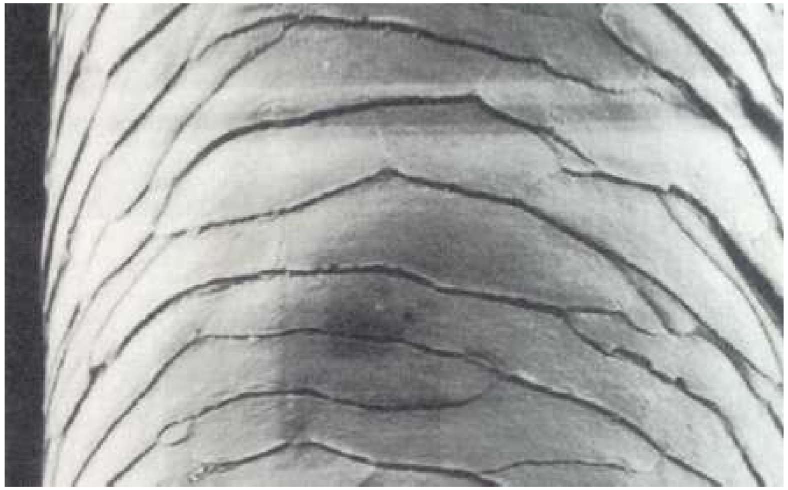
- **Phong reflection model**
  - Constant ambient light
  - Diffuse lighting
  - Specular lighting

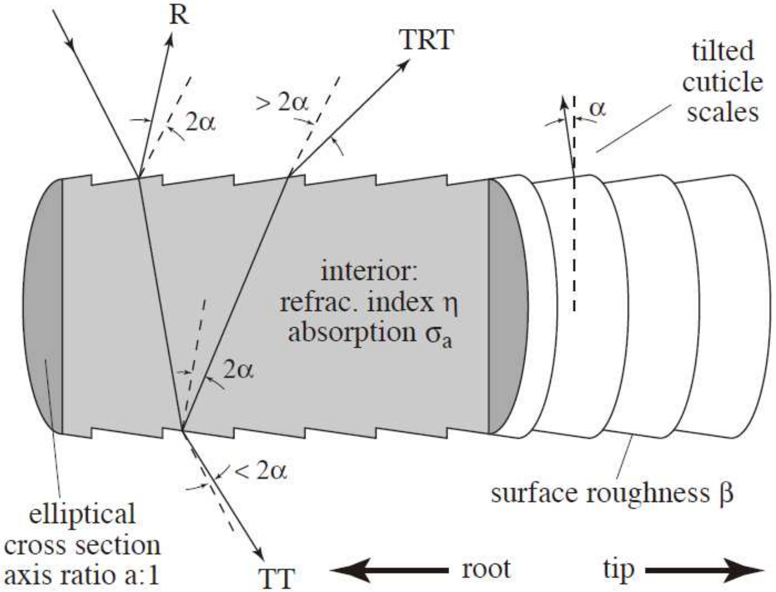
$$I_{\mathrm{p}} = k_{\mathrm{a}} i_{\mathrm{a}} + \sum_{m \in \text{lights}} (k_{\mathrm{d}} (\hat{L}_m \cdot \hat{N}) i_{m,\mathrm{d}} + k_{\mathrm{s}} (\hat{R}_m \cdot \hat{V})^{\alpha} i_{m,\mathrm{s}})$$



Hair Material Property

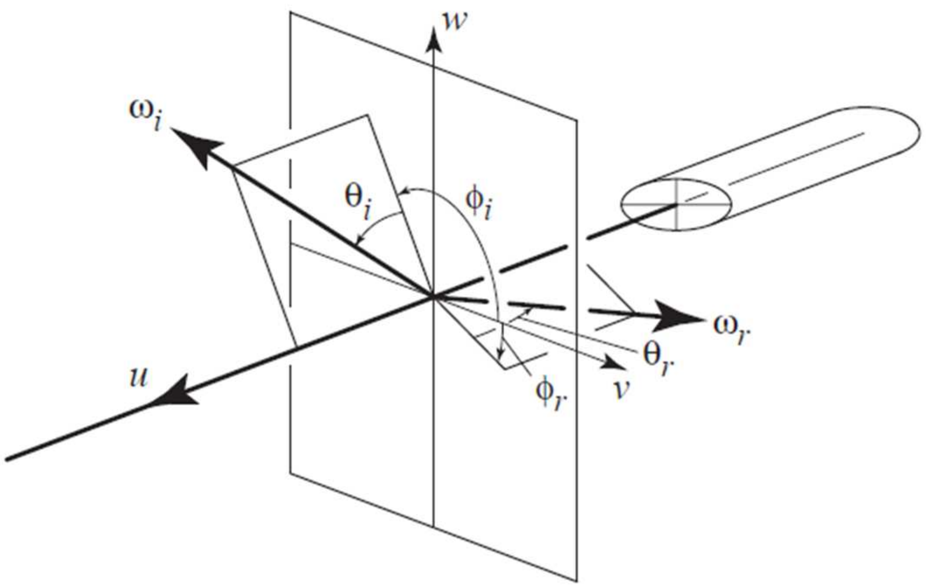
- **The close-look of hair structure**





Light Scattering in Hair

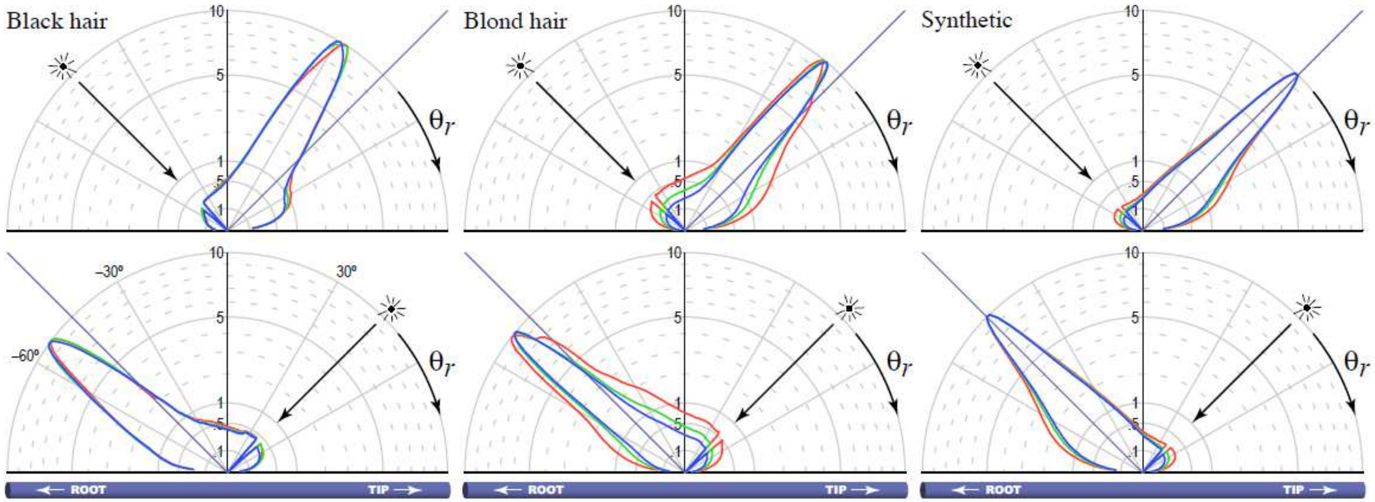
- **Basic notation**



- Direction of illumination ω_i and scattering ω_r
- Normal plane

Light Scattering in Hair

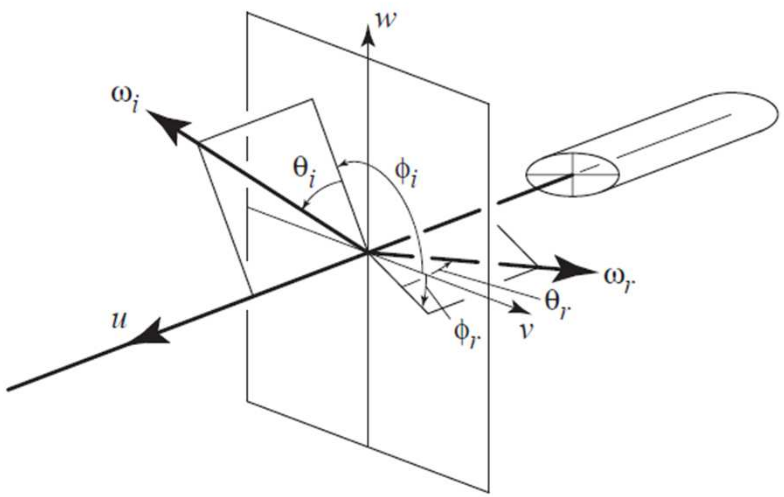
- **Formulation**
  - The scattering integral $\bar{L}_r(\omega_r) = D \int S(\omega_i, \omega_r) L_i(\omega_i) \cos\theta_i \, d\omega_i$

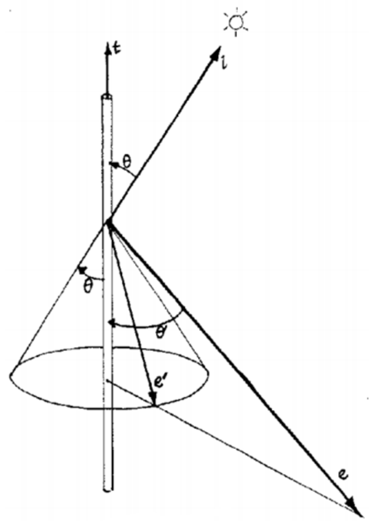


Kajiya-Kay Hair Rendering Model

- **Formulation**

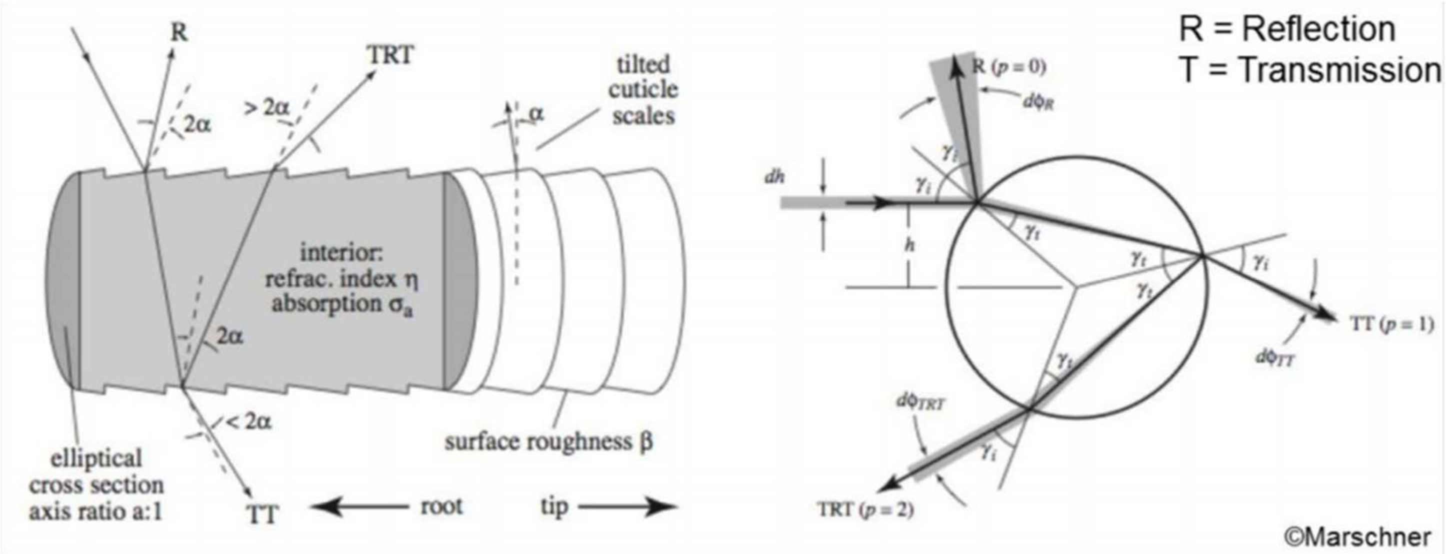
$$S(\theta_i, \phi_i, \theta_r, \phi_r) = k_d + k_s \frac{\cos^p(\theta_r + \theta_i)}{\cos(\theta_i)}$$





Marschner's Hair Rendering Model

- **Consider both reflectance and transmission**



Marschner's Hair Rendering Model

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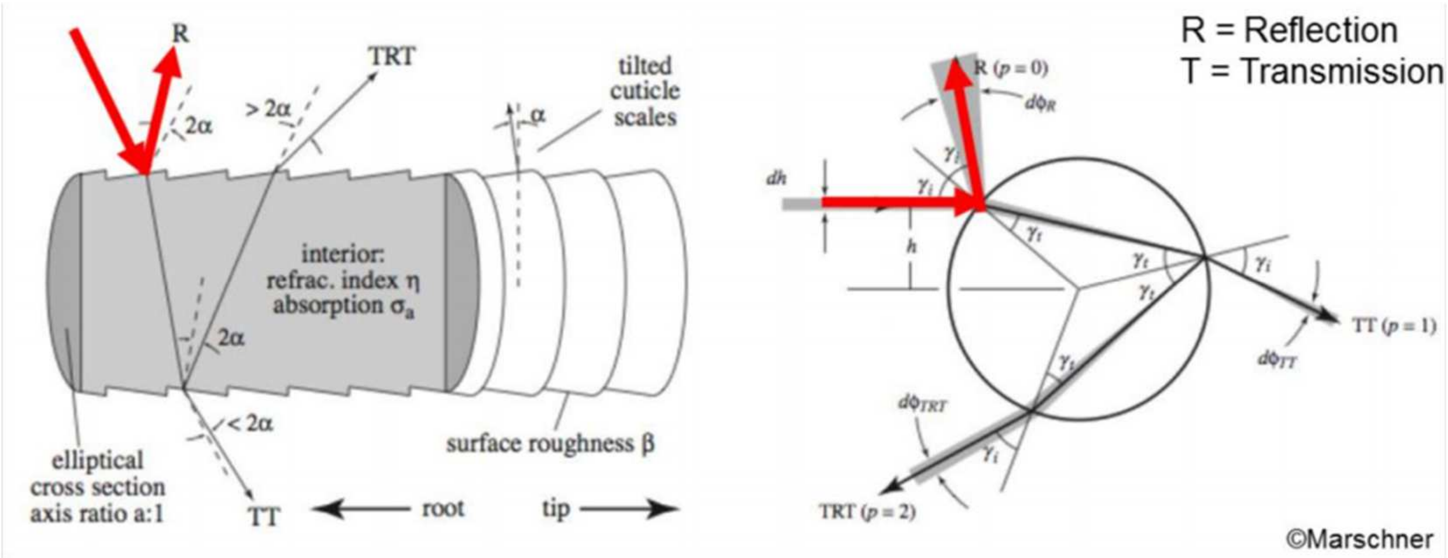

Marschner's Hair Rendering Model

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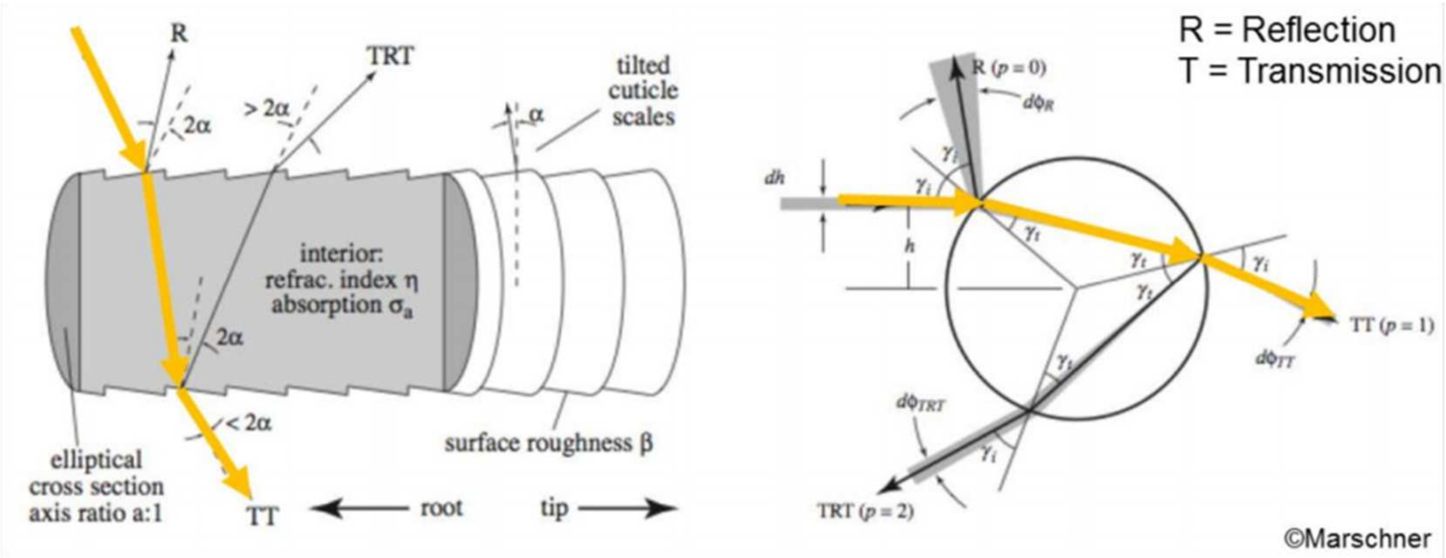
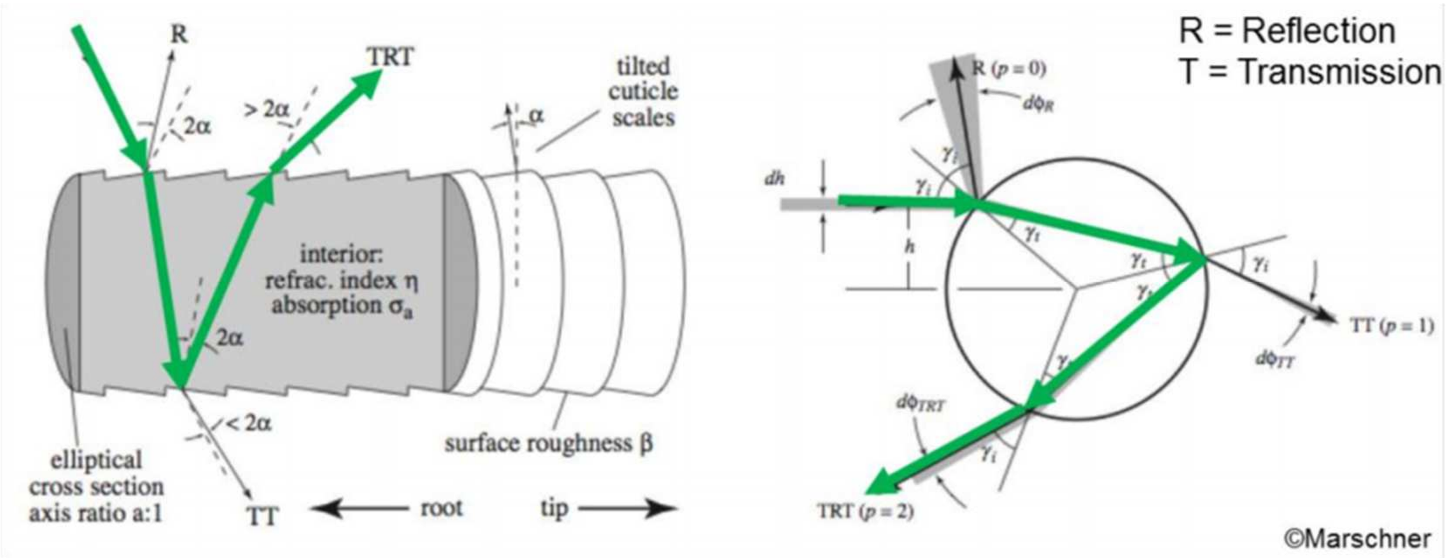


Marschner's Hair Rendering Model

- **Considering both reflectance and transmission**



Marschner's Hair Rendering Model

Marschner’s Hair Rendering Model

(a) R component

(b) TRT component

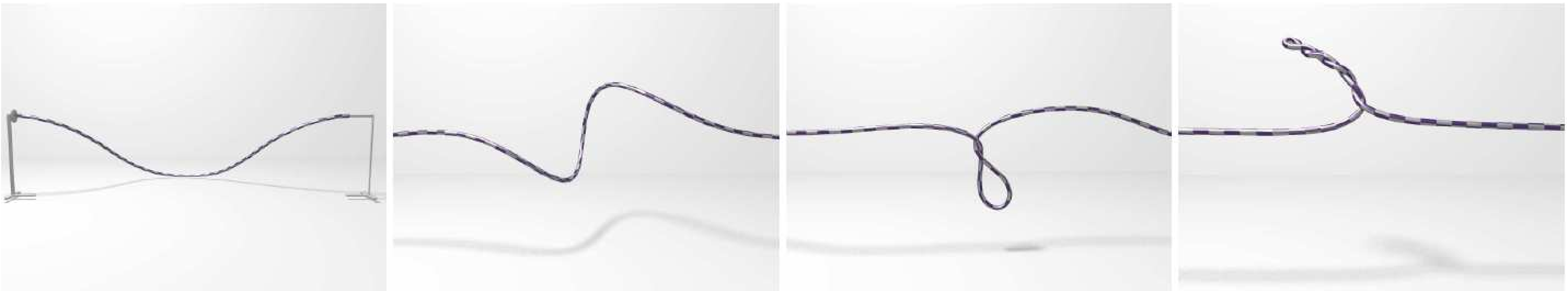
(c) Full model

(d) (e) Full model with different parameter

IV. Elastic Rod

Dynamic looping of a rod

- **A rod under torsional strain**
  - Material torsion
  - Robust handling of self-contacts



Cosserat Rod Elements for Dynamic Simulation

- **Physically based deformation model**
  - For one-dimensional elastic objects with torsion
  - Inspired by the Cosserat theory of elastic rods
  - From flexible structures (threads, ropes or hair strands) to stiff objects with intrinsic bending and torsion (springs or wires)

- **Energy-based formulation**
  - Continuous kinetic, potential and dissipation energy
  - Discretization by employing finite element methods

Cosserat theory of elastic rods

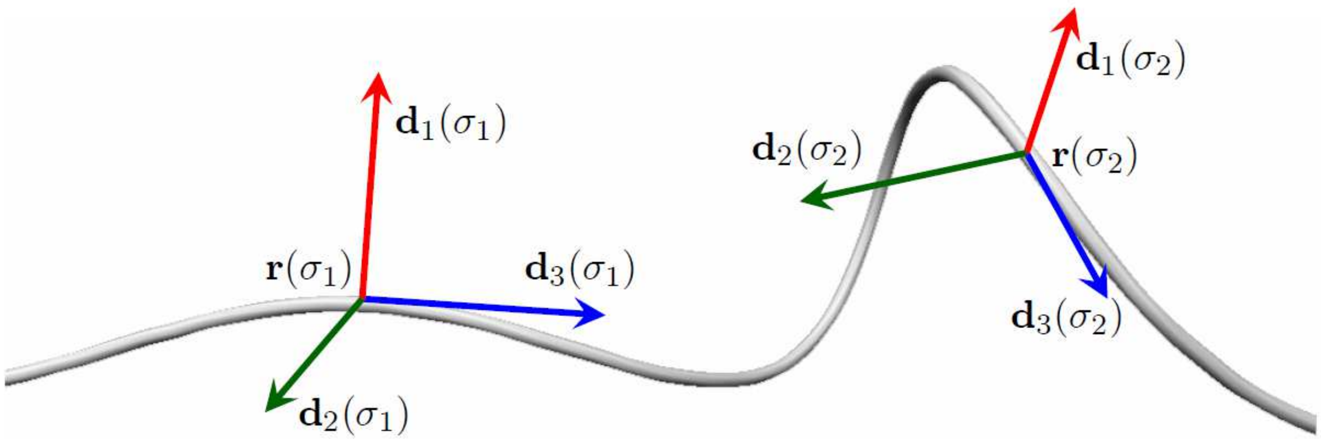
- **Representation of rods**
  - Thought of as a long and thin deformable body
  - Characterized by the centerline

$$\mathbf{r}(\sigma) = (r_x(\sigma), r_y(\sigma), r_z(\sigma))^{\mathsf{T}} \qquad \sigma \in [0, 1]$$

  - Stretching of the centerline **r** at σ
    - Spatial derivative: $\|\mathbf{r}'(\sigma)\|$
    - Unstretched (after normalization): $\|\mathbf{r}'\| = 1$

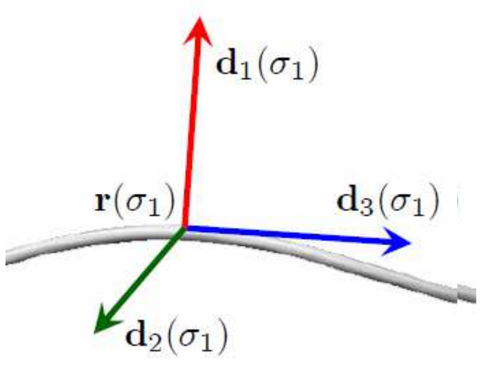
Cosserat theory of elastic rods

- **Representation of rods**
  - Configuration of the rod is defined by its centerline



Cosserat theory of elastic rods

- **Darboux vector**
  - From differential geometry

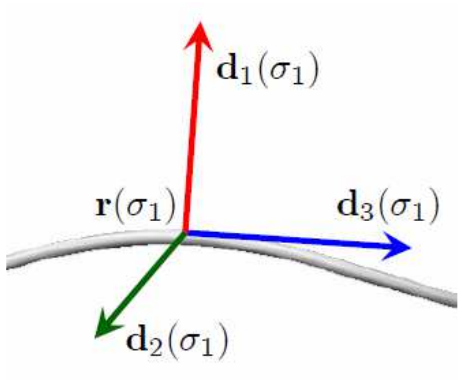


$$\mathbf{d}_k' = \boxed{\mathbf{u}} \times \mathbf{d}_k, \qquad k = 1, 2, 3$$

  - Strain rates for the bending and torsion can be expressed as

$$u_k = \mathbf{u} \cdot \mathbf{d}_k, \qquad k = \underbrace{1, 2}_{\text{bending}}, \underbrace{3}_{\text{torsion}}$$

Cosserat theory of elastic rods



- **Temporal derivative of the centerline**
  - Translational velocity of the mass point

$$\dot{\mathbf{r}}(\sigma)$$

  - The angular velocity related to the temporal derivatives of the directors

$$\dot{\mathbf{d}}_k = \omega \times \mathbf{d}_k, \qquad k = 1, 2, 3$$

    - Angular velocity of each direction

$$\omega_k = \omega \cdot \mathbf{d}_k$$

Cosserat theory of elastic rods

- **Representation of rotation**
  - Euler angles
    - Suffer from singularities (e.g., gimbal lock)

  - Quaternion representation

$$\mathbf{q} = (q_1, q_2, q_3, q_4)^\top \text{ with } q_i \in \mathbb{R}$$

    - Only unit quaternions represent pure rotations (normalization)

$$||\mathbf{q}|| = 1$$

Cosserat theory of elastic rods

- **The directors in terms of the quaternion**

$$\mathbf{d}_1 = \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 \\ 2(q_1 q_2 + q_3 q_4) \\ 2(q_1 q_3 - q_2 q_4) \end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix} 2(q_1 q_2 - q_3 q_4) \\ -q_1^2 + q_2^2 - q_3^2 + q_4^2 \\ 2(q_2 q_3 + q_1 q_4) \end{pmatrix}$$

$$\mathbf{d}_3 = \begin{pmatrix} 2(q_1 q_3 + q_2 q_4) \\ 2(q_2 q_3 - q_1 q_4) \\ -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{pmatrix}$$

Cosserat theory of elastic rods

- **Strain rates in the local frame**

$$u_k = \frac{2}{||\mathbf{q}||^2}\mathbf{B}_k\mathbf{q}\cdot\mathbf{q}'$$

- **Angular velocity components in the local frame**

$$\omega_k = \frac{2}{||\mathbf{q}||^2}\mathbf{B}_k\mathbf{q}\cdot\dot{\mathbf{q}} \qquad \omega_k^0 = \frac{2}{||\mathbf{q}||^2}\mathbf{B}_k^0\mathbf{q}\cdot\dot{\mathbf{q}}$$

B and $\mathbf{B}^0$ are skew symmetric matrices

Cosserat theory of elastic rods

- **Energy formulation**
  - Potential energy
    - Energy V_s of the stretch deformation(neglecting shear)

$$V_s = \frac{1}{2}\int_0^1 K_s(\|\mathbf{r}'\| - 1)^2 d\sigma$$

    - Bending energy V_b follows from the strain rates

$$V_b = \frac{1}{2}\int_0^1 \sum_{k=1}^{3} K_{kk}\left(\frac{2}{\|\mathbf{q}\|^2}\mathbf{B}_k\mathbf{q}\cdot\mathbf{q}' - \hat{u}_k\right)^2 d\sigma \qquad u_k = \frac{2}{\|\mathbf{q}\|^2}\mathbf{B}_k\mathbf{q}\cdot\mathbf{q}'$$

stiffness tensor

Cosserat theory of elastic rods

- **Energy formulation**
  - Kinetic energy
    - Translational energy T_t of the centerline

$$T_t = \frac{1}{2}\int_0^1 \rho\pi r^2 \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,d\sigma$$

    - Rotational energy follows from the angular velocities

$$T_r = \frac{1}{2}\int_0^1 \sum_{k=1}^{3} I_{kk}\Big(\frac{2}{\|\mathbf{q}\|^2}\mathbf{B}_k\mathbf{q}\cdot\dot{\mathbf{q}}\Big)^2 d\sigma \qquad \omega_k = \frac{2}{\|\mathbf{q}\|^2}\mathbf{B}_k\mathbf{q}\cdot\dot{\mathbf{q}}$$

inertia tensor

Cosserat theory of elastic rods

- **Energy formulation**
  - Dissipation energy
    - Internal friction energy, damping velocity

$$D_t = \frac{1}{2}\int_0^1 \gamma_t \mathbf{v}^{(rel)} \cdot \mathbf{v}^{(rel)} d\sigma$$

$$\mathbf{v}^{(rel)} = \frac{1}{||\mathbf{r}'||^2}\mathbf{r}'(\dot{\mathbf{r}}' \cdot \mathbf{r}')$$

projected relative translational velocity

    - Rotational dissipation energy from relative angular velocity

$$D_r = \frac{1}{2}\int_0^1 \gamma_r \omega_0' \cdot \omega_0' d\sigma$$

Cosserat theory of elastic rods

- **Lagrangian equation of motion**
  - Dynamic equilibrium configuration of an elastic rod
    - A critical point of the Lagrangian

$$L = T - V + D$$

    - Subject to the holonomic constraints

$$\mathbf{C}_p \equiv \frac{\mathbf{r}'}{||\mathbf{r}'||} - \mathbf{d}_3 = 0$$

$$C_q \equiv ||\mathbf{q}||^2 - 1 = 0$$

Cosserat theory of elastic rods

- **Lagrangian equation of motion**
  - Employing calculus of variations by introducing Lagrangian multipliers

$$\lambda \in \mathbb{R}^3 \text{ and } \mu \in \mathbb{R}$$

  - Lagrangian equation of motion for an elastic rod

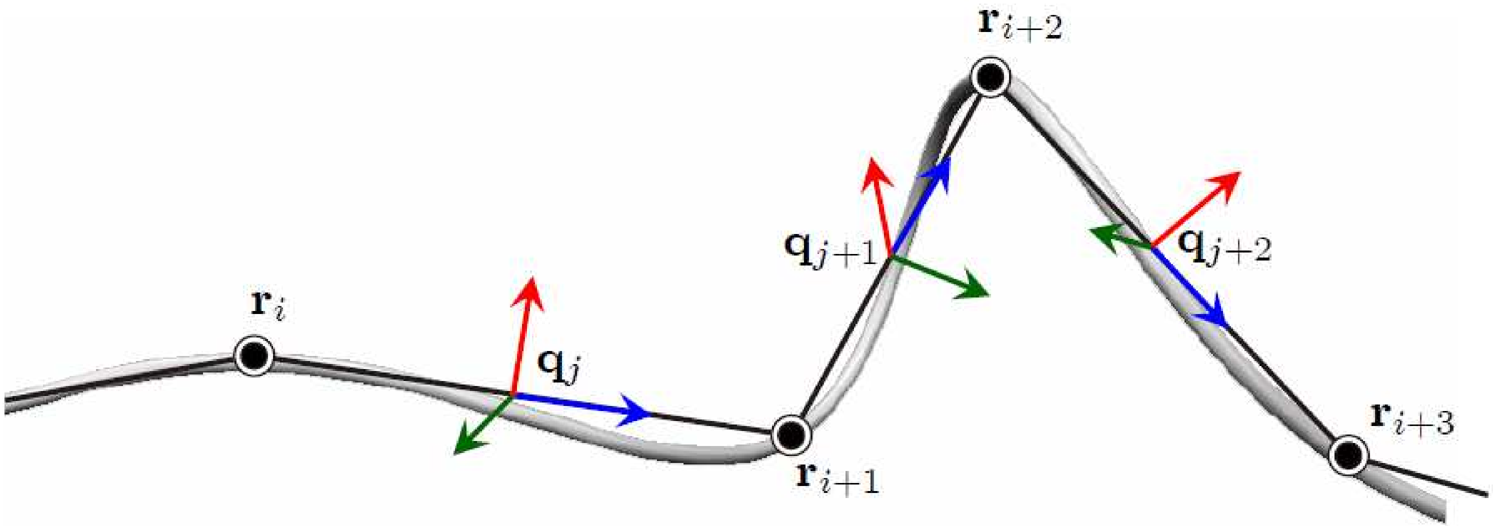
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{g}_i} - \frac{\partial T}{\partial g_i} + \frac{\partial V}{\partial g_i} + \frac{\partial D}{\partial \dot{g}_i} + \lambda \cdot \frac{\partial \mathbf{C}_p}{\partial g_i} + \mu \frac{\partial C_q}{\partial g_i} = \int_0^1 \mathbf{F}_e d\sigma$$

$$g_i \in \{r_x, r_y, r_z, q_1, q_2, q_3, q_4\}$$

external forces and torques

Cosserat theory of elastic rods

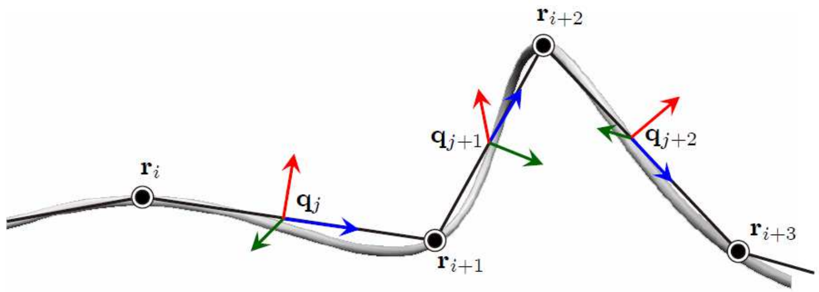
- **Discrete energy formulation**
  - Discretize the rod into elements



Cosserat theory of elastic rods

- **Discrete energy formulation**
  - Discrete spatial derivative of the centerline

$$\mathbf{r}_i' = \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\|\mathbf{r}_{i+1} - \mathbf{r}_i\|}$$



  - By assuming a high stretch stiffness

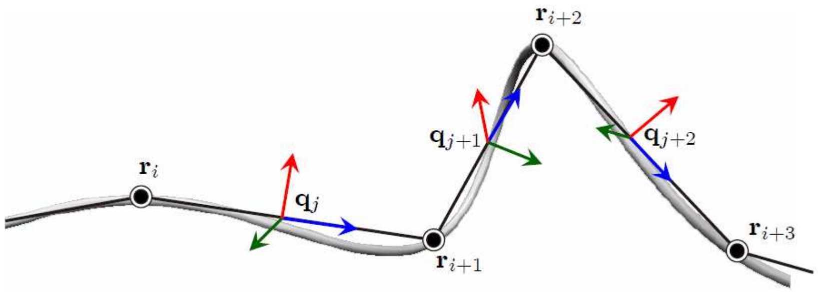
$$\mathbf{r}_i' \approx \frac{1}{l_i}(\mathbf{r}_{i+1} - \mathbf{r}_i) \qquad l_i = \|\mathbf{r}_{i+1}^0 - \mathbf{r}_i^0\|$$

initial positions of the mass points

Cosserat theory of elastic rods

- **Discrete energy formulation**
  - Discrete spatial derivative of the quaternion for orientation

$$\mathbf{q}'_j \approx \frac{1}{l_j}(\mathbf{q}_{j+1} - \mathbf{q}_j)$$

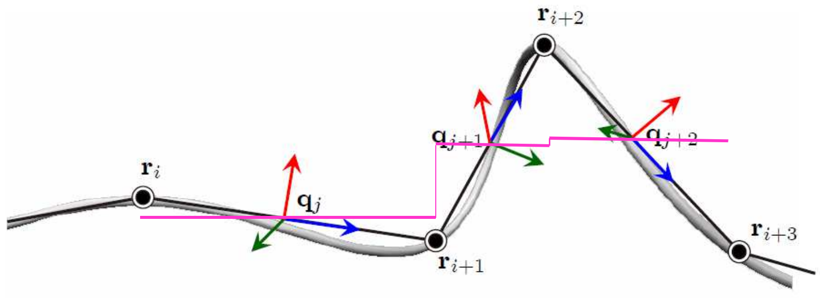


$$l_j = \frac{1}{2}(\|\mathbf{r}^0_{i+2} - \mathbf{r}^0_{i+1}\| + \|\mathbf{r}^0_{i+1} - \mathbf{r}^0_i\|)$$

Cosserat theory of elastic rods

- **Discrete energy formulation**
  - Interpolate the displacements within the elements
    - Employ constant shape functions

$$\bar{\mathbf{r}}_i(\xi) = \bar{\mathbf{r}}_i = \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_{i+1})$$
$$\bar{\mathbf{q}}_j(\xi) = \bar{\mathbf{q}}_j = \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_{j+1})$$



- Higher-order shape functions increase the visual plausibility
  - Require significantly more computations

Cosserat theory of elastic rods

- **Discrete energies**
  - Expression for the stretch energy

$$
\begin{aligned}
V_s[i] &= \frac{1}{2}\int_0^{l_i} K_s(\|\mathbf{r}_i'\| - 1)^2 d\xi \\
&= \frac{1}{2} l_i K_s\left(\frac{1}{l_i}\sqrt{(\mathbf{r}_{i+1} - \mathbf{r}_i)\cdot(\mathbf{r}_{i+1} - \mathbf{r}_i)} - 1\right)^2
\end{aligned}
$$

  - Expression for the bending energy

$$
\begin{aligned}
V_b[j] &= \frac{1}{2}\int_0^{l_j} \sum_{k=1}^{3} K_{kk}\Big(\underbrace{\frac{2}{\|\bar{\mathbf{q}}_j\|^2}}_{\approx 2} \mathbf{B}_k \bar{\mathbf{q}}_j \cdot \mathbf{q}_j' - \hat{u}_k\Big)^2 d\xi \\
&= \frac{l_j}{2}\sum_{k=1}^{3} K_{kk}\left(\mathbf{B}_k(\mathbf{q}_j + \mathbf{q}_{j+1})\cdot\frac{1}{l_j}(\mathbf{q}_{j+1} - \mathbf{q}_j) - \hat{u}_k\right)^2
\end{aligned}
$$

Cosserat theory of elastic rods

- **Discrete potential energy**
  - Kinetic translational energy per centerline element

$$
\begin{aligned}
T_t[i] &= \frac{1}{2}\int_0^{l_i} \rho\pi r^2 \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i d\xi \\
&= \frac{1}{8}\rho\pi r^2 l_i(\dot{\mathbf{r}}_{i+1} + \dot{\mathbf{r}}_i)\cdot(\dot{\mathbf{r}}_{i+1} + \dot{\mathbf{r}}_i)
\end{aligned}
$$

  - Rotational kinetic energy per centerline element

$$
\begin{aligned}
T_r[j] &= \frac{1}{2}\int_0^{l_j} \sum_{k=1}^{3} I_{kk}(\underbrace{\frac{2}{\|\bar{\mathbf{q}}_j\|^2}}_{\approx 2}\mathbf{B}_k\bar{\mathbf{q}}_j \cdot \dot{\bar{\mathbf{q}}}_j)^2 d\xi \\
&= \frac{l_j}{8}\sum_{k=1}^{3} I_{kk}(\mathbf{B}_k(\mathbf{q}_j + \mathbf{q}_{j+1})\cdot(\dot{\mathbf{q}}_j + \dot{\mathbf{q}}_{j+1}))^2
\end{aligned}
$$

Cosserat theory of elastic rods

- **Discrete kinetic energy**
  - Translational dissipation energy per centerline element

$$D_t[i] = \frac{1}{2}\int_0^{l_i} \gamma_t \mathbf{v}_i^{(rel)} \cdot \mathbf{v}_i^{(rel)} d\xi = \frac{l_i}{2}\gamma_t \mathbf{v}_i^{(rel)} \cdot \mathbf{v}_i^{(rel)}$$

    - Projected relative translational velocity

$$\mathbf{v}_i^{(rel)} = \underbrace{\frac{1}{\|\mathbf{r}_i'\|^2}}_{\approx 1} \mathbf{r}_i'(\dot{\mathbf{r}}_i' \cdot \mathbf{r}_i') = \frac{1}{l_i^3}(\mathbf{r}_{i+1} - \mathbf{r}_i)\Big((\dot{\mathbf{r}}_{i+1} - \dot{\mathbf{r}}_i) \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)\Big)$$

Cosserat theory of elastic rods

- **Discrete kinetic energy**
  - Rotational dissipation energy per centerline element

$$D_r[j] = \frac{1}{2}\int_0^{l_j} \gamma_r \sum_{k=1}^{3} \left(\frac{1}{l_j} 2\mathbf{B}_k^0 \mathbf{q}_{j+1}\dot{\mathbf{q}}_{j+1} - \frac{1}{l_j} 2\mathbf{B}_k^0 \mathbf{q}_j \dot{\mathbf{q}}_j\right)^2 d\xi$$

$$= \frac{2}{l_j}\gamma_r \sum_{k=1}^{3} \left(\mathbf{B}_k^0 \mathbf{q}_{j+1}\dot{\mathbf{q}}_{j+1} - \mathbf{B}_k^0 \mathbf{q}_j \dot{\mathbf{q}}_j\right)^2$$

    - Assuming $\|\mathbf{q}_j\| \approx 1$

Numerical solution

- **Assembly of the equations of motion**
  - Substituting the discrete energies per element into

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{g}_i} - \frac{\partial T}{\partial g_i} + \frac{\partial V}{\partial g_i} + \frac{\partial D}{\partial \dot{g}_i} + \lambda \cdot \frac{\partial \mathbf{C}_p}{\partial g_i} + \mu \frac{\partial C_q}{\partial g_i} = \int_0^1 \mathbf{F}_e d\sigma$$

  - Writing for all elements results in a system of equations of the form

$$\mathbf{M}\ddot{\mathbf{g}} + \mathbf{f}(\dot{\mathbf{g}}, \mathbf{g}) = (\mathbf{F}_e\ \tau_e)^{\mathsf{T}}$$

Conjugate gradient method, expensive!

Contain nonlinear terms

conforms to a nonlinear stiffness function

Numerical solution

- **Implementation of the constraints**
  - Look at the formulation again

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{g}_i} - \frac{\partial T}{\partial g_i} + \frac{\partial V}{\partial g_i} + \frac{\partial D}{\partial \dot{g}_i} + \boxed{\lambda \cdot \frac{\partial \mathbf{C}_p}{\partial g_i} + \mu \frac{\partial C_q}{\partial g_i}} = \int_0^1 \mathbf{F}_e d\sigma$$

Lagrange multipliers to enforce constraints

  - Penalty method (efficiency)
    - Formulated a constraint as an energy function $E(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$
    - The constraint is satisfied if

$E(\mathbf{x}) = 0$ $\Rightarrow$ Constraint forces **F**

Numerical solution

- **Implementation of the constraints**
  - Employ energy constraints from penalty method (potential energy)
    - Parallel constraint

$$\mathbf{C}_p \equiv \frac{\mathbf{r}'}{||\mathbf{r}'||} - \mathbf{d}_3 = 0 \quad \Rightarrow \quad E_p = \frac{1}{2}\int_0^1 \kappa\left(\frac{\mathbf{r}'}{||\mathbf{r}'||} - \mathbf{d}_3\right)\cdot\left(\frac{\mathbf{r}'}{||\mathbf{r}'||} - \mathbf{d}_3\right)d\sigma$$

$$\begin{aligned} E_p[i] &= \frac{1}{2}\int_0^{l_i} \kappa\left(\frac{\mathbf{r}_i'}{||\mathbf{r}_i'||} - \mathbf{d}_3(\mathbf{q}_i)\right)\cdot\left(\frac{\mathbf{r}_i'}{||\mathbf{r}_i'||} - \mathbf{d}_3(\mathbf{q}_i)\right)d\xi \\ &= \frac{l_i}{2}\kappa\left(\frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{||\mathbf{r}_{i+1} - \mathbf{r}_i||} - \mathbf{d}_3(\mathbf{q}_i)\right)\cdot\left(\frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{||\mathbf{r}_{i+1} - \mathbf{r}_i||} - \mathbf{d}_3(\mathbf{q}_i)\right) \end{aligned}$$

    - Rotational constraint (coordinate projection)

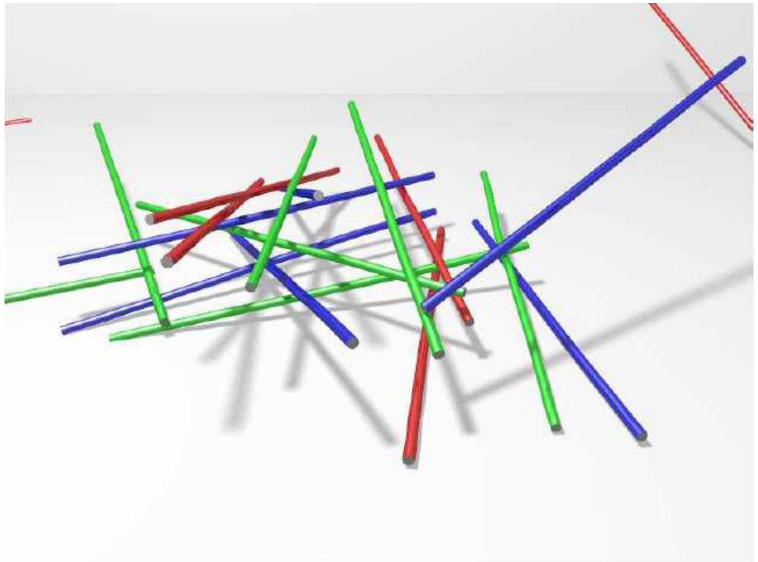
$$C_q \equiv ||\mathbf{q}||^2 - 1 = 0 \quad \Rightarrow \quad \mathbf{q}_i \leftarrow \frac{\hat{\mathbf{q}}_i}{||\hat{\mathbf{q}}_i||}$$

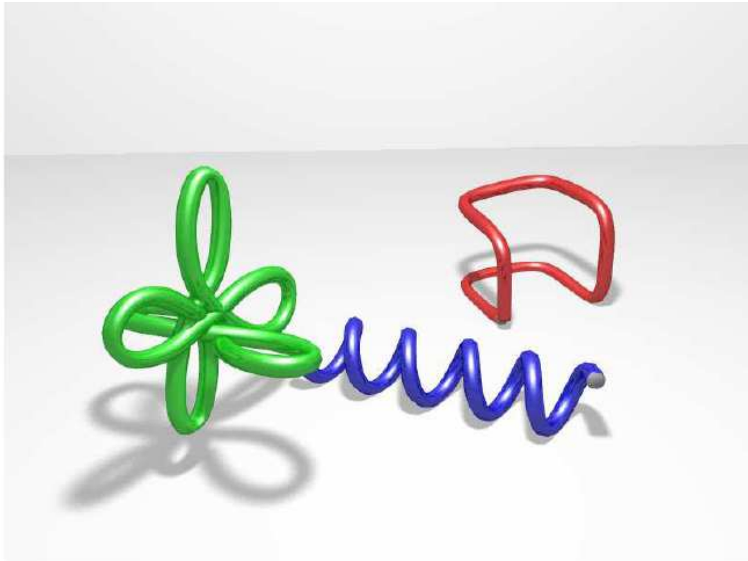
Results

- **Discrete rods**

Results

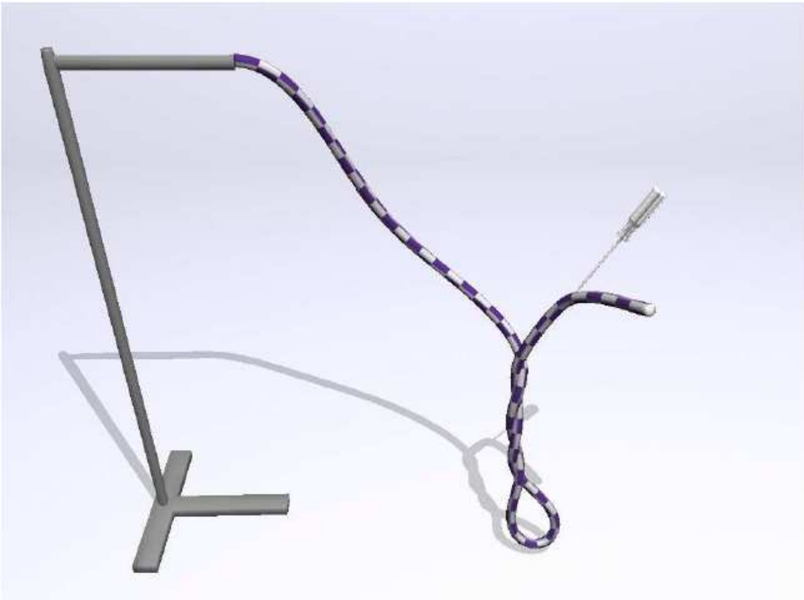
- **Pre-shaped stiff objects**

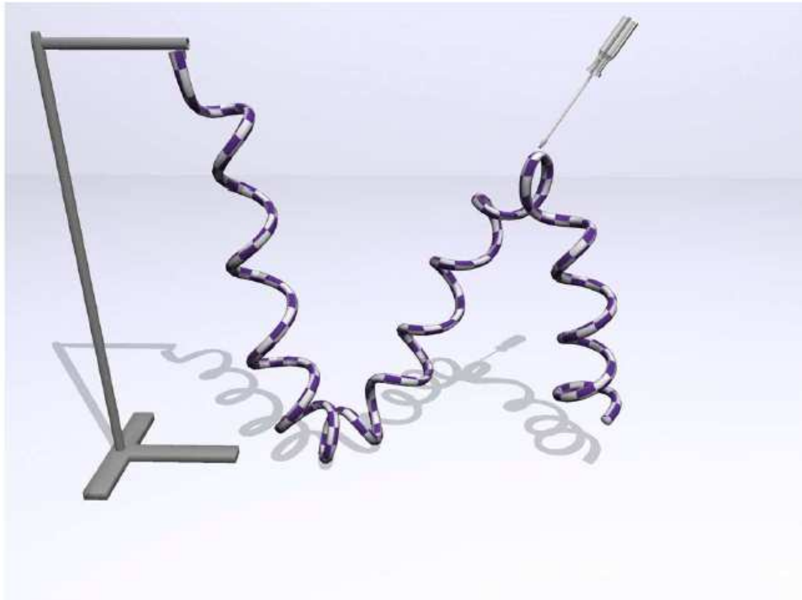




Results

- **Virtual interactive ropes and threads**

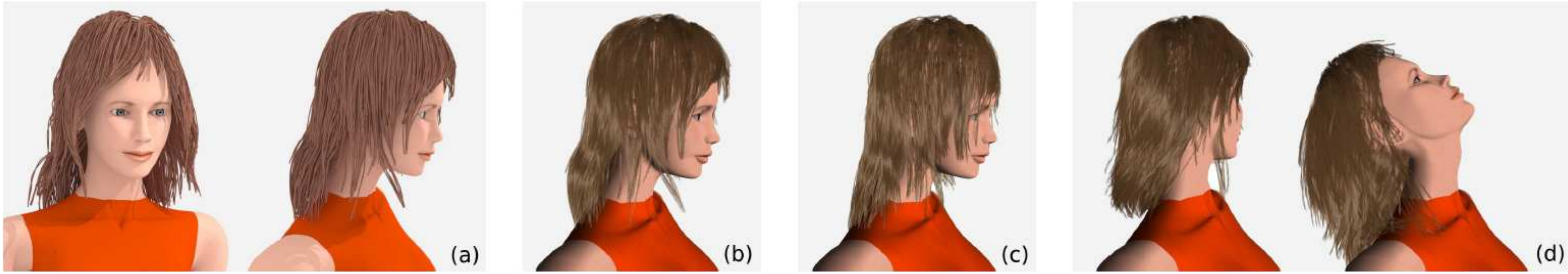




II. Inverse Hair Dynamics

Inverse Dynamic Hair Modeling with Frictional Contact

- **Recover the hair dynamics from geometric styling**
  - Recover frictional contact
  - Strictly convex second-order cone quadratic program



Inverse Dynamic Hair Modeling with Frictional Contact

INRIA and Laboratoire Jean Kuntzmann (Grenoble University, CNRS), France

Inverse Dynamic Hair Modeling with Frictional Contact

Alexandre Derouet-Jourdan
Florence Bertails-Descoubes
Gilles Daviet
Joëlle Thollot

Next Lecture: Soft-Body Simulation – Cloth I