



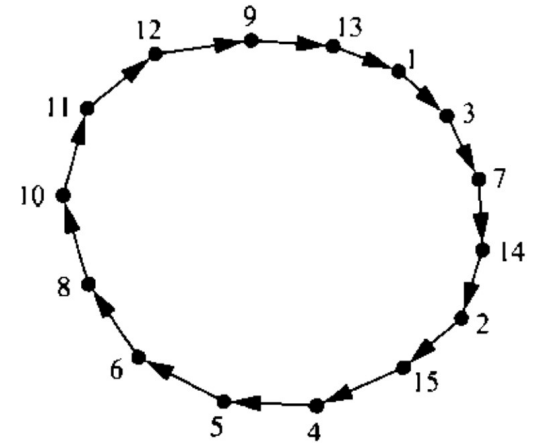
PRAM 2

Graph algorithms

CS121 Parallel Computing
Fall 2023

Coloring a cycle

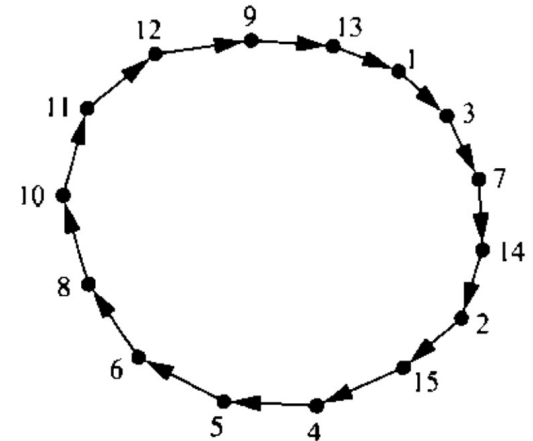
- Given a graph $G=(V,E)$, a k -coloring of G is a mapping $c: V \rightarrow \{0,1, \dots, k-1\}$ s.t. $c(i) \neq c(j)$ whenever $(i,j) \in E$.
- We give a super fast algorithm for 3-coloring a (directed) cycle of n nodes.
 - If n is odd, any coloring uses at least 3 colors.
- Coloring the cycle is a form of symmetry breaking.
- For any node v , let $S(v)$ be the node after v .
- The main subroutine is the following.
 - Initially, color every node by its node ID.
 - Consider the binary representation of $c(v)$ for a node v .
 - Let k be the least significant digit in which $c(v)$ and $c(S(v))$ differ.
 - Set $c'(v)=2^k+c(v)_k$, where $c(v)_k$ is the k 'th digit of $c(v)$.



v	c	k	c'
1	0001	1	2
3	0011	2	4
7	0111	0	1
14	1110	2	5
2	0010	0	0
15	1111	0	1
4	0100	0	0
5	0101	0	1
6	0110	1	3
8	1000	1	2
10	1010	0	0
11	1011	0	1
12	1100	0	0
9	1001	2	4
13	1101	2	5

Coloring a cycle

- **Claim** If c is a valid coloring, then so is c' .
- **Proof** Since c is a valid coloring, then $c(v) \neq c(S(v))$, so k exists.
 - Suppose $c'(v) = c'(u)$, for some v and $u = S(v)$.
 - Then $c'(v) = 2k + c(v)_k$ and $c'(u) = 2l + c(u)_l$ for some k and l .
 - Since $c'(v) = c'(u)$, then $k = l$, because $c(v)_k, c(u)_l < 2$.
 - But then $c(v)_k = c(u)_k$, contradicting the definition of k .



v	c	k	c'
1	0001	1	2
3	0011	2	4
7	0111	0	1
14	1110	2	5
2	0010	0	0
15	1111	0	1
4	0100	0	0
5	0101	0	1
6	0110	1	3
8	1000	1	2
10	1010	0	0
11	1011	0	1
12	1100	0	0
9	1001	2	4
13	1101	2	5



Coloring a cycle

- To analyze the time complexity, suppose in some round the max number of bits to represent any color is t .
- Then the max number of bits to represent any color in the next round is $\lceil \log t \rceil + 1$, because any color in the next round is $\leq 2t + 1$.
 - So the number of bits used to represent a color decreases from t to $\lceil \log t \rceil + 1$ in each round.
- Let $\log^{(i)} x = \log(\log^{(i-1)} x)$, i.e. we apply the log function i times to x .
- Let $\log^* x = \min\{i \mid \log^{(i)} x \leq 1\}$ be the number of times we have to take log's until a value becomes ≤ 1 .
 - $\log^* x$ is incredibly small. In fact, $\log^* x \leq 6$ for all $x \leq 2^{65536}!$

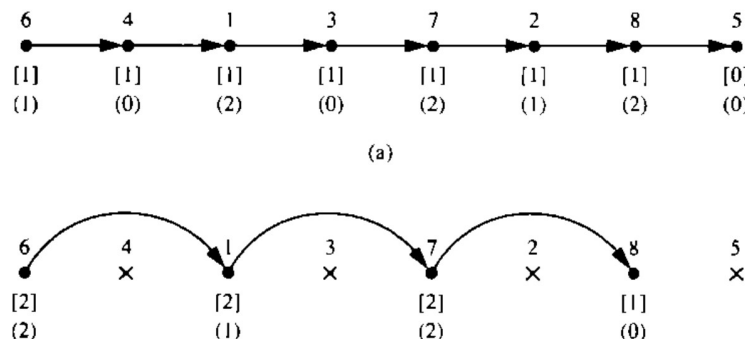


Coloring a cycle

- Since the number of bits to represent a color in the first round is $\log n$, then in $O(\log^* n)$ rounds, we can represent any color using $O(1)$ bits.
 - In fact, we can apply the subroutine until we use 6 colors in a round.
 - With 6 colors, need 3 bits to represent a color. So in the next round, colors are between 0 and $2^3-1=7$, and we again use up to 6 colors.
- To decrease the number of colors from 6 to 3, we run 3 more rounds.
 - In round i , take any node colored using color $i+2$ and color it using the min possible color in $\{0,1,2\}$, i.e. the min color not used by its neighbors.
- In total, we 3-color the ring in $O(\log^* n)$ rounds, using $O(n \log^* n)$ work.
- The algorithm can be modified to produce a 3-coloring in $O(\log n)$ time and $O(n)$ work.

Independent set on line

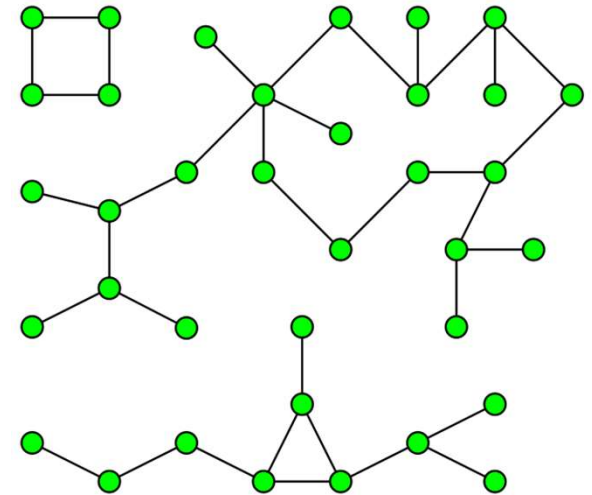
- **Thm** Given a k coloring of a line graph with n nodes, we can compute an independent set of size $\Omega(n/k)$ in $O(1)$ time.
- **Proof** Every node has a color from 1 to k .
 - Take the nodes whose colors are local minima as the indep. set S .
 - No two nodes in S are neighbors.
 - S can be computed in $O(1)$ time.
 - Consider two consecutive nodes $u, v \in S$.
 - Since u, v are local minima and consecutive, the colors between u, v first increase, then decrease.
 - Thus, there are at most $2k - 3$ nodes between u and v .
 - So there are $\leq 2k - 3$ nodes between any two consecutive nodes in S , and so $|S| \geq \frac{n}{2k-3}$.
- Thus, compute independent set of size $\Omega(n)$ on line with n nodes in $O(\log n)$ time by computing a 3 coloring of the line in $O(\log n)$ time, then computing an independent set of size $\Omega(n/3) = \Omega(n)$ in $O(1)$ time.



- The colors of the nodes are shown in parentheses.
- Nodes 4, 3, 2, 5 are local minima.

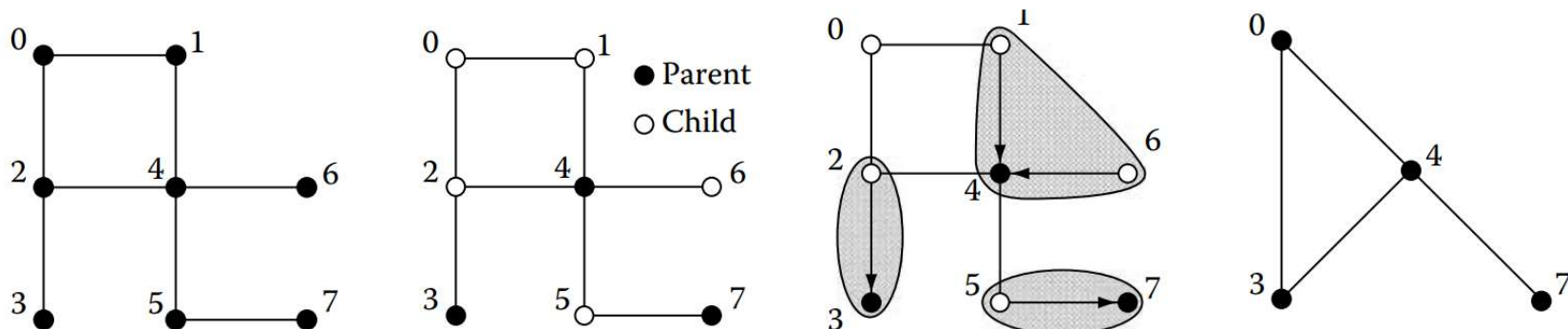
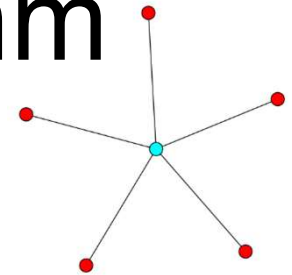
Connected components

- Given an undirected graph, partition it into maximal sets of nodes that are connected to each other.
- Can be solved sequentially in $O(m+n)$ time using BFS / DFS.
 - m is number of edges, n is number of vertices.
- However, no efficient BFS / DFS PRAM algorithms known.
- Instead, use graph contractions.
 - In each phase, merge (contract) a set of connected nodes into a supernode.
 - Form a contracted graph on the supernodes, then apply algorithm recursively.
 - Eventually each connected component is contracted to one node.
 - Many different algorithms, depending on which nodes they contract.



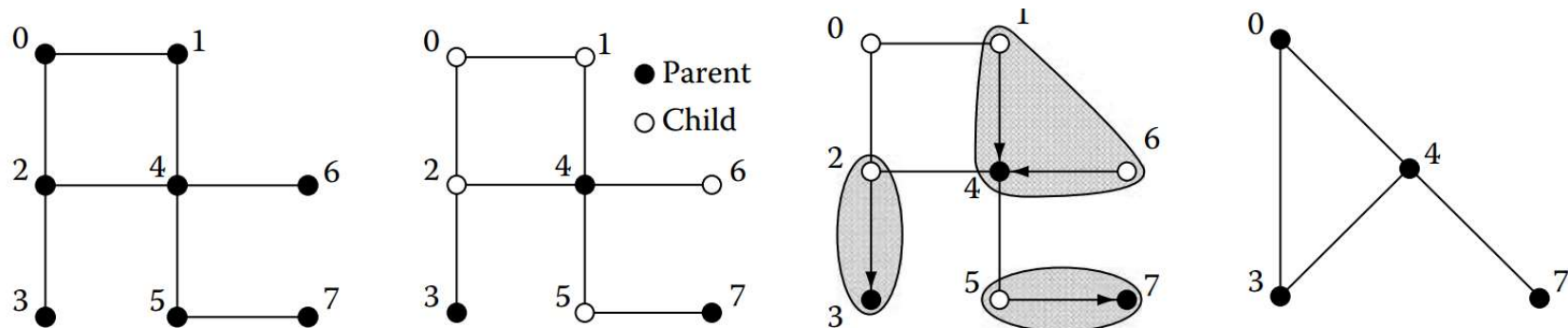
Randomized parallel algorithm

- Break graph into star graphs and contract each star.
- In each phase, do the following steps in parallel.
 - Every node flips a coin and chooses to be a parent or child node.
 - Each child node points to a parent node it's connected to.
 - Now have a set of stars, with the parent nodes as the centers.
 - If child not connected to any parent node, it forms its own star.
 - Contract each star to its center, then apply algorithm recursively.
 - Label all nodes in star by the label of the parent.
 - Keep the edges between differently labeled nodes.
 - After recursion returns, each child with a parent again takes parent's label, which might have changed.



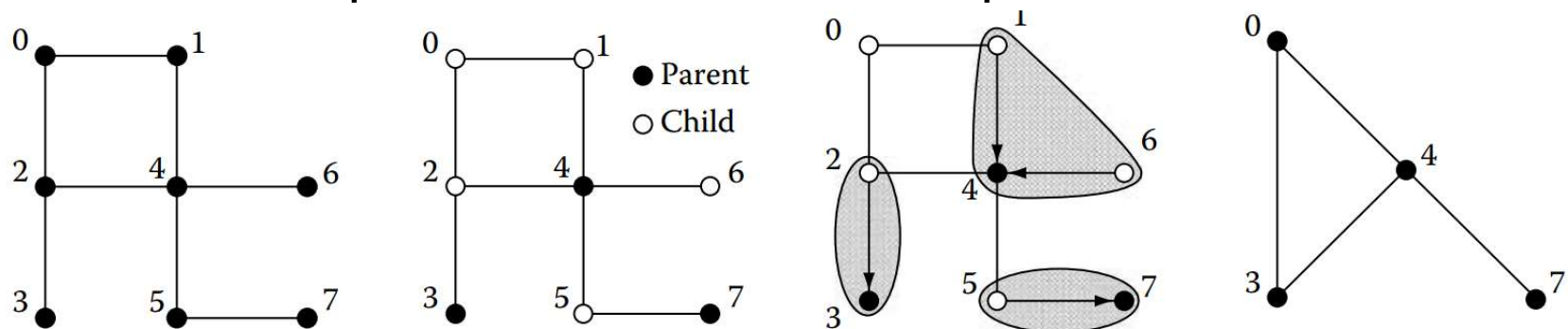
Implementation

- Use $n+m$ processors, one processor for each vertex and edge.
 - Call these V and E procs, resp.
 - Each V proc has a label which can change over time.
 - Each E proc responsible for edge (u,v) , where u, v are V procs.
 - V and E procs may become inactive over time.
- In each phase, each active V proc flips a coin to decide if it's a child or parent proc.
- Each active E proc (u,v) checks if u is a child proc and v is a parent (or vice versa).
 - If so, it sets u 's label to v (or v 's label to u).
 - Another E proc (u,v') could set u 's label to v' . In this case, either the v or v' write succeeds.



Implementation

- Each parent V proc, or child V proc whose label didn't change (i.e. it had no parent), stays active.
 - Other V procs become inactive.
- Each active E proc (u,v) where u, v have different labels stays active.
 - Other E procs become inactive.
 - From now on, E will be responsible for V processes (u',v') , where u' and v' are the labels of u and v , resp.
- The active V and E procs run the algorithm recursively.
- After recursion returns, inactive E procs (u,v) (where u is the child) write v 's label to u .
- At end, all V procs in a connected component have same label.



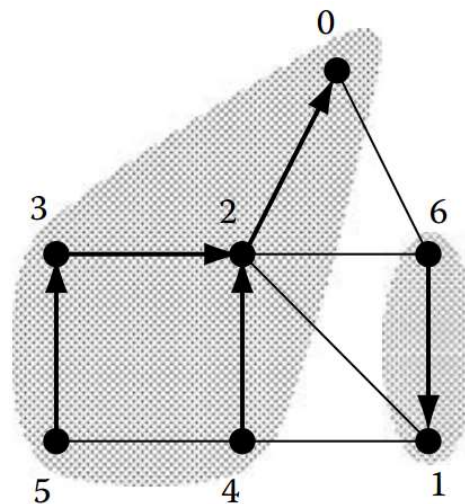
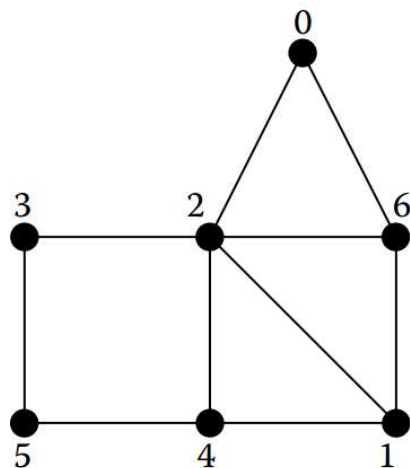


Complexity

- Key fact is that number of active V procs decreases by $1/4$ fraction in expectation each phase.
 - A V proc becomes inactive if it's a child node and one its neighbors is a parent node.
 - The former probability is $1/2$, and the latter is $\geq 1/2$.
 - Thus each V proc becomes inactive with probability $\geq 1/4$.
- With high probability, after $O(\log n)$ phases, there's only one V proc and recursion ends.
- Each phase takes $O(1)$ time, and does $O(m+n)$ work.
- Total time is $O(\log n)$, total work is $O((m+n) \log n)$.
 - This algorithm isn't work efficient.
 - There exist work efficient randomized CC algorithms.

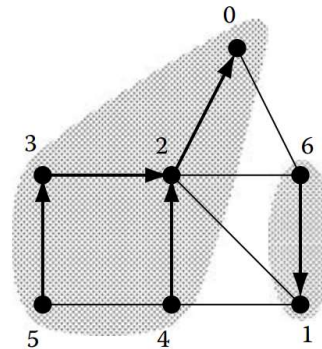
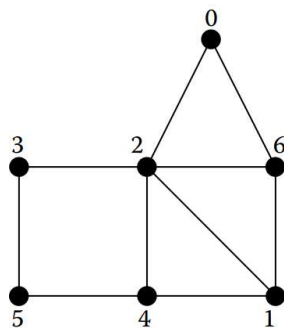
Deterministic parallel algorithm

- Again work in phases, with following parallel steps.
 - Each node points to a neighbor with lower ID.
 - This breaks graph into a directed forest.
 - Contract each forest to the lowest ID node using pointer jumping.
 - Recurse on contracted graph.



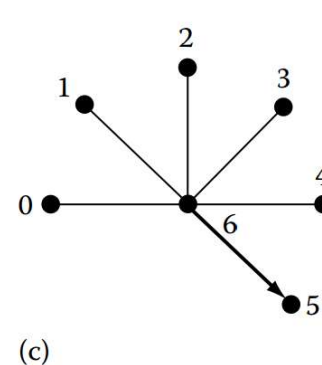
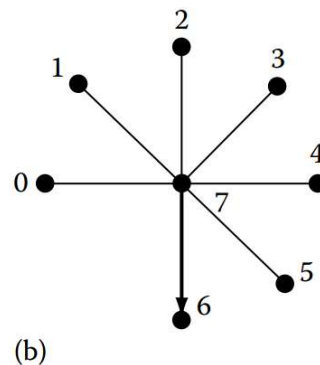
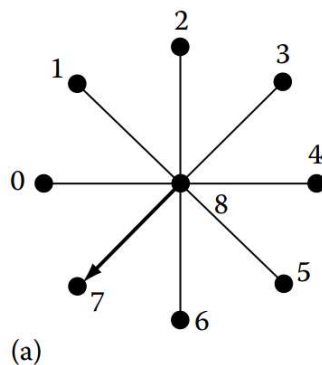
Implementation

- As before, use $n+m$ procs, for V and E .
 - But now, V procs always active. E procs may become inactive.
- Each E proc (u,v) checks if $u < v$, and if so sets v 's label to u .
 - Again, conflicts resolved arbitrarily.
- V procs then apply pointer jumping on the labels, taking the label of the proc it points to.
- Each active E proc (u,v) where u, v have different labels stays active. Other E procs become inactive.
- The V procs and active E procs run algorithm recursively.
 - Note that in recursive call, all V procs apply pointer jumping.



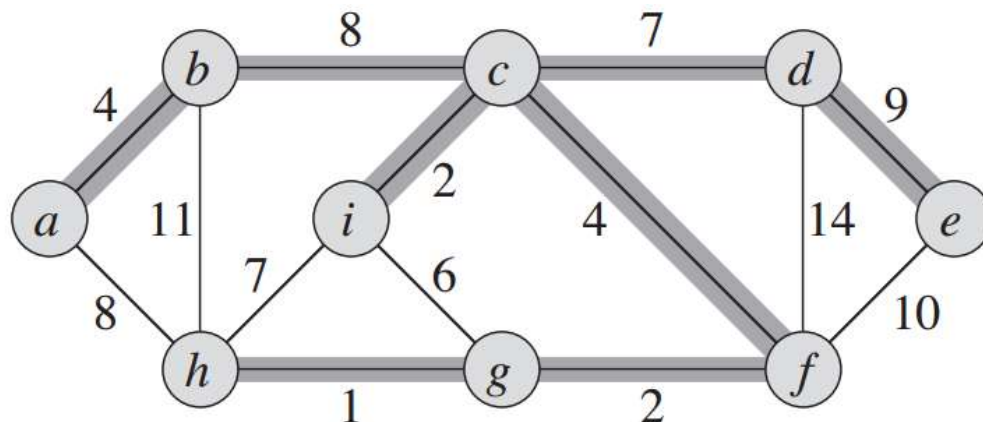
Complexity

- The basic algorithm may take $O(n)$ time on the graph below.
- But notice that if we made nodes point to higher neighbors, the graph would be solved in $O(1)$ time.
- In each phase, if we consider either having nodes point to smaller neighbors, or pointing to higher neighbors.
 - If v doesn't point to any nbr in "high \rightarrow low" round, then it's smaller than all nbrs. So in "low \rightarrow high" round, it points to some nbr.
 - In one of these cases, $\geq n/2$ nodes point to another node, and are contracted.
- Thus the algorithm finishes in $O(\log n)$ phases.
- Each phase does pointer jumping, using $O(\log n)$ time and $O(n)$ work.
- Total time is $O(\log^2 n)$, and work is $O((m+n) \log n)$.



Minimum spanning tree

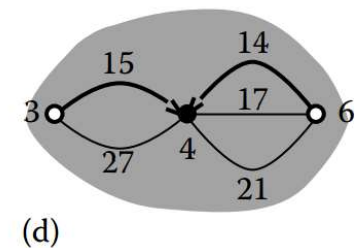
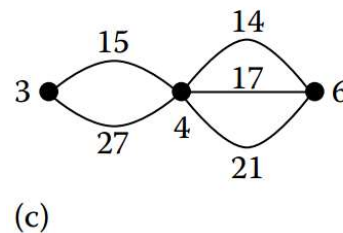
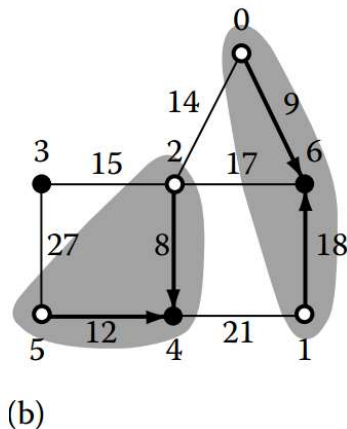
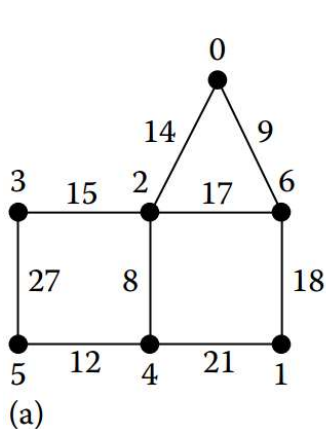
- Given an undirected graph with edge weights, an MST is a connected subgraph containing all the vertices, which has minimum total weight.
- Can be solved in $O(m + n \log n)$ time sequentially by a greedy algorithm.
- Key property is that for any set of vertices W , the minimum cost edge from W to $V \setminus W$ is in the MST.
 - So for any vertex v , min cost edge containing v is in MST.
- Will describe a parallel MST algorithm based on the randomized parallel algorithm for connected component.



Source: Introduction to Algorithms, Cormen et al.

Randomized parallel algorithm

- Each node randomly chooses to be a parent or child node.
- Each child node u finds min weight incident edge (u,v) , and points to v if v is parent.
 - This forms a set of stars with parents as centers.
 - If v isn't a parent, u forms its own star.
- Contract each star to the parent, and run algorithm recursively.
- How do we find min weight incident edge in $O(1)$ time?
 - One possibility is to use priority CRCW.
 - Presort the edges by nondecreasing weight. Each edge processor's priority is its value.
 - So when each E proc (u,v) writes to u , min weight edge wins.





Complexity

- At least $1/4$ of vertex processors become inactive each phase in expectation.
 - Given a node u and min weight edge (u,v) , there's $1/4$ probability u is child and v is parent.
- Thus, there are $O(\log n)$ phases with high probability.
 - Finding min weight incident edge takes $O(1)$ time after presorting edge weights.
- Total time is $O(\log n)$, total work is $O((m+n) \log n)$.