Computer Animation & Physical Simulation

Lecture 11: Soft-Body Simulation – Cloth II

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General Problem for Mass-Spring Model

- **Physical consistency**
  - Cannot recover the real underlying physics
  - Could be problematic when handling unstructured triangular mesh

- **Mesh refinement**
  - Cannot converge to the correct solution

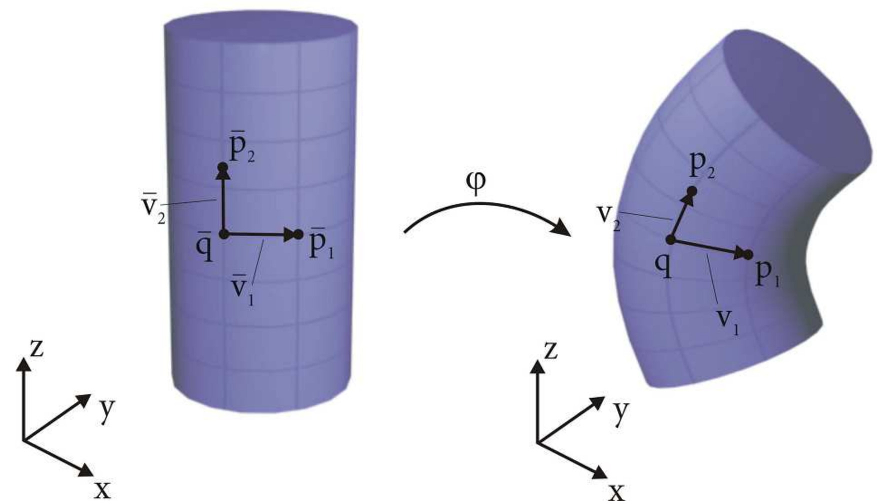
I. Continuum Models

Deformation Measures

- **Approximate the fabric as a continuous medium**
  - If the regions of each polygon contains a sufficiently large number of woven structures
  - Definition of deformation

$$\varphi : \Omega \subset \mathbf{R}^3 \rightarrow \mathbf{R}^3$$

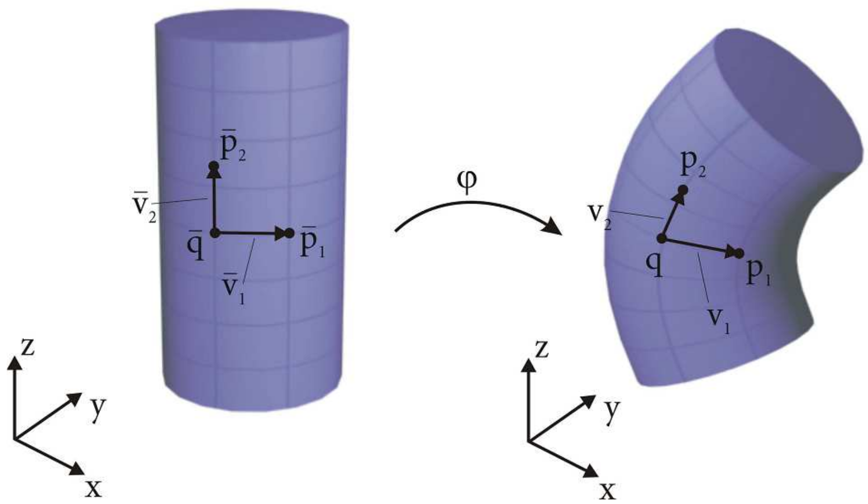
$$\varphi = \bar{\varphi} + u = id + u : \Omega \rightarrow \mathbf{R}^3$$



Deformation Measures

- **Derivation**
  - Consider two vector pairs in the un-deformed and deformed configuration

$$\bar{\mathbf{v}}_i = \bar{\mathbf{p}}_i - \bar{\mathbf{q}} \qquad\qquad \mathbf{v}_i = \mathbf{p}_i - \mathbf{q}$$



    - Taylor series expansion

$$\begin{aligned}
\mathbf{v}_i &= \varphi(\bar{\mathbf{q}} + \bar{\mathbf{v}}_i) - \varphi(\bar{\mathbf{q}}) \\
&= \varphi(\bar{\mathbf{q}}) + \nabla\varphi(\bar{\mathbf{q}}) \cdot \bar{\mathbf{v}}_i + O(\bar{\mathbf{v}}_i^2) - \varphi(\bar{\mathbf{q}}) \\
&\approx \nabla\varphi(\bar{\mathbf{q}})\bar{\mathbf{v}}_i = (\nabla u(\bar{\mathbf{q}}) - \mathrm{id})\bar{\mathbf{v}}_i
\end{aligned}$$

Deformation Measures

- **Deformation definition**
  - Deformation gradient

$$F = \nabla \varphi$$

  - A general deformation measure

$$\mathbf{v}_1 \cdot \mathbf{v}_2 - \bar{\mathbf{v}}_1 \cdot \bar{\mathbf{v}}_2 = \bar{\mathbf{v}}_1 \cdot (\nabla \varphi^T \nabla \varphi - id) \cdot \bar{\mathbf{v}}_2 \quad + \quad \varphi = \bar{\varphi} + u = id + u : \Omega \rightarrow \mathbf{R}^3$$

$$\varepsilon_G = \frac{1}{2}(\nabla \varphi^T \nabla \varphi - id) = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T \nabla u)$$

Symmetric Green strain tensor

Linear approximation

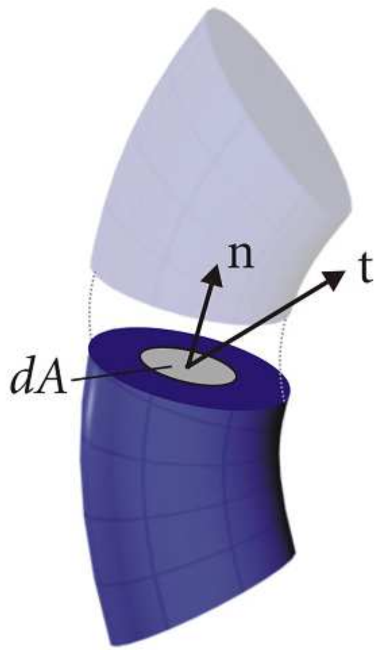
$$\varepsilon_C = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

linear Cauchy strain tensor

Internal Stress

- **The traction vector t**

$$\mathbf{t} = \lim_{dA \to 0} \frac{d\mathbf{f}}{dA}$$



- Cauchy stress tensor

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$$

Equilibrium and Dynamic Equations

- **Equilibrium state**
  - Internal and external forces are in equilibrium

$$\int_{\partial V} \mathbf{t}\, da + \int_V \mathbf{f}\, dv = \int_{\partial V} \boldsymbol{\sigma}\mathbf{n}\, da + \int_V \mathbf{f}\, dv = 0$$

$$\Downarrow$$

$$\mathrm{div}\, \boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = 0$$

- **Dynamic state**

$$\mathrm{div}\, \boldsymbol{\sigma}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{f}(\mathbf{x}) = \rho\ddot{\mathbf{x}}$$

Constitutive Relations

- **In general**
  - The relationship between stress and strain can be of high complexity

- **Hyperelastic material**
  - Stresses in a body depend only on its current state of deformation

- **Linear elasticity**

$$\sigma = \mathcal{C} : \varepsilon \qquad\qquad \sigma_{ij} = \mathcal{C}_{ijkl}\varepsilon_{kl}$$

Constitutive Relations

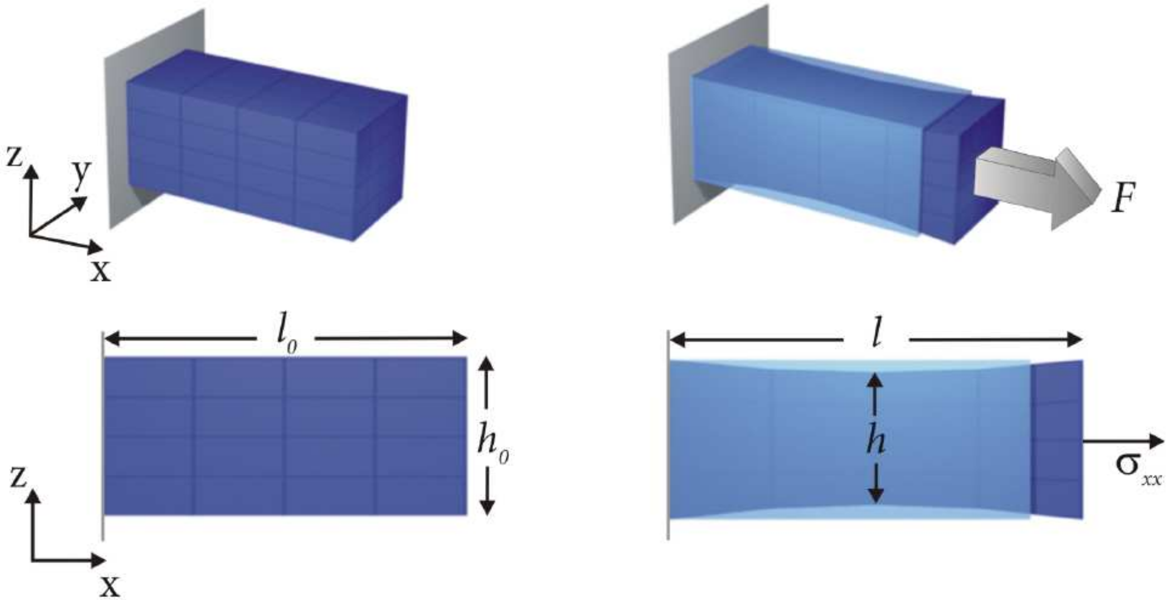
- **Linear-elastic isotropic material**
  - Governed by only two independent constants

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl}$$

    - λ and μ are the Lamé constants
    - Young modulus: $$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$
    - Poisson ratio: $$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

Constitutive Relations

- **Meaning of Young modulus and Poisson ratio**



$$\frac{\sigma_{xx}}{\varepsilon_{xx}} = E$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \frac{\nu}{E}\sigma_{xx}$$

Energy(Variational) Formulation

- **Total potential energy**

$$\Pi = \Lambda - W$$

  - Strain energy $\Lambda = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \varepsilon \; dV = \frac{1}{2}\int_V \varepsilon^T \mathbf{D} \varepsilon \; dV$
  - Work $W = \int_V \mathbf{u} \cdot \mathbf{f}_b \; dV + \int_\Gamma \mathbf{u} \cdot \mathbf{f}_s \; dS + \sum_i \mathbf{u}_i \cdot \mathbf{p}_i$
  - Determine the displacement

$$\delta\Pi(\mathbf{u}) = 0 \;, \qquad \text{for all variations } \delta\mathbf{u}$$

Energy(Variational) Formulation

- **Equilibrium state**

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

stiffness matrix

- **Dynamic state**

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) + \mathbf{f} = 0$$

mass matrix

damping matrix

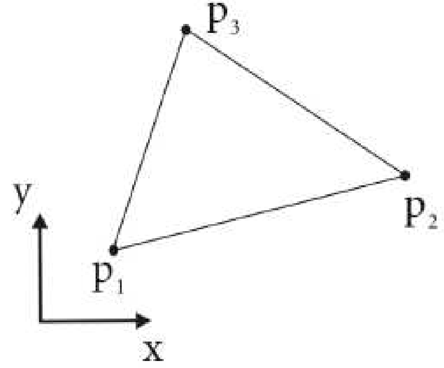
Plane Stress Analysis

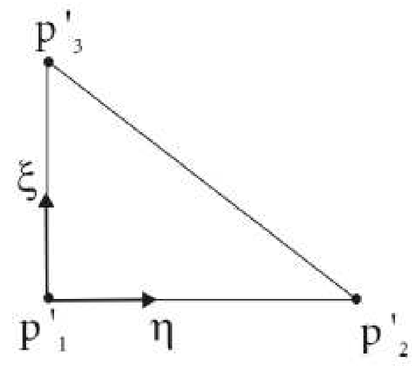
- **Approximation**
  - Basis function interpolation within a triangle element

$$\mathbf{u} = \sum_i N_i \tilde{\mathbf{u}}_i$$

$$N_1 = 1 - \xi - \eta, \quad N_2 = \xi, \quad N_3 = \eta$$

$$\frac{\partial N_i}{\partial x_j} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x_j} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x_j}$$





Plane Stress Analysis

- **Approximation**
  - Cauchy strain Over a single triangular element

$$\varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \mathbf{u} = \sum_{i=0}^{3} \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N^i}{\partial x} \end{bmatrix} \tilde{\mathbf{u}}_i = \sum_{i=0}^{3} \mathbf{B}_i \tilde{\mathbf{u}}_i$$

  - Stiffness matrix

$$\mathbf{K} = \sum_i \int_{\Omega_i} \mathbf{B}_i^T \mathbf{C} \mathbf{B}_i \, d\Omega = \sum_i \mathbf{B}_i^T \mathbf{C} \mathbf{B}_i t A_i$$

II. Triangular Springs from Continuum Models

General Idea

- **How to compute forces on discrete particles?**
  - Energy formulations similar to continuum models
  - Possess equivalence to a continuum model on discrete particles

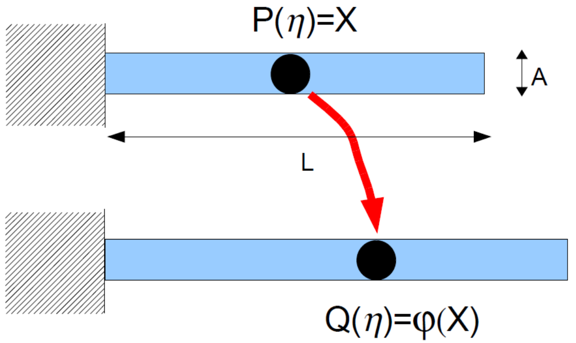
  - Equivalence between a continuum formulation of the membrane energy and the energy of a set of *triangular biquadratic springs*

II.a Springs and 1D Elasticity

Linear Elastic Axially Loaded Bar

- **Setup**
  - The bar is assumed to have constant cross-sectional area **A**
  - The center of each cross-sections being on a straight line
  - The rest configuration of the bar is $\Omega \subset \mathbb{R}^3$
  - After applying an axial load, each center of cross-sections **X** is moved into a new position

$$\mathbf{\Phi}(\mathbf{X}) \in [0, \mathbf{\Phi}(L)]$$



Linear Elastic Axially Loaded Bar

- **Derivation**
  - An infinitesimal material segment of length dx located around **X** is deformed into a segment of length

$$\frac{d\mathbf{\Phi}}{dx}(X)\,dx$$

  - Stretch ratio: $s = \frac{d\mathbf{\Phi}}{dx}$
    - Equal 1: the deformation does not entail any local stretching

Linear Elastic Axially Loaded Bar

- **Derivation**
  - The strain
    - A quantity that measures how different is this stretch ratio from the value 1
    - Positive implying extension of the material
    - Negative implying contraction
  - A family of possible strain functions

$$\epsilon(C) = \left\{ \begin{array}{ll} \frac{1}{\alpha}\left(s^{\alpha} - 1\right) & \text{if } \alpha \neq 0 \\ \log(s) & \text{if } \alpha = 0 \end{array} \right\}$$

- $\alpha = 1, \epsilon = s - 1$: engineering strain
- $\alpha = 2 \ \epsilon = 1/2(s^2 - 1)$: Green-Lagrange strain
- $\alpha = 0$: Henky or natural strain

Linear Elastic Axially Loaded Bar

- **Derivation**
  - Hypothesis of a linear elastic material
    - Stress at each point is proportional to the strain at that point

$$\sigma = \lambda\epsilon$$

      - λ: Stiffness parameter
    - The work W to bring the material cross section around **X** to $\mathbf{\Phi}(\mathbf{X})$ is

$$W = \frac{1}{2}\sigma\epsilon$$

    - Total energy required to deform the bar

$$W_\Omega = \int_\Omega \frac{1}{2}\sigma\epsilon \, dV = \frac{\lambda A}{2\alpha^2}\int_0^L \left((\frac{d\mathbf{\Phi}}{dx})^\alpha - 1 \right)^2 d\mathbf{X}$$

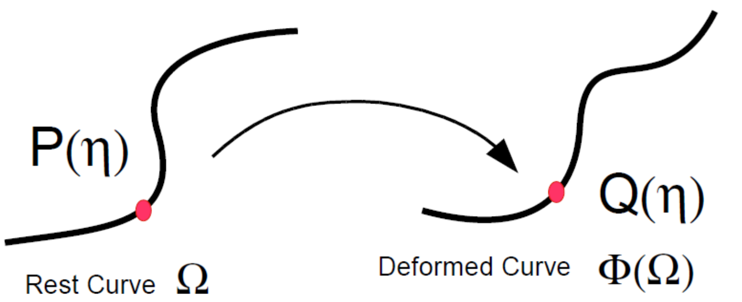
Linear Elastic Axially Loaded Bar

- **Derivation**
  - If the segment [0,L] is parameterized by a function

$$\mathbf{X} = P(\eta), \eta \in [a,b] \subset \mathbb{R}$$

  - Its deformed segment by a function

$$\mathbf{\Phi}(\mathbf{X}) = Q(\eta), \eta \in [a,b] \subset \mathbb{R}$$



    - The deformation function can be written as

$$\mathbf{\Phi}(\mathbf{X}) = Q(P^{-1}(\mathbf{X}))$$

    - Total energy can be expressed as

$$W_\Omega = \frac{\lambda A}{2\alpha^2} \int_a^b \left((\frac{dQ}{d\eta})^\alpha - (\frac{dP}{d\eta})^\alpha \right)^2 \left(\frac{dP}{d\eta} \right)^{(1-2\alpha)} d\eta$$

Stretching Energy of Deformable Curves

- **Consider a curved bar**
  - A curved bar Ω of cross-section **A** embedded in a Euclidean space $\mathbb{R}^d,\ d > 1$
  - The center line curve parameterized as $\mathbf{P}(\eta)$
  - Deformed into another curve parameterized as $\mathbf{Q}(\eta)$

$$W_\Omega = \frac{\lambda A}{2\alpha^2}\int_a^b \left((\frac{dQ}{d\eta})^\alpha - (\frac{dP}{d\eta})^\alpha \right)^2 \left(\frac{dP}{d\eta}\right)^{(1-2\alpha)} d\eta$$

$$\frac{dP}{d\eta} \Rightarrow \|\frac{d\mathbf{P}}{d\eta}\|$$

$$W_\Omega = \frac{\lambda A}{2\alpha^2}\int_a^b \left(\|\frac{d\mathbf{Q}}{d\eta}\|^\alpha - \|\frac{d\mathbf{P}}{d\eta}\|^\alpha \right)^2 \left\|\frac{d\mathbf{P}}{d\eta}\right\|^{(1-2\alpha)} d\eta$$

If the parameter η is the arc length of the reference curve

$$W_\Omega = \frac{\lambda A}{2\alpha^2}\int_\Omega \left(\|\frac{d\mathbf{Q}}{d\eta}\|^\alpha - 1 \right)^2 d\eta$$

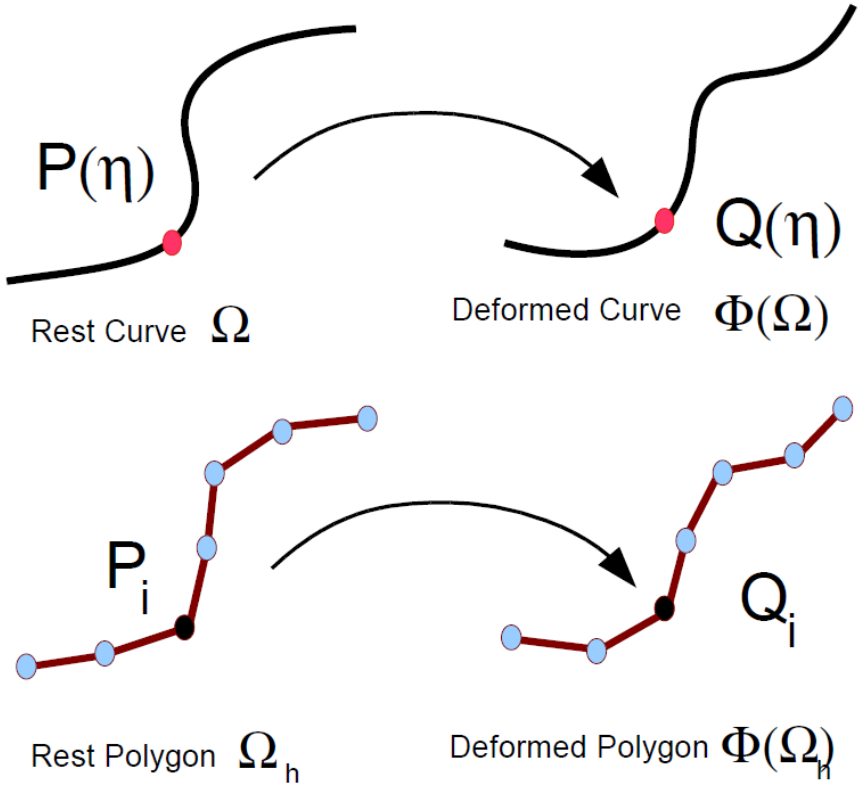
Finite Element Discretization

- **Rayleigh-Ritz approach**
  - Equivalent to Galerkin weighted residual method
  - Rely on the variational form (strain energy)
    - More convenient to derive symmetric analytical expressions
    - First discretize the stretching energy, then apply the principle of minimum potential energy
- **Approximation**
  - The reference curve can be approximated with a set of line segments

$$\Omega^h = \bigcup_{i=1,..,N} S_i = [\mathbf{P}_i, \mathbf{P}_{i+1}]$$

Finite Element Discretization

- **Approximation**
  - Each reference segment $[\mathbf{P}_i, \mathbf{P}_{i+1}]$ is deformed into segment $[\mathbf{Q}_i, \mathbf{Q}_{i+1}]$



Finite Element Discretization

- **Parameterization**
  - Each segment in the reference/deformed configuration can be parameterized as a linear interpolation

$$\mathbf{P}(\eta) = (1-\eta)\mathbf{P}_i + \eta\mathbf{P}_{i+1}$$
$$\mathbf{Q}(\eta) = (1-\eta)\mathbf{Q}_i + \eta\mathbf{Q}_{i+1}$$

  - Simplified notations
    - Length of the rest segment $L_i = ||\mathbf{P}_i\mathbf{P}_{i+1}||$
    - Length of the deformed segment $l_i = ||\mathbf{Q}_i\mathbf{Q}_{i+1}||$

Finite Element Discretization

- **Stretching energy**
  - Stretching energy of each line segment

$$W_\Omega = \frac{\lambda A}{2\alpha^2} \int_a^b \left(\| \frac{d\mathbf{Q}}{d\eta} \|^\alpha - \| \frac{d\mathbf{P}}{d\eta} \|^\alpha \right)^2 \left\| \frac{d\mathbf{P}}{d\eta} \right\|^{(1-2\alpha)} d\eta$$

$$\Downarrow$$

$$W_{\Omega_h}(S_i) = \frac{\lambda A L_i^{(1-2\alpha)}}{2\alpha^2} (l_i^\alpha - L_i^\alpha)^2$$

  - Engineering (quadratic) strain $\alpha = 1$
    - The stretching energy therefore corresponds to <u>the energy of a spring of stiffness</u> $\lambda A / L_i$

Finite Element Discretization

- **Stretching energy**
  - Stretching energy of each line segment
    - Green-Lagrange strain $\alpha = 2$
      - The stretching energy is a biquadratic function of the deformed segment length and of the deformed position

$$W_{\Omega_h}(S_i) = \frac{\lambda A}{8L_i^3}\left(l_i^2 - L_i^2\right)^2$$

      - This expression resembles a (quadratic) spring energy but with squared lengths
      - Coined as a tensile <u>*biquadratic spring*</u>
    - Rewritten of stretching energy $W_{\Omega_h}(S_i) = \lambda A L_i \; w(s)$

Finite Element Discretization

- **Study on engineering and biquadratic springs**

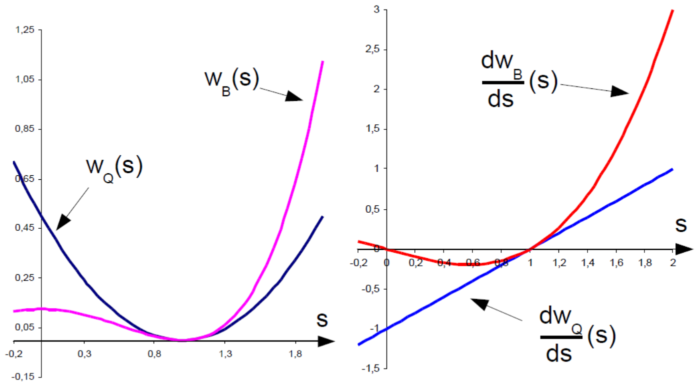
$$W_{\Omega_h}(S_i) = \lambda A L_i \; w(s)$$

  - Engineering (quadratic) spring

$$w(s) = w_Q(s) = 1/2(1-s)^2$$

  - Biquadratic springs

$$w(s) = w_B(s) = 1/8(1-s^2)^2$$



  - Analysis
    - Extension: biquadratic springs being far more stiffer than quadratic springs
    - Compression: both have their unphysical aspects

Linear Elasticity : Small Displacements Hypothesis

- **Even more restrictive hypothesis**
  - Assume that all vertex displacement is small compared to the edge length

$$\mathbf{U}_i = \mathbf{Q}_i - \mathbf{P}_i$$

    - For small displacements, approximation

$$l_i^\alpha \approx L_i^\alpha + \alpha L_i^{(\alpha-2)} (\mathbf{U}_{i+1} - \mathbf{U}_i) \cdot (\mathbf{P}_{i+1} - \mathbf{P}_i)$$

$$\Downarrow$$

$$(s^\alpha - 1)/\alpha \approx (\mathbf{U}_{i+1} - \mathbf{U}_i) \cdot (\mathbf{P}_{i+1} - \mathbf{P}_i)/L_i^2$$

$$W_{\Omega_h}(S_i) = \frac{\lambda A L_i^{(1-2\alpha)}}{2\alpha^2} (l_i^\alpha - L_i^\alpha)^2 \Rightarrow W_{\Omega_h}^{\text{Linear}}(S_i) = \frac{\lambda A}{L_i^3} ((\mathbf{U}_{i+1} - \mathbf{U}_i) \cdot (\mathbf{P}_{i+1} - \mathbf{P}_i))^2$$

Linear elastic model

II.b Membrane Energy on Triangle Meshes

Membrane Energy

- **The energy to deform a piece of cloth**
  - Membrane energy
    - Characterize the resistance to *in-plane* stretching
    - Generalize the stretching energy for curves
  - Bending energy
    - Measures the resistance to change in the surface normal orientation
  - We consider $\mathbf{\Phi}(\Omega)$
    - A two-dimensional domain $\Omega \subset \mathbb{R}^2$ being deformed into another domain
    - Right Cauchy-Green deformation tensor

$$\mathbf{C} = \nabla\mathbf{\Phi}^T \nabla\mathbf{\Phi}$$

Membrane Energy

- **Green-Lagrange strain tensor**
  - Defined based on Cauchy-Green deformation tensor

$$\mathbf{E} = 1/2(\mathbf{C} - \mathbf{I})$$

    - Invariant to translations or rotations
    - Appropriate for describing deformations under large displacements
  - Assumption
    - Isotropic St Venant Kirchhoff membrane
      - Linear relationship between stress and strain
      - Density of membrane energy

$$W(\mathbf{X}) = \frac{\lambda}{2}(\mathrm{tr}\mathbf{E})^2 + \frac{\mu}{2}\mathrm{tr}\mathbf{E}^2$$

λ and μ : Lamé coefficients of the material

Membrane Energy

- **Isotropic St Venant Kirchhoff membrane**
  - Lamé coefficients of the material
    - Related to the physically meaningful Young modulus and Poisson coefficient
      - Young modulus E: quantify the stiffness of the material
      - Poisson coefficient μ: characterize the material compressibility
    - Relation

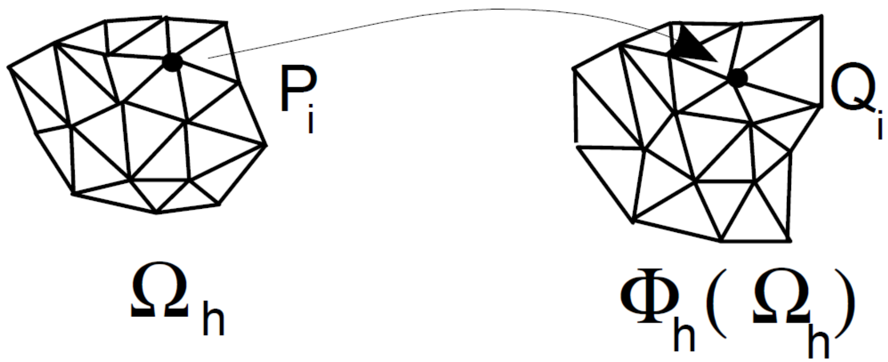
$$\lambda = \frac{E\nu}{1-\nu^2} \qquad \mu = \frac{E(1-\nu)}{1-\nu^2}$$

  - Total membrane energy $W_\Omega = \int_\Omega W(\mathbf{X})\, d\Omega = \int_\Omega \left(\frac{\lambda}{2} (\mathrm{tr}\mathbf{E})^2 + \frac{\mu}{2} \mathrm{tr}\mathbf{E}^2 \right)\, d\Omega$

Deformation Function on a Linear Triangle

- **Domain discretization**
  - The integration of first derivatives of the deformation function
    - Discretized $\mathbf{\Phi}$ into a simplicial surface
      - A set of triangles $\{T_i\}, i = \{1, .., p\}$
      - A set of vertices $\{\mathbf{P}_i\}, i \in \{1, .., n\}$
    - Linear triangle element for discretization

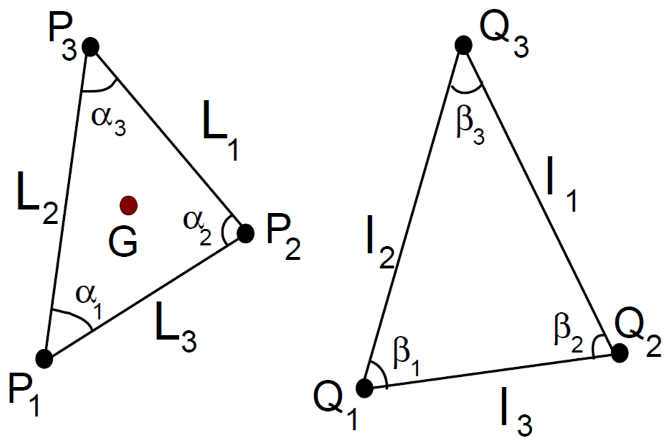
$$\mathbf{Q}_i = \mathbf{\Phi}_h(\mathbf{P}_i)$$



Deformation Function on a Linear Triangle

- **Deformation of a single triangle**

$$\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\} \longrightarrow \{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$$



  - Definition
    - Area of the rest/deformed triangle: $\mathcal{A}_P$ (resp. $\mathcal{A}_Q$)
    - Edge lengths: l_i (resp. L_i)
    - Three angles: α_i (resp. β_i)
  - Parameterization of point inside a triangle

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{1x} & \mathbf{P}_{2x} & \mathbf{P}_{3x} \\ \mathbf{P}_{1y} & \mathbf{P}_{2y} & \mathbf{P}_{3y} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = [\overline{\mathbf{P}}] \ \ \overline{\mathbf{E}}$$

Deformation Function on a Linear Triangle

- **Deformation of a single triangle**
  - Inverse relation defines the barycentric coordinates

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{1x} & \mathbf{D}_{1y} & \eta_1^0 \\ \mathbf{D}_{2x} & \mathbf{D}_{2y} & \eta_2^0 \\ \mathbf{D}_{3x} & \mathbf{D}_{3y} & \eta_3^0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [\overline{\mathbf{D}}] \ \overline{\mathbf{X}}$$

- $\mathbf{D}_i$ is the ith *shape vector* of triangle T_P
- η_i^0 is the ith barycentric coordinate of the origin of the coordinate frame

Deformation Function on a Linear Triangle

- **Deformation of a single triangle**
  - Introduce the centroid **G**

$$\eta_i = 1/3$$

    - The barycentric coordinates

$$\mathbf{X} = \sum_{i=1}^{3} \eta_i(\mathbf{X})\ \mathbf{P}_i = \sum_{i=1}^{3} \left(\frac{1}{3} + \mathbf{D}_i \cdot (\mathbf{X} - \mathbf{G}) \right)\ \mathbf{P}_i$$

    - Deformation function

$$\mathbf{\Phi}(\mathbf{X}) = \sum_{i=1}^{3} \eta_i(\mathbf{X})\ \mathbf{Q}_i = \sum_{i=1}^{3} \left(\frac{1}{3} + \mathbf{D}_i \cdot (\mathbf{X} - \mathbf{G}) \right)\ \mathbf{Q}_i$$

    - Recall the membrane energy

$$\mathbf{C} = \nabla\mathbf{\Phi}^T \nabla\mathbf{\Phi} \qquad \mathbf{E} = 1/2(\mathbf{C} - \mathbf{I}) \qquad W(\mathbf{X}) = \frac{\lambda}{2}(\mathrm{tr}\mathbf{E})^2 + \frac{\mu}{2}\mathrm{tr}\mathbf{E}^2$$

Invariants of Green Lagrange Strain Tensor

- **Rewrite the invariants**
  - Two invariants (with respect to translation and rotation)

$$\mathrm{t}r\mathbf{E} \qquad \mathrm{t}r\mathbf{E}^2 \qquad \mathbf{E} = 1/2(\mathbf{C} - \mathbf{I})$$

    - As a function of the two triangles shape

$$\nabla\mathbf{\Phi} = \left[\frac{\partial\mathbf{\Phi}_i}{\partial x_j}\right] = \sum_{i=1}^{3}\mathbf{Q}_i \otimes \mathbf{D}_i$$

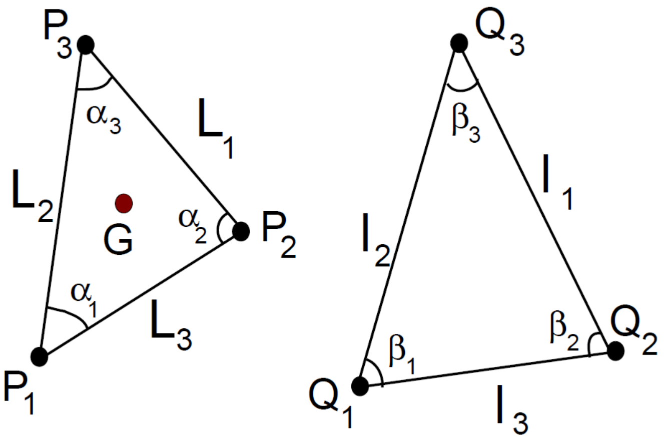
$$\mathbf{C} = \nabla\mathbf{\Phi}^T\nabla\mathbf{\Phi} = \sum_{i=1}^{3}\sum_{j=1}^{3}(\mathbf{Q}_i \cdot \mathbf{Q}_j)(\mathbf{D}_i \otimes \mathbf{D}_j) \qquad \mathrm{t}r\mathbf{E} = 1/2(\mathrm{t}r\mathbf{C} - 2)$$

$$\Downarrow$$

$$\mathrm{t}r\mathbf{C} = \sum_{i=1}^{3}\sum_{j=1}^{3}(\mathbf{Q}_i \cdot \mathbf{Q}_j)(\mathbf{D}_i \cdot \mathbf{D}_j)$$

Invariants of Green Lagrange Strain Tensor

- **Rewrite the invariants**
  - After a set of derivations



$$\mathrm{t}r\mathbf{C} = \frac{1}{2\mathcal{A}_P}\left(l_1^2 \cot\alpha_1 + l_2^2\cot\alpha_2 + l_3^2\cot\alpha_3\right)$$

$$\boxed{\mathrm{t}r\mathbf{E}} = \frac{(l_1^2 - L_1^2)\cot\alpha_1 + (l_2^2 - L_2^2)\cot\alpha_2 + (l_3^2 - L_3^2)\cot\alpha_3}{2\mathcal{A}_P}$$

$$\mathrm{t}r\mathbf{E}^2 = (\mathrm{t}r\mathbf{E})^2 - 2\det\mathbf{E} = \frac{1 + 2\mathrm{t}r\mathbf{E} + 2(\mathrm{t}r\mathbf{E})^2 - \det\mathbf{C}}{2}$$

$$\det\mathbf{\Phi} = \mathcal{A}_Q/\mathcal{A}_P \qquad \Delta^2 l_i = (l_i^2 - L_i^2)$$

$$\boxed{\mathrm{t}r\mathbf{E}^2} = \frac{\sum_{i\neq j} 2\Delta^2 l_i \Delta^2 l_j - \sum_{i=1}^{3}(\Delta^2 l_i)^2}{64\mathcal{A}_P^2}$$

Membrane Energy and Triangular Biquadratic Springs

- **Total energy from energy density**

$$W(\mathbf{X}) = \frac{\lambda}{2}(\mathrm{tr}\mathbf{E})^2 + \frac{\mu}{2}\mathrm{tr}\mathbf{E}^2$$

$$\Downarrow$$

$$\begin{aligned} W_{TRBS}(T_P) &= \int_{T_P} W(\mathbf{X})\, d\mathbf{X} = \mathcal{A}_P W(\mathbf{G}) \\ &= \sum_{i=1}^{3} \frac{(\Delta^2 l_i)^2 (2\cot^2\alpha_i(\lambda+\mu)+\mu)}{64\mathcal{A}_P} + \\ &\quad \sum_{i\neq j} \frac{2\Delta^2 l_i \Delta^2 l_j (2\cot\alpha_i \cot\alpha_j(\lambda+\mu)-\mu)}{64\mathcal{A}_P} \end{aligned}$$

$$\Delta^2 l_i = (l_i^2 - L_i^2)$$

Membrane Energy and Triangular Biquadratic Springs

- **TRiangular Biquadratic Springs (TRBS)**

$$W_{TRBS}(T_P) = \sum_{i=1}^{3} \frac{(\Delta^2 l_i)^2 (2\cot^2 \alpha_i (\lambda + \mu) + \mu)}{64 \mathcal{A}_P} + \sum_{i \neq j} \frac{2\Delta^2 l_i \Delta^2 l_j (2 \cot \alpha_i \cot \alpha_j (\lambda + \mu) - \mu)}{64 \mathcal{A}_P}$$

  - First tem
    - Energy of three tensile biquadratic springs, preventing edges from stretching
  - Second term
    - Angular biquadratic springs, preventing changes in vertex angles

Membrane Energy and Triangular Biquadratic Springs

- **TRiangular Biquadratic Springs (TRBS)**
  - Rewriting the equations

$$W_{TRBS}(T_P) = \sum_{i=1}^{3} \frac{k_i^{T_P}}{4}(l_i^2 - L_i^2)^2 + \sum_{i \neq j} \frac{c_k^{T_P}}{2}(l_i^2 - L_i^2)(l_j^2 - L_j^2)$$

$$k_i^{T_P} = \frac{2\cot^2\alpha_i(\lambda+\mu)+\mu}{16\mathcal{A}_P} = \frac{E(2\cot^2\alpha_i + 1 - \nu)}{16(1-\nu^2)\mathcal{A}_P}$$ Tensile stiffness

$$c_k^{T_P} = \frac{2\cot\alpha_i\cot\alpha_j(\lambda+\mu)-\mu}{16\mathcal{A}_P} = \frac{E(2\cot\alpha_i\cot\alpha_j + \nu - 1)}{16(1-\nu^2)\mathcal{A}_P}$$ Angular stiffness

Force Computation

- **Apply Rayleigh-Ritz analysis**
  - Triangular surface should evolve by minimizing its membrane energy
  - Along the opposite derivative of that energy with respect to the nodes of the system

$$\mathbf{F}_i^{TRBS}(T_P) = -\left(\frac{\partial W(T_P)}{\partial \mathbf{Q}_i}\right)^T = \sum_{j \neq i} k_k^{T_P} \Delta^2 l_k (\mathbf{Q}_j - \mathbf{Q}_i) + \sum_{j \neq i} (c_j^{T_P} \Delta^2 l_i + c_i^{T_P} \Delta^2 l_j)(\mathbf{Q}_j - \mathbf{Q}_i)$$

Cloth Animation

(a) TRBS, ν = 0.95 (b) TRBS, ν = 0.2 (c) TRQS, ν = 0.95 (d) TRQS, ν = 0.2 (e) Springs, ν = 0.3

With damping: $$(\Delta^2 l_i)^{\mathrm{Damped}} = \Delta^2 l_i + \zeta(\mathbf{v}_j - \mathbf{v}_k) \cdot (\mathbf{Q}_j - \mathbf{Q}_k)$$

III. Data-Driven Constitutive Model for Cloth Simulation

Data Driven Approach

- **Previous cloth simulation approach**
  - Use linear and isotropic elastic models with manually selected stiffness parameters
  - Do not allow differentiating the behavior of distinct cloth materials
  - More realistically animate cloth?
    - Data-driven approach
    - Measure cloth parameters through optimization
    - Create natural and realistic clothing wrinkles and shapes for a range of different materials

Piecewise Linear Elastic Model

- **Planar stretching model**
  - Assumption
    - The scale of threads and their interweaving patterns are significantly smaller than elastic behaviors
    - Linear anisotropic model obtained by generalizing Hooke's law

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{uu} \\ \sigma_{vv} \\ \sigma_{uv} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{uu} \\ \varepsilon_{vv} \\ \varepsilon_{uv} \end{bmatrix} = \mathbf{C}\boldsymbol{\varepsilon}$$

stiffness tensor matrix

Piecewise Linear Elastic Model

- **Planar stretching model**
  - Experimental observation
    - Woven composite fabrics are orthotropic
    - When the local coordinate system is aligned with the warp-weft directions, **C** can be simplified

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$$

    - Shearing resistance is separable from the stretching resistance

Piecewise Linear Elastic Model

- **Planar stretching model**
  - Formulation
    - Instead of treating **C** as a constant matrix, formulate **C** as a piecewise linear function **C(ε)** of the strain tensor **ε**

$$\boldsymbol{\varepsilon} = [\varepsilon_{uu}, \varepsilon_{vv}, \varepsilon_{uv}]^{\mathrm{T}}$$

      - Its values are not intuitive enough to demonstrate the actual deformation
      - Re-parameterize using eigenvalue decomposition

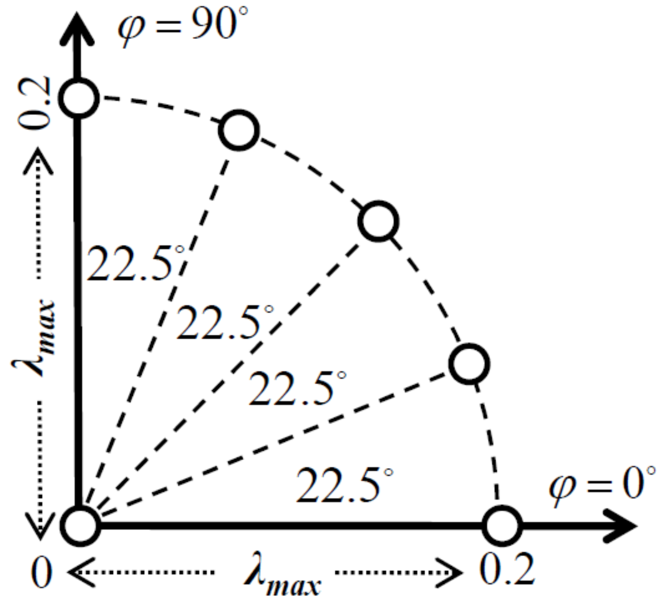
$$2\begin{bmatrix} \varepsilon_{uu} & \varepsilon_{uv} \\ \varepsilon_{uv} & \varepsilon_{vv} \end{bmatrix} = \mathbf{R}_{\varphi}^{\mathrm{T}} \begin{bmatrix} (\lambda_{\max}+1)^2 - 1 & 0 \\ 0 & (\lambda_{\min}+1)^2 - 1 \end{bmatrix} \mathbf{R}_{\varphi}$$

Piecewise Linear Elastic Model

- **Planar stretching model**
  - Formulation
    - Experimental notice
      - λ_{min} has significantly less influence on **C**
    - Simplify the parameterization using only

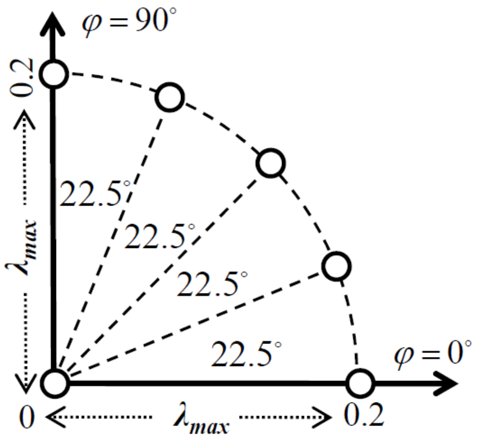
λ_{max} φ

$$\mathbf{C}(\lambda_{max}, \varphi)$$



Piecewise Linear Elastic Model

- **Planar stretching model**
  - Formulation
    - Each data point contains four parameters, c_{11}, c_{12}, c_{22} and c_{33}



$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$$

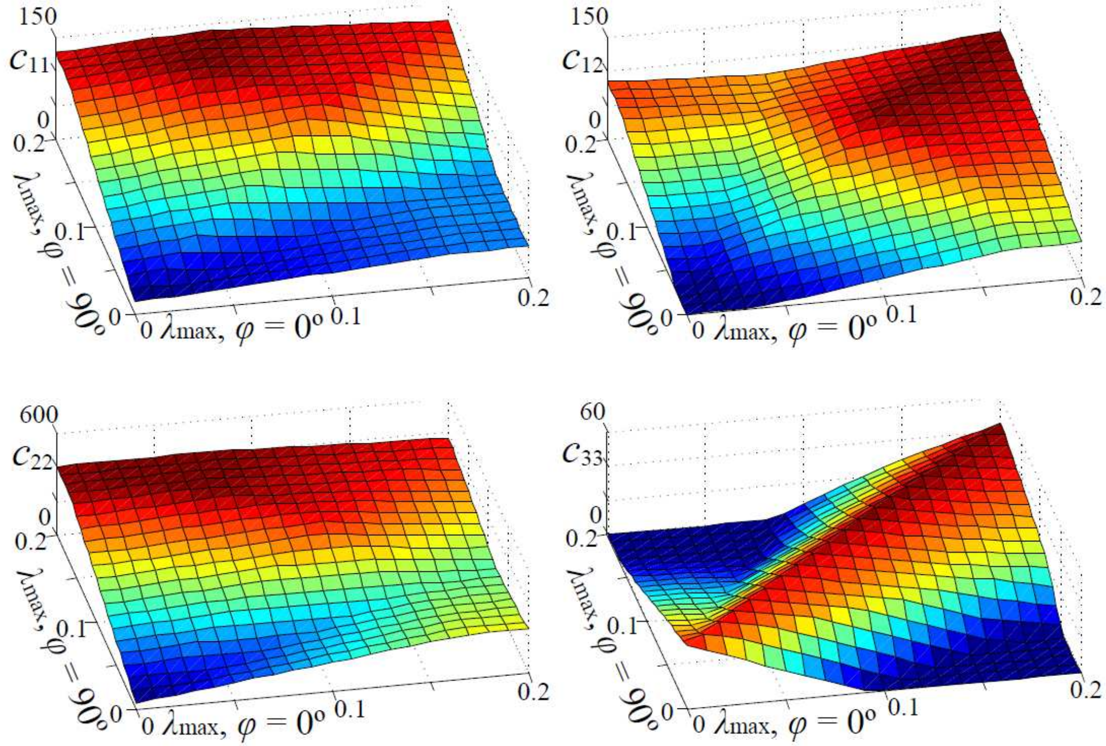
Full stretching model has 24 parameters

Linearly interpolating data points

The interpolated result is a stiffness tensor

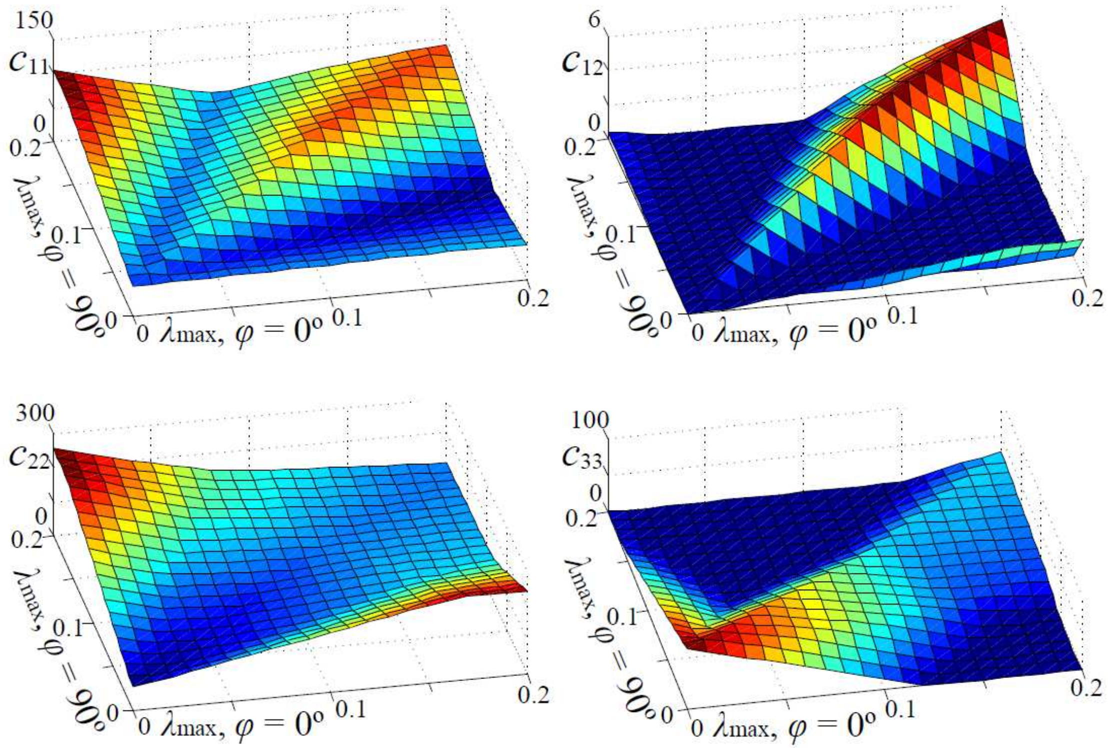
Piecewise Linear Elastic Model

- **Planar stretching model**
  - The four stiffness parameters for the Gray Interlock material shown in the polar space



Piecewise Linear Elastic Model

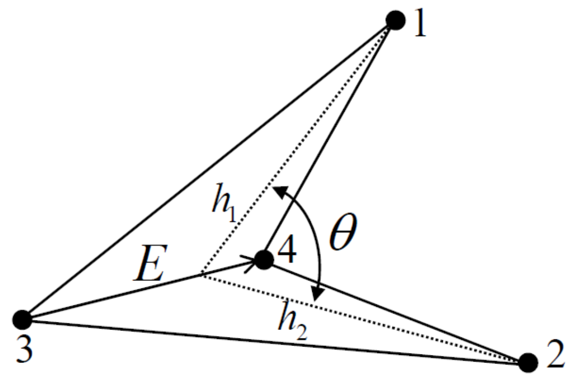
- **Planar stretching model**
  - The four stiffness parameters for the Ivory Rib Knit material shown in the polar space



Piecewise Linear Elastic Model

- **Bending model**
  - How to measure bending
    - From the bending force model [Bridson, 2003]

$$F_i = k \sin(\theta/2)\,(h_1 + h_2)^{-1}\,|E|\,u_i$$



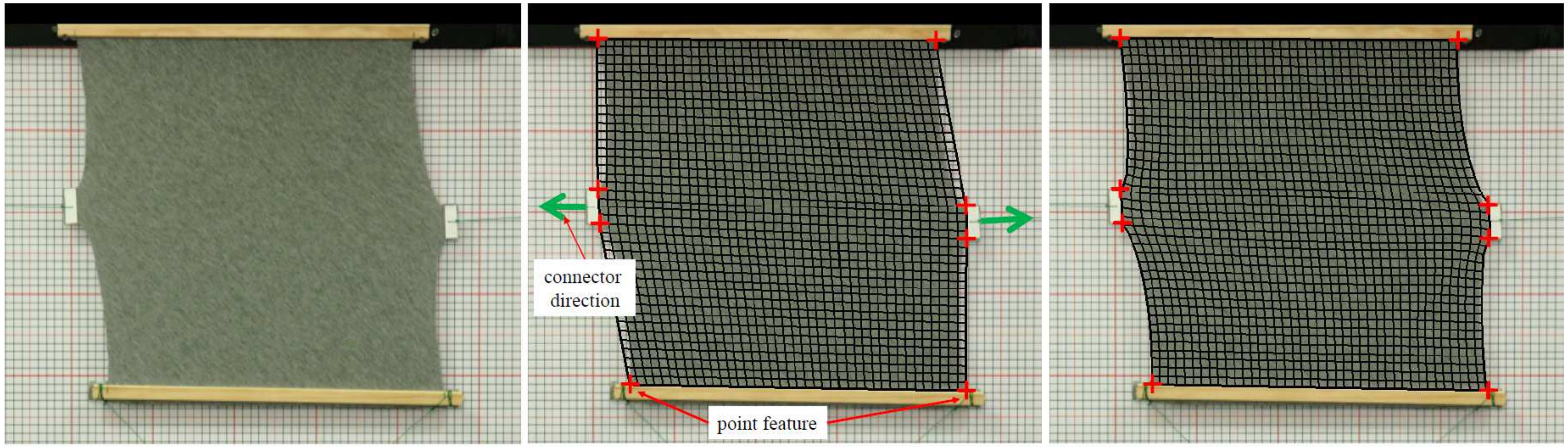
    - Piecewise linear k according to

$$\alpha = \sin(\theta/2)(h_1 + h_2)^{-1}$$

      - Approximate half of the curvature in a simple fashion
      - Linear interpolation in the polar space

Measurement

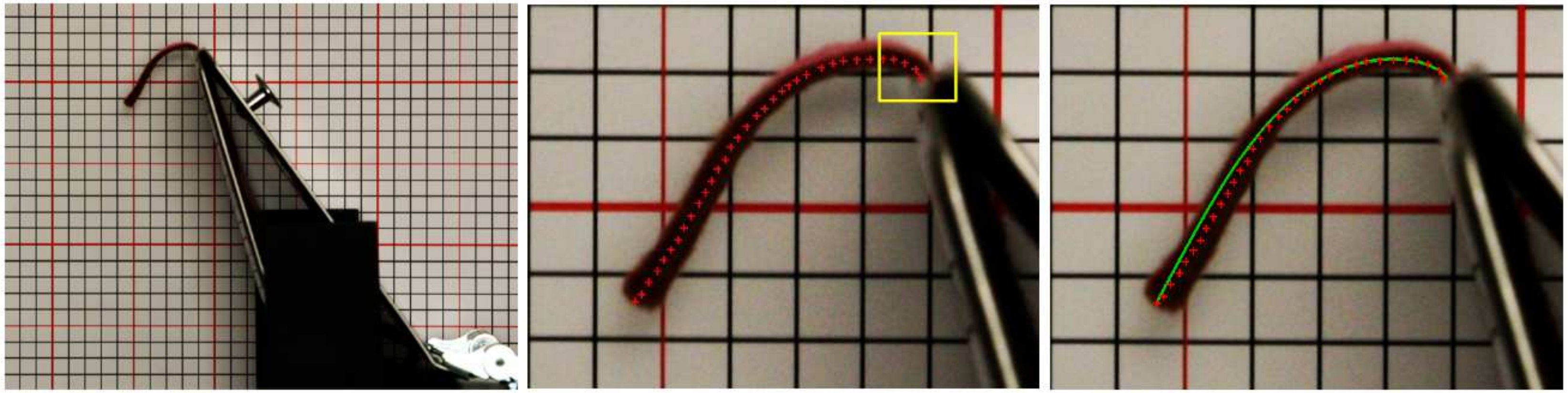
- **Stretching measurement**



(a) Original photograph (b) Point features (in red) and initialized cloth mesh (c) Final cloth mesh

Measurement

- **Bending measurement**



(a) Original photograph (b) Point features (in red) (c) Features and simulated curve (in green)

Optimization

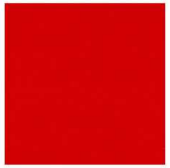
- **Problem formulation**
  - Let f_i^* be shape features captured from the i-th test
  - $f_i(p_0, p_1, ..., p_n)$ be corresponding features generated by cloth simulation
  - Our goal
    - Find optimal parameters to minimize captured and simulated features

$$\{p_0, p_1, \cdots, p_n\} = \underset{\{p_0, p_1, \cdots, p_n\}}{\arg\min} \sum_{i=1}^{T} w_i \left\| f_i^* - f_i(p_0, p_1, \cdots, p_n) \right\|$$

$$w_i = \min\left(\|f_{rest} - f_i\|^{-1},\ 10^6 \right)$$

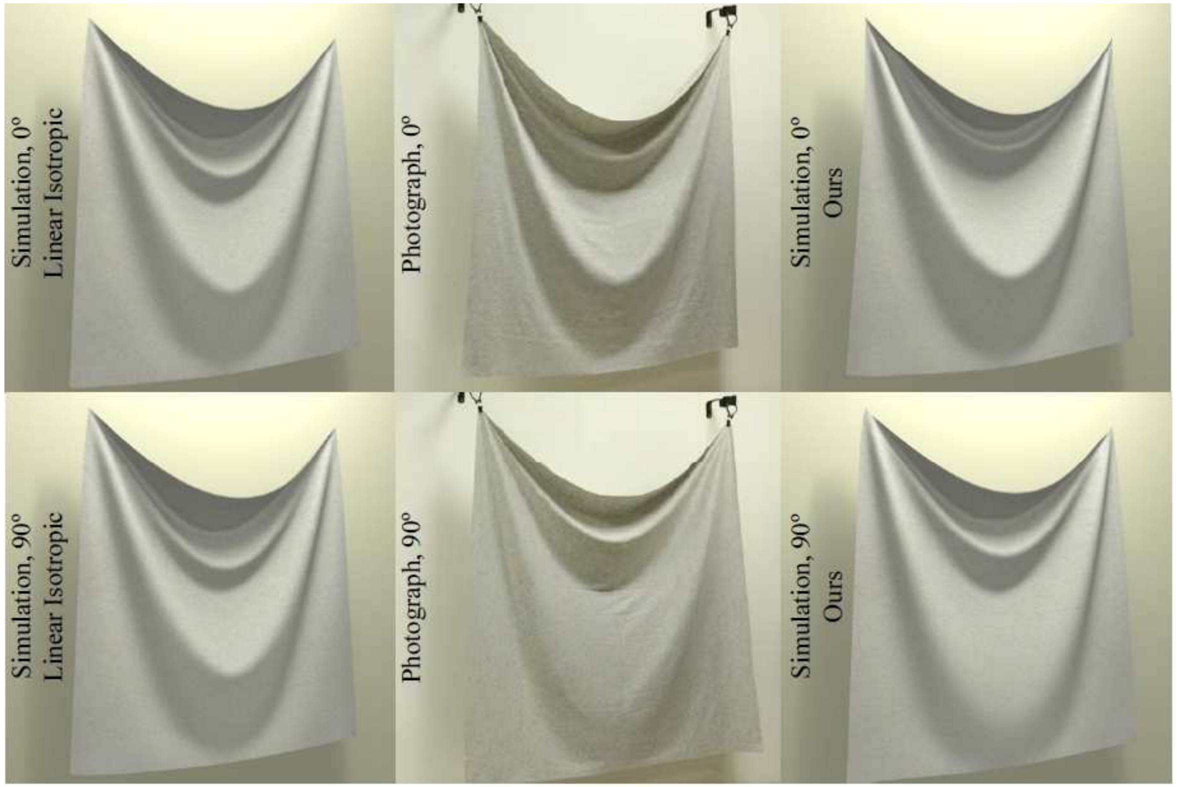
Database

- **An elastic model database of ten different cloth materials**

					
Color & Name	Ivory Rib Knit	Pink Ribbon Brown	White Dots on Black	Navy Sparkle Sweat	Camel Ponte Roma
Composition	95% Cotton 5% Spandex	100% Polyester	100% Polyester	96% Polyester 4% Spandex	60% Polyester 40% Rayon
Common Usage	Underwear	Blanket	Tablecloth	Sweater	Jacket
Density (kg/m^2)	0.276	0.228	0.128	0.224	0.284
Error (mm)	1.92	1.66	2.42	3.24	1.67
					
Color & Name	Gray Interlock	11oz Black Denim	White Swim Solid	Tango Red Jet Set	Royal target
Composition	60% Cotton 40% Polyester	99% Cotton 1% Spandex	87% Nylon 13% Spandex	100% Polyester	65% Cotton 35% Polyester
Common Usage	T-shirt	Jeans	Swimsuit	Fashion dress	Pants
Density (kg/m^2)	0.187	0.324	0.204	0.113	0.220
Error (mm)	2.58	2.30	1.57	2.06	0.89

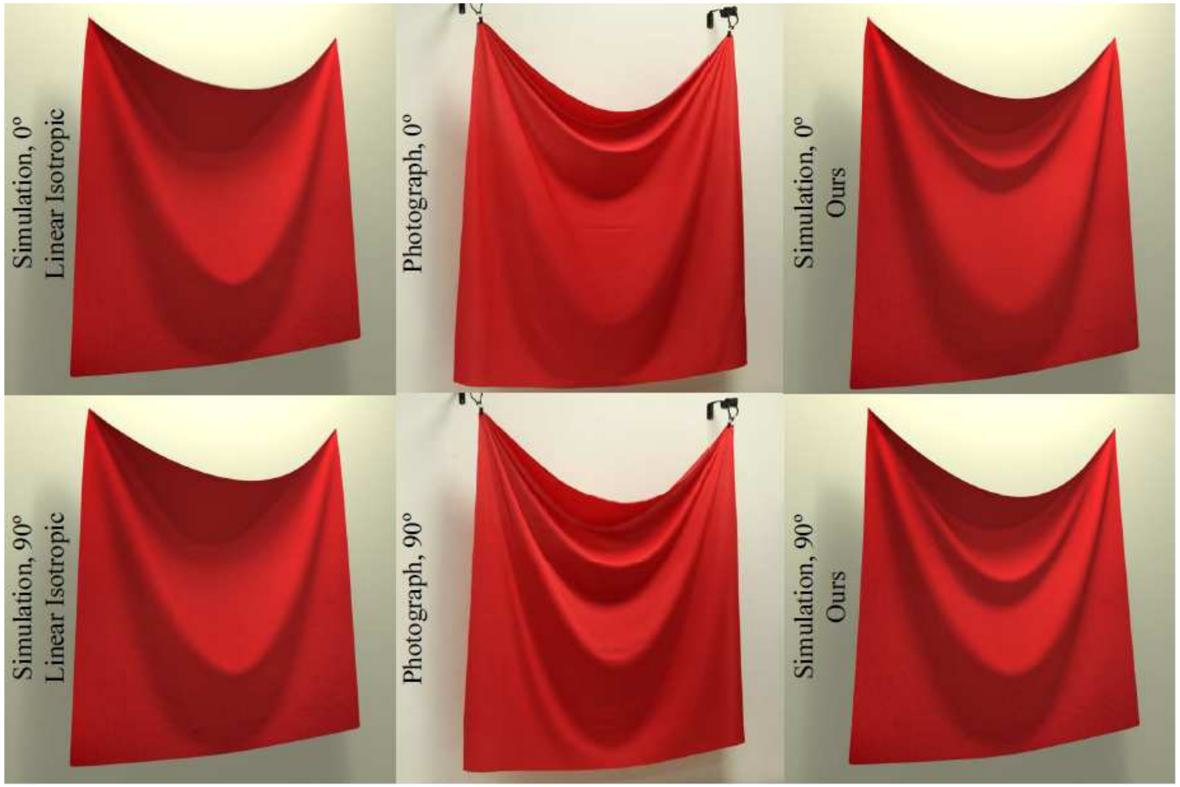
Cloth Animation Results

- **Gray interlock**



Cloth Animation Results

- **Tango red jet set**

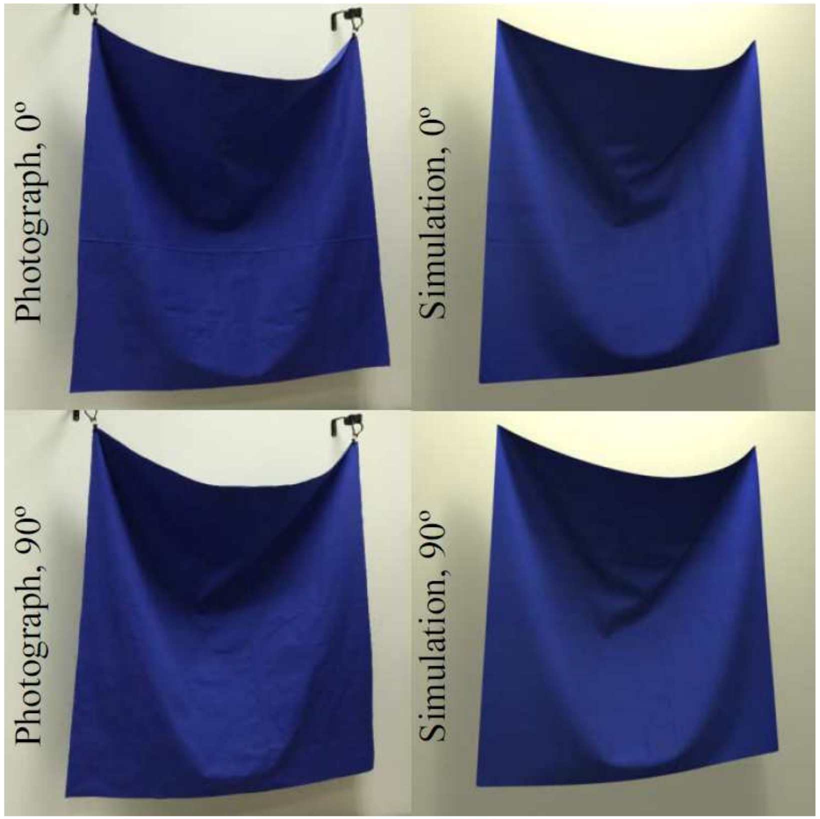


Cloth Animation Results

- **Pink ribbon brown**

Cloth Animation Results

- **Royal target**



Cloth Animation Results

- **Cloth draping over a sphere**

Data-Driven Elastic Models for Cloth: Modeling and Measurement

SIGGRAPH 2011

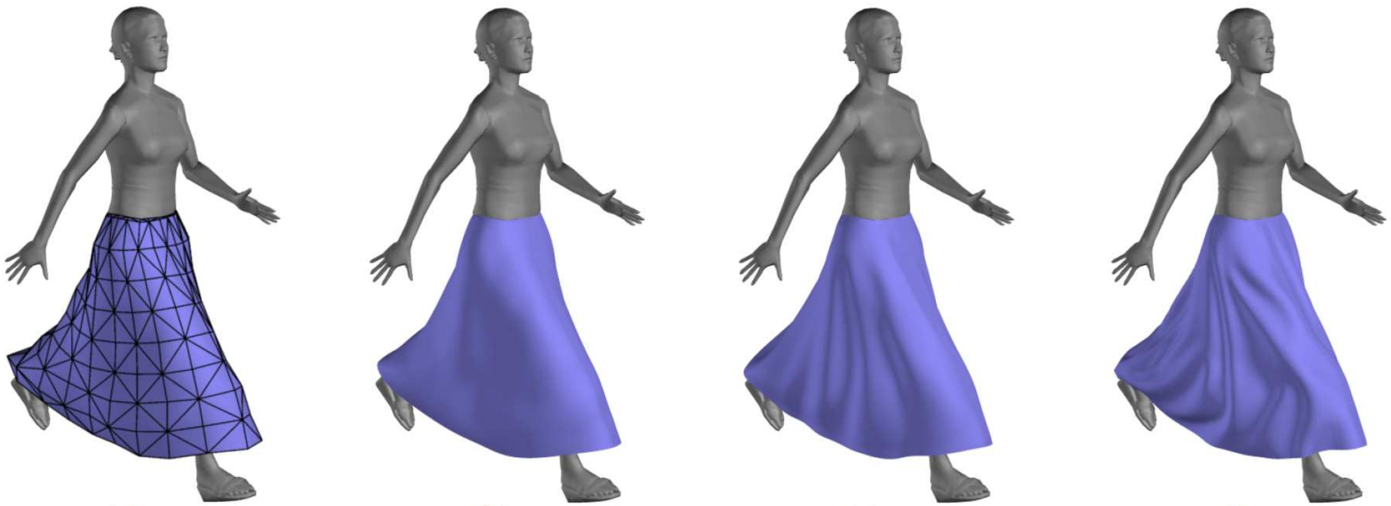
Huamin Wang
James F. O'Brien
Ravi Ramamoorthi

University of California, Berkeley

IV. Data-Driven Up-sampling for Cloth Simulation

Physics-Inspired Upsampling for Cloth Simulation in Games

- **Given coarse and fine simulations**
  - Learning the up-sampling operator to add dynamic details



(a) (b) (c) (d)

Real-Time Cloth Simulation

Physics-Inspired Upsampling
for Cloth Simulation in Games

SIGGRAPH 2011

Disney Interactive Studios

Ladislav Kavan
Dan Gerszewski
Adam W. Bargteil
Peter-Pike Sloan

The University of Utah

Next Lecture : Soft-Body Simulation – Deformable Solids