上海科技大学 ShanghaiTech University

L3. Common topics in tomography

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L3. Common topics in tomography

❑Fourier Series & Transform

- Fourier Series (1D & 2D), with example
- Fourier Transform (1D), with examples
- Properties of Fourier Transform

❑Signal Sampling & Processing

- Nyquist sampling theory
- Down sampling & Up-sampling

❑Image Space & k Space

- Fourier Transform (2D), with examples
- Spatial frequency, K space, frequency space/domain
- Rotation in image space and frequency space
- Filtering in frequency domain & image domain

Fourier Series (Real form)

Fourier, doing heat transfer work, demonstrated that any **periodic function** can be approximated as a linear composition of **sine waves**:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(2\pi nt) + b_n \sin(2\pi nt)]$$

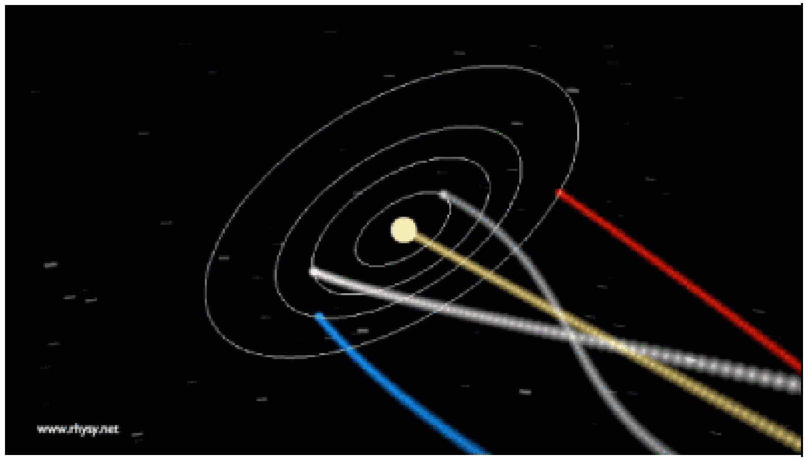
$$\frac{a_0}{2} = \int_0^1 f(t)\,dt$$

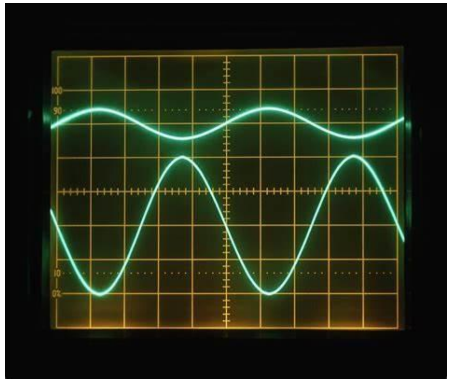
$$a_n = 2\int_0^1 f(t)\cos(2\pi n t)\,dt$$

$$b_n = 2\int_0^1 f(t)\sin(2\pi n t)\,dt$$

Note the period T=1, b.c $0 \le t \le 1$.

Sine waves or sinusoidal waves appear to be the most fundamental representation of many things in nature.





Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Fourier Series (Complex Form)

$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi int}$$

$$c_n = \int_0^1 e^{-2\pi int} f(t)\, dt$$

$$\hat{f}(n) = \int_0^1 e^{-2\pi int} f(t)\, dt$$

Note the period T=1， b.c $0 \le t \le 1$.

Example: Fourier series of square function

Let's calculate the Fourier coefficients. The function is

$$\boxed{f(t) = \begin{cases} +1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \end{cases}}$$

Note the period T=1,
b.c. $0 \leq t \leq 1$.

and then extended to be periodic of period 1. The zeroth coefficient is the average value of the function on $0 \leq t \leq 1$. Obviously this is zero. For the other coefficients we have

$$\begin{aligned} \hat{f}(n) &= \int_0^1 e^{-2\pi i n t} f(t)\, dt \\ &= \int_0^{1/2} e^{-2\pi i n t}\, dt - \int_{1/2}^1 e^{-2\pi i n t}\, dt \\ &= \left[-\frac{1}{2\pi i n} e^{-2\pi i n t}\right]_0^{1/2} - \left[-\frac{1}{2\pi i n} e^{-2\pi i n t}\right]_{1/2}^1 = \frac{1}{\pi i n}\left(1 - e^{-\pi i n}\right) \end{aligned}$$

We should thus consider the *infinite* Fourier series

$$\sum_{n \neq 0} \frac{1}{\pi i n}\left(1 - e^{-\pi i n}\right) e^{2\pi i n t}$$

Only Odd Terms Remain!

We can write this in a simpler form by first noting that

$$1 - e^{-\pi i n} = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

so the series becomes

$$\sum_{n \text{ odd}} \frac{2}{\pi i n} e^{2\pi i n t} \,.$$

Now combine the positive and negative terms and use

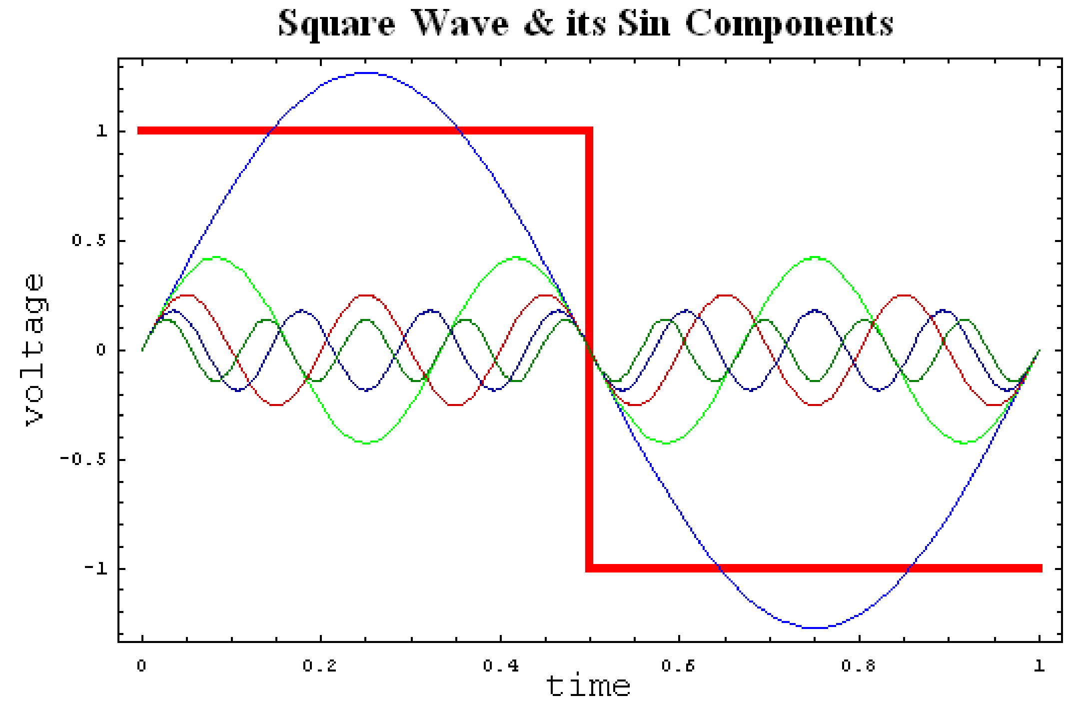
$$e^{2\pi i n t} - e^{-2\pi i n t} = 2i \sin 2\pi n t \,.$$

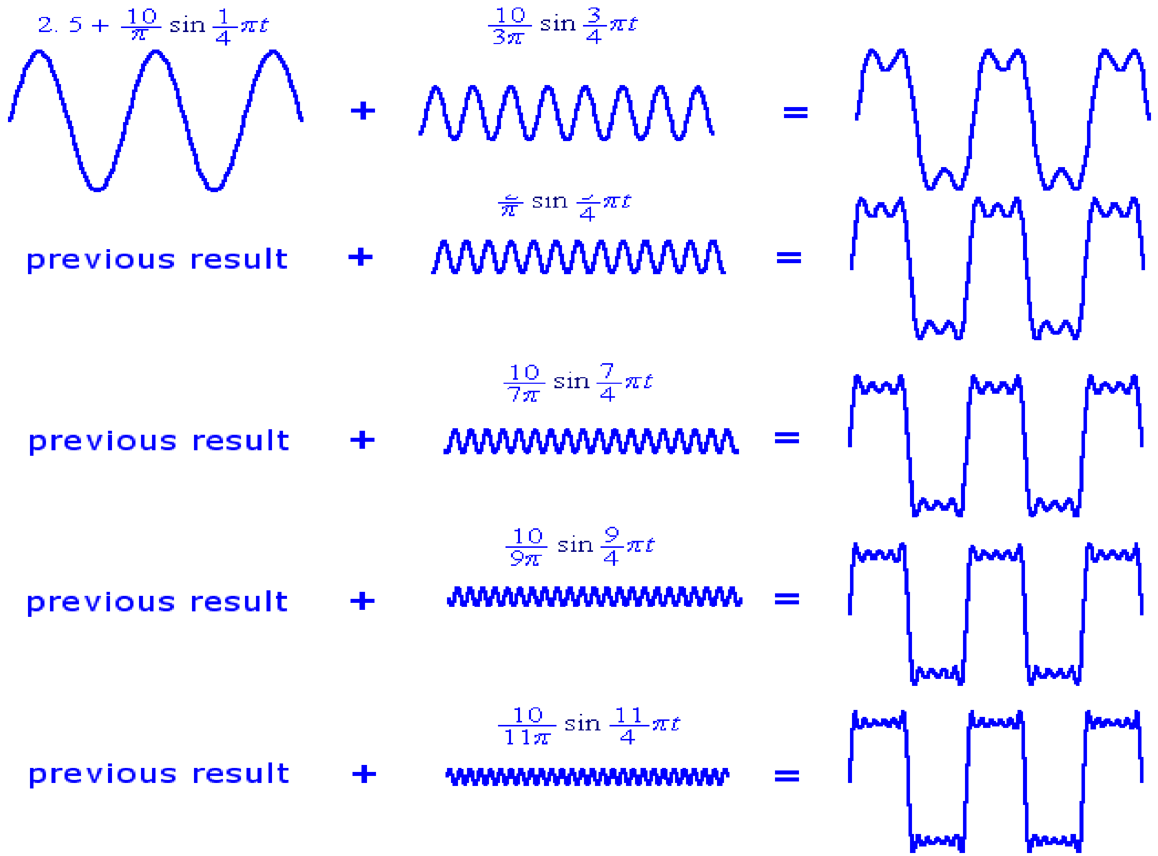
Substituting this into the series and writing $n = 2k + 1$, our final answer is

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi(2k+1)t \,.$$

(Note that the function $f(t)$ is *odd* and this jibes with the Fourier series having only sine terms.)

A Square function is a SUM of many sinusoidal component functions!





When the period T does not equal to one

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{i2\pi n\left(\frac{t}{T}\right)}, \quad t\in[0,T] \quad or \quad t\in[-T/2,T/2]$$

where

$$c_n=\frac{1}{T}\int_0^T e^{-i2\pi n\left(\frac{t}{T}\right)} f(t)dt \quad or \quad c_n=\frac{1}{T}\int_{-T/2}^{T/2} e^{-i2\pi n\left(\frac{t}{T}\right)} f(t)dt.$$

2D Fourier Series

$$F(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi(x\frac{m}{M}+y\frac{n}{N})}$$

$$f(m,n) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} F(x,y)e^{j2\pi(x\frac{m}{M}+y\frac{n}{N})}$$

Fourier Transform & Inverse Fourier Transform

$$\begin{cases} \hat{f}(u) = F(f(t)) = \int_{-\infty}^{\infty} e^{-i2\pi ut} f(t)dt\,, \\ f(t) = F^{-1}(\hat{f}(u)) = \int_{-\infty}^{\infty} e^{i2\pi ut} \hat{f}(u)du\,. \end{cases}$$

$$f(t) \Leftrightarrow \hat{f}(u)$$

Fourier Transform is an extension of Fourier Series with an infinite period.

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} e^{-i2\pi n\left(\frac{t}{T}\right)} f(t)dt \right] e^{i2\pi n\left(\frac{t}{T}\right)}$$

Note T does not necessarily equal to 1!

$$= \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} e^{-i2\pi n\left(\frac{t}{T}\right)} f(t)dt \right] e^{i2\pi n\left(\frac{t}{T}\right)} \frac{1}{T}$$

$$= \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-i2\pi ut} f(t)dt \right] e^{i2\pi n\left(\frac{t}{T}\right)} \frac{1}{T}$$

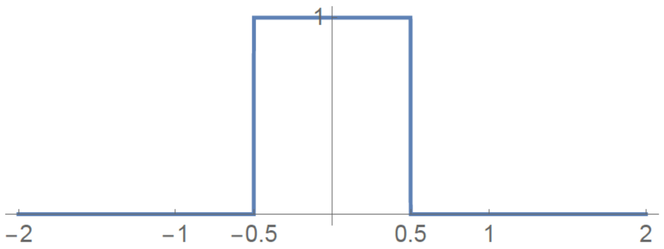
(since u=n/T)

$$= \sum_{n=-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi n\left(\frac{t}{T}\right)} \frac{1}{T} = F^{-1}\left(\hat{f}(u)\right) = \int_{-\infty}^{\infty} \hat{f}(u) e^{i2\pi ut} du$$

(since du=1/T)

Example 1: Gate Function

Gate function is sometimes called hat function, rectangular function, boxcar function.

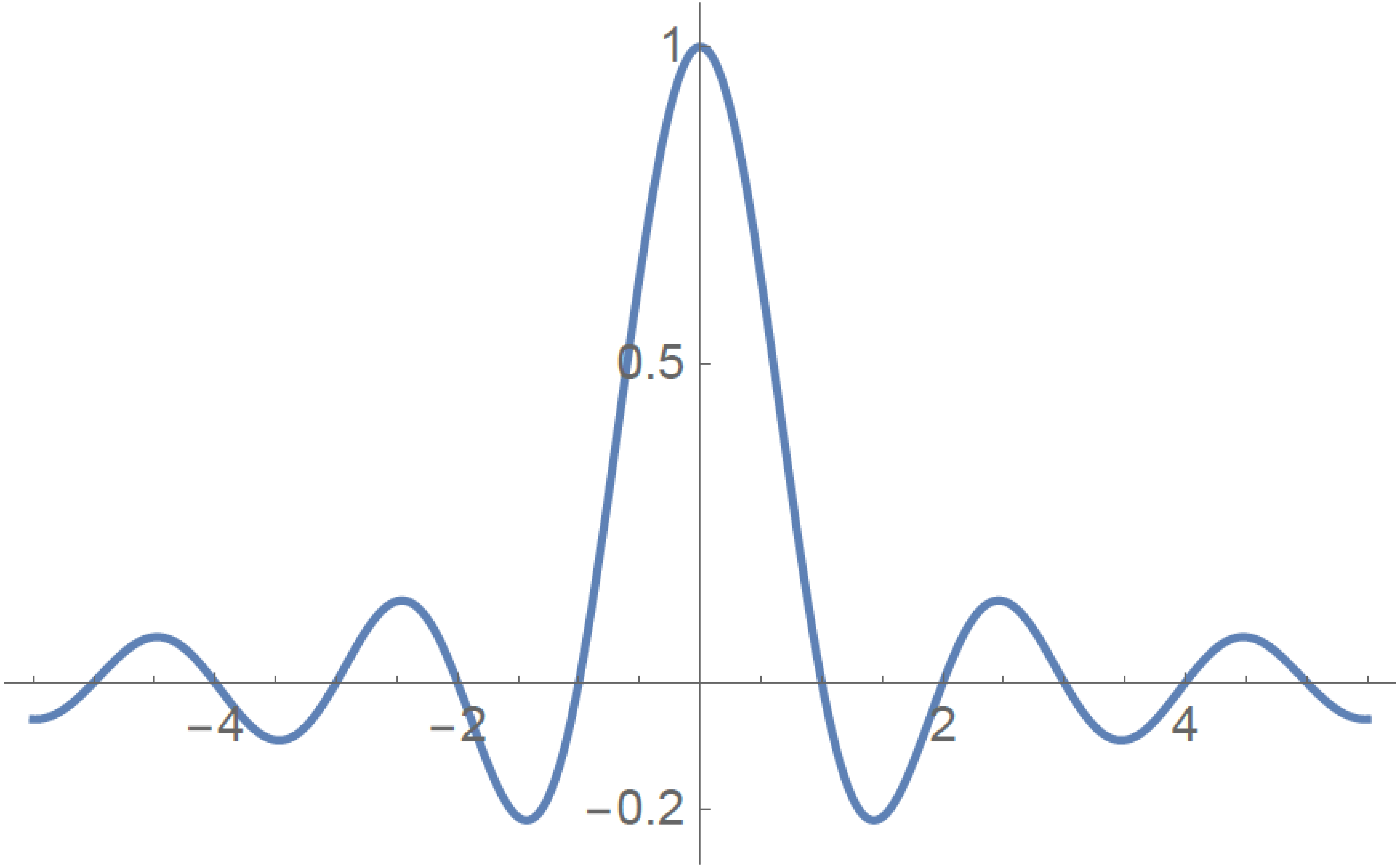


As an example, let us compute the Fourier transform of the gate function:

$$\Pi(t) \equiv \begin{cases} 1, & |t| < 1/2; \\ 0, & |t| \geq 1/2. \end{cases}$$

$$\hat{\Pi}(u) = \int_{-\infty}^{\infty} e^{-i2\pi ut} f(t)dt = \int_{-1/2}^{1/2} e^{-i2\pi ut} dt = \frac{\sin(\pi u)}{\pi u}.$$ A more

Sinc Function

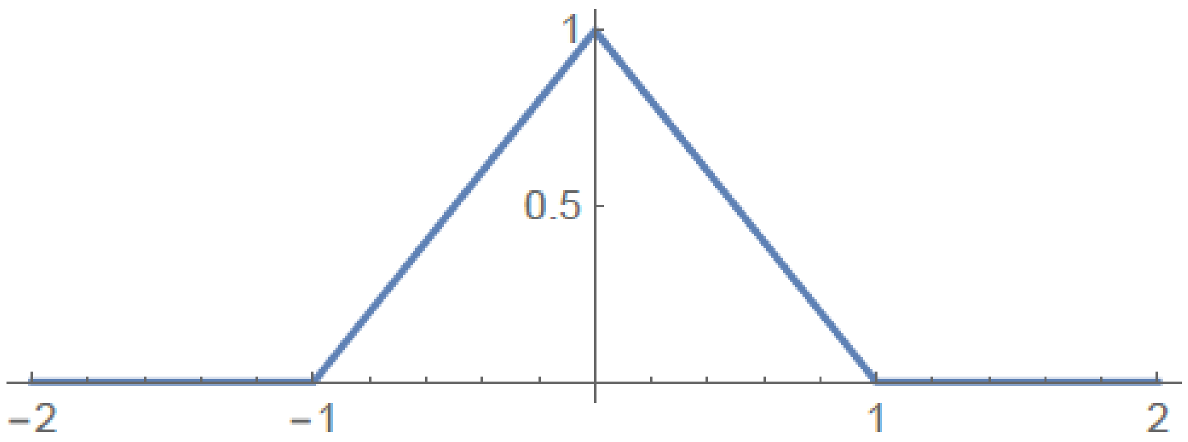


Example 2: Triangle Function

The triangle function Consider next the "triangle function", defined by

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here's the graph.

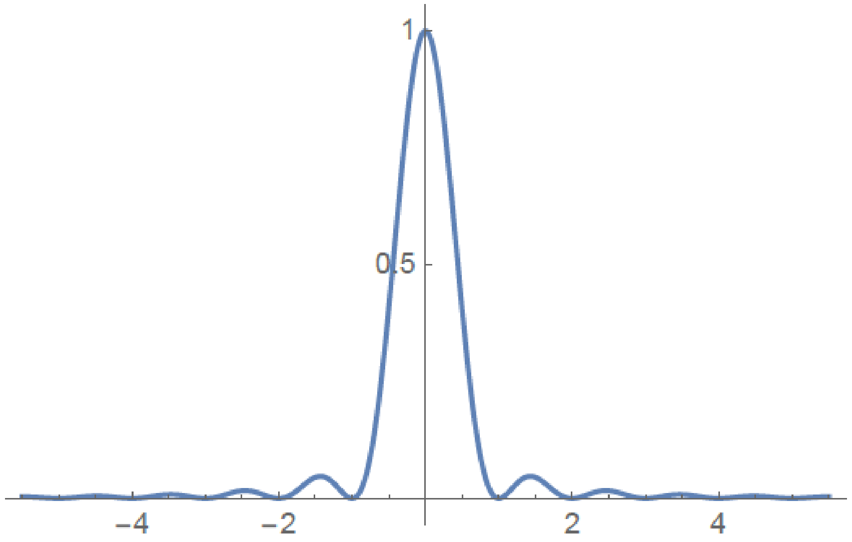


Sinc^2

$$\begin{aligned}\mathcal{F}\Lambda(s) &= \int_{-\infty}^{\infty} \Lambda(x)e^{-2\pi isx}\,dx = \int_{-1}^{0}(1+x)e^{-2\pi isx}\,dx + \int_{0}^{1}(1-x)e^{-2\pi isx}\,dx \\ &= \left(\frac{1+2i\pi s}{4\pi^2 s^2} - \frac{e^{2\pi is}}{4\pi^2 s^2}\right) - \left(\frac{2i\pi s - 1}{4\pi^2 s^2} + \frac{e^{-2\pi is}}{4\pi^2 s^2}\right) \\ &= -\frac{e^{-2\pi is}(e^{2\pi is}-1)^2}{4\pi^2 s^2} = -\frac{e^{-2\pi is}(e^{\pi is}(e^{\pi is}-e^{-\pi is}))^2}{4\pi^2 s^2} \\ &= -\frac{e^{-2\pi is}e^{2\pi is}(2i)^2\sin^2\pi s}{4\pi^2 s^2} = \left(\frac{\sin\pi s}{\pi s}\right)^2 = \operatorname{sinc}^2 s.\end{aligned}$$

It's no accident that the Fourier transform of the triangle function turns out to be the square of the Fourier transform of the rect function. It has to do with convolution, an operation we have seen for Fourier series and will see anew for Fourier transforms in the next chapter.

The graph of $\operatorname{sinc}^2 s$ looks like:



Fourier Transform of a Gaussian Function is a Gaussian Function.

$s(t) = e^{-\pi t^2}$

$$S(f) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i 2\pi (\omega_0 + f) t} \, dt$$

Expanding this equation

$$S(f) = \int_{-\infty}^{\infty} e^{-\pi \left(t^2 + i 2 f t\right)} \, dt$$

$$S(f) = \int_{-\infty}^{\infty} e^{-\pi (t + i f)^2} e^{-\pi f^2} \, dt$$

Rearranging and changing the variable of integration u = t + i f. (I have to admit that I am a little rusty on changing variables in integration.)

$$S(f) = e^{-\pi f^2} \int_{-\infty}^{\infty} e^{-\pi u^2} \, d u$$

Integration of $\int_{-\infty}^{\infty} e^{-\pi u^2} \, d u$ is equal to 1.

$$S(f) = e^{-\pi f^2}$$

Properties of Fourier Transform

- Linearity
- Scaling Theorem
- Shift Theorem
- Convolution Theorem
- Rotation Theorem
- Parseval’s Theorem/Rayleigh’s Theorem/energy theorem

Linearity

One of the simplest and most frequently invoked properties of the Fourier transform is that it is linear (operating on functions). This means:

$$\boxed{\begin{aligned} \mathcal{F}(f+g)(s) &= \mathcal{F}f(s) + \mathcal{F}g(s) \\ \mathcal{F}(\alpha f)(s) &= \alpha \mathcal{F}f(s) \quad \text{for any number } \alpha \text{ (real or complex).} \end{aligned}}$$

The linearity properties are easy to check from the corresponding properties for integrals. For example:

$$\begin{aligned} \mathcal{F}(f+g)(s) &= \int_{-\infty}^{\infty} (f(x)+g(x))e^{-2\pi isx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx + \int_{-\infty}^{\infty} g(x)e^{-2\pi isx}\,dx = \mathcal{F}f(s) + \mathcal{F}g(s)\,. \end{aligned}$$

! Fourier Transforms of different functions may be added together.

Example of Linearity

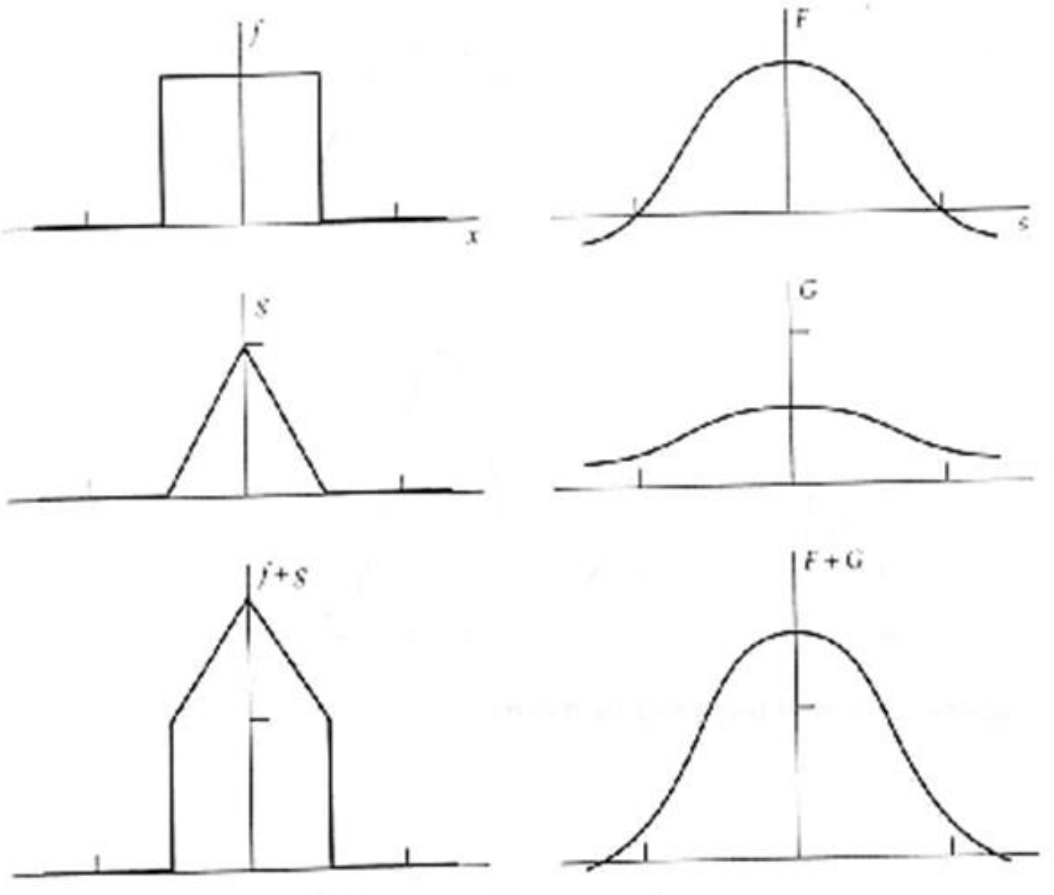


Fig. 6.5 The addition theorem $f + g \supset F + G$.

Scaling Theorem

We assume that $a \neq 0$. First suppose $a > 0$. Then

$$\int_{-\infty}^{\infty} f(at)e^{-2\pi ist}\,dt = \int_{-\infty}^{\infty} f(u)e^{-2\pi is(u/a)}\frac{1}{a}\,du$$

(substituting $u = at$; the limits go the same way because $a > 0$)

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(u)e^{-2\pi i(s/a)u}\,du = \frac{1}{a}\mathcal{F}f\left(\frac{s}{a}\right)$$

If $a < 0$ the limits of integration are reversed when we make the substitution $u = ax$:

$$\int_{-\infty}^{\infty} f(at)e^{-2\pi ist}\,dt = \frac{1}{a}\int_{+\infty}^{-\infty} f(u)e^{-2\pi is(u/a)}\,du$$

$$= -\frac{1}{a}\int_{-\infty}^{+\infty} f(u)e^{-2\pi i(s/a)u}\,du$$

(flipping the limits back introduces a minus sign)

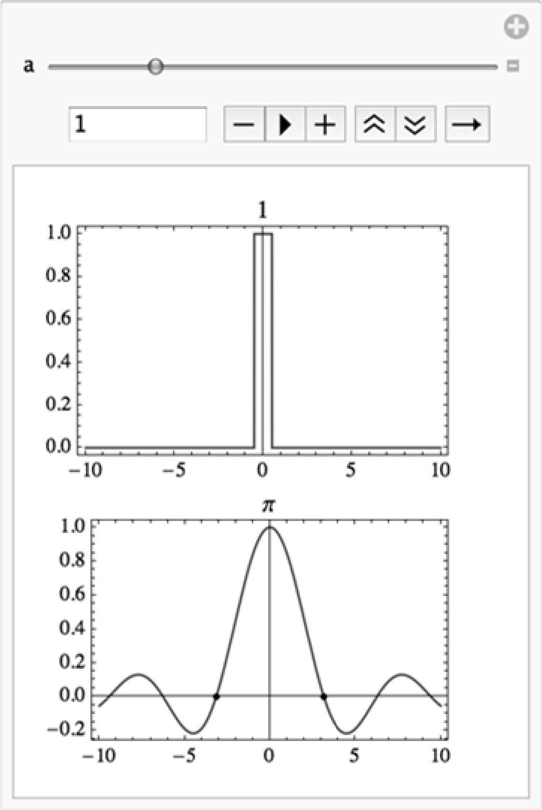
$$= -\frac{1}{a}\mathcal{F}f\left(\frac{s}{a}\right).$$

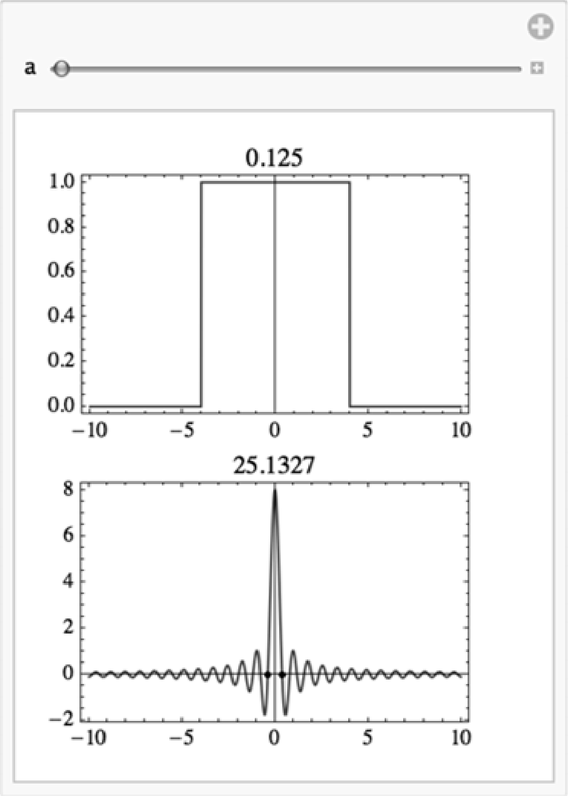
Since $-a$ is positive when a is negative ($-a = |a|$), we can combine the two cases and present the stretch theorem in its full glory:

- If $f(t) \rightleftharpoons F(s)$ then $f(at) \rightleftharpoons \frac{1}{|a|}F\left(\frac{s}{a}\right)$.

! if the signal is scaled in one domain, inverse scaling occurs in the Fourier domain.

Example of Scaling





Shift Theorem

To compute the Fourier transform of $f(t-b)$ for any constant b, we have

$$\int_{-\infty}^{\infty} f(t-b)e^{-2\pi ist}\,dt = \int_{-\infty}^{\infty} f(u)e^{-2\pi is(u+b)}\,du$$

(substituting $u = t - b$; the limits still go from $-\infty$ to ∞)

$$= \int_{-\infty}^{\infty} f(u)e^{-2\pi isu}e^{-2\pi isb}\,du$$

$$= e^{-2\pi isb}\int_{-\infty}^{\infty} f(u)e^{-2\pi isu}\,du$$

($e^{-2\pi isb}$ comes out of the intergral because it doesn't depend on u)

$$= e^{-2\pi isb}\hat{f}(s).$$

The best notation to capture this property is probably the pair notation, $f \rightleftharpoons F$.[8] Thus:

- $\boxed{\text{If } f(t) \rightleftharpoons F(s) \text{ then } f(t-b) \rightleftharpoons e^{-2\pi isb}F(s).}$

! if a function is translated in one domain, this is a phase shift in Fourier domain.

Convolution Theorem

A convolution is an integral that expresses the amount of overlap of one function **g(t)** as it is shifted over another function **f(t)**. It therefore "blends" one function with another.

$$f * g \equiv \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} g(\tau)\, f(t-\tau)\, d\tau$$

Convolution in imaging is a complicated mathematical procedure that results in image blurring.

! Convolution in one domain is equal to a multiplication in its Fourier domain.

$$\mathrm{g(x)} * \mathrm{h(x)} \overset{\mathcal{FT}}{\Longleftrightarrow} \mathrm{G(s)\ H(s)}$$

$$\mathcal{FT}\{\mathrm{g(x)} * \mathrm{h(x)}\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u)\, h(x-u)\, du \right) e^{-i2\pi s x}\, dx$$

Changing the order of integration.

$$\mathcal{FT}\{\mathrm{g(x)} * \mathrm{h(x)}\} = \int_{-\infty}^{\infty} g(u) \left(\int_{-\infty}^{\infty} h(x-u)\, e^{-i2\pi s x}\, dx \right) du$$

Notice that the convolution is a Fourier Transform of h shifted by u.

$$\mathcal{FT}\{\mathrm{g(x)} * \mathrm{h(x)}\} = \int_{-\infty}^{\infty} g(u)\, H(s)\, e^{-i2\pi s u}\, du$$

Rearranging terms and bringing H(s) outside the integration.

$$\mathcal{FT}\{\mathrm{g(x)} * \mathrm{h(x)}\} = \left(\int_{-\infty}^{\infty} g(u)\, e^{-i2\pi s u}\, du \right) \mathrm{H(s)}$$

Performing the second integration along u

$$\mathcal{FT}\{\mathrm{g(x)} * \mathrm{h(x)}\} = \mathrm{G(s)\ H(s)}$$

Rotation Theorem

For $f(\vec{r}) \Leftrightarrow \hat{f}(\vec{u})$, if $\vec{r}\,' = R_{\vec{\theta}}\vec{r}$ where $R_{\vec{\theta}}$ is an orthonormal rotation matrix specified by the angular vector $\vec{\theta}$, then we have $f(R_{\vec{\theta}}\vec{r}) \Leftrightarrow \hat{f}(R_{\vec{\theta}}\vec{u})$.

Proof:

$$\hat{f}(\vec{u}) = \int_{R^N} f(\vec{r})e^{-i2\pi(\vec{r}\cdot\vec{u})}d\vec{r} = \int_{R^N} f(\vec{r})e^{-i2\pi(\vec{r}^{\,t}\vec{u})}d\vec{r}\,.$$

Then,

$$\hat{f}(R_{\vec{\theta}}\vec{u}) = \int_{R^N} f(\vec{r})e^{-i2\pi((\vec{r}^{\,t}R_{\vec{\theta}})\vec{u})}d\vec{r} = \int_{R^N} f(\vec{r})e^{-i2\pi((R_{\vec{\theta}}^t\vec{r})^t\vec{u})}d\vec{r}\,.$$

Let $\vec{r}\,' = R_{\vec{\theta}}^t\vec{r}$, the subscribe "$t$" means the transport of a matrix, we have $\vec{r} = R_{\vec{\theta}}\vec{r}\,'$ and

$$\hat{f}(R_{\vec{\theta}}\vec{u}) = \int_{R^N} f(R_{\vec{\theta}}\vec{r}\,')e^{-i2\pi((\vec{r}\,')^t\vec{u})}d\vec{r}\,',$$ that is,

$$\boxed{f(R_{\vec{\theta}}\vec{r}) \Leftrightarrow \hat{f}(R_{\vec{\theta}}\vec{u})\,.}$$

! Rotation in one domain is equivalent to rotation in Fourier domain.

Parseval's (Identity) Theorem

Also known as Rayleigh's Theorem, energy theorem

复数z的模和共轭:

$$|z| = z\bar{z}$$

$$|x + i\, y| \equiv \sqrt{x^2 + y^2}\,.$$

$|z|$ is the **modulus** (模) of the complex number z, also called the **complex norm**.

$|z|^2$ is called **absolute square**.

$\bar{z}$ is **conjugate** (共轭) of z. Sometimes conjugation is represented by star as z^*.

$$\int_{-\infty}^{\infty} |g(x)|^2\, dx = \int_{-\infty}^{\infty} |G(s)|^2\, ds$$

Proof:
$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt = \int_{-\infty}^{\infty} f(t)\overline{\left(\int_{-\infty}^{\infty} e^{2\pi i s t}\mathcal{F}g(s)\, ds\right)}\, dt$$

(applying Fourier inversion to g and $\mathcal{F}g$, that's the key)

$$= \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} e^{-2\pi i s t}\overline{\mathcal{F}g(s)}\, ds\right) dt$$

(complex conjugation doing what it does)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\mathcal{F}g(s)}e^{-2\pi i s t}\, ds\, dt$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-2\pi i s t}\, dt\right)\overline{\mathcal{F}g(s)}\, ds$$

(changing the order of integration and swapping double integrals and iterated integrals)

$$= \int_{-\infty}^{\infty} \mathcal{F}f(s)\overline{\mathcal{F}g(s)}\, ds.$$

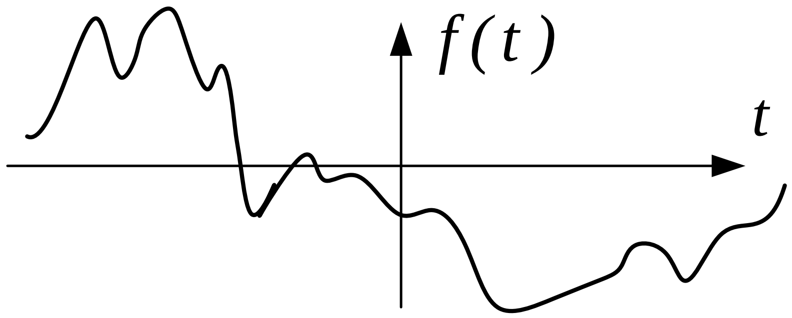
L4. Common topics in 3D tomography

❑Fourier Series and Transform
- Fourier Series (1D & 2D), with example
- Fourier Transform (1D &2D), with examples
- Properties of Fourier Transform

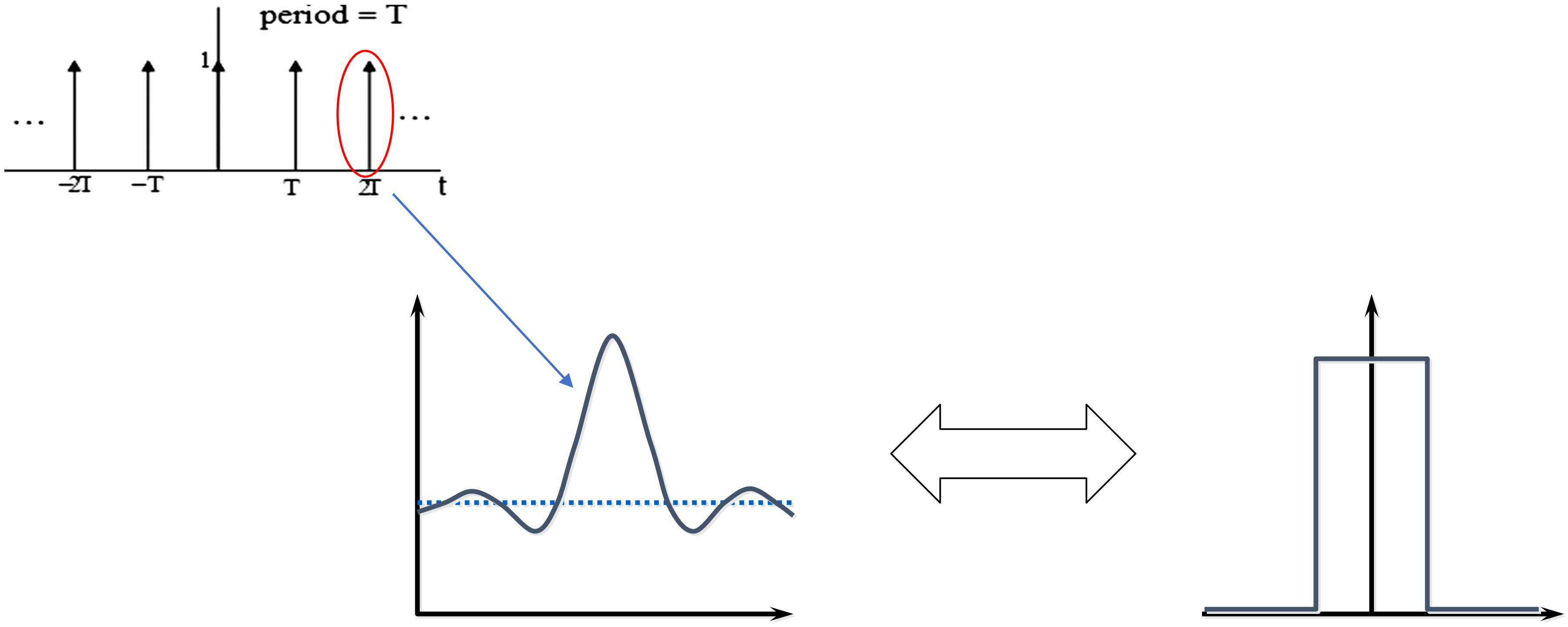
❑Signal Sampling & Processing
- Nyquist sampling theory
- Down sampling & Up-sampling
- Signal and Noise
- Filtering

❑Image Space and its k Space
- Spatial frequency
- K space, frequency space/domain
- Filtering in frequency domain

Analog to Digital

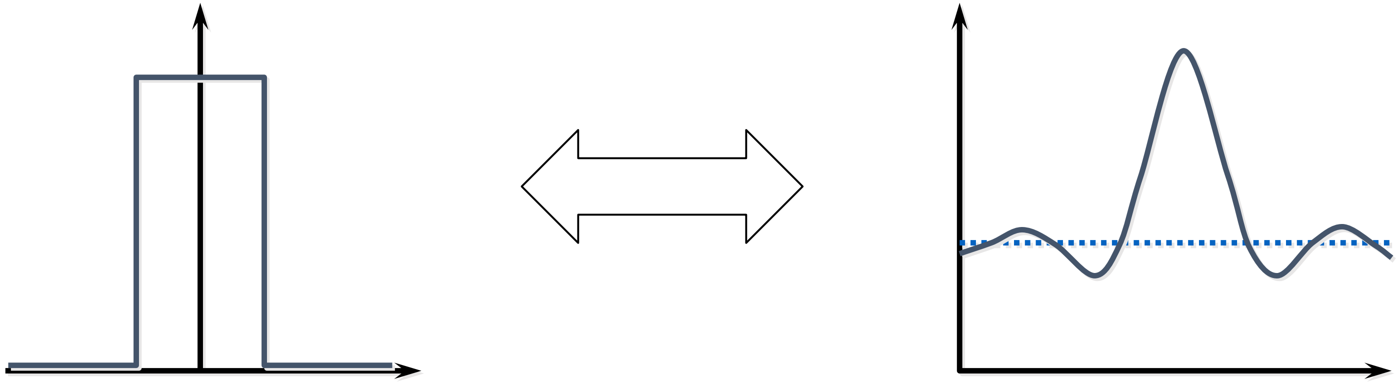


Ideal Sampling Filter



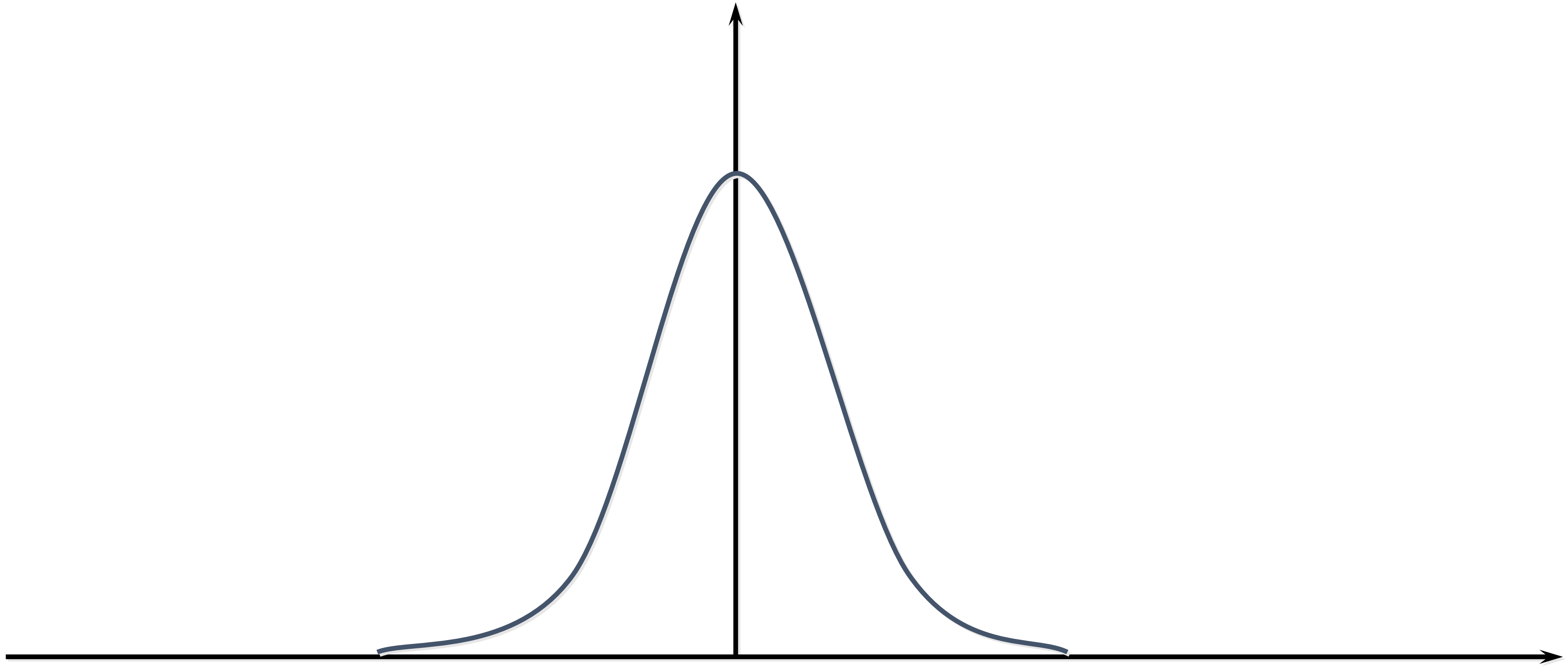
Delta function sampling (filter) is not physically possible, the ideal sampling (filter) is a **sinc function**, with infinite ringing.

Cheap Sampling Filter



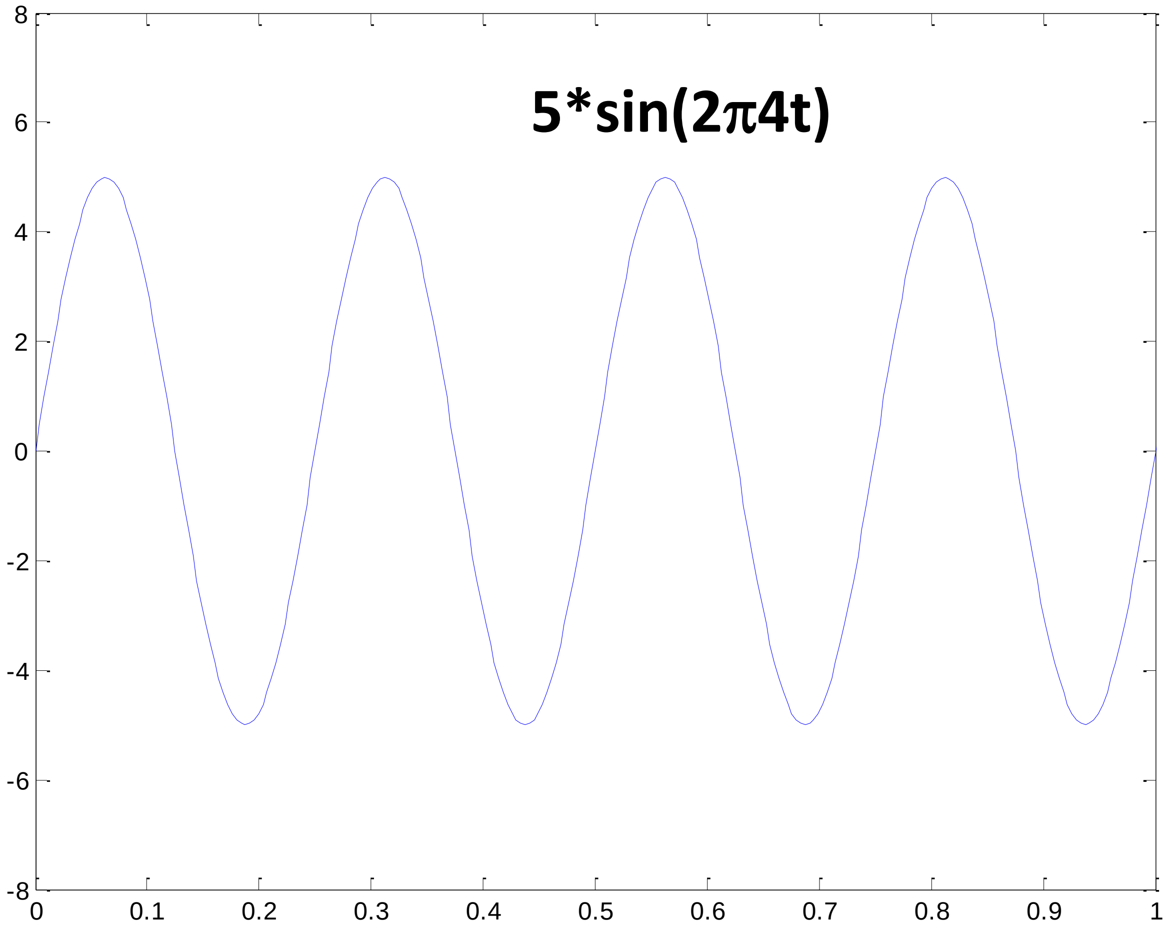
It is a **sinc function in the frequency domain**, with infinite ringing.

Gaussian Sampling Filter



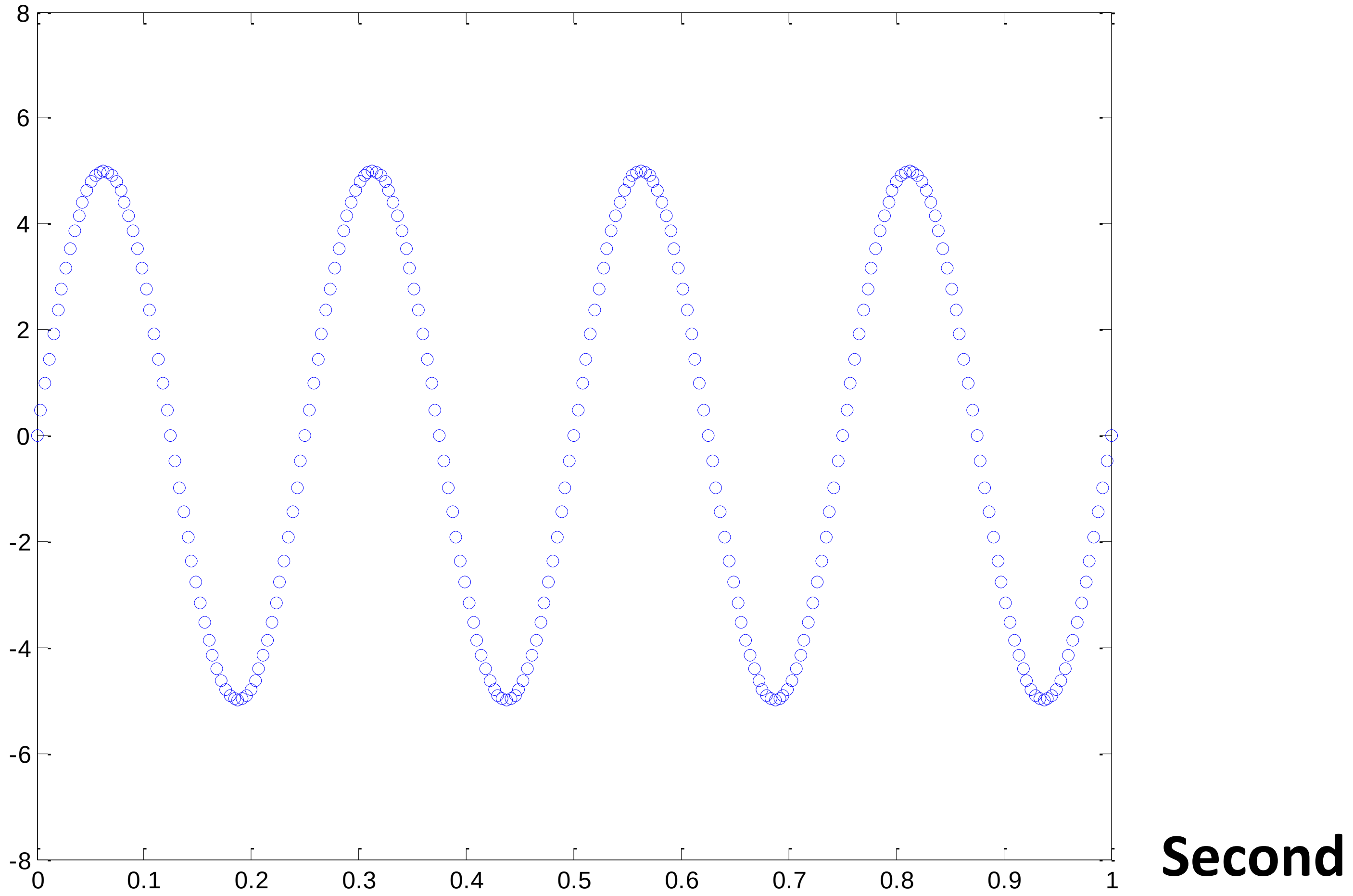
- Fourier transform of Gaussian = Gaussian
- Good compromise as a sampling filter

Continuous Wave



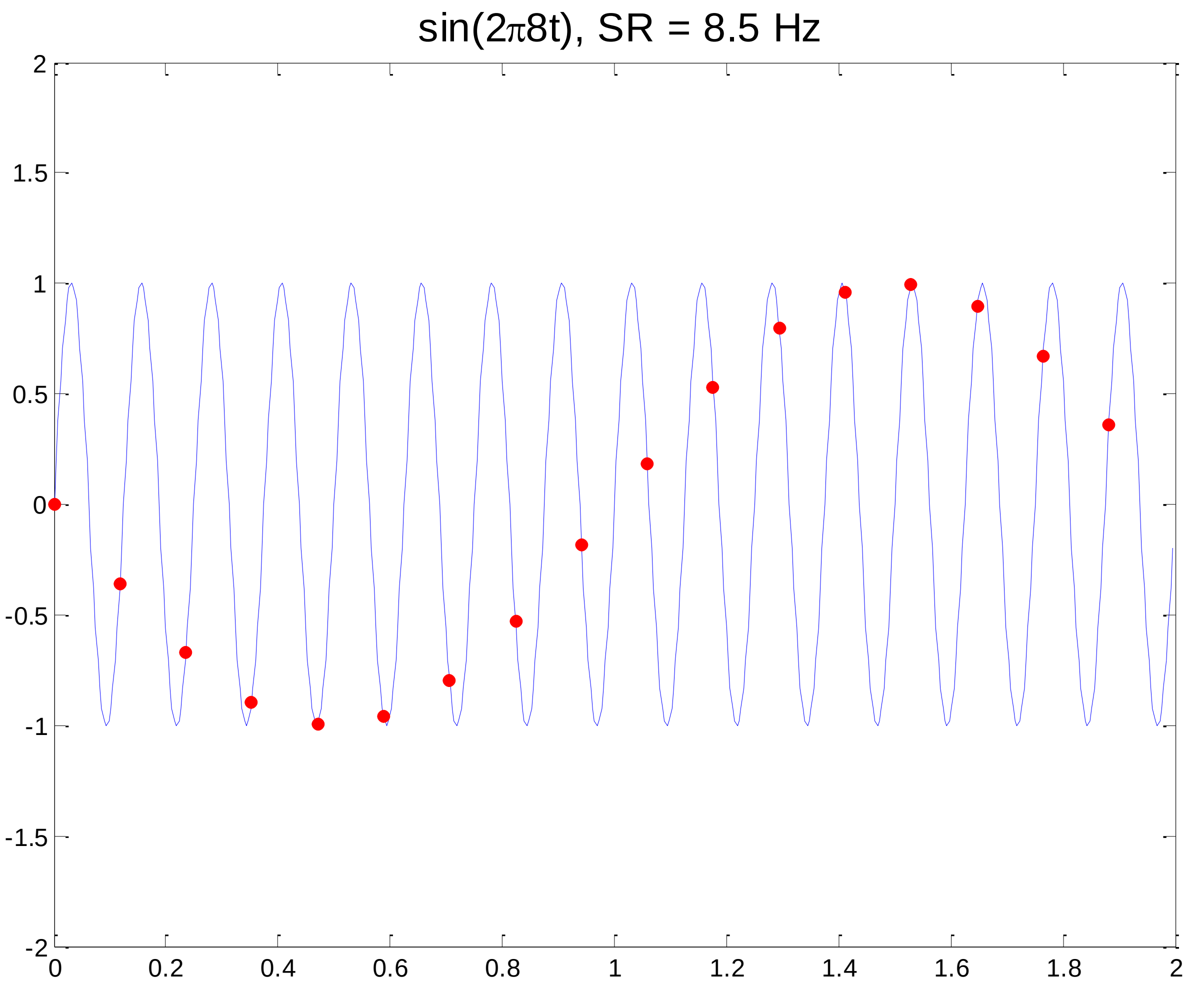
Second

Well Sampled



Original Signal Frequency = 4 Hz, Sampling Rate = 256 Samples/s

Under-sampled

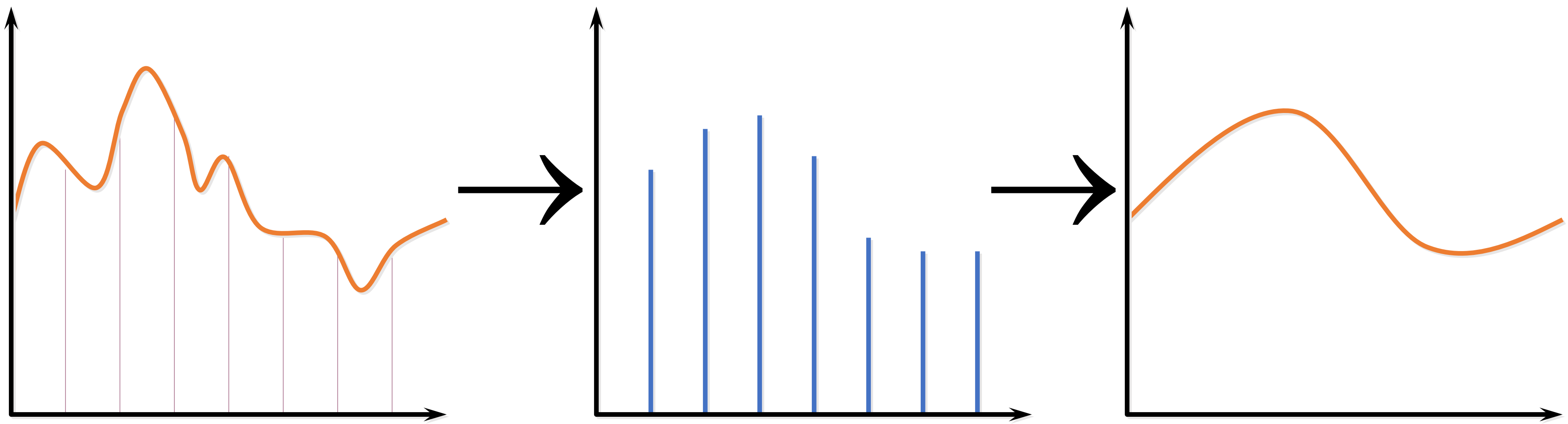


Original signal cannot be accurately reconstructed from under-sampled signal!

Aliasing Problem

Original signal

Sampled signal with Aliasing problem; it looks like another signal.



Aliasing: - An effect that causes different signals to become indistinguishable when sampled.

How to faithfully sample a signal?

Nyquist Sampling Theorem

- Nyquist Sampling Theorem says:

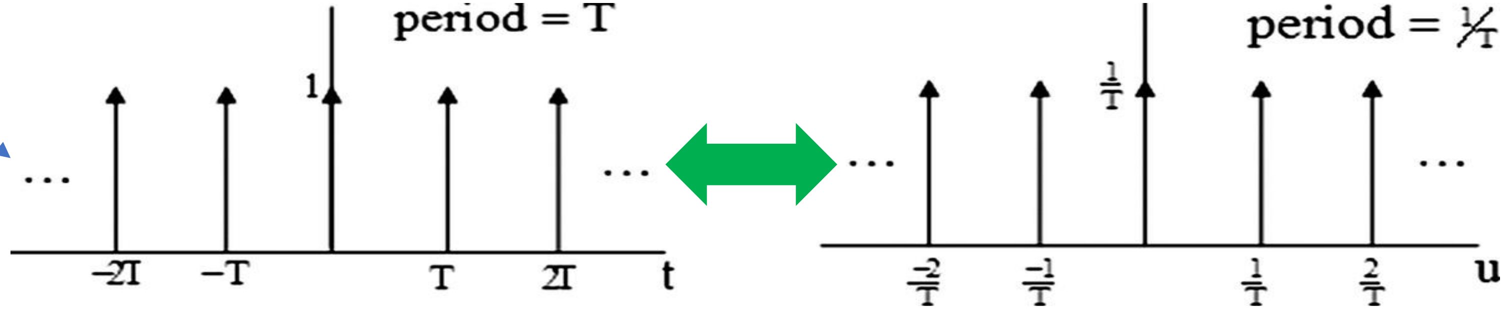
  *- the sampling frequency should be **at least twice** the highest frequency contained in the signal.*

Fourier Transform of a Comb Function is a Comb Function.

$$s(t) = \sum_n \delta(t - nT) \quad \leftrightarrow \quad S(f) = \frac{1}{T}\sum_n \delta\left(f - \frac{n}{T}\right)$$

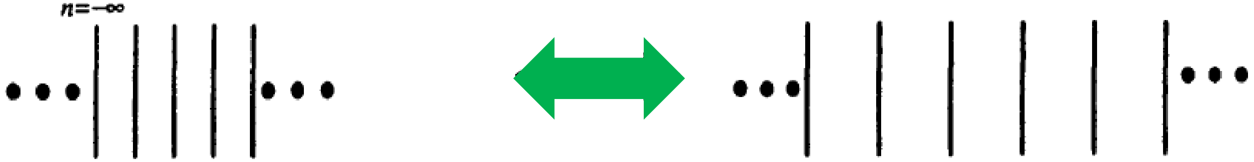
Comb Function

where we used $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0)$.



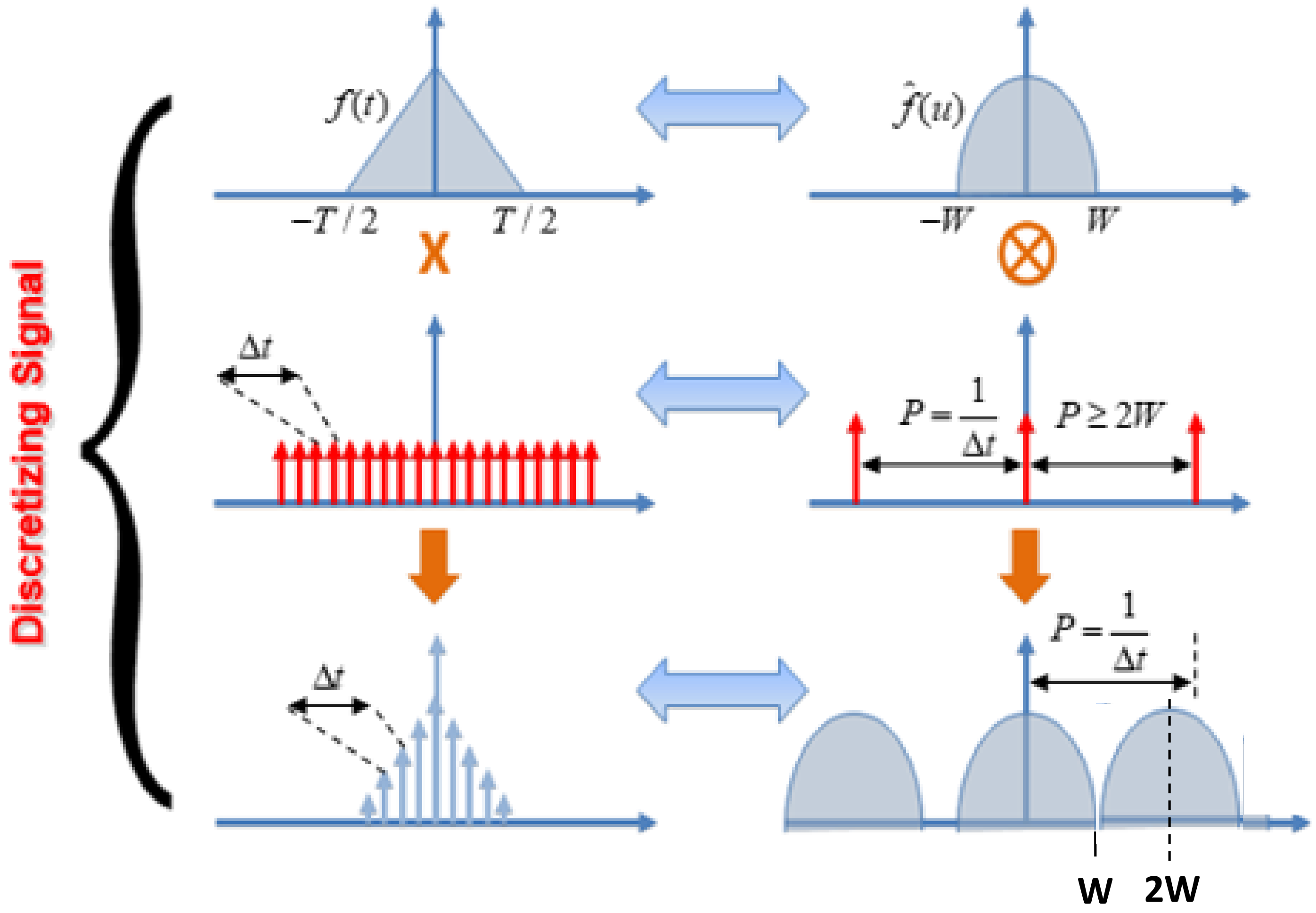
$$\sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \qquad\qquad \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{\Delta t}\right)$$

Comb



Comb

Discretizing Signal



W: highest/maximum frequency of Fourier signal

P: sampling frequency

If P < 2W, two Fourier signals cannot be separated → Aliasing

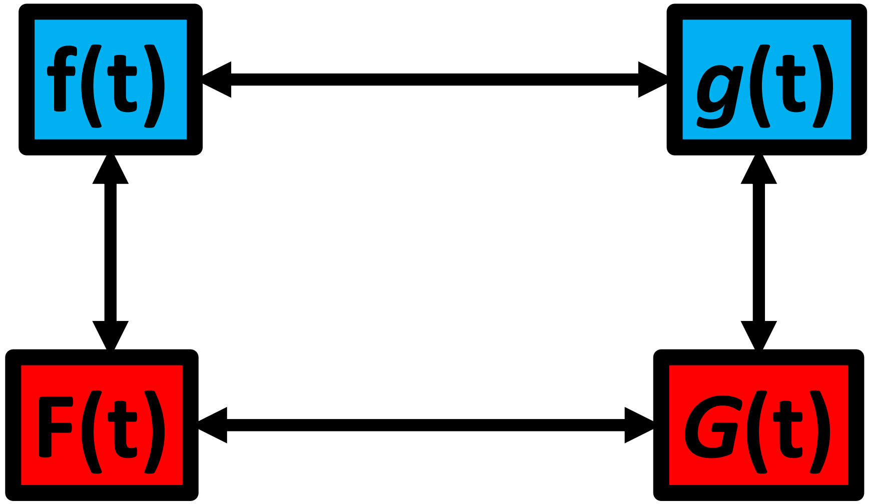
Original signal × comb function = discretized signal

Fourier signal ⊗ comb function = Fourier signal

Big Picture: From Continuous to Discrete

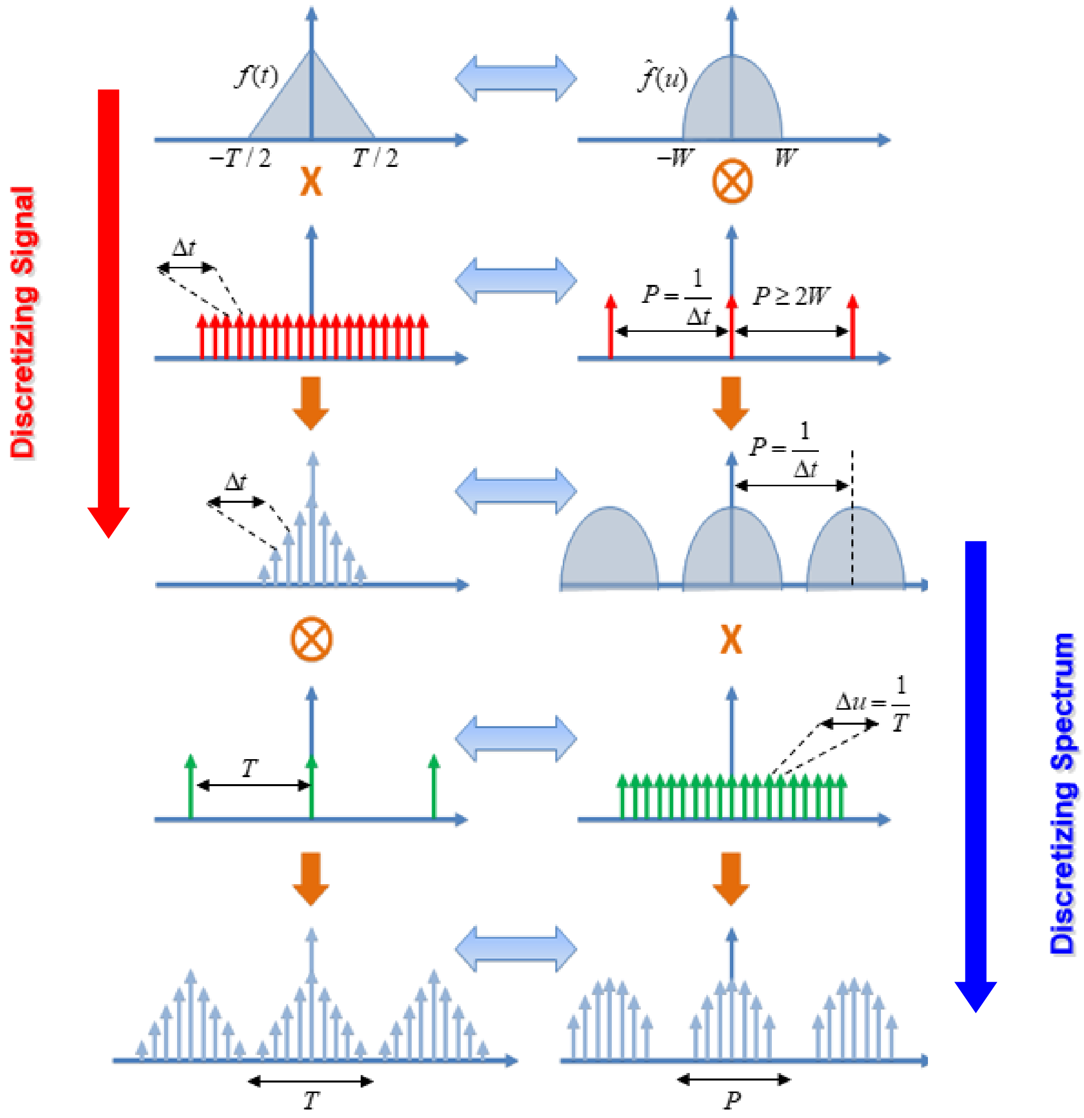
Start with a signal $f(t)$ and its Fourier transform $\mathcal{F}f(s)$, each a function of a continuous variable. We want to:

- Find a discrete version of $f(t)$ that's a reasonable approximation to $f(t)$.
- Find a discrete version of $\mathcal{F}f(s)$ that's a reasonable approximation to $\mathcal{F}f(s)$.
- Find a way that the discrete version of $\mathcal{F}f(s)$ is related to the discrete version of $f(t)$ that's a reasonable approximation to the way $\mathcal{F}f(s)$ is related to $f(t)$.



Continuous **Discrete**

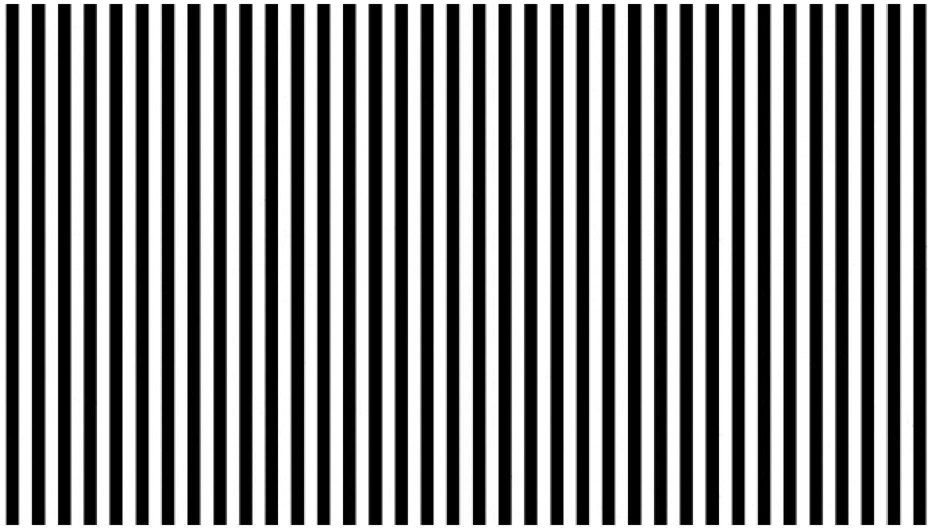
Big Picture



Discretizing Signal
$f(t)$
$-T/2$
$T/2$
X
$\hat{f}(u)$
$-W$
W
$\otimes$
Δt
$P=\frac{1}{\Delta t}$
$P \geq 2W$
Δt
$P=\frac{1}{\Delta t}$
$\otimes$
X
T
$\Delta u=\frac{1}{T}$
Discretizing Spectrum
T
P

Spatial Sampling

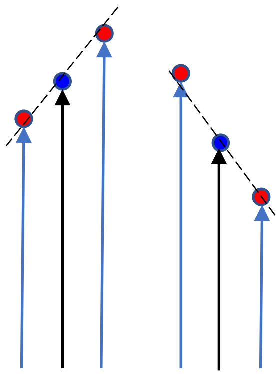
Bar Pattern Phantom with period = 1000 um



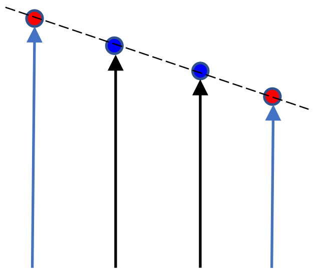
What is the least spatial sampling frequency that the imaging system should provide?

Digital Downsampling & Upsampling

downsampling



upsampling



Same philosophy can be extended to 2D too!

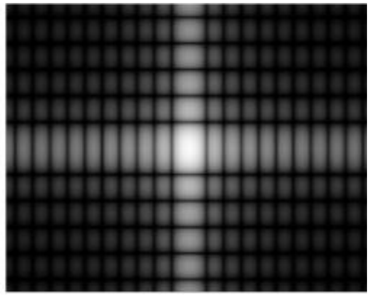
2D Fourier Transform

$$F(u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

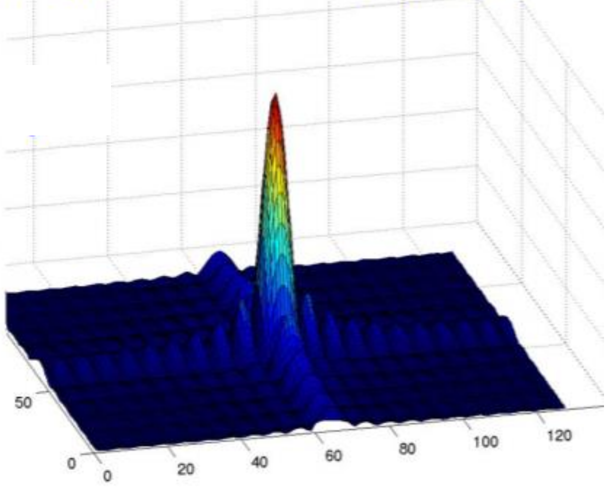
Example: 2D Rectangle Function

Rectangle of Sides X and Y, Centered at Origin



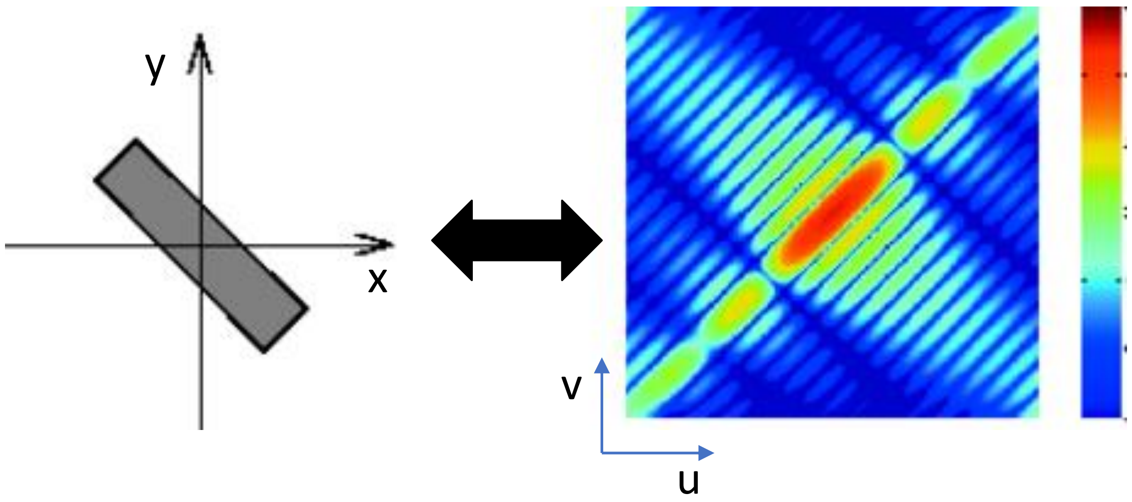
$f(x,y)$ $|F(u,v)|$

$$
\begin{aligned}
F(u,v) &= \int\int f(x,y)e^{-j2\pi(ux+vy)}dxdy,\\
&= \int_{-X/2}^{X/2} e^{-j2\pi ux}dx \int_{-Y/2}^{Y/2} e^{-j2\pi vy}dy,\\
&= \left[\frac{e^{-j2\pi ux}}{-j2\pi u}\right]_{-X/2}^{X/2} \left[\frac{e^{-j2\pi vy}}{-j2\pi v}\right]_{-Y/2}^{Y/2},\\
&= \frac{1}{-j2\pi u}\left[e^{-juX} - e^{juX}\right]\frac{1}{-j2\pi v}\left[e^{-jvY} - e^{jvY}\right],\\
&= XY\left[\frac{sin(\pi Xu)}{\pi Xu}\right]\left[\frac{sin(2\pi Yv)}{\pi Yv}\right]\\
&= XY\mathrm{sinc}(\pi \mathrm{Xu})\mathrm{sinc}(\pi \mathrm{Yv}).
\end{aligned}
$$



$|F(u,v)|$

Image space and K-space



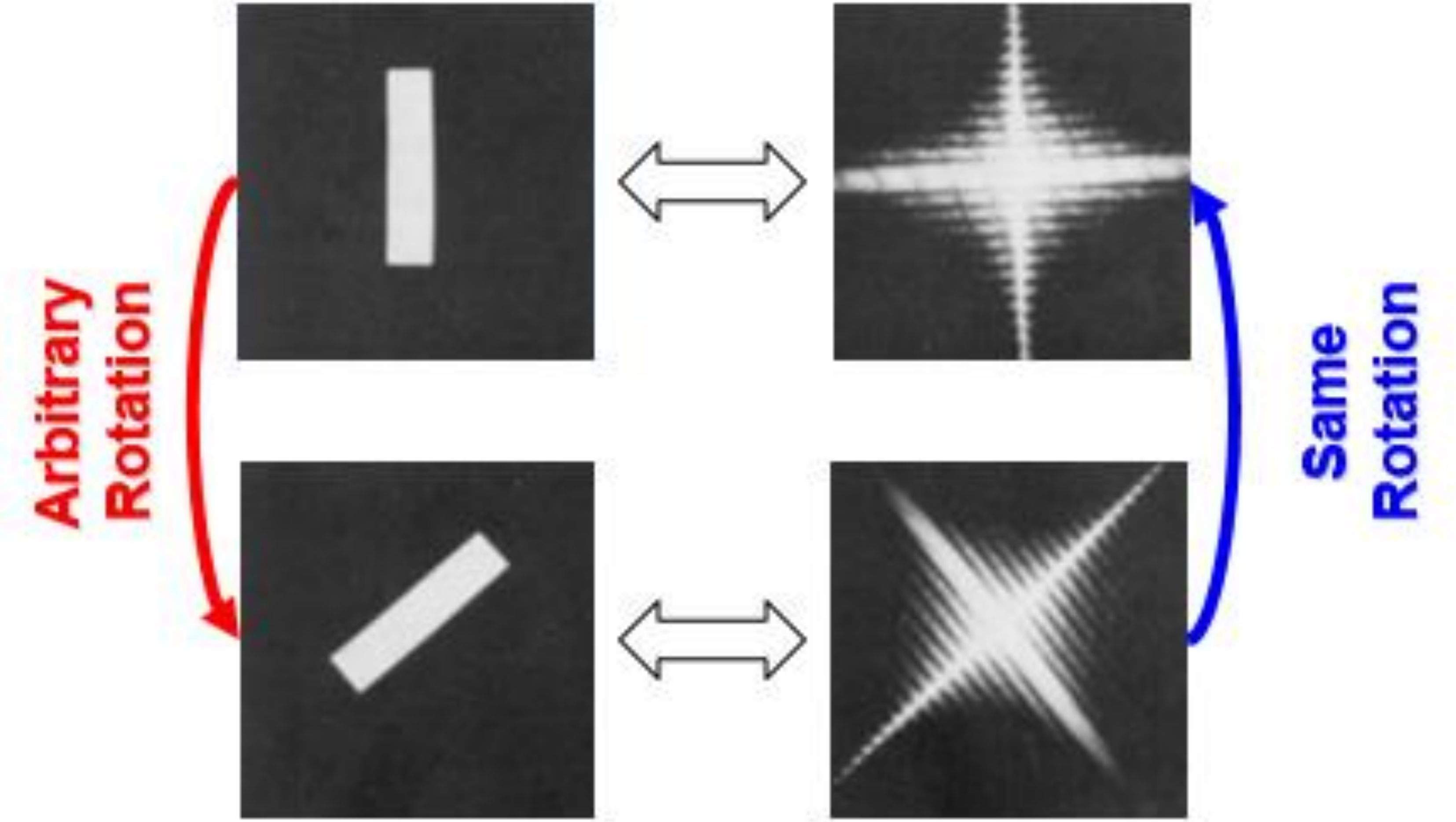
(x, y) 坐标系

(u, v) 坐标系

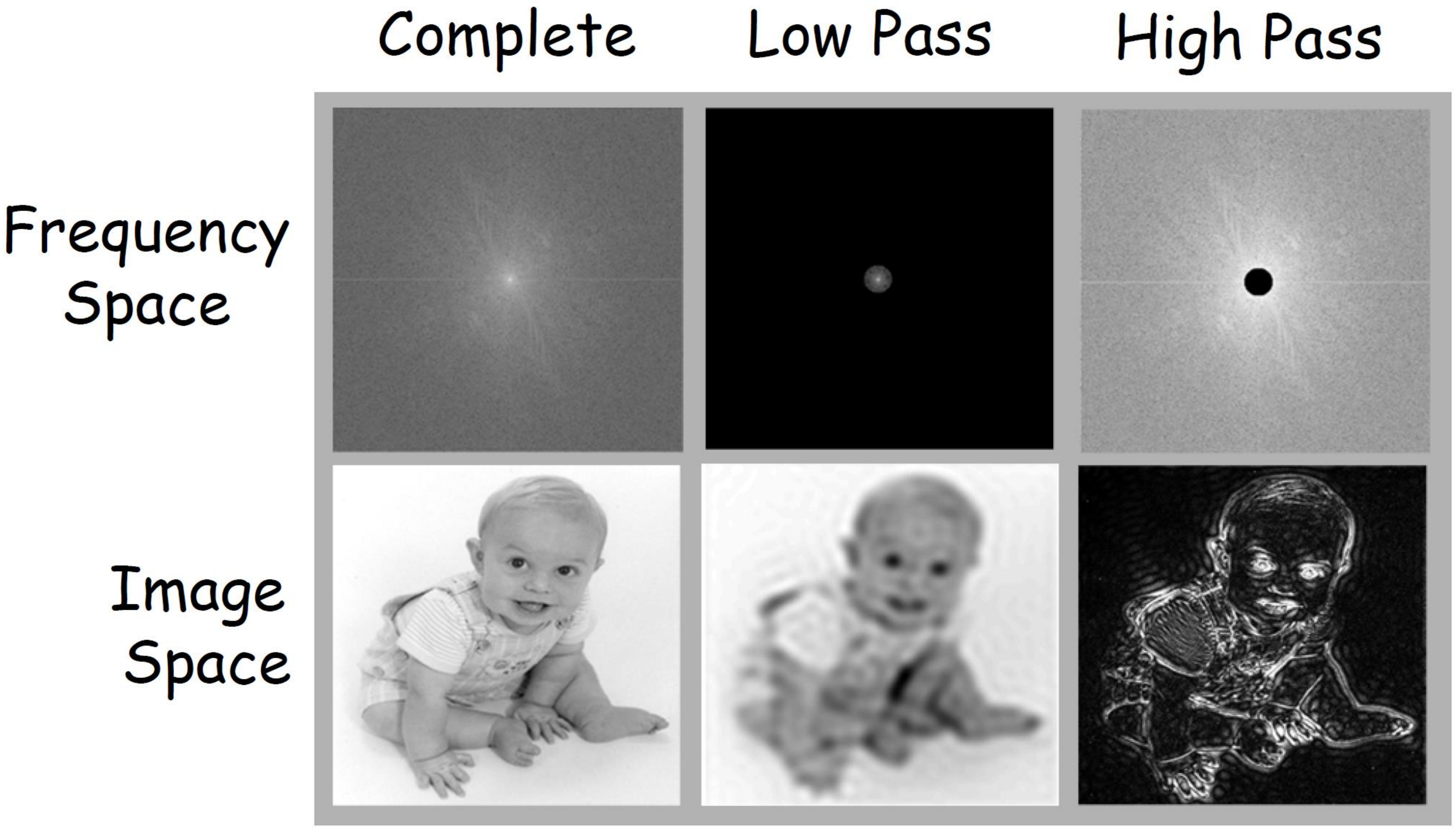
Spatial dimension in spatial domain

Spatial frequency in k-space (i.e. frequency space/domain)

Rotation in image domain is equivalent to Rotation in frequency domain!

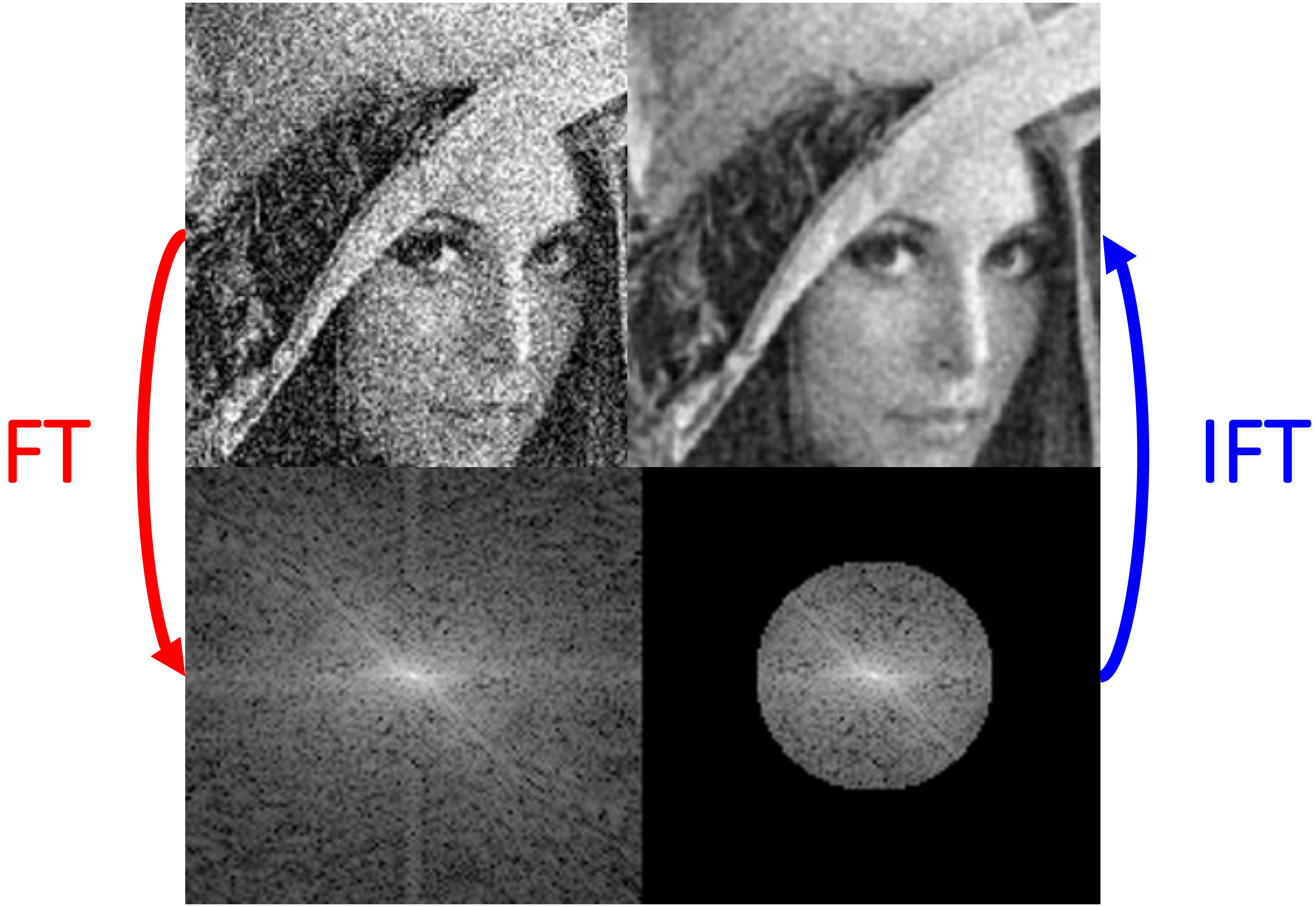


Filtering is convolution in image space and simple multiplication in frequency space!



Filtering is done in **frequency** space!

Noise Suppression with low pass filtering



Filtering in Image Space

'kernel'

original image

1	5	3	5	4	6
4	3	32	5	6	9
6	10	4	8	8	7

*

filter

1/12	1/12	1/12
1/12	4/12	1/12
1/12	1/12	1/12

=

filtered image

1	5	3	5	4	6
4	a	b	c	d	9
6	10	4	8	8	7

$$\left\{\begin{array}{l}
a=(1)(1/12)+(5)(1/12)+(3)(1/12)+(4)(1/12)+(3)(4/12)+(32)(1/12)+(6)(1/12)+(10)(1/12)+(4)(1/12)=6.4 \\
b=(5)(1/12)+(3)(1/12)+(5)(1/12)+(3)(1/12)+(32)(4/12)+(5)(1/12)+(10)(1/12)+(4)(1/12)+(8)(1/12)=14.3 \\
c=(3)(1/12)+(5)(1/12)+(4)(1/12)+(32)(1/12)+(5)(4/12)+(6)(1/12)+(4)(1/12)+(8)(1/12)+(8)(1/12)=7.5 \\
d=(5)(1/12)+(4)(1/12)+(6)(1/12)+(5)(1/12)+(6)(4/12)+(9)(1/12)+(8)(1/12)+(8)(1/12)+(7)(1/12)=6.3
\end{array}\right\}$$

1	5	3	5	4	6
4	6.4	14.3	7.5	6	9
6	10	4	8	8	7

filtered image