1. Consider a hemi-sphere specified by the *spherical coordinate system* using parameters θ and ϕ, where $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi)$. Using two uniform random variables $\xi_1, \xi_2 \in U(0,1)$, derive the mapping f, g

$$\begin{cases} \theta = f(\xi_1) \\ \phi = g(\xi_2), \end{cases}$$

based on which the random variables θ and ϕ can generate *cosine-weighted samples* on the surface of the hemisphere, i.e., $p(\omega) = C \cdot \cos(\theta)$ where $C \in \mathbb{R}$ is a constant which should be determined by calculation.

Answer: Since $\sin\theta \cdot p(\omega) = p(\theta, \phi)$, then $p(\theta, \phi) = C \sin\theta \cos\theta$. To sample $p(\theta, \phi)$, we first generate a uniform sample on ϕ, obtaining $\phi = 2\pi\xi_2$. Then

$$p(\theta \mid \phi) = C' \cdot \sin\theta \cos\theta$$

while its CDF is

$$P(\theta) = \int_0^\theta C' \cdot \sin t \cos t \, \mathrm{d}t = \sin^2\theta.$$

From the inverse of the CDF, we can construct

$$\begin{cases} \theta = \arcsin\left(\sqrt{\xi_1}\right) \\ \phi = 2\pi\xi_2 \end{cases}$$

2. Consider THE RENDERING EQUATION written in directional form (the position $\mathbf{p}$ is neglected)

$$L_o(\theta_o, \phi_o) = L_e(\theta_o, \phi_o) + \int_0^{2\pi} \left(\mathrm{d}\phi_i \int_0^{\pi/2} \sin\theta_i \,\mathrm{d}\theta_i \cdot f_r(\theta_o, \theta_i, \phi_o, \phi_i) L_i(\theta_i, \phi_i) \cos\theta_i \right).$$

Suppose that f_r and L_i are both analytical, $f_r = \pi^{-1}$ and $L_i = |\cos\theta_i|$, derive the *Monte Carlo Estimator* $I(\theta, \phi)$ that estimates the L_o above using the random variables θ and ϕ obtained from the previous question. Can this estimator be more efficient by altering the mapping of f and g? You can give your reasoning by derivations.

Answer: This question only require you to copy down the integral content and the PDF, as the *Monte Carlo Integrator* of the integral

$$I = \int_A f(x) \,\mathrm{d}\mu(x)$$

is

$$\langle I \rangle = \frac{f(X)}{p_\mu(X)}$$

where X is arbitrary random variable that spans the domain.

This way, to estimate L_o, the estimator can be written as

$$\begin{aligned} \langle L_o \rangle &= L_e(\theta_o, \phi_o) + \frac{\sin\theta \cdot \pi^{-1} \cos^2\theta}{p(\theta, \phi) = \pi^{-1} \sin\theta \cos\theta} \\ &= L_e(\theta_o, \phi_o) + \cos\theta. \end{aligned}$$

where all terms are either constant or a basic random variable.

When $\langle I \rangle$ is a constant, the estimator is most efficient.