CS240 Algorithm Design and Analysis

Lecture 16

Lower Bounds

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Last Time – What you need to know

- Gradient descent
- Maximum Cut
  - **Theorem.** Let (A, B) be a locally optimal partition and let (A*, B*) be the optimal partition. Then $w(A, B) \geq \frac{1}{2} \Sigma_e w_e \geq \frac{1}{2} w(A^*, B^*)$
  - **Big-improvement-flip algorithm.** Only choose a node which, when flipped, increases the cut value by at least $\frac{2\varepsilon}{n} w(A, B)$
  - **Claim.** Big-improvement-flip algorithm terminates after $O(\varepsilon^{-1} n \log W)$ flips, where $W = \Sigma_e w_e$.
    - Each flip improves cut value by at least a factor of $(1 + \varepsilon/n)$
    - After n/ε iterations the cut value improves by a factor of 2
    - Cut value can be doubled at most $\log_2 W$ times.

if x >= 1, $(1+1/x)^x >= 2$

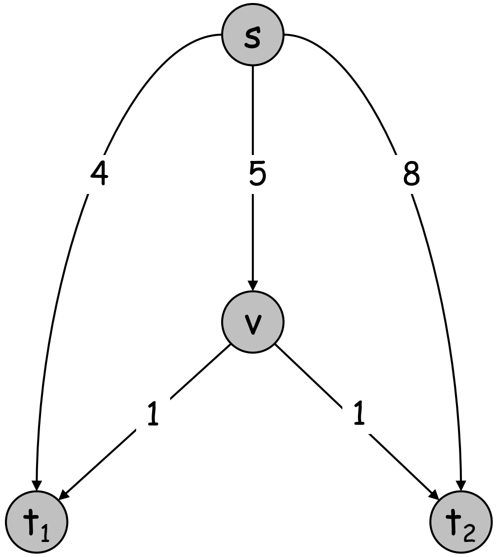
Nash Equilibrium

Best response dynamics. Each agent is continually prepared to improve its solution in response to changes made by other agents.

Nash equilibrium. Solution where no agent has an incentive to switch.

Ex:

- Two agents start with outer paths.
- Agent 1 has no incentive to switch paths (since $4 < 5 + 1$), but agent 2 does (since $8 > 5 + 1$).
- Once this happens, agent 1 prefers middle path (since $4 > 5/2 + 1$).
- Both agents using middle path is a Nash equilibrium.

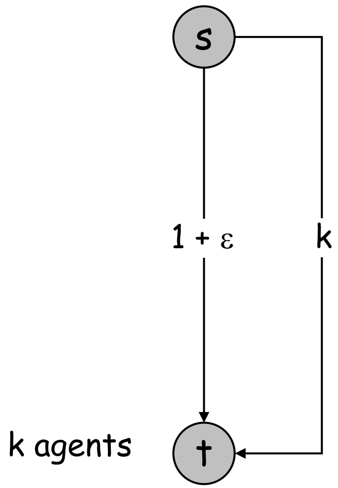


Note. Best response dynamics may not terminate since no single objective function is being optimized.

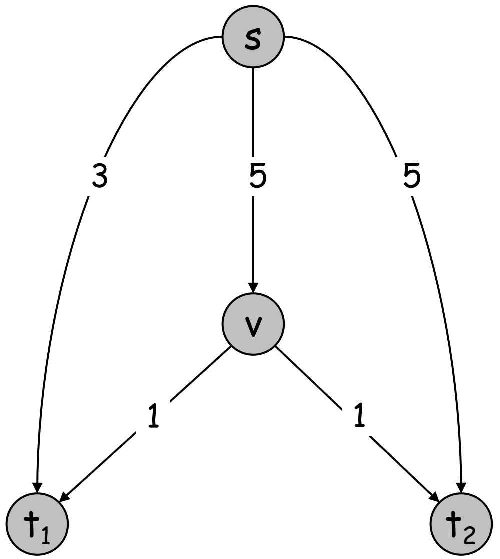
Social Optimum

Social optimum. Minimizes total cost to all agents.

Observation. In general, there can be many Nash equilibria. Even when it's unique, it does not necessarily equal the social optimum.



Social optimum = $1 + \varepsilon$
Nash equilibrium A = $1 + \varepsilon$
Nash equilibrium B = k



Social optimum = 7
Unique Nash equilibrium = 8

Price of Stability

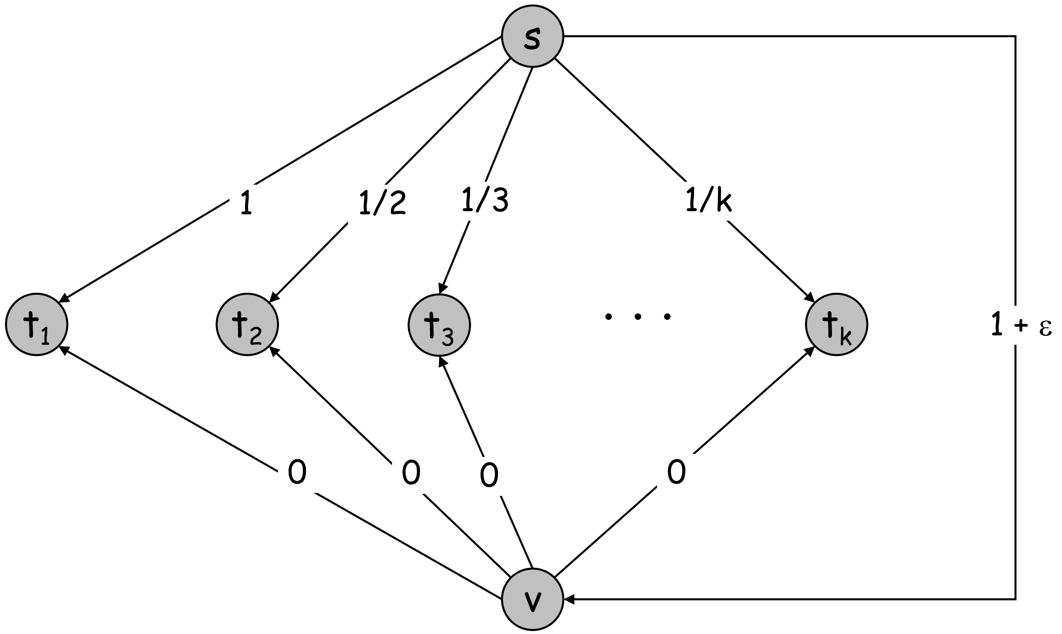
Price of stability. Ratio of best Nash equilibrium to social optimum.

Fundamental question. What is price of stability?

Ex: Price of stability = $\Theta(\log k)$.

- **Social optimum:** Everyone takes bottom paths.
- **Unique Nash equilibrium:** Everyone takes top paths.
- **Price of stability:** $H(k) / (1 + \varepsilon)$.

$1 + 1/2 + \ldots + 1/k$



Finding a Nash Equilibrium

Theorem. The following algorithm terminates with a Nash equilibrium (but its running time may be exponential).

```
Best-Response-Dynamics(G, c) {
   Pick a path for each agent
   while (not a Nash equilibrium) {
      Pick an agent i who can improve by switching paths
      Switch path of agent i
   }
}
```

$H(0) = 0$

$H(k)=\sum_{i=1}^{k} \frac{1}{i}$

Pf. Consider a set of paths $P_1, \ldots, P_k$.

- Let x_e denote the number of paths that use edge e.
- Let $\Phi(P_1, \ldots, P_k) = \sum_{e \in E} c_e \cdot H(x_e)$ be a potential function.
- Since there are only finitely many sets of paths, it suffices to show that Φ strictly decreases in each step.

Finding a Nash Equilibrium

Pf. (continued)

- Consider agent j switching from path P_j to path P_j'.
- Agent j switches because

$$\underbrace{\sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1}}_{\text{newly incurred cost}} < \underbrace{\sum_{e \in P_j - P_j'} \frac{c_e}{x_e}}_{\text{cost saved}}$$

- Φ increases by

$$\sum_{f \in P_j' - P_j} c_f \left[H(x_f + 1) - H(x_f) \right] = \sum_{f \in P_j' - P_j} \frac{c_f}{x_f + 1}$$

- Φ decreases by

$$\sum_{e \in P_j - P_j'} c_e \left[H(x_e) - H(x_e - 1) \right] = \sum_{e \in P_j - P_j'} \frac{c_e}{x_e}$$

- Thus, net change in Φ is negative. ▪

Bounding the price of stability

Claim. Let $C(P_1, \ldots, P_k)$ denote the total cost of selecting paths $P_1, \ldots, P_k$.

For any set of paths $P_1, \ldots, P_k$, we have

$$C(P_1, \ldots, P_k) \;\leq\; \Phi(P_1, \ldots, P_k) \;\leq\; H(k) \cdot C(P_1, \ldots, P_k)$$

Pf. Let x_e denote the number of paths containing edge e.

- Let E^+ denote set of edges that belong to at least one of the paths.

$$C(P_1, \ldots, P_k) \;=\; \sum_{e \in E^+} c_e \;\leq\; \underbrace{\sum_{e \in E^+} c_e \, H(x_e)}_{\Phi(P_1, \ldots, P_k)} \;\leq\; \sum_{e \in E^+} c_e \, H(k) \;=\; H(k)\; C(P_1, \ldots, P_k)$$

Bounding the price of stability

Theorem. There is a Nash equilibrium for which the total cost to all agents exceeds that of the social optimum by at most a factor of H(k).

Pf.

- Let (P_1^*, …, P_k^*) denote set of socially optimal paths.
- Run best-response dynamics algorithm starting from P*.
- Since Φ is monotone decreasing $\Phi(P_1, \ldots, P_k) \le \Phi(P_1^*, \ldots, P_k^*)$.

$$C(P_1, \ldots, P_k) \underset{\substack{\uparrow \\ \text{previous claim} \\ \text{applied to P}}}{\le} \Phi(P_1, \ldots, P_k) \le \Phi(P_1^*, \ldots, P_k^*) \underset{\substack{\uparrow \\ \text{previous claim} \\ \text{applied to P*}}}{\le} H(k) \cdot C(P_1^*, \ldots, P_k^*)$$

Lower Bounds

Upper and Lower Bounds

What is the minimum resources (time, space, etc.) needed to solve a problem?

Consider sorting n numbers.

- Insertion sort takes $O(n^2)$ time.
  - This puts an upper bound of $O(n^2)$ on the time to sort n numbers.
- Merge sort takes $O(n \log n)$ time.
  - This puts an upper bound of $O(n \log n)$ on the time to sort n numbers.

We want to make the upper bound as low as possible, i.e., solve the problem faster.

Suppose an algorithm A solves problem X in $f(n)$ time when input size is n.

- Then $f(n)$ is an upper bound on the complexity of X.

Upper and Lower Bounds

What about the least amount of time to solve X?

Suppose we know that any algorithm that solves X takes at least g(n) time, when X has size n.

- Then g(n) is a lower bound on the complexity of X.

If the lower bound g(n) is large, it means problem X is hard to solve.

- **Ex** NP-Hard problems are hard because they (probably) have super-polynomial lower bounds.

To show a lower bound, we need to give a proof.

- Usually, we show if an algorithm takes too little time, it must sometimes produce the wrong answer

The lower bound for a problem depends on the computational model.

- If a model has very powerful primitive operations, then algorithms can run faster, and the lower bound is smaller.

If the complexity of an algorithm for problem X matches the lower bound for problem X, the algorithm is optimal, and the lower bound is tight.

A Warm-up

Say we want to find the larger of two numbers x and y.

- We can do this with 1 comparison, so this is an upper bound.
- What's the lower bound? Do we need at least 1 comparison? Can we do 0 comparisons?
- No. Suppose an algorithm doesn't compare x and y.
  - So basically, the algorithm declares either x or y to be bigger, without looking at them.
- Say the algorithm declares x bigger. Then let's set y > x.
  - Algorithm won't notice this, because it doesn't compare x and y.
  - So algorithm still declares x is bigger, which is wrong.
  - This type of argument is called indistinguishability and is frequently used when proving lower bounds.
  - Same argument if algorithm always declares y bigger without comparing.
- Hence, any algorithm must do at least 1 comparison, so 1 is a lower bound.

Outline

We'll prove lower bounds for the following problems.

- Merging two lists.
- Finding the max.
- Finding the max and min.
- Sorting n numbers.

Merging Two Lists

How many comparisons needed to merge two lists of size n into sorted order?

- During the execution, the algorithm can compare some input elements a and b, and get back response "a<b", "a=b" or "a>b".

If the lists are sorted, 2n-1 comparisons is an upper bound.

Let's prove this is also a lower bound.

Let the input lists be $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$, and suppose $a_1<b_1<a_2<b_2<\ldots<a_n<b_n$.

- So the algorithm must output $a_1,b_1,a_2,b_2,\ldots,a_n,b_n$.

When comparing some a_i and b_j, it gets back the following response:

- $a_i < b_j$ if $i \leq j$.
- $a_i > b_j$ if $i > j$.

We show any algorithm has to perform $\geq 2n-1$ comparisons to merge the two lists.

- This gives a 2n-1 time lower bound on merging, since a merging algorithm must correctly merge any two input lists, including the two lists above.

Merging Two Lists

Claim Any correct algorithm must compare a_i to b_i, for every i.

- Suppose not; say the algorithm doesn't compare a_1 to b_1.
- Now, if the input was actually $b_1<a_1<a_2<b_2<\ldots<a_n<b_n$, then the algorithm still outputs $a_1,b_1,\ldots,a_n,b_n$, which is wrong.
  - Because the algorithm doesn't compare a_1 and b_1, it can't distinguish the new input from the original.
- Same argument if algorithm doesn't compare a_i to b_i, for any i.
- So, algorithm does n comparisons of this type.

Claim Any correct algorithm must compare b_i to a_{i+1}, for every $i<n$.

- If not, then say it doesn't compare b_1 to a_2. Then it can't distinguish original input from input $a_1<a_2<b_1<b_2<\ldots<a_n<b_n$, and will give wrong answer.
- Thus, n-1 comparisons of this type.

So, any algorithm must do at least 2n-1 comparisons.

So 2n-1 is a lower bound on the complexity to merge into sorted order.

Finding the Max

How many comparisons to find the largest number in an unsorted array of n distinct numbers.

Upper bound: n-1.

Lower bound: also n-1.

To prove this, we'll keep track of what information the algorithm learns at it executes.

- Say algorithm never compared some element to any other element.
  - Then the algorithm doesn't know anything about this element. It could be the max, or not the max.
  - Thus, the algorithm can't correctly output the max without comparing this element to some others.
- Say there are two elements, and both are larger than every element they've been compared to.
  - Then either one of them could be the max.
  - So algorithm can't output the max without comparing these two elements.

Let's formalize this intuition.

Finding the Max

At any stage of the algorithm A, give every array element one of 3 colors, white, blue or red.

- White means this element has never been compared to any other element.
- Blue means this element is bigger than all the elements it's been compared to.
- Red means this element was smaller than some element it was compared to.
- Let w_k, b_k, r_k be number of white, blue and red elements after A has done k comparisons.
  - So initially, $w_0=n$ and $b_0=r_0=0$.

We'll show that for any k, $w_k+b_k \geq n-k$.

We'll show that as long as $w_k+b_k > 1$, A can't terminate.

Hence, **when A terminates, we have $w_k+b_k \leq 1$, and A must have done $k \geq n-1$ comparisons.**

Finding the Max

Claim For any k, $w_k+b_k \geq n-k$.

Proof By induction on k. Claim holds for k=0.

- For larger k, consider the k'th comparison. It must either be between 2 white elements (WW case), a white and blue element (WB case), a white and red (WR), 2 reds (RR), 2 blues (BB), red and blue (RB).
- Do a case-by-case analysis.
- WW: Make the first element > second element.
  - This is possible, because both elements are white, so neither have been in any comparisons, so they can be in either order.
  - After comparison, first element becomes blue, second element red.
  - Number of whites decreases by 2, blues increases by 1.
  - By induction, $w_{k-1}+b_{k-1} \geq n-k+1$. Also, $w_k=w_{k-1}-2$, and $b_k=b_{k-1}+1$. So $w_k+b_k \geq n-k$.
- WB: Make the first element < second element.
  - This is possible, since first element hasn't been in any comparisons.
  - So first element becomes red, second remains blue.
  - So $w_k=w_{k-1}-1$, $b_k=b_{k-1}$, so $w_k+b_k \geq n-k$.

Finding the Max

WR: Make the first element > second element. First element becomes blue, second stays red.

- So $w_k = w_{k-1} - 1$, $b_k = b_{k-1} + 1$, so $w_k + b_k \geq n-k+1 > n-k$.

RR: Make first element > second element. Both elements stay red.

- $w_k + b_k = w_{k-1} + b_{k-1} \geq n-k+1 > n-k$.

BB: Make first element > second element. First one stays blue, second becomes red.

- $w_k + b_k = w_{k-1} + b_{k-1} - 1 \geq n-k$.

RB: Make first element < second element. Both elements stay same color.

- $w_k + b_k = w_{k-1} + b_{k-1} \geq n-k+1 > n-k$.

Hence $w_k + b_k \geq n-k$ by induction.

Finding the Max

Claim Suppose after making k comparisons, we have $w_k+b_k>1$. Then A cannot terminate.

Proof Say A terminates, and outputs a value x as the max.

- Since $w_k+b_k>1$, either $w_k \geq 1$, or $b_k>1$.
- If $w_k \geq 1$, then there's a white element y that's never been compared to x (or any other elt).
  - Make y > x. Then the algorithm is wrong.
- If $b_k>1$, then there are at least 2 blue elements.
  - x must be a blue element.
    - If x is red, it's not max.
    - x is not white, by above.
  - Take another blue element z. x and z were never compared.
    - If they had been, either x or z would have turned red.
  - Make z > x. Then the algorithm is wrong.

All together, since A can't terminate as long as $w_k+b_k>1$, then $k \geq n-1$ when A terminates.

So, A does $\geq n-1$ comparisons.

Finding the Max and Min

- How many comparisons does it take to find the max and min elements in an unsorted array A of n distinct numbers.
- Upper bound 1: 2n-2 comparisons.
- Upper bound 2: 3n/2-2 comparisons.
  - Pair up the elements, A[1] and A[2], A[3] and A[4], etc.
  - Compare the elements in each pair (n/2 comps total).
  - Put all the bigger elements in a temp array Big, put all the smaller elements in temp array Small.
    - Big and Small each have size n/2.
  - Find the max element in Big and output it as max of A (n/2-1 comparisons).
  - Find the min element in Small and output it as the min of A (n/2-1 comparisons).
- Lower bound: 3n/2-2 comparisons!

Finding the Max and Min

- Intuition for proof is similar to one for max.
- At any stage of algorithm, give each array element one of 4 colors, white, blue, red and purple, representing what the algorithm knows about the element.
  - White means this element has never been compared against any other element.
  - Blue means this element is bigger than all the elements it's been compared against.
  - Red means this element was smaller than every element it was compared against.
  - Purple means this element was bigger than some element(s) it was compared to, and smaller than some other(s).

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Finding the Max and Min

- To terminate, algorithm must eliminate all white elements, since these could be the min or max.
- Also, algorithm can only leave one blue and one red.
  - Else either of two blues can be max, either of two reds can be min.
- As comparisons happen, algorithm gets more info, and elements change color, e.g., from white to blue, red to purple, etc.
- Too few comparisons means the algorithm doesn't have time to eliminate all whites, and all but 1 blue and red.
- Proof keeps track of number of whites, blues and reds after some number of comparisons.

Finding the Max and Min

- Label each comparison by its type.
  - E.g., WW is comparison between two white elements.
  - There are 10 types, WW, WB, WR, WP, BB, BR, BP, RR, RP, PP.
- Denote the number of comparisons of type WW by ww, number of WB comps by wb, etc., 10 numbers total.
- Let w, b, r denote number of whites, blues and reds, resp., at some stage of the algorithm.

Finding the Max and Min

- Claim 1 When A terminates, w=0 and b=r=1.
- Proof Say A outputs x as max, y as min.
  - Neither x nor y can be white, since we can make a white element be neither max nor min.
  - If there is a white element z when A terminates, we can make z > x, and A is wrong. So w=0.
  - x must be a blue element, as in the finding max proof.
  - If there's another blue element z, then x and z weren't compared, so we can make z > x, and A is wrong. So b=1.
  - y must be a red element.
  - If there's another red element z, then we can make z < y, and A is wrong. So r=1.

Finding the Max and Min

- The table states what happens when each type of comparison occurs. Similar to the case analysis in finding max proof.
  - Ex If WW occurs, make the first element > second element (denoted $E_1>E_2$), so these elements become blue and red (BR).
  - Ex If WB occurs, we make the first element < second element (denoted $E_1<E_2$), so the elements become red and blue (RB).

Comparison type	Result	Comparison type	Result
WW	$E_1>E_2$, BR	BR	$E_1>E_2$, BR
WB	$E_1<E_2$, RB	BP	$E_1>E_2$, BP
WR	$E_1>E_2$, BR	RR	$E_1<E_2$, RP
WP	$E_1>E_2$, BP	RP	$E_1<E_2$, RP
BB	$E_1>E_2$, BP	PP	$E_1<E_2$, PP

Finding the Max and Min

- **Claim 2** At any stage of the algorithm, we have
  - $w = n - 2ww - rw - bw - pw$.
  - $b = ww + rw + pw - bb$.
  - $r = ww + bw - rr$.
- **Proof** These follow just by counting w,b,r using the table on the previous page.
  - For w, there are initially n whites. Each WW comparison removes 2 whites. Each RW, BW or PW comparison removes 1 white.
  - For b, each WW, RW or PW comparison creates 1 blue element. Each BB comparison removes 1 blue.
  - For r, each WW, BW comparison creates 1 red element. Each RR removes 1 red.

Finding the Max and Min

- **Theorem** Any algorithm performs at least 3n/2-2 comparisons.
- **Proof** The total number of comparisons is $C = ww + wb + wr + wp + bb + br + bp + rr + rp + pp$.
  - By claims 1 and 2, when A terminates, we have $2ww + rw + bw + pw = n, bb = ww + rw + pw - 1, rr = ww + bw - 1$.
  - So $bb + rr \; = \; 2ww + rw + bw + pw - 2 \; = \; n - 2$.
  - $C \geq \; ww + wb + wr + wp + bb + rr$
    $= \; ww + wb + wr + wp + n - 2$
    $= \; n - ww + n - 2$
    $= 2n - 2 - ww$.
  - $ww \leq \; n/2$, because each WW comp decreases number of whites by 2, and there are only n whites.
  - So $C \geq 3n/2 - 2$.

Sorting

- How many comparisons are needed to sort n numbers?
- Upper bound: O(n log n) using merge sort.
- Lower bound: $\Omega(n \log n)$.
- To prove the lower bound, we first need a model for how a comparison-based sorting algorithm works.
  - This is called the decision tree model.
- The lower bound is not valid in other models.
  - If an algorithm can do things besides comparing two numbers, e.g., look at the digits of a number, it can sort faster than $\Omega(n \log n)$ time.
  - Lower bounds can be very sensitive to the computational model.

Decision Trees

- In this model, in each step, algorithm can only compare a pair of numbers x, y.
- Based on result of the comparison, it decides next pair of numbers to compare.
  - An execution of the algorithm is a sequence of comparisons, each comparison determined by result of previous comparison.
- When the algorithm terminates, it outputs a permutation representing the sorted order of the input.
- The complexity of the algorithm is the most number of comparisons it does before terminating.

Decision Trees

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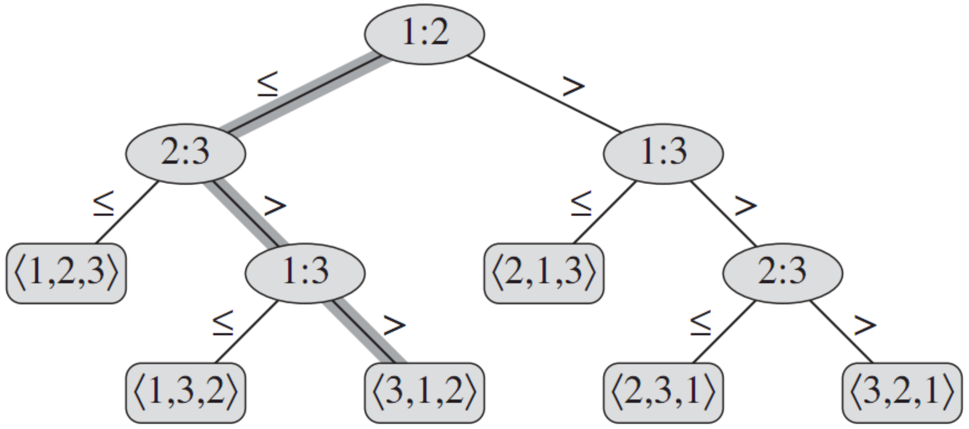
- Model behavior of the algorithm by a binary tree.
  - Each internal node is a pair of number x,y to compare.
  - If x≤y, go to left child. If x>y, go to right child.
  - Each leaf represents an output and is labeled with a permutation representing the sorted order of the inputs.
- An execution is simply a path from root to a leaf.
  - At any node, the algorithm has obtained some info from the comparisons it's done.
  - It uses this info to decide the next comparison to do.
  - Eventually, it obtains enough info to generate an output.
- Complexity of algorithm is the length of the longest root-leaf path.



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Lower Bound for Sorting

- Given n numbers as input, they can be in n! different orders.
- Given an input order, algorithm must output that order.
  - The decision tree of algorithm must have a leaf labeled with that order.
  - The decision tree has $\geq$ n! leafs.
- The height of decision tree is h.
  - The complexity of the algorithm is h.
  - Since decision tree is binary, it has $\leq 2^h$ leaves.
- So $2^h \geq$ (# leaves of decision tree) $\geq$ n!, and so $h \geq \log_2(n!)$.
  - $\log_2(n!) = \log_2 n + \log_2(n-1) + \cdots + \log_2 1 \geq$
    $\log_2 n + \log_2(n-1) + \cdots + \log_2(n/2) \geq$
    $\frac{n}{2}(\log_2 n - 1) = \Omega(n \log n)$.
- Proved the algorithm does $\Omega(n \log n)$ comparisons.

Next Time:
Amortized analysis