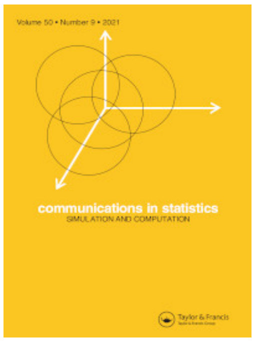

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Communications in Statistics - Simulation and Computation

ISSN: 0361-0918 (Print) 1532-4141 (Online) Journal homepage: www.tandfonline.com/journals/lssp20

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COMMUNICATIONS IN STATISTICS - SIMULATION AND COMPUTATION®
2021, VOL. 50, NO. 9, 2743–2756
https://doi.org/10.1080/03610918.2019.1612911

Modeling and asymptotic analysis of insurance company performance

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ABSTRACT
We consider a classical Cramér-Lundberg model with dividends. It is additionally supposed that the claim amounts have exponential distribution. Moreover, we are interested in a barrier dividend strategy with Parisian implementation delay. That means, the payment is made only if the company surplus stays above the barrier at least during time interval of length *h*. The mean expected discounted dividends paid before Parisian ruin are chosen as objective function. Optimization is carried out. The results are compared with those obtained previously by the authors for the no-delay case. Statistical estimation, stability problems and simulation are tackled as well.

ARTICLE HISTORY
Received 27 December 2018
Accepted 24 April 2019

KEYWORDS
Dividends with Parisian implementation delay; Parisian ruin; Statistical estimates; Simulation

MATHEMATICS SUBJECT CLASSIFICATION (2010)
62F12; 91B30; 90C31

1. Introduction

New problems have arisen in actuarial sciences during the last twenty years, see, e.g., Bulinskaya (2017a) and 446 references therein. This period is characterized by interplay of insurance and finance, unification of reliability and cost approaches, see, e.g., Bulinskaya (2017b), as well as, consideration of complex systems. Sophisticated mathematical tools are used for analysis and optimization of insurance systems including dividends, reinsurance and investment.

A dividend is a payment made by a corporation to its shareholders, usually as a distribution of profits. It was B. de Finetti who introduced the dividends study in actuarial mathematics. He argued, in de Finetti (1957), that under net profit condition the insurance company surplus could become infinite as time grows. This being unrealistic it is necessary to choose a dividend strategy. There exist a lot of articles devoted to dividends payment, see, e.g., Avanzi (2009) and Albrecher and Thonhauser (2009) for the survey of results published earlier. Expected discounted dividends paid before ruin are usually taken as objective function (risk measure), see the classical textbooks by Bühlman (1970) and Gerber (1979), as well as, the article by Sethi, Derzko, and Lehoczky (1991). The barrier strategy is the most popular one although it was established in Azcue and Muler (2005) that the optimal dividend strategy is not always barrier. Many ramifications of this strategy were proposed nowadays, see, e.g., Drekic, Woo, and Xu (2018) and references therein.

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Our aim is to consider a generalization of the strategies treated in Dassios and Wu (2009) and Bulinskaya and Shigida (2018) focusing our attention on problems of parameters statistical estimation and simulation.

The article is organized as follows. In Sec. 2 we give the model description and preliminary results necessary for our investigation. Objective function $\mathrm{E}V(x,b)$, the present value of total expected dividends until Parisian ruin under barrier strategy with Parisian implementation delay, is calculated in Sec. 3. It is compared in Sec. 4 with a similar expression until the classical ruin given in Dassios and Wu (2009). Statistical estimates of the objective function are proposed in Sec. 5. Numerical results obtained by application of Python are provided as well. Conclusion and further research directions are contained in Sec. 6.

2. Model description and preliminary results

We consider the standard Cramér-Lundberg model:

$$X_t = X_0 + ct - \sum_{i=1}^{N_t} C_i, \quad t \geq 0, \tag{1}$$

where X_t is the insurance company surplus at time t, c is the rate of premium accumulation, $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity λ and the claim amounts $\{C_i\}_{i=1}^{\infty}$ are independent identically distributed (i.i.d.) random variables (r.v.'s) independent of N_t. Assume additionally that the claims distribution is exponential with parameter α.

For a positive constant r let ν_r^+ and ν_r^- be the positive and negative roots of the equation

$$-r + c\nu_r + \lambda\left(\frac{\alpha}{\nu_r + \alpha} - 1\right) = 0.$$

Explicitly,

$$\begin{aligned} \nu_r^+ &= \frac{\sqrt{(c\alpha + r + \lambda)^2 - 4c\alpha\lambda} - (c\alpha - r - \lambda)}{2c}, \\ \nu_r^- &= \frac{-\sqrt{(c\alpha + r + \lambda)^2 - 4c\alpha\lambda} - (c\alpha - r - \lambda)}{2c}. \end{aligned} \tag{2}$$

Let d and h take nonnegative real values and $l_i \in \mathbf{R}^1$, $i = 1, 2$. Everywhere, l_1 is assumed to be less than l_2. From the article Dassios and Wu (2009) we also take the notation

$$R_{x,l_1,l_2} = \inf\{t > 0 | X_0 = x, X_t \notin (l_1, l_2)\} \quad \text{for } l_1 \leq x < l_2 \tag{3}$$

and the linear system of equations

$$\begin{aligned} e^{\nu_r^+ x} &= \frac{\alpha e^{\nu_r^+ l_1}}{\alpha + \nu_r^+} E\left[e^{-rR_{x,l_1,l_2}} 1_{\{X_{R_{x,l_1,l_2}} \leq l_1\}}\right] + e^{\nu_r^+ l_2} \mathrm{E}\left[e^{-rR_{x,l_1,l_2}} 1_{\{X_{R_{x,l_1,l_2}} = l_2\}}\right], \\ e^{\nu_r^- x} &= \frac{\alpha e^{\nu_r^- l_1}}{\alpha + \nu_r^-} \mathrm{E}\left[e^{-rR_{x,l_1,l_2}} 1_{\{X_{R_{x,l_1,l_2}} \leq l_1\}}\right] + e^{\nu_r^- l_2} \mathrm{E}\left[e^{-rR_{x,l_1,l_2}} 1_{\{X_{R_{x,l_1,l_2}} = l_2\}}\right], \end{aligned}$$

which quickly yields the formulas

$$\begin{aligned}
\mathrm{E}\left[e^{-rR_{x,l_1,l_2}}1_{\left\{X_{R_{x,l_1,l_2}}\le l_1\right\}}\right] &= \frac{e^{\nu_r^+(x-l_2)}-e^{\nu_r^-(x-l_2)}}{\dfrac{\alpha e^{\nu_r^+(l_1-l_2)}}{\alpha+\nu_r^+}-\dfrac{\alpha e^{\nu_r^-(l_1-l_2)}}{\alpha+\nu_r^-}},\\
\mathrm{E}\left[e^{-rR_{x,l_1,l_2}}1_{\left\{X_{R_{x,l_1,l_2}}=l_2\right\}}\right] &= \frac{(\alpha+\nu_r^+)e^{\nu_r^+(x-l_1)}-(\alpha+\nu_r^-)e^{\nu_r^-(x-l_1)}}{(\alpha+\nu_r^+)e^{\nu_r^+(l_2-l_1)}-(\alpha+\nu_r^-)e^{\nu_r^-(l_2-l_1)}}.
\end{aligned} \tag{4}$$

Denote

$$\begin{aligned}
\tau_{x,l_2} &= \inf\{t\ge 0|X_0=x, X_t=l_2\},\\
T_{x,l_1,d} &= \inf\{t\ge 0|X_0=x, X_t \text{ has been } <l_1 \text{ for at least } d\},\\
F_{x,l_2,h} &= \inf\{t\ge 0|X_0=x, X_t \text{ has been } \ge l_2 \text{ for at least } h\}.
\end{aligned}$$

Throughout the article, we will often use the fact that X_t has independent increments and is translation invariant, so, the strong Markov property is applicable. Also, $X_t-\mathrm{E}[X_t]$ is a martingale (because it is a process with independent increments whose mean value is constant), and the optimal stopping theorem is applicable.

For further reasoning, let U_i be the ith excursion above l_1 and V_i the ith excursion below l_1. All of these random variables are independent (by the strong Markov property). If $x\ge l_1$, U_1 has a distribution different from that of other $\{U_i\}_{i=2}^{\infty}$, which are identically distributed, so are $\{V_i\}_{i=1}^{\infty}$. If $x<l_1$, V_1 has a distribution different from that of other $\{V_i\}_{i=2}^{\infty}$, which are identically distributed, so are $\{U_i\}_{i=1}^{\infty}$. In any case, let p_1 be the density of U_2 and p_2 the density of V_2 (p_1 and p_2 do not depend on l_1). If $x\ge l_1$, let $g_1^{x-l_1}$ be the density of U_1 (this way, g_1^x does not depend on l_1), otherwise let $g_2^{x-l_1}$ be the density of V_1 (g_2^x also does not depend on l_1).

Lemma 1. *For* $l_1\le x<l_2$

$$\begin{aligned}
\mathrm{E}\left[e^{-r\tau_{x,l_2}}1_{\{\tau_{x,l_2}<T_{x,l_1,d}\}}\right] &= \frac{(\alpha+\nu_r^+)e^{\nu_r^+(x-l_1)}-(\alpha+\nu_r^-)e^{\nu_r^-(x-l_1)}}{(\alpha+\nu_r^+)e^{\nu_r^+(l_2-l_1)}-(\alpha+\nu_r^-)e^{\nu_r^-(l_2-l_1)}}\\
&+\frac{\nu_r^+-\nu_r^-}{(\alpha+\nu_r^+)e^{\nu_r^+(l_2-l_1)}-(\alpha+\nu_r^-)e^{\nu_r^-(l_2-l_1)}}\\
&\times\frac{\left(e^{\nu_r^+(x-l_2)}-e^{\nu_r^-(x-l_2)}\right)\int_0^d e^{-rs}p_2(s)\ ds}{\left(\dfrac{\alpha e^{\nu_r^+(l_1-l_2)}}{\alpha+\nu_r^+}-\dfrac{\alpha e^{\nu_r^-(l_1-l_2)}}{\alpha+\nu_r^-}\right)-\left(e^{\nu_r^+(l_1-l_2)}-e^{\nu_r^-(l_1-l_2)}\right)\int_0^d e^{-rs}p_2(s)\ ds}.
\end{aligned}\tag{5}$$

For $x<l_1$

$$\begin{aligned}
\mathrm{E}\left[e^{-r\tau_{x,l_2}}1_{\{\tau_{x,l_2}<T_{x,l_1,d}\}}\right] &= \frac{\nu_r^+-\nu_r^-}{(\alpha+\nu_r^+)e^{\nu_r^+(l_2-l_1)}-(\alpha+\nu_r^-)e^{\nu_r^-(l_2-l_1)}}\times\left(\frac{\alpha e^{\nu_r^+(l_1-l_2)}}{\alpha+\nu_r^+}-\frac{\alpha e^{\nu_r^-(l_1-l_2)}}{\alpha+\nu_r^-}\right)\\
&\times\frac{\int_0^d e^{-rs}g_2^{x-l_1}(s)\ ds}{\left(\dfrac{\alpha e^{\nu_r^+(l_1-l_2)}}{\alpha+\nu_r^+}-\dfrac{\alpha e^{\nu_r^-(l_1-l_2)}}{\alpha+\nu_r^-}\right)-\left(e^{\nu_r^+(l_1-l_2)}-e^{\nu_r^-(l_1-l_2)}\right)\int_0^d e^{-rs}p_2(s)\ ds}.
\end{aligned}\tag{6}$$

Proof. Denote

$$A_i = \{\tau_{x,l_2} \text{ is achieved during the } i\text{th excursion above } l_1 \text{ before } T_{x,l_1,d}\}.$$

First we consider the case $l_1 \le x < l_2$. Let Bl_i denote the event that during the ith excursion above l_1 (whose length is U_i) the process X_t does not reach the level l_2, Ab_i the event that it does, $\tilde{U}_i$ is the time until it reaches l_2 during the ith excursion (the length of the excursion above l_1 but below l_2, let it equal U_i on Bl_i).

Then it is obvious that $\mathrm{E}[e^{-r\tau_{x,l_2}} 1_{A_i}]$ is equal to

$$\mathrm{E}\left[e^{-rU_1} 1_{\mathrm{Bl}_1} e^{-rV_1} 1_{\{V_1<d\}} \dots e^{-rU_{i-1}} 1_{\mathrm{Bl}_{i-1}} e^{-rV_{i-1}} 1_{\{V_{i-1}<d\}} e^{-r\tilde{U}_i} 1_{\mathrm{Ab}_i}\right].$$

Now we apply the strong Markov property, recall that all V_i are i. i. d., all U_i except U_1 are i. i. d., and use the notation (3).

Then we have, for $i \ge 2$,

$$\mathrm{E}[e^{-r\tau_{x,l_2}} 1_{A_i}] = \mathrm{E}\left[e^{-rR_{x,l_1,l_2}} 1_{\{X_{R_{x,l_1,l_2}} \le l_1\}}\right] \mathrm{E}\left[e^{-rV_1} 1_{\{V_1<d\}}\right]^{i-1} \\ \times \mathrm{E}\left[e^{-rR_{l_1,l_1,l_2}} 1_{\{X_{R_{l_1,l_1,l_2}} \le l_1\}}\right]^{i-2} \mathrm{E}\left[e^{-rR_{l_1,l_1,l_2}} 1_{\{X_{R_{l_1,l_1,l_2}} = l_2\}}\right],$$

and for $i = 1$

$$\mathrm{E}[e^{-r\tau_{x,l_2}} 1_{A_1}] = \mathrm{E}\left[e^{-rR_{x,l_1,l_2}} 1_{\{X_{R_{x,l_1,l_2}} = l_2\}}\right].$$

Using (4) and applying the simple identity

$$\mathrm{E}\left[e^{-r\tau_{x,l_2}} 1_{\{\tau_{x,l_2} < T_{x,l_1,d}\}}\right] = \sum_{i=1}^{\infty} \mathrm{E}[e^{-r\tau_{x,l_2}} 1_{A_i}],$$

we get (5).

If $x < l_1$ then, similarly,

$$\mathrm{E}[e^{-r\tau_{x,l_2}} 1_{A_i}] = \mathrm{E}\left[e^{-rV_1} 1_{\{V_1<d\}}\right] \mathrm{E}\left[e^{-rV_2} 1_{\{V_2<d\}}\right]^{i-1} \\ \times \mathrm{E}\left[e^{-rR_{l_1,l_1,l_2}} 1_{\{X_{R_{l_1,l_1,l_2}} \le l_1\}}\right]^{i-1} \mathrm{E}\left[e^{-rR_{l_1,l_1,l_2}} 1_{\{X_{R_{l_1,l_1,l_2}} = l_2\}}\right].$$

Again, recalling (4) and summing up for $i = 1, 2, \dots$, we get (6). □

Now let us introduce another level $l_0 < l_1$.

Theorem 1. *The following relation is valid*

$$\mathrm{E}\left[e^{-rF_{l_1,l_1,h}} 1_{\{F_{l_1,l_1,h} < T_{l_1,l_0,d}\}}\right] = \frac{e^{-rh}\bar{P}_1(h)}{1 - \mathbf{A} \times \int_0^h e^{-rs} p_1(s)\, ds},$$

where

$$\mathbf{A} = \frac{\alpha e^{\nu_r^+(l_1-l_0)} - \alpha e^{\nu_r^-(l_1-l_0)}}{(\alpha+\nu_r^+)e^{\nu_r^+(l_1-l_0)} - (\alpha+\nu_r^-)e^{\nu_r^-(l_1-l_0)}}$$
$$+ \frac{\nu_r^+ - \nu_r^-}{(\alpha+\nu_r^+)e^{\nu_r^+(l_1-l_0)} - (\alpha+\nu_r^-)e^{\nu_r^-(l_1-l_0)}} \times \left(\frac{\alpha}{\alpha+\nu_r^+} - \frac{\alpha}{\alpha+\nu_r^-}\right)$$
$$\times \frac{\int_0^d e^{-rs} p_2(s)\ ds}{\left(\frac{\alpha e^{\nu_r^+(l_0-l_1)}}{\alpha+\nu_r^+} - \frac{\alpha e^{\nu_r^-(l_0-l_1)}}{\alpha+\nu_r^-}\right) - \left(e^{\nu_r^+(l_0-l_1)} - e^{\nu_r^-(l_0-l_1)}\right)\int_0^d e^{-rs} p_2(s)\ ds}.$$

Proof. Put

$$B_i = \left\{F_{l_1,l_1,h} \text{ is achieved during the } i\text{th excursion above } l_1 \text{ before } T_{l_1,l_0,d}\right\}.$$

Let NR_i denote the event that the process X_t does not stay below l_0 for at least d during the ith excursion below l_1 (whose length is V_i). So, it can get below l_0, but it spends less than d there each time.

It is obvious that $\mathrm{E}[e^{-rF_{l_1,l_1,h}} 1_{B_i}]$ is given by

$$\mathrm{E}\left[e^{-rU_1} 1_{\{U_1<h\}} e^{-rV_1} 1_{\mathrm{NR}_1} \dots e^{-rU_{i-1}} 1_{\{U_{i-1}<h\}} e^{-rV_{i-1}} 1_{\mathrm{NR}_{i-1}} e^{-rh} 1_{\{U_i>h\}}\right].$$

Again, applying the strong Markov property, we can rewrite that as

$$\mathrm{E}\left[e^{-rF_{l_1,l_1,h}} 1_{B_i}\right] = \mathrm{E}\left[e^{-rU_1} 1_{\{U_1<h\}}\right]^{i-1} \mathrm{E}\left[e^{-rV_1} 1_{\{V_1<T_{l_1,l_0,d}\}}\right]^{i-1} e^{-rh}\bar{P}_1(h). \qquad (7)$$

Here, only the term $\mathrm{E}[e^{-rV_1} 1_{\{V_1<T_{l_1,l_0,d}\}}]$ requires some effort to calculate. We know that the overshoot below l_1 is an exponential random variable ζ with parameter α. Then this term equals

$$\mathrm{E}\left[e^{-r\tau_{l_1-\zeta,l_1}} 1_{\{\tau_{l_1-\zeta,l_1}<T_{l_1-\zeta,l_0,d}\}}\right].$$

For any fixed $\zeta = y$ we already know the answer (see Lemma 1), so we just need to integrate this result multiplied by the density of ζ, which is $\alpha e^{-\alpha y}$. It turns out quite easy if we use the fact

$$\int_0^\infty g_2^{-x}(s)\alpha e^{-\alpha x}\ dx = p_2(s),$$

which is a consequence of the overshoot being exponential (the ith excursion below zero starts at $-\zeta$, where ζ is an exponential random variable with parameter α, $i \geq 2$).

After the integration, we get

$$\mathrm{E}\left[e^{-r\tau_{l_1-\zeta,l_1}} 1_{\{\tau_{l_1-\zeta,l_1}<T_{l_1-\zeta,l_0,d}\}}\right] = \mathbf{A}.$$

Other terms in (7) are known to us, so summing up by $i = 1, 2, \dots$ we obtain the desired result. □

Remark. The proved theorem generalizes Theorem 1 in Bulinskaya and Shigida (2018).

Theorem 2. *The function* $\mathrm{E}[e^{-rF_{l_1,l_1,h}}X_{F_{l_1,l_1,h}}1_{\{F_{l_1,l_1,h}<T_{l_1,l_0,d}\}}]$ *is given by*

$$\mathrm{E}\left[e^{-rF_{l_1,l_1,h}}1_{\{F_{l_1,l_1,h}<T_{l_1,l_0,d}\}}\right]\times\left(\frac{\frac{1}{\alpha}+\left(c-\frac{\lambda}{\alpha}\right)\int_0^h sp_1(s)\ ds}{\bar{P}_1(h)}+l_1+ch-\frac{\lambda h+1}{\alpha}\right).$$

Proof. Let NR_i denote the event that X_t does not stay below l_0 for at least d during the ith excursion below l_1 (the duration of this excursion is V_i). Then it is possible to write $\mathrm{E}[e^{-rF_{l_1,l_1,h}}X_{F_{l_1,l_1,h}}1_{B_i}]$ as follows

$$\mathrm{E}\left[e^{-rU_1}1_{\{U_1<h\}}e^{-rV_1}1_{\mathrm{NR}_1}\ldots e^{-rU_{i-1}}1_{\{U_{i-1}<h\}}e^{-rV_{i-1}}1_{\mathrm{NR}_{i-1}}e^{-rh}X_{\sum_{k=1}^{i-1}(U_k+V_k)+h}1_{\{U_i>h\}}\right].$$

Moreover,

$$\begin{aligned}X_{\sum_{k=1}^{i-1}(U_k+V_k)+h}&=X_{\sum_{k=1}^{i-1}(U_k+V_k)+h}-X_{\sum_{k=1}^{i-1}(U_k+V_k)}+X_{\sum_{k=1}^{i-1}(U_k+V_k)}\\&=X_{\sum_{k=1}^{i-1}(U_k+V_k)+h}-X_{\sum_{k=1}^{i-1}(U_k+V_k)}+l_1.\end{aligned}$$

Then, by the strong Markov property we get $\mathrm{E}[e^{-rF_{l_1,l_1,h}}X_{F_{l_1,l_1,h}}1_{B_i}]$ in the form

$$\begin{aligned}&\mathrm{E}\left[e^{-rU_1}1_{\{U_1<h\}}e^{-rV_1}1_{\mathrm{NR}_1}\ldots e^{-rU_{i-1}}1_{\{U_{i-1}<h\}}e^{-rV_{i-1}}1_{\mathrm{NR}_{i-1}}e^{-rh}\right]\\&\times\mathrm{E}[X_h1_{\{U_1>h\}}]=\frac{\mathrm{E}\left[e^{-rF_{l_1,l_1,h}}1_{B_i}\right]}{\bar{P}_1(h)}\mathrm{E}[X_h1_{\{U_1>h\}}],\end{aligned}$$

which means

$$\mathrm{E}\left[e^{-rF_{l_1,l_1,h}}X_{F_{l_1,l_1,h}}1_{\{F_{l_1,l_1,h}<T_{l_1,l_0,d}\}}\right]=\frac{\mathrm{E}\left[e^{-rF_{l_1,l_1,h}}1_{\{F_{l_1,l_1,h}<T_{l_1,l_0,d}\}}\right]}{\bar{P}_1(h)}\mathrm{E}[X_h1_{\{U_1>h\}}].$$

We already know $\mathrm{E}[e^{-rF_{l_1,l_1,h}}1_{\{F_{l_1,l_1,h}<T_{l_1,l_0,d}\}}]$ from Theorem 1, so now we have the task of calculating $\mathrm{E}[X_h1_{\{U_1>h\}}]$.

Recall that $X_t-(c-\frac{\lambda}{\alpha})t$ is a martingale with mean l_1 (because it is a process with independent increments whose mean is constant). Obviously, $U_1\wedge h$ is a bounded stopping time ($a\wedge b$ means the minimum of a and b). By the optimal stopping theorem

$$\begin{aligned}l_1&=\mathrm{E}\left[X_{U_1\wedge h}-\left(c-\frac{\lambda}{\alpha}\right)(U_1\wedge h)\right]\\&=\mathrm{E}[X_{U_1}1_{\{U_1<h\}}]+\mathrm{E}[X_h1_{\{U_1>h\}}]-\left(c-\frac{\lambda}{\alpha}\right)\left(\mathrm{E}[U_11_{\{U_1<h\}}]+hP\{U_1>h\}\right).\end{aligned}$$

Now using $\mathrm{E}[X_{U_1}1_{\{U_1<h\}}]=\mathrm{E}[X_{U_1}]P\{U_1<h\}=(l_1-\frac{1}{\alpha})P\{U_1<h\}$ and solving this equation with respect to $\mathrm{E}[X_h1_{\{U_1>h\}}]$, we get

$$\mathrm{E}[X_h1_{\{U_1>h\}}]=\frac{1}{\alpha}+\left(c-\frac{\lambda}{\alpha}\right)\mathrm{E}[U_11_{\{U_1<h\}}]+\left(l_1+ch-\frac{\lambda h+1}{\alpha}\right)\bar{P}_1(h).$$

This concludes the proof of the theorem. □

3. The expectation of total dividend payments

Now we consider a company whose capital at time t (before dividends payment) is X_t. In order to formulate a new dividend strategy with payment delay introduced in

Dassios and Wu (2009) we need some notation. The initial surplus is $X_0 = x$, where $0 \le x < b$. We set $g^X_{b,t} = \sup\{s \le t | X_s \le b\}$. Let

$$\tau^X_0 = \inf\{t \ge 0 | X_t = b\}$$

be the first time X hits the barrier b and let

$$\tau^X_i = \inf\left\{t \ge \tau^X_{i-1} | 1_{\left\{X_t > X_{\tau^X_{i-1}}\right\}}\left(t - g^X_{X_{\tau^X_{i-1}},t}\right) \ge h\right\}$$

be the first time after τ^X_{i-1} when the length of the excursion above $X_{\tau^X_{i-1}}$ reaches h. We assume that dividends are paid only if the surplus stayed above the barrier b during the time interval of length h. Then the excess is immediately paid out and the surplus starts from the level b. The modified surplus process (taking into account dividend payments):

$$Y_t = X_t 1_{\left\{0 \le t < \tau^X_0\right\}} + \sum_{i=0}^{\infty}\left(X_t - X_{\tau^X_i} + b\right)1_{\left\{\tau^X_i \le t < \tau^X_{i+1}\right\}}. \tag{8}$$

Define the time of Parisian ruin

$$T_d = \inf\left\{t > 0 | 1_{\{Y_t < 0\}}\left(t - d^Y_t\right) \ge d\right\}, \quad \text{where } d^Y_t = \sup\{s < t | Y_s \ge 0\}.$$

The present value of the total dividend payments before Parisian ruin of Y is defined by

$$V(x,b) = \sum_{i=1}^{\infty} e^{-r\tau^X_i}\left(X_{\tau^X_i} - X_{\tau^X_{i-1}}\right)1_{\left\{\tau^X_i \le T_d\right\}}. \tag{9}$$

Obviously, $\mathrm{E}V(x,b)$ depends not only on initial surplus x and dividends barrier b. However other parameters (d, h, r, λ, α) are omitted to simplify notation.

Also, denote $\mathrm{NR}[t_1, t_2)$ the event that there is no moment $t_1 \le t < t_2$ when the surplus has stayed below zero for at least d. Then we can rewrite (9) as

$$V(x,b) = \sum_{i=1}^{\infty} e^{-r\tau^X_i}\left(X_{\tau^X_i} - X_{\tau^X_{i-1}}\right)1_{\mathrm{NR}\left[0,\tau^X_i\right)}.$$

Our task is to find the optimal barrier b^* maximizing the expectation $\mathrm{E}[V(x,b)]$. By the strong Markov property we can write $\mathrm{E}[V(x,b)]$ as

$$\begin{aligned} &\mathrm{E}\left[e^{-r\tau^X_0}1_{\mathrm{NR}\left[0,\tau^X_0\right)}\right]\mathrm{E}\left[\sum_{i=1}^{\infty} e^{-r\left(\tau^X_i - \tau^X_0\right)}\left(X_{\tau^X_i} - X_{\tau^X_{i-1}}\right)1_{\mathrm{NR}\left[\tau^X_0,\tau^X_i\right)}\right] \\ &= \mathrm{E}\left[e^{-r\tau^X_0}1_{\mathrm{NR}\left[0,\tau^X_0\right)}\right]\mathrm{E}[V(b)]. \end{aligned} \tag{10}$$

For now, let us concentrate on finding $\mathrm{E}[V(b)]$ where $V(b) = V(0,b)$.

$$\begin{aligned} \mathrm{E}[V(b)] &= \mathrm{E}\left[e^{-r\left(\tau^X_1 - \tau^X_0\right)}\left(X_{\tau^X_1} - X_{\tau^X_0}\right)1_{\mathrm{NR}\left[\tau^X_0,\tau^X_1\right)}\right] \\ &\quad + \mathrm{E}\left[e^{-r\left(\tau^X_1 - \tau^X_0\right)}1_{\mathrm{NR}\left[\tau^X_0,\tau^X_1\right)}\right]\mathrm{E}\left[\sum_{i=2}^{\infty} e^{-r\left(\tau^X_i - \tau^X_0\right)}\left(X_{\tau^X_i} - X_{\tau^X_{i-1}}\right)1_{\mathrm{NR}\left[\tau^X_0,\tau^X_i\right)}\right] \\ &= \mathrm{E}\left[e^{-r\left(\tau^X_1 - \tau^X_0\right)}\left(X_{\tau^X_1} - X_{\tau^X_0}\right)1_{\mathrm{NR}\left[\tau^X_0,\tau^X_1\right)}\right] \\ &\quad + \mathrm{E}\left[e^{-r\left(\tau^X_1 - \tau^X_0\right)}1_{\mathrm{NR}\left[\tau^X_0,\tau^X_1\right)}\right]\mathrm{E}[V(b)]. \end{aligned}$$

Solving this simple linear equation with respect to $\mathrm{E}[V(b)]$, we get

$$\mathrm{E}[V(b)] = \frac{\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}\left(X_{\tau_1^X}-X_{\tau_0^X}\right)1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]}{1-\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]}.$$

Hence, recalling (10)

$$\mathrm{E}[V(x,b)] = \mathrm{E}\left[e^{-r\tau_0^X}1_{\mathrm{NR}\left[0,\tau_0^X\right)}\right]\frac{\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}\left(X_{\tau_1^X}-X_{\tau_0^X}\right)1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]}{1-\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]}. \tag{11}$$

But we have already found each term in (11) in the previous section.

4. Analysis of $\mathrm{E}[V(x,b)]$

Now it is possible to establish that the expected present value of dividends until the Parisian ruin ($d>0$) is greater than that until the classical ruin ($d=0$).

Corollary 1. *For any $d>0$, $\mathrm{E}[V(x,b)]>\mathrm{E}[V(x,b)]|_{d=0}$.*

Proof. Using (5) from Lemma 1 with $l_1=0, l_2=b$, we get immediately

$$\begin{aligned}\mathrm{E}\left[e^{-r\tau_0^X}1_{\mathrm{NR}\left[0,\tau_0^X\right)}\right] &= \mathrm{E}\left[e^{-r\tau_{x,b}}1_{\{\tau_{x,b}<T_{x,0,d}\}}\right] = \frac{(\alpha+\nu_r^+)e^{\nu_r^+x}-(\alpha+\nu_r^-)e^{\nu_r^-x}}{(\alpha+\nu_r^+)e^{\nu_r^+b}-(\alpha+\nu_r^-)e^{\nu_r^-b}}\\ &+\frac{\nu_r^+-\nu_r^-}{(\alpha+\nu_r^+)e^{\nu_r^+b}-(\alpha+\nu_r^-)e^{\nu_r^-b}}\\ &\times\frac{\left(e^{\nu_r^+(x-b)}-e^{\nu_r^-(x-b)}\right)\int_0^d e^{-rs}p_2(s)\ ds}{\left(\frac{\alpha e^{-\nu_r^+b}}{\alpha+\nu_r^+}-\frac{\alpha e^{-\nu_r^-b}}{\alpha+\nu_r^-}\right)-\left(e^{-\nu_r^+b}-e^{-\nu_r^-b}\right)\int_0^d e^{-rs}p_2(s)\ ds}.\end{aligned} \tag{12}$$

Obviously, if $d=0$, this equals

$$\frac{(\alpha+\nu_r^+)e^{\nu_r^+x}-(\alpha+\nu_r^-)e^{\nu_r^-x}}{(\alpha+\nu_r^+)e^{\nu_r^+b}-(\alpha+\nu_r^-)e^{\nu_r^-b}}.$$

Furthermore,

$$\begin{aligned}&\left(\frac{\alpha e^{-\nu_r^+b}}{\alpha+\nu_r^+}-\frac{\alpha e^{-\nu_r^-b}}{\alpha+\nu_r^-}\right)-\left(e^{-\nu_r^+b}-e^{-\nu_r^-b}\right)\int_0^d e^{-rs}p_2(s)\ ds\\ &<\left(\frac{\alpha e^{-\nu_r^+b}}{\alpha+\nu_r^+}-\frac{\alpha e^{-\nu_r^-b}}{\alpha+\nu_r^-}\right)-\left(e^{-\nu_r^+b}-e^{-\nu_r^-b}\right)\int_0^\infty e^{-rs}p_2(s)\ ds\\ &=\left(\frac{\alpha e^{-\nu_r^+b}}{\alpha+\nu_r^+}-\frac{\alpha e^{-\nu_r^-b}}{\alpha+\nu_r^-}\right)-\left(e^{-\nu_r^+b}-e^{-\nu_r^-b}\right)\frac{\alpha}{\alpha+\nu_r^+}<0.\end{aligned}$$

We used the following chain of equalities:

$$\int_0^\infty e^{-rs}p_2(s)\ ds = 2c\alpha\left(\sqrt{(c\alpha+r+\lambda)^2-4c\alpha\lambda}+(c\alpha+r+\lambda)\right)^{-1} = \frac{\alpha}{\alpha+\nu_r^+},$$

the first of which can be found, for example, in Bulinskaya and Shigida (2018), and the second is easy to check utilizing the explicit form (2).

We conclude that $\mathrm{E}[e^{-r\tau_0^X}1_{\mathrm{NR}[0,\tau_0^X)}]$ is greater for $d>0$ than for $d=0$, taking into account inequality $e^{\nu_r^+(x-b)}-e^{\nu_r^-(x-b)}<0$.

According to Theorem 1:

$$\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right] = \mathrm{E}\left[e^{-rF_{b,b,h}}1_{\{F_{b,b,h}<T_{b,0,d}\}}\right] = \frac{e^{-rh}\bar{P}_1(h)}{1-\mathbf{A}\times\int_0^h e^{-rs}p_1(s)\ ds.},$$

where

$$\begin{aligned}\mathbf{A} = {} & \frac{\alpha e^{\nu_r^+b}-\alpha e^{\nu_r^-b}}{(\alpha+\nu_r^+)e^{\nu_r^+b}-(\alpha+\nu_r^-)e^{\nu_r^-b}} \\ & + \frac{\nu_r^+-\nu_r^-}{(\alpha+\nu_r^+)e^{\nu_r^+b}-(\alpha+\nu_r^-)e^{\nu_r^-b}}\times\left(\frac{\alpha}{\alpha+\nu_r^+}-\frac{\alpha}{\alpha+\nu_r^-}\right) \\ & \times\frac{\int_0^d e^{-rs}p_2(s)\ ds}{\left(\frac{\alpha e^{-\nu_r^+b}}{\alpha+\nu_r^+}-\frac{\alpha e^{-\nu_r^-b}}{\alpha+\nu_r^-}\right)-\left(e^{-\nu_r^+b}-e^{-\nu_r^-b}\right)\int_0^d e^{-rs}p_2(s)\ ds}.\end{aligned}$$

Again, obviously,

$$\mathbf{A}|_{d=0} = \frac{\alpha e^{\nu_r^+b}-\alpha e^{\nu_r^-b}}{(\alpha+\nu_r^+)e^{\nu_r^+b}-(\alpha+\nu_r^-)e^{\nu_r^-b}}.$$

Also, we know that

$$\left(\frac{\alpha}{\alpha+\nu_r^+}-\frac{\alpha}{\alpha+\nu_r^-}\right)<0$$

and

$$\left(\frac{\alpha e^{-\nu_r^+b}}{\alpha+\nu_r^+}-\frac{\alpha e^{-\nu_r^-b}}{\alpha+\nu_r^-}\right)-\left(e^{-\nu_r^+b}-e^{-\nu_r^-b}\right)\int_0^d e^{-rs}p_2(s)\ ds<0.$$

Hence, we can conclude that

$$\mathbf{A}>\mathbf{A}|_{d=0}$$

if $d>0$. This quickly gives

$$\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]>\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]\Big|_{d=0}.$$

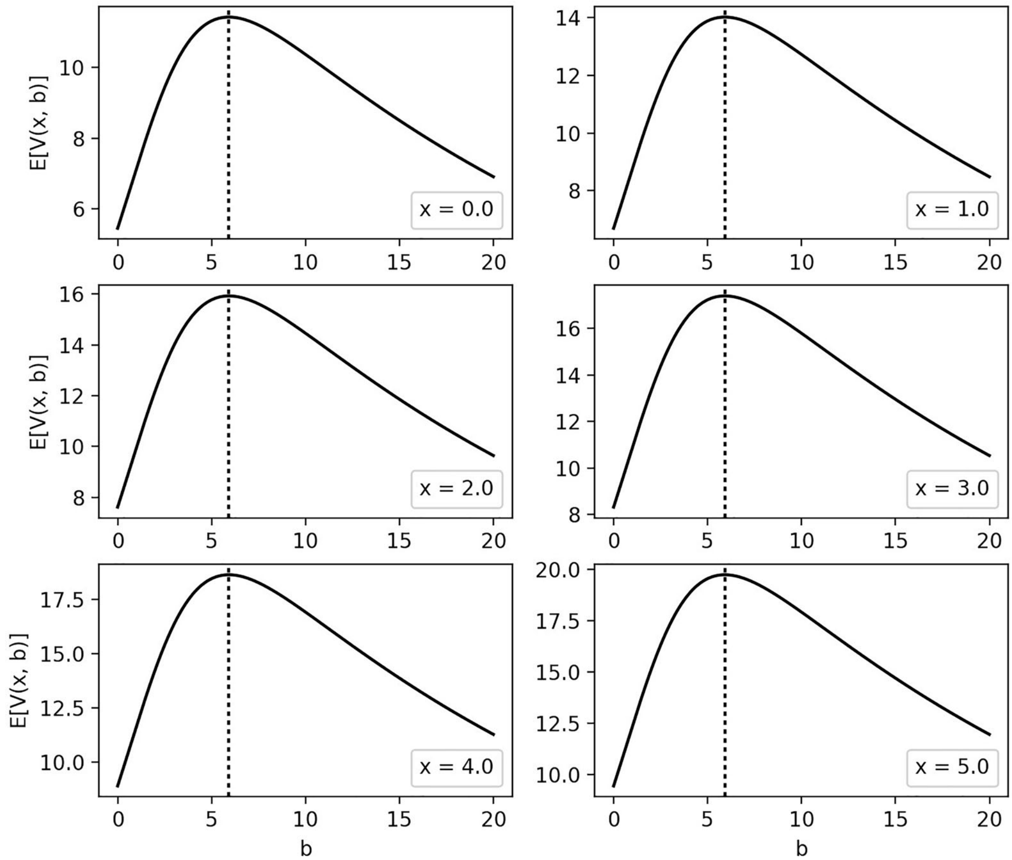

Figure 1. Form of $\mathrm{E}V(x,b)$ as function of b for fixed x.

Finally, applying Theorem 2 we can write:

$$\begin{aligned}
&\mathrm{E}\left[e^{-r\left(\tau_1^X-\tau_0^X\right)}\left(X_{\tau_1^X}-X_{\tau_0^X}\right)1_{\mathrm{NR}\left[\tau_0^X,\tau_1^X\right)}\right]\\
&=\mathrm{E}\left[e^{-rF_{b,b,h}}X_{F_{b,b,h}}1_{\{F_{b,b,h}<T_{b,0,d}\}}\right]-b\cdot\mathrm{E}\left[e^{-rF_{b,b,h}}1_{\{F_{b,b,h}<T_{b,0,d}\}}\right]\\
&=\mathrm{E}\left[e^{-rF_{b,b,h}}1_{\{F_{b,b,h}<T_{b,0,d}\}}\right]\times\left(\frac{\frac{1}{\alpha}+\left(c-\frac{\lambda}{\alpha}\right)\int_0^h sp_1(s)\ ds}{\bar{P}_1(h)}+ch-\frac{\lambda h+1}{\alpha}\right).
\end{aligned}$$

The net profit condition leads to the inequality

$$\left(\frac{\frac{1}{\alpha}+\left(c-\frac{\lambda}{\alpha}\right)\int_0^h sp_1(s)\ ds}{\bar{P}_1(h)}+ch-\frac{\lambda h+1}{\alpha}\right)>0.$$

Thus, it can be seen that

$$\mathrm{E}[V(x,b)]>\mathrm{E}[V(x,b)]|_{d=0}.$$

□

5. Optimal barrier and statistical estimates

The explicit expression of the function $\mathrm{E}[V(x,b)]$ seems very complicated for analytical investigation. So, the numerical results were obtained at first. An analysis of the model under consideration was conducted using the Python programing language. In Figure 1 we provide 6 graphs of the expected discounted total dividend payment as a function of b (for $c=10.0$, $\lambda=5.4, \alpha=1.0$, $d=0.1$, $h=0.3$, $r=0.2$ and $x=0,1,2,3,4,5$). It can be seen from those graphs that the optimal barrier b^* maximizing the expectation (see the vertical dashed line) does not depend on x. The function was analyzed in Python using the `scipy` library and the graphs were obtained with the `matplotlib` library.

Thus, taking the derivative of the function $\mathrm{E}[V(x,b)]$ with respect to b and finding the point b^* where it equals zero, we can find the global maximum of this function.

Using the expression (12), it can be shown that $\mathrm{E}[e^{-r\tau_0^X}1_{\mathrm{NR}[0,\tau_0^X)}]$ has the form $C(x)f(b)$ where the function $f(b)$ does not depend on x and the factor

$$\begin{aligned}C(x)=&\left(\alpha+\nu_r^+\right)\left(\alpha-\left(\alpha+\nu_r^-\right)\int_0^d e^{-rs}p_2(s)ds\right)e^{\nu_r^+x}\\&-\left(\alpha+\nu_r^-\right)\left(\alpha-\left(\alpha+\nu_r^+\right)\int_0^d e^{-rs}p_2(s)ds\right)e^{\nu_r^-x}\end{aligned}$$

does not depend on b. Due to (11) this means that the partial derivative with respect to b of the expected total dividend payment has such a form as well:

$$\frac{\partial}{\partial b}\mathrm{E}[V(x,b)]=C(x)\tau(b),$$

where $\tau(b)$ does not depend on x. This means that the optimal barrier b^*, satisfying

$$\tau(b^*)=0,$$

does not depend on x either.

If $b^*>0$ (which does not follow from the net profit condition), for all $0\le x<b^*$ it is the optimal barrier (the barrier which maximizes the expected total dividend payment).

Also, a simulation of the process Y_t itself was carried out. We generated a large sample of independent exponential random variables with parameter λ, which are treated as intervals between claims, and of independent exponential random variables with parameter α which represent claim amounts. Random samples were generated using the standard module random in `Python`.

After that, the formulas (1) and (8) translated into code were applied directly to get the simulation of our model. Also, the formula (9) (with $T_d\wedge t$ instead of T_d) was used to calculate the total (discounted to the moment $t=0$) dividend payment up to t. The simulation of the two processes (Y_t itself and the process $V_t(x,b)$ of dividend payments up to t) is shown in Figure 2. (Note that the upper picture is a magnification of the lower left corner of the lower picture.) The horizontal line (which is close to zero) denotes the dividend barrier $b=10$, the black curve fluctuating around it is Y_t. The horizontal dashed line shows the expectation of the total dividend payment, the vertical dashed line marking the time of Parisian ruin. It is clear that $V_t(x,b)$ (gray step-

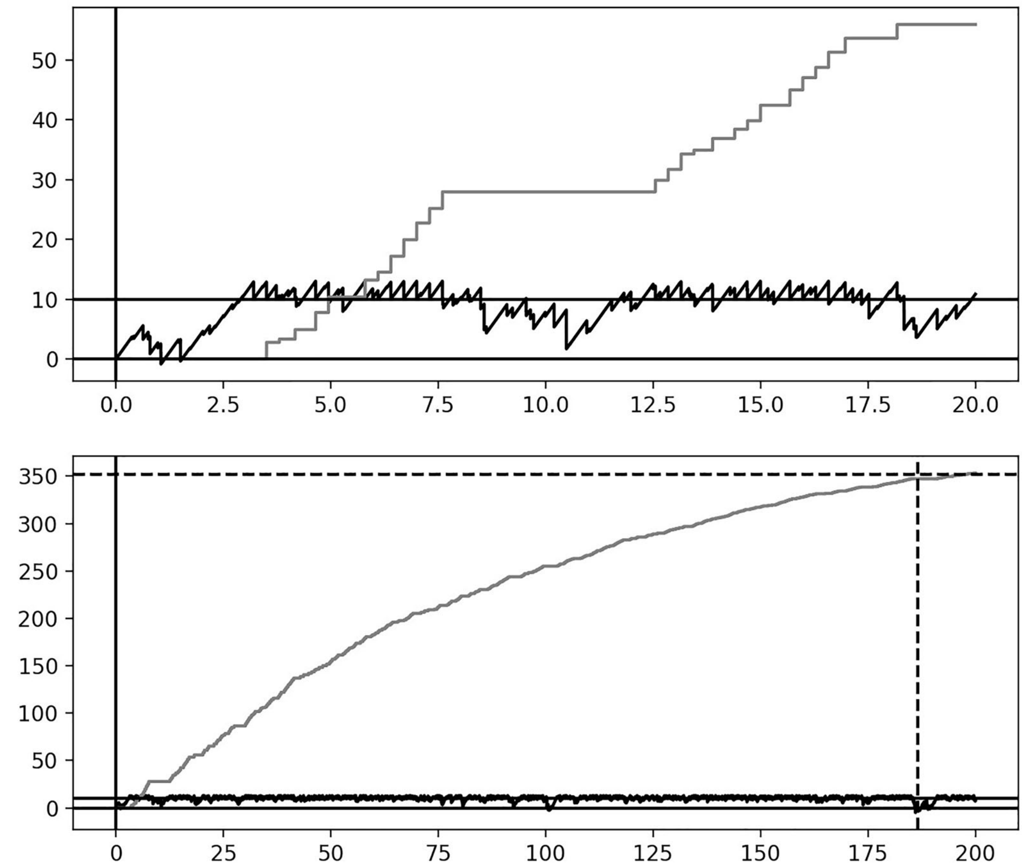

Figure 2. Simulation of the process Y_t and dividends $V_t(x, b)$.

function) is close to $\mathrm{E}[V(x, b)]$ at time $t = T_d$ of Parisian ruin. Parameters here are as follows $\lambda = 5.4$, $c = 10.0$, $\alpha = 1.0$, $x = 0.0$, $d = 0.5$, $h = 0.3$, $r = 0.01$.

Another problem is incomplete information about system parameters α and λ. Hence, we considered the estimate of the target function $\mathrm{E}[V(x, b)]$ based on observations up to the moment t. If T_n means the time of the nth claim, then

$$\hat{\lambda} = \left(\frac{T_n}{n}\right)^{-1}, \quad \hat{\alpha} = \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^{-1} \tag{13}$$

are simple estimates of the parameters of the two exponential sequences. Calculating the target function (10) with these estimates instead of the real values of the parameters, we obtain a strongly consistent estimate of this function (because it is continuous).

For each time t we can do that for estimates (13) with $n = N_t$. This way, the graphs in Figure 3 were obtained. The horizontal dashed line denotes the real value of $\mathrm{E}[V(x, b)]$. The convergence of the estimate (black curve) to the real value is clear in these pictures. As above, the vertical dashed line shows the time of Parisian ruin (if it is absent, ruin during the considered interval does not occur). The gray curve is the graph of $V_t(x, b)$ (dividends payment in simulation).

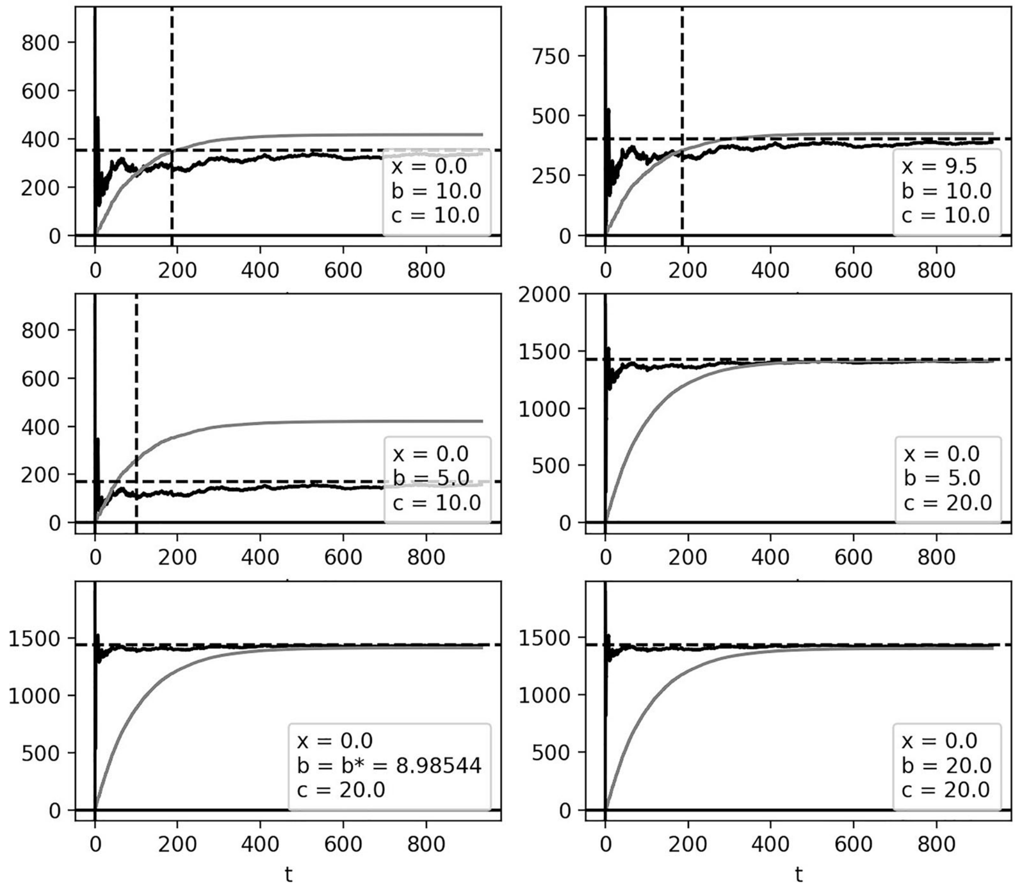

Figure 3. Convergence of the expected dividends estimate.

6. Conclusion and further research directions

In this article we investigated the results of application of the dividends strategy with Parisian implementation delay and a fixed barrier. The explicit expression of expected discounted dividends until the Parisian ruin is obtained. Due to its complicated form we were unable to find the optimal barrier in a closed form. However it is not difficult to perform this procedure numerically. Moreover, being interested in application of the obtained results we posed a new problem, namely, how to predict the expected dividends $\mathrm{E}[V(x,b)]$ on the base of the company surplus observation during a given time interval $[0,t]$. More precisely, we used the statistical estimates of the parameters instead of the exact values and established the convergence with probability 1 of the estimates of expected dividends to the real value. The simulation was carried out as well. The next step is to use more sophisticated estimates of parameters and establish the (asymptotic) unbiasedness (as $t\to\infty$) and L^2-consistency of expected dividends estimates. Another research direction is investigation of the system stability to small fluctuations of parameters and perturbations of underlying processes using the methods presented in Rachev et al. (2013), Saltelli et al. (2008) and their ramifications, see, e.g., Bulinskaya and Gusak (2016, 2018).

Acknowledgments

The authors would like to thank two anonymous reviewers for their helpful suggestions on improvement of presentation.

Funding

The research was partially supported by the Russian Foundation for Basic Research under grant 17-01-00468.

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