ECO 3302 – Intermediate Macroeconomics

Lecture 7: The Solow Model

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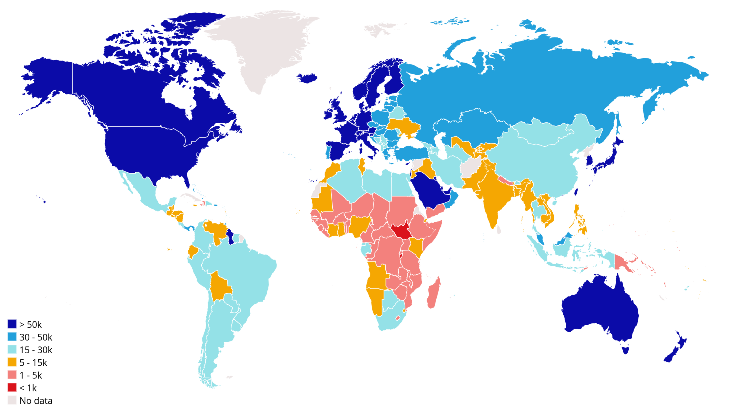
Introduction to the Solow model

- **Last lecture:** learned to make cross-country income comparisons
- **Today:** try to understand why some countries much richer than others



Introduction

- **Our background**:
  - Saw that production factors and technology determine an economy's income:

$$Y = F(K, L; A)$$

  - Differences in income across countries and over time come from differences in capital, labor, and technology
  - **Static focus**: given technology and factors, study output at given point in time

- **Today and next lectures, our focus becomes dynamic**: explain why GDP grows over time and why it grows much faster for some countries
  - We do this with Solow growth model

Environment and Assumptions

Environment: Households

- **Closed economy with single good**
- **Discrete time** running to an infinite horizon ($t = 0, 1, 2, \ldots$)
- **Large number of households that will not be optimizing!**
  - Main difference between Solow and Neoclassical growth models
- To simplify analysis, assume:
  - All households identical $\rightarrow$ economy admits **representative household**
  - **Households save constant exogenous fraction $s \in (0, 1)$ of disposable income**
    - This is the no-optimizing part on households side
      (ie, households don't decide how much to save)
    - It both simplifies and limits analysis
      (eg, can't study effects of tax increase on savings and growth)
  - **Households own all factors of production and supply labor inelastically**

Environment: Firms

- Assume *all* firms have same production function $\rightarrow$ **representative firm**
- **Aggregate production function**:

$$Y_t = F[K_t, L_t, A_t] \tag{1}$$

  - Y: final good (think of it as units of real GDP)
  - K: capital stock (ie, machines, buildings,...)
  - L: labor (eg, population size, labor force size, total hours worked, ...)
  - A: total factor productivity (TFP)
- A is "efficiency" shifter of production function: $\uparrow A \implies \uparrow Y$ with given K, L

A1: Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \qquad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \qquad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0$$

Moreover, F exhibits constant returns to scale (CRS) in K and L:

$$F(\lambda K, \lambda L) = \lambda Y, \qquad \text{for any } \lambda > 0$$

Environment: Technology

A2: Inada Conditions

F satisfies:

$$F(0,L,A)=0 \quad \text{and} \quad F(K,0,A)=0 \quad \text{for } K,L,A>0 \qquad \text{(essential inputs)}$$
$$\lim_{K\to 0} F_K(K,L,A)=\infty \quad \text{and} \quad \lim_{K\to\infty} F_K(K,L,A)=0, \qquad \text{for all } L,A>0$$
$$\lim_{L\to 0} F_L(K,L,A)=\infty \quad \text{and} \quad \lim_{L\to\infty} F_L(K,L,A)=0, \qquad \text{for all } K,A>0$$

- Inada (boundary) conditions ensure the existence of *interior equilibria*
- Assumptions $A1$ and $A2$ are "**neoclassical technology assumptions**"
    - Give rise to neoclassical production function

Environment: Market structure, endowments, market clearing

- Assume **competitive markets and market clearing**
  (ie, agents are price-takers and prices clear markets)

- **Households own all labor and supply it inelastically**
  (ie, all labor put to work unless its rental price is 0!)

- Endowment of labor assumed equal to population size, $\overline{L}_t$

- Labor market clearing condition given by

$$L_t = \overline{L}_t, \qquad \text{for } t = 0, 1, 2, \ldots$$

- Rental price of labor, the *wage rate*, denoted W_t. Then it must be satisfied,

$$L_t \leq \overline{L}_t, \; W_t \geq 0, \qquad \text{and} \qquad [L_t - \overline{L}_t] W_t = 0$$

  Employed labor must be lower than population size, wage must be non-negative, and either the labor market clears or the wage is zero

Environment: Market structure, endowments, market clearing

- **Households own all capital and rent it to firms** at rental rate R_t
- Capital market clearing condition given by

$$K_t = \overline{K}_t, \qquad \text{for } t = 0, 1, 2, \dots$$

- **We take initial capital endowments** $K_0 \geq 0$ **as given**
- **Capital depreciates at exponential rate** $\delta \in (0, 1)$
  - 1 unit of capital today is equivalent to $1 - \delta$ units tomorrow
- **Loss of capital affects interest rates** (return to savings of households):

$$r_t = R_t - \delta$$

Environment: Firm optimization

- Problem of the representative firm is to choose K, L to maximize profits:

$$\max_{K_t \geq 0, L_t \geq 0} \quad \Pi_t \equiv \underbrace{P_t Y_t}_{\text{revenues}} - \underbrace{W_t L_t - R_t K_t}_{\text{costs}}$$

$$\text{s.t.} \quad Y_t = F[K_t, L_t, A_t]$$

- Important to notice:

1. Maximization problem imposes competitive markets
   (firms take factor prices W_t and R_t as given)

2. **Can normalize price of final good, $P_t = 1$, in all periods**
   (final output is the *numeraire*)

3. Can substitute Y into the profits equation Π

Firm optimization

Hence, can write:

$$\max_{K_t \geq 0, L_t \geq 0} \quad \Pi_t \equiv \underbrace{F[K_t, L_t, A_t]}_{\text{revenues}} - \underbrace{W_t L_t - R_t K_t}_{\text{costs}}$$

- Given properties of F (A1–A2), we can take **FOCs to obtain (interior) solution**:

$$\frac{\partial \Pi_t}{\partial K_t} = 0 \qquad \Longrightarrow \qquad R_t = F_K[K_t, L_t, A_t] \tag{2}$$

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \qquad \Longrightarrow \qquad W_t = F_L[K_t, L_t, A_t] \tag{3}$$

  With competitive markets, rental rates equal to marginal (revenue) products

- **Solving for K_t and L_t we can derive demand for capital and labor**

Fundamental law of motion

- K depreciates exponentially at rate δ and grows with investment $I \geq 0$.
  Capital's law of motion:

$$\underbrace{K_{t+1}}_{\text{capital tomorrow}} = \underbrace{(1-\delta)K_t}_{\text{today's undepreciated capital}} + \underbrace{I_t}_{\text{today's investment}} \tag{4}$$

- **In closed economy, abstracting from the government:**

$$Y_t = C_t + I_t, \tag{5}$$

  where C_t denotes consumption.

- Whatever is not consumed is invested:

$$S_t = I_t = Y_t - C_t$$

Fundamental law of motion

- **Solow's behavioral rule**: HHs save constant fraction $s \in (0, 1)$ of income:

$$S_t = I_t = sY_t \tag{6}$$

$$C_t = (1 - s)Y_t \tag{7}$$

- Equation (4) can be rewritten as

$$\begin{aligned} K_{t+1} &= (1 - \delta)K_t + sY_t \\ &= (1 - \delta)K_t + sF[K_t, L_t, A_t] \end{aligned} \tag{8}$$

- This equation is the *fundamental law of motion* in Solow's growth model
- Equation (8) together with the laws of motion of L_t and A_t describe the equilibrium in the Solow growth model

Equilibrium

Definition of Equilibrium

- **Solow model combines features of Keynesian models** (ie, behavioral rules) **and modern macro approaches** (ie, optimization & market clearing)

Definition of Equilibrium

In the basic Solow model for a given sequence of $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial capital stock K_0, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, rental rates and wages $\{K_t, Y_t, C_t, R_t, W_t\}_{t=0}^{\infty}$ such that K_t satisfies equation (8), Y_t is given by equation (1), C_t is given by equation (7), and R_t and W_t are given by equations (2) and (3), respectively.

- **Equilibrium is an "entire path" of allocations and prices**, not a static object!

Theoretical Analysis

National accounting in the Solow model

- **Remember Euler's theorem?** With it and FOCs from firm's problem, we can now establish our first result!

Simplified version of Euler's Theorem

Suppose $A1$ holds and markets are competitive. Then, in the equilibrium of the Solow model, **firms make no profits and the following equation holds**:

$$Y_t = W_t L_t + R_t K_t \tag{9}$$

- **Factor shares** obtained dividing both sides of (9) by Y_t:

$$\Lambda_{Lt} + \Lambda_{Kt} = 1 \quad \Longrightarrow \quad \Lambda_{Kt} = 1 - \Lambda_{Lt}$$

Equilibrium without population growth and technological progress

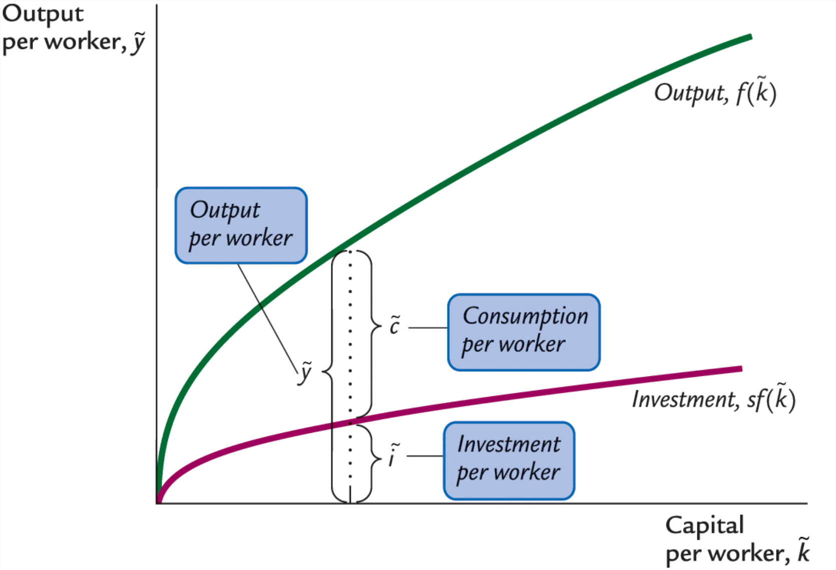
- **Two simplifying assumptions**:
  1. No population growth: $L_t = L > 0$ for all $t = 0, 1, 2, \ldots$
  2. No technological progress: $A_t = A > 0$ for all $t = 0, 1, 2, \ldots$

- **Study economy in "per capita" terms**. Using tildes to denote per-capita vars:

$$
\begin{aligned}
\text{Capital-labor ratio :} \qquad & \tilde{k}_t := \frac{K_t}{L} \\
\text{Output per capita :} \qquad & \tilde{y}_t := Y_t / L \\
& = F\left[\frac{K_t}{L}, 1, A\right] \qquad \text{(by CRS)} \\
& = f(\tilde{k}_t, A)
\end{aligned}
$$

Since there is no technological progress, A is constant and can be omitted.
Ie, $\tilde{y}_t = f(\tilde{k}_t)$ (Capital-labor ratio entirely determines output per capita in this economy)

Graphical representation: per-capita production function and investment



Equilibrium without population growth and technological progress

- Assume Cobb–Douglas production function:

$$Y_t = AK_t^{\alpha} L^{1-\alpha}, \qquad \alpha \in (0, 1)$$

(Clearly satisfies neoclassical technology assumptions A1–A2)

- Expressing output in per capita terms:

$$\tilde{y}_t = A\tilde{k}_t^{\alpha}$$

- FOC wrt $\tilde{k}$ yields rental rate of capital:

$$R_t = \frac{\mathrm{d}\tilde{y}_t}{\mathrm{d}\tilde{k}_t} = \alpha A\tilde{k}_t^{\alpha-1}$$

- Wage obtained applying Euler's theorem:

$$\begin{aligned} W_t &= \tilde{y}_t - R_t\tilde{k}_t \\ &= (1-\alpha)A\tilde{k}_t^{\alpha} \end{aligned}$$

Equilibrium without population growth and technological progress

- Same results with original production function
- Rental rate of capital:

$$
\begin{aligned}
R_t = \frac{\partial Y_t}{\partial K_t} &= \alpha A K_t^{\alpha-1} L^{1-\alpha} \\
&= \alpha A (\tilde{k}_t L)^{\alpha-1} L^{1-\alpha} \qquad \text{(using } K_t = \tilde{k}_t L\text{)} \\
&= \alpha A \tilde{k}_t^{\alpha-1}
\end{aligned}
$$

- Wage:

$$
\begin{aligned}
W_t = \frac{\partial Y_t}{\partial L} &= (1-\alpha) A K_t^{\alpha} L^{-\alpha} \\
&= (1-\alpha) A \tilde{k}_t^{\alpha}
\end{aligned}
$$

(Can verify Euler's theorem using these W_t and R_t)

Equilibrium without population growth and technological progress

- Recall fundamental role of capital: $\tilde{y}_t = f(\tilde{k}_t)$
- Can also rewrite capital's law of motion in per-capita terms:

$$
\begin{aligned}
& K_{t+1} = (1-\delta)K_t + sF(K_t, L, A) \\
\iff \quad & \frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\frac{F(K_t, L, A)}{L} \\
\iff \quad & \tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + sf(\tilde{k}_t)
\end{aligned} \tag{10}
$$

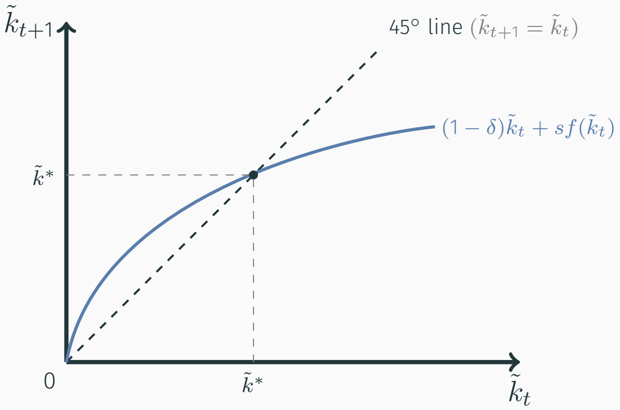
Definition of steady-state equilibrium

A **steady-state equilibrium without technological progress & population growth** is an equilibrium path in which $\tilde{k}_t = \tilde{k}^*$ for all t (ie, constant capital-labor ratio)

- Economy will approach steady-state (ss) equilibrium over time

Finding the steady-state capital per worker

- In steady state, capital per worker doesn't grow: $\tilde{k}_t = \tilde{k}^*$ for all t
- Can find steady-state capital per worker k^* using capital's law of motion



Finding the steady-state capital per worker

- In steady state, capital per worker doesn't grow: $\tilde{k}_t = \tilde{k}^*$ for all t
- Can find steady-state capital per worker k^* using capital's law of motion:

$$\tilde{k}^* = (1-\delta)\tilde{k}^* + sf(\tilde{k}^*)$$

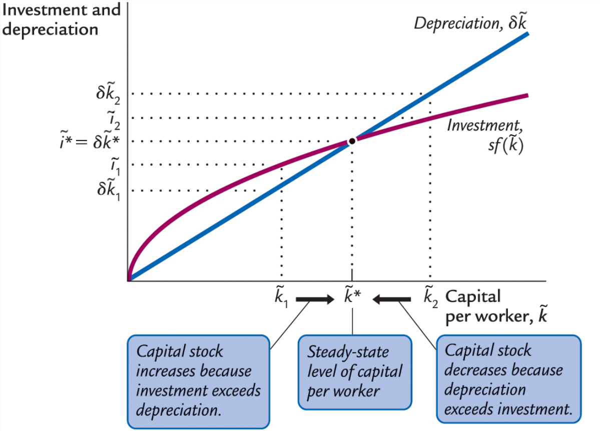
- Solution is a fixed point to this equation. **Two candidates:**
  1. $\tilde{k}^* = 0$ has no economic interest (we rule it out by assumption $\tilde{k}_0 > 0$)
  2. $\tilde{k}^* > 0$: positive interesection of $(1-\delta)\tilde{k} + sf(\tilde{k})$ curve with 45° line
- **Solution characterization**:

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta}{s} \quad \Longleftrightarrow \quad \underbrace{sf(\tilde{k}^*)}_{\text{ss investment}} = \underbrace{\delta\tilde{k}^*}_{\text{ss depreciation}}$$

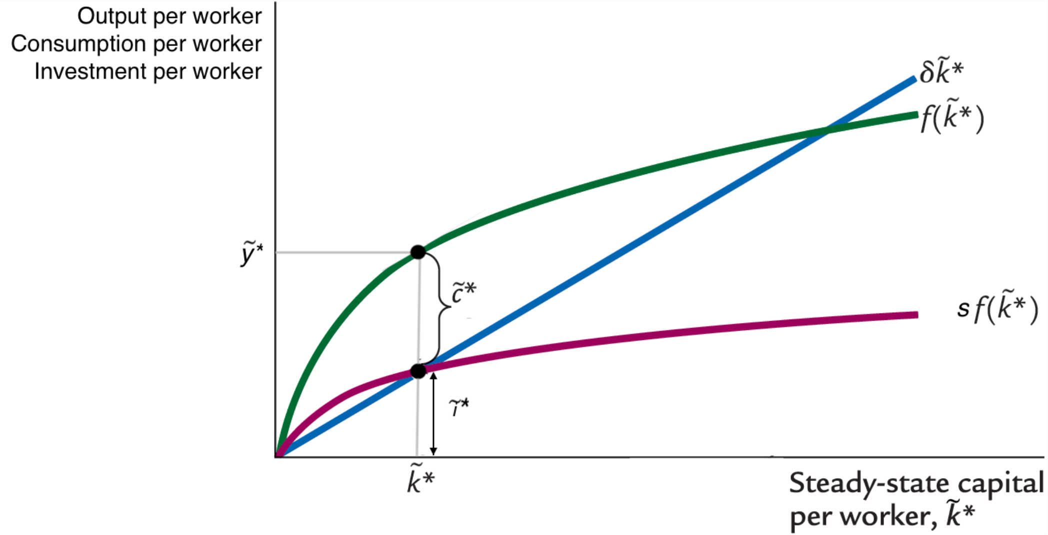
Steady-state capital per worker, investment, and depreciation

In steady state, capital per worker doesn't grow: $\tilde{k}(t) = \tilde{k}^*$ for all t

This only happens when investment ($\tilde{i} = sf(\tilde{k})$) equals depreciation ($\delta\tilde{k}$)



Steady-state equilibrium without technological progress and population growth



Equilibrium characterization

Under assumptions $A1$ and $A2$, there exists a unique steady-state equilibrium in the Solow model where the capital-labor ratio $\tilde{0} < k^* < \infty$ satisfies

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta}{s}$$

output per capita is

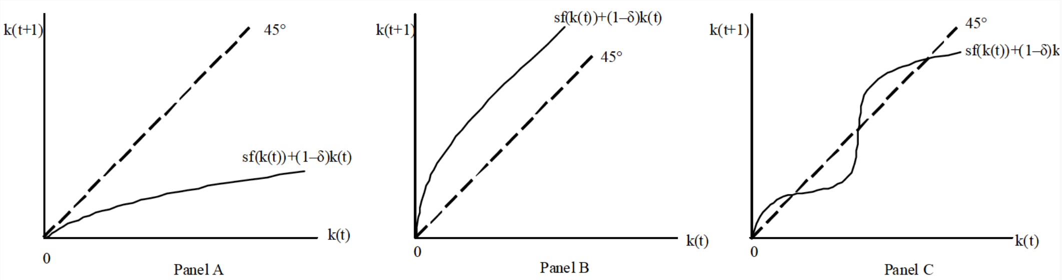
$$\tilde{y}^* = f(\tilde{k}^*) \tag{11}$$

and consumption per capita is

$$\tilde{c}^* = (1 - s)f(\tilde{k}^*) \tag{12}$$

- Existence and uniqueness of equilibrium guaranteed by assumptions $A1$–$A2$

Examples where existence and uniqueness of equilibrium fails



- **Panels A and B**: No equilibrium with $\tilde{k}^* > 0$ (violate A2, Inada conditions)
- **Panel C**: Multiple equilibria where $\tilde{k}^* > 0$ (violates A1, decreasing MPs)

Comparative statics

- Countries with higher saving rates ($\uparrow s$) and better technologies ($\uparrow A$) will have higher capital-labor ratios ($\uparrow \tilde{k}^*$) and will be richer ($\uparrow \tilde{y}^*$)
- Countries with greater technological depreciation ($\uparrow \delta$) will have lower capital-labor ratios ($\downarrow \tilde{k}^*$) and will be poorer ($\downarrow \tilde{y}^*$)
- The same is true for consumption $\tilde{c}^*$ (since it is a linear function of output per worker)

The golden rule

- There is unique saving rate, s_{gold}, that maximizes steady-state consumption
  - This savings rate found by taking the derivative of $\tilde{c}^*$ wrt s
  - **Step 1**. Write steady-state $\tilde{c}^*$ as function of s:

$$\begin{aligned}\tilde{c}^*(s) &= (1-s)f(\tilde{k}^*(s)) \\ &= f(\tilde{k}^*(s)) - \delta\tilde{k}^*(s) \qquad \text{(using } sf(\tilde{k}^*) = \delta\tilde{k}^*\text{)}\end{aligned}$$

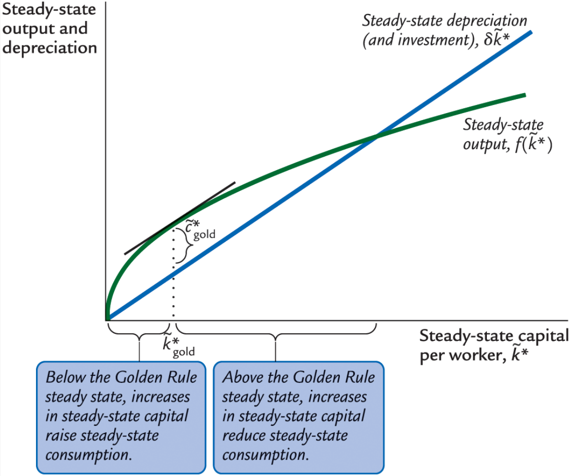
  - **Step 2**. Differentiate wrt s:

$$\frac{\partial \tilde{c}^*(s)}{\partial s} = \left[f'(\tilde{k}^*(s)) - \delta\right]\frac{\partial \tilde{k}^*}{\partial s}$$

- The golden rule of savings states that the saving rate s_{gold} must be such that

$$\frac{\partial c^*(s_{\text{gold}})}{\partial s} = 0$$

The golden-rule savings rate maximizes steady-state consumption



The golden rule

- The golden rule savings rate s_{gold} is such that, at that rate, steady-state consumption is maximized. That is,

$$\frac{\partial c^*(s_{\text{gold}})}{\partial s} = 0$$

- Following result follows:

Result

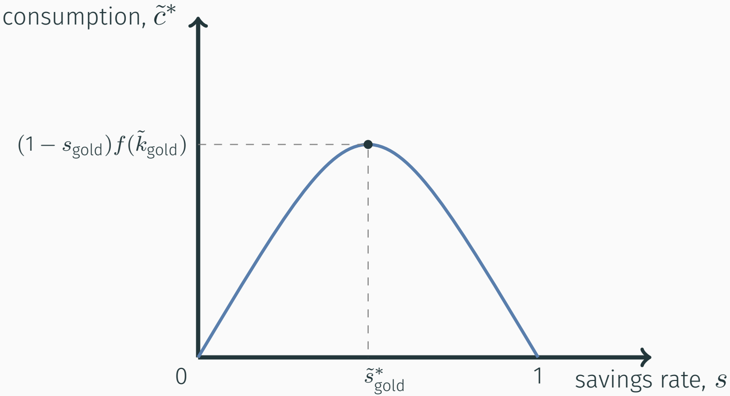
In the basic Solow growth model, the highest level of steady-state consumption is reached at s_{gold}, and the corresponding steady-state capital level $\tilde{k}^*_{\text{gold}}$ such that

$$f'(\tilde{k}^*_{\text{gold}}) = \delta \tag{13}$$

The golden rule

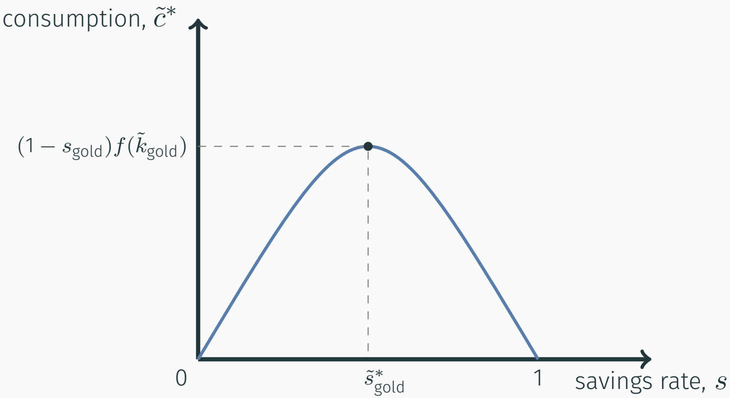
Highest level of steady-state consumption reached at s_{gold}:

$$\tilde{c}^*(s_{\text{gold}}) = (1 - s_{\text{gold}})f(\tilde{k}_{\text{gold}})$$



The golden rule

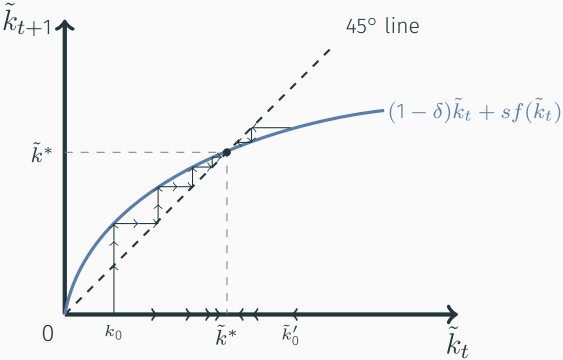
When economy is below $\tilde{k}^*_{\text{gold}}$, a higher saving rate will increase consumption;
Above $\tilde{k}^*_{\text{gold}}$, consumption can be raised by saving less (*dynamic ineffiency*)



Transitional dynamics

- Equilibrium path refers to entire path of capital stock, output, consumption and factor prices
- To see how the equilibrium path looks like we need to study the transitional dynamics of equation (10), starting with arbitrary capital-labor ratio, $\tilde{k}_0 > 0$
- We are often interested only in the steady state equilibrium
- Let's look at transitional dynamics graphically...

Transitional dynamics



- **Capital deepening**: Starting at $\tilde{k}_0 < \tilde{k}^*$, economy grows until $\tilde{k}^*$, so that capital-labor ratio increases (and also income per capita)
- If economy instead starts at $\tilde{k}_0' > \tilde{k}^*$, economy decumulates capital until $\tilde{k}^*$, so capital-labor ratio decreases (and so does income per capita)

An Example: Cobb–Douglas Technology

- Suppose $Y_t = AK_t^{\alpha}L^{1-\alpha}$, where $\alpha \in (0,1)$
- In steady state:

$$
\begin{aligned}
\tilde{y}^* &= f(\tilde{k}^*) \\
&= A{\tilde{k}^*}^{\alpha} \\
&= A\left\{\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}\right\}^{\alpha} && (\tilde{k}^* \text{ from solving } \Delta\tilde{k}^* = 0) \\
&= A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{c}^* &= (1-s)\tilde{y}^* \\
&= (1-s)A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

Summary of Solow model without population growth and tech. progress

- **Basic Solow model has very nice properties:**
  - Unique and stable steady state
  - Simple comparative statics
- **...but so far has no growth**: in steady state, there is no growth in capital-labor ratio ($\tilde{k}^*$) and no growth in output per capita ($\tilde{y}^*$)
- **Solow model without technological progress can only explain growth during the transition phase** (when $\tilde{k} < \tilde{k}^*$)
  - Growth slows down and eventually comes to a halt!
- **Although not in most desirable manner, Solow's model can account for sustained growth with *exogenous* technological change**

Equilibrium with population growth

- One tweak to basic model: add exogenous population growth

$$L_{t+1} = (1+n)L_t \qquad \Longleftrightarrow \qquad \frac{\Delta L_{t+1}}{L_t} = n$$

  where $n \geq 0$ is the population growth rate and $L_0 > 0$ given

- Fundamental law of capital accumulation:

$$K_{t+1} = (1-\delta)K_t + sY_t \tag{14}$$

- By definition, capital-labor ratio and output per capita:

$$\tilde{k}_t = \frac{K_t}{L_t} \qquad \text{and} \qquad \tilde{y}_t = \frac{Y_t}{L_t}$$

Equilibrium with population growth

- Taking logs and differentiating both sides of $\tilde{k}_t = K_t/L_t$ wrt time yields: details

$$
\begin{aligned}
\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} &= \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t} \\
&= \frac{sY_t - \delta K_t}{K_t} - n && \left(\text{using } \Delta K_{t+1} = sY_t - \delta K_t \text{ and } \Delta L_{t+1}/L_t = n\right) \\
&= s\frac{Y_t}{K_t} - (\delta + n) \\
&= s\frac{\tilde{y}_t L_t}{\tilde{k}_t L_t} - (\delta + n) && \left(\text{using } K_t = \tilde{k}_t L_t \text{ and } Y_t = \tilde{y}_t L_t\right) \\
&= s\frac{\tilde{y}(t)}{\tilde{k}(t)} - (\delta + n) && (15)
\end{aligned}
$$

Equilibrium with population growth

- Rearrange equation (15) to get law of motion in per capital terms:

$$\Delta \tilde{k}_{t+1} = s\tilde{y}_t - (\delta + n)\tilde{k}_t \tag{16}$$

- **Three remarks**:
  - Investment per worker ($s\tilde{y}$) increases capital per worker ($\tilde{k}$)
  - Depreciation (δ) and population growth (n) reduce capital per worker ($\tilde{k}$)
  - Previous version of Solow model nested here for special case $n = 0$
- Can solve model with these equations and definition of steady state:
  - Graphically
  - Analytically

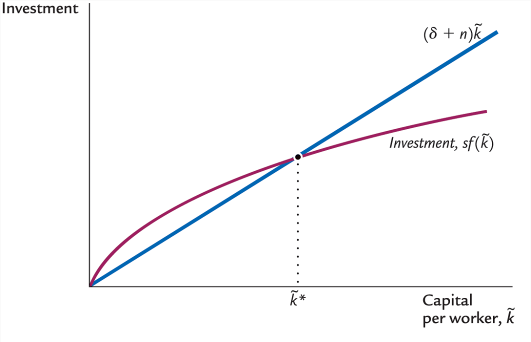
Equilibrium with population growth: Graphical solution

- In steady state, the capital-labor ratio is constant:

$$\Delta \tilde{k}_{t+1} = 0 \quad \Longrightarrow \quad s\tilde{y}^* = (\delta + n)\tilde{k}^*$$

- The steady-state level of investment per capita $(s\tilde{y}^*)$ is $(\delta + n)\tilde{k}^*$

  Steady-state investment makes up for depreciated capital and population growth



Equilibrium with population growth: Analytical solution

- In steady state, the capital-labor ratio is constant:

$$\Delta \tilde{k}_{t+1} = 0 \quad \Longrightarrow \quad \frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta + n}{s}$$

- With Cobb–Douglas production function:

$$\begin{aligned} \tilde{y}^* &= A{\tilde{k}^*}^{\alpha} \\ &= A\left(\frac{sA}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} \\ &= A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

$$\tilde{c}^* = (1 - s)\tilde{y}^*$$

Summary of Solow model without technological progress

- **Adding population growth hasn't changed basic features of the model**:
  - Unique and stable steady state
  - Simple comparative statics
- **New insights**:
  - **To replenish the capital-labor ratio, investment must be determined by considering** both depreciation of physical assets and **population growth**
  - **Richer countries have** higher savings/investment rates, higher levels of technology, and **lower** depreciation- and **population growth rates**
- **Same old problems.** Solow model can still not explain sustained growth
  - No per-capita growth once economy reaches steady state: Y grows, $\tilde{y}$ doesn't
  - Growth in per-capita terms slows down and eventually comes to a halt!

The growth slowdown

- Recall capital accumulation equation (15):

$$\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} = s\frac{\tilde{y}_t}{\tilde{k}_t} - (\delta + n)$$

- With Cobb-Douglas production function:

$$\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} = sA\tilde{k}_t^{\alpha-1} - (\delta + n)$$

- **Growth rate of capital-labor ratio $\tilde{k}$ declines over time as $\tilde{k}$ rises since $\alpha < 1$**

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{k}}\left(\frac{\Delta \tilde{k}}{\tilde{k}}\right) = (\alpha - 1)sA\tilde{k}^{\alpha-2} < 0$$

- Given that $\tilde{y}$ is proportional to $\tilde{k}$, $\tilde{y}_t = f(\tilde{k}_t)$, the same is true for $\tilde{y}$

The growth slowdown: Graphical representation

The further below (above) the economy is from the steady state $\tilde{k}^*$,
the faster the economy grows (de-grows)

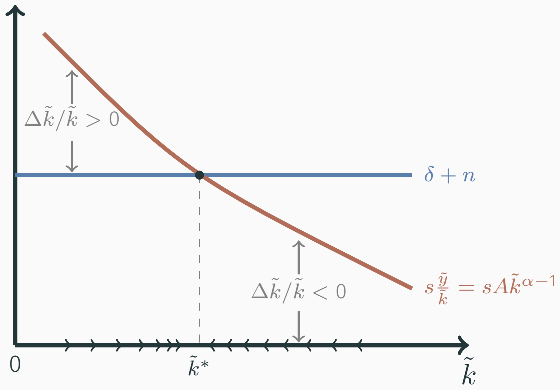


Figure 1: Transitional dynamics with Cobb–Douglas function and no tech. change

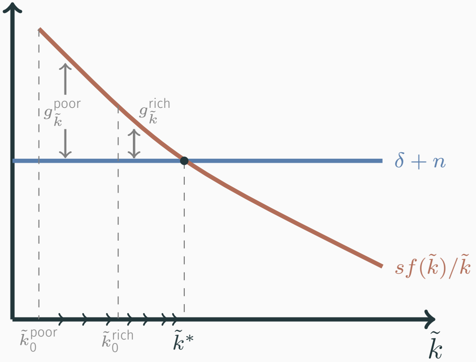
Convergence

- Using a Cobb–Douglas production function, we established that the further below an economy is from its steady state, the faster it grows
- **This result is also true for more general (neoclassical) production functions**: smaller values of $\tilde{k}$ associated with larger values of $\tilde{g}_k$
- **Does this result mean that economies with lower capital per worker tend to grow faster? Or, in other words, that there is convergence across countries?**
- It depends!
  - **If economies structurally similar** (ie, same values for s, n, δ, A and same technology f), **then yes**: they share same steady-state values $\tilde{k}^*$ and $\tilde{y}^*$
  - **If economies not structurally similar, they don't need to converge**: they have different steady-state values $\tilde{k}^*$ and $\tilde{y}^*$

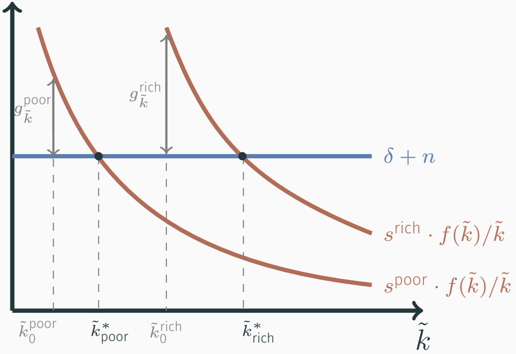
Conditional convergence

- **Consider two economies**—one rich, one poor—**that are structurally similar** (ie, same s, n, δ, A, f), **but have different initial conditions**: $\tilde{k}_0^{\text{rich}} > \tilde{k}_0^{\text{poor}} > 0$

- Model predicts less-advanced economy will exhibit higher growth rate $g_{\tilde{k}}$



- **Consider two economies**—one rich, one poor—**that aren't structurally similar** (in that $s^{\text{rich}} > s^{\text{poor}}$) **and have different initial conditions:** $\tilde{k}_0^{\text{rich}} > \tilde{k}_0^{\text{poor}} > 0$
- **Model predicts rich country grows further apart from poor country if rich country further away from its steady state than poor country**; ie, $g_{\tilde{k}}^{\text{rich}} > g_{\tilde{k}}^{\text{poor}}$



Conditional vs. absolute convergence

- Two types of convergence:
  - **Conditional convergence**: economies with lower starting values of capital-labor ratio will exhibit higher per-capita growth rates and will thereby tend to catch up with initially richer countries when economies are structurally similar
  - **Absolute convergence**: poorer countries tend to grow faster than richer ones even when they are not structurally similar
- Because absolute convergence is a less restrictive form of convergence, it is harder to observe it in practice
- Let's see if the data supports any of these notions of convergence!

Empirical evidence supports conditional convergence, not absolute convergence

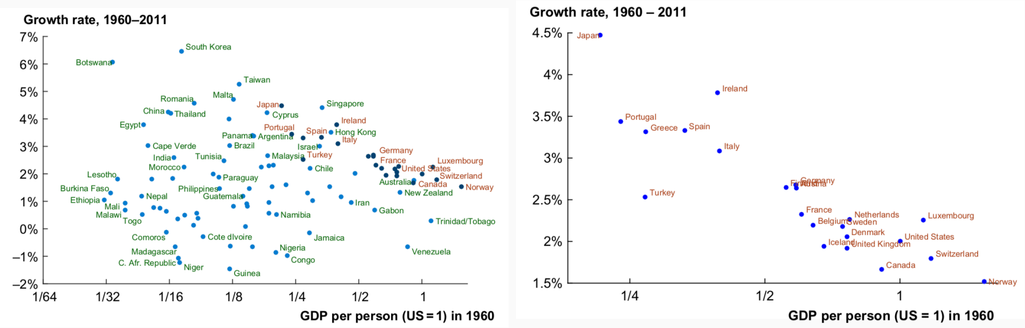


Figure 2: Convergence across countries, 1960–2011. All countries (left), OECD economies (right)

Empirical evidence supports conditional convergence

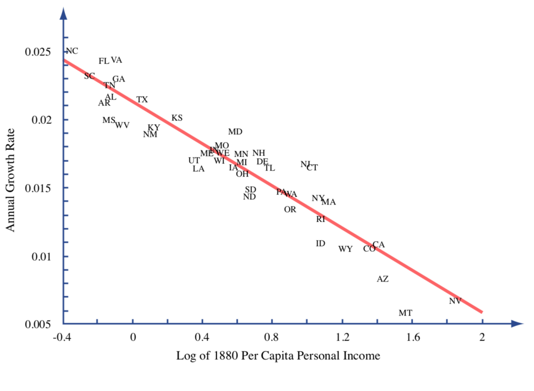


Figure 3: Convergence across US states, 1880–2000

Empirical evidence supports conditional convergence

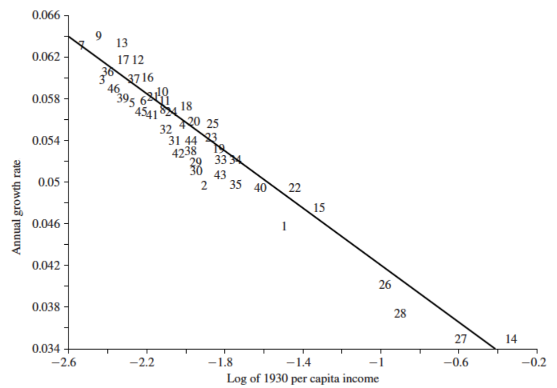


Figure 4: Convergence across Japanese prefectures, 1930–1990

Adding technological progress

- Adding pop. growth to basic model doesn't solve fundamental problem:
  - Model can still not account for sustained growth
    - Once steady state reached, output per capita $\tilde{y}^*$ doesn't grow
  - This is at odds with empirical evidence
    - With given inputs, society produces more output per capita today than in past: $\tilde{y}(\bar{\tilde{k}})_{2024} > \tilde{y}(\bar{\tilde{k}})_{1800}$
- To address this problem, we add technological progress to model

Adding technological progress

- Two tweaks to basic model:
  - Population growth: $L_{t+1} = (1 + n)L_t$, where $L_0 > 0$
  - Technological progress: $A_{t+1} = (1 + g_A)A_t$, where $A_0 > 0$
- **Importantly, we treat tech. progress as exogenous ("manna from heaven"):** economic agents cannot influence it!
  - Again, this is at odds with reality: technology developed with R&D
  - But it is a useful first step
  - More advanced models feature endogenous tech. progress
    (Romer 1990, Gorssman–Helpman 1991, Aghion–Howitt 1992, Jones 1995, Acemoglu 2002, ...)
  - But these models are beyond our scope

Adding technological progress: BGP and Harrod-neutral tech. progress

- How to introduce technology A into production function F?
  - **Three options**: Hicks-neutral, Solow-neutral, Harrod-neutral tech. progress
  - Standard approach is to choose one that generates **balanced growth path (BGP)**
  - **BGP**: allocation where output grows at a constant rate and capital-output ratio, interest rate, and factor shares remain constant (in line with Kaldor facts)
- **Important result** (due to Uzawa 1961) **establishes that only Harrod-neutral or labor-augmenting technological progress can generate balanced growth**
- Thus, **we now consider production functions**:

$$Y_t = F\left[K_t, A_t L_t\right]$$

Equilibrium with population growth and technological progress

- Capital's fundamental law of motion:

$$\Delta K_{t+1} = sF\left[K_t, A_t L_t\right] - \delta K_t$$

- **Convenient to analyze economy in "effective" or "efficiency" units of labor.**
  Capital-labor ratio in effective units (using **hat notation**):

$$\hat{k}_t := \frac{K_t}{A_t L_t}$$

- Taking logs, differentiating wrt time, and approximating: details

$$\frac{\Delta \hat{k}_{t+1}}{\hat{k}_t} = \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta A_{t+1}}{A_t} - \frac{\Delta L_{t+1}}{L_t}$$
$$= \frac{\Delta K_{t+1}}{K_t} - g_A - n$$

Equilibrium with population growth and technological progress

- Output per effective unit of labor:

$$\hat{y}_t := \frac{Y_t}{A_t L_t} = F\left[\frac{K_t}{A_t L_t}, 1\right] \equiv f(\hat{k}_t)$$

- Income per capita:

$$\tilde{y}_t := \frac{Y_t}{L_t} = A_t \hat{y}_t = A_t f(\hat{k}_t)$$

- Now, even if $\hat{y}_t$ constant, income per capita $\tilde{y}_t$ grows because A_t grows!

Equilibrium with population growth and technological progress

- With technological progress, we no longer look for steady state but for ***balanced growth path***, where income per capita grows at constant rate
  - Transformed variables $\hat{y}_t$, $\hat{k}_t$ remain constant
  - So BGP can be thought of as steady state of transformed model
  - This explains why in such models economists use “BGP” and “steady state” interchangeably

- Back to our expression:

$$\begin{aligned}
\frac{\Delta \hat{k}_{t+1}}{\hat{k}_t} &= \frac{\Delta K_{t+1}}{K_t} - g_A - n \\
&= \frac{sY_t - \delta K_t}{K_t} - g_A - n && \text{(substituting } \Delta K_{t+1}\text{)} \\
&= s\frac{Y(t)}{K(t)} - (\delta + g_A + n)
\end{aligned}$$

Equilibrium with population growth and technological progress

- We can now use $\hat{k} \equiv K/(AL)$ to write:

$$\frac{\Delta \hat{k}_{t+1}}{\hat{k}_t} = s\frac{Y_t}{\hat{k}_t A_t L_t} - (\delta + g_A + n)$$

$$= s\frac{\hat{y}(t)}{\hat{k}(t)} - (\delta + g_A + n) \qquad \text{(using } \tilde{y} = Y/(AL)\text{)}$$

- We can now write the law of motion for capital in effective units:

$$\Delta \hat{k}_{t+1} = s\hat{y}_t - (\delta + g_A + n)\hat{k}_t \tag{17}$$

- **Three remarks**:
  - Investment in effective units ($s\hat{y}$) increases effective capital per worker ($\hat{k}$)
  - Depreciation (δ), technological progress (g_A), and population growth (n) reduce effective capital per worker ($\hat{k}$)
  - Previous model versions nested here when $g_A = 0$ and/or $n = 0$

Equilibrium with population growth and technological progress

- Law of motion for capital in effective units:

$$\Delta\hat{k}_{t+1} = s\hat{y}_t - (\delta + g_A + n)\hat{k}_t$$

- A steady state or BGP is now defined as an equilibrium in which the effective capital-labor ratio, $\hat{k}_t$ is constant overtime—that is, $\Delta\hat{k}_{t+1} = 0$

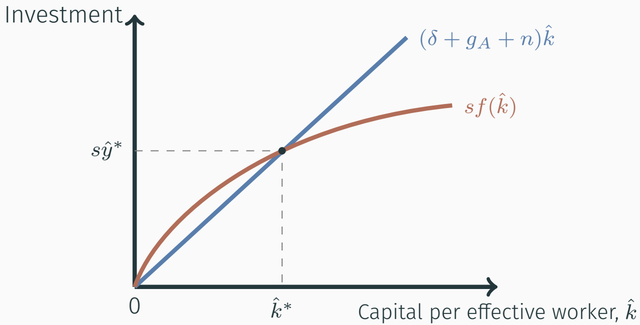
- Let's solve the model:
  - Graphically
  - Analytically

- In steady state:

$$\Delta \hat{k}_{t+1} = 0 \qquad \Longrightarrow \qquad s\hat{y}^* = (\delta + g_A + n)\hat{k}^*$$

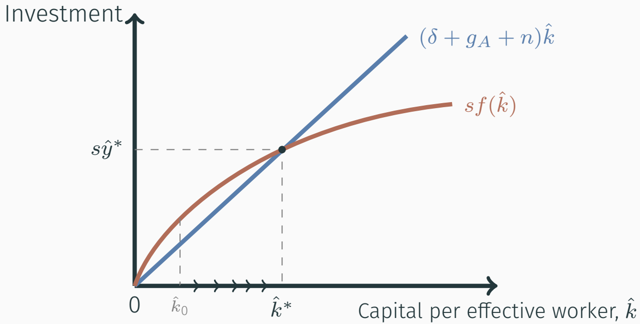
- The steady-state level of investment in effective units ($s\hat{y}^*$) is $(\delta + g_A + n)\hat{k}^*$

  Steady-state investment makes up for depreciated capital, pop. growth, and tech. progress



Equilibrium with population growth and technological progress

- **Transitional dynamics**: If economy is below (above) its steady state, the effective capital-labor ratio will increase (decline) until $\hat{k}^*$ is reached
- Once steady-state effective capital-labor ratio $\hat{k}^*$ is reached, economy grows along balanced growth path



Example: Cobb–Douglas production

- With Cobb–Douglas production $\Delta\hat{k}_{t+1} = 0$ yields:

$$s\hat{y}^* = (\delta + g_A + n)\hat{k}^* \qquad \Longleftrightarrow \qquad s(\hat{k}^*)^\alpha = (\delta + g_A + n)\hat{k}^*$$

- Solving for $\hat{k}^*$:

$$\hat{k}^* = \left(\frac{s}{\delta + g_A + n}\right)^{\frac{1}{1-\alpha}}$$

- Substituting $\hat{k}^*$ into $\hat{y}^*$:

$$\hat{y}^* = \left(\frac{s}{\delta + g_A + n}\right)^{\frac{\alpha}{1-\alpha}}$$

Example: Cobb–Douglas production

- Output per capita along BGP

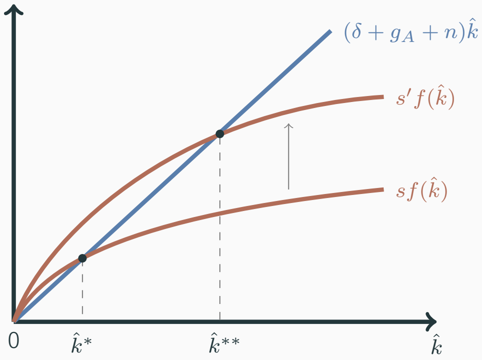
$$
\begin{aligned}
\tilde{y}_t^* &= A_t \hat{y}^* \\
&= A_t \left(\frac{s}{\delta + g_A + n} \right)^{\frac{\alpha}{1-\alpha}} \\
&= (1 + g_A)^t A_0 \times \left(\frac{s}{\delta + g_A + n} \right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

depends on initial level of technology A_0 and time t, as well as on savings rate s, depreciation rate δ, rate of tech. progress g_A, and rate of pop. growth n

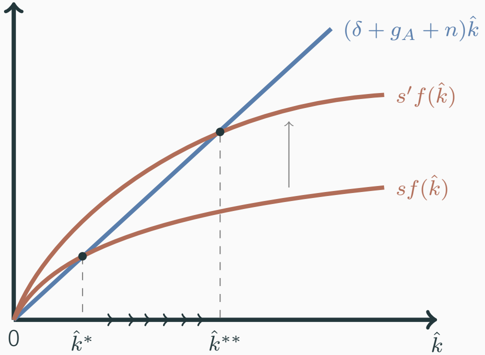
- **Level vs. growth effects**: Changes in investment, depreciation, or pop. growth rates affect long-run level of output per capita, not its long-run growth rate

Comparative dynamics

- **Shocks to the investment rate**: an increase (decrease) from s to s' moves the economy to a higher (lower) steady state $\hat{k}^{**}$



- **Shocks to the investment rate**: an increase (decrease) from s to s' moves the economy to a higher (lower) steady state $\hat{k}^{**}$
  - At initial $\hat{k}^*$, investment exceeds amount needed to keep $\hat{k}^*$ constant, so $\hat{k}$ rises
  - Increase in s raises the growth rate *temporarily* (along the transition to $\hat{k}^{**}$)



Comparative dynamics

- Prior to increase in savings rate s, output per worker grew at rate g_A
- When savings rate increases at t^*, output per capita $\tilde{y}$ grows more rapidly until economy reaches new steady state, when growth rate returns to g_A

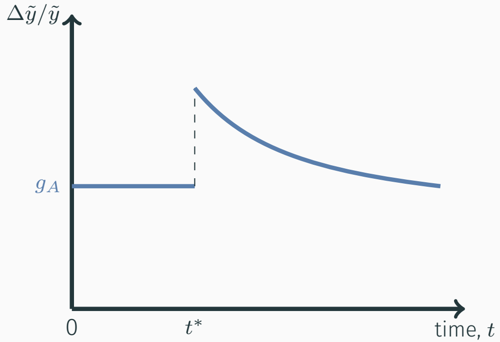


Figure 5: Effect of an increase in the savings rate on growth rate of output per capita

- Policy changes don't have long-run growth effects, only level effects

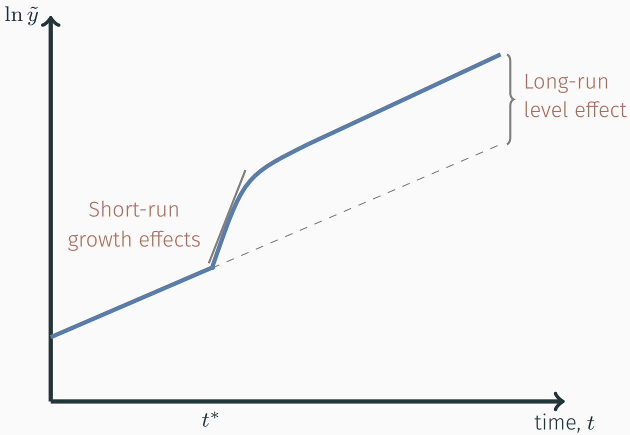
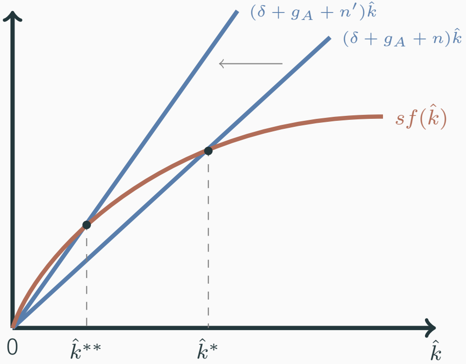


Figure 6: Level effects in the Solow model

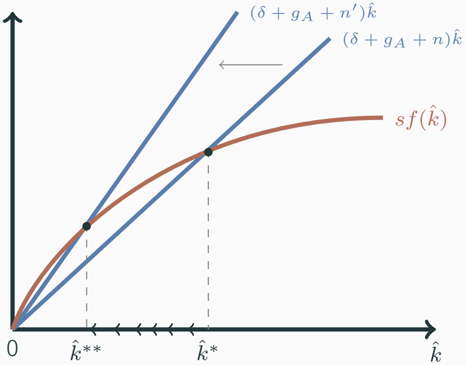
Comparative dynamics

- **Shocks to the population growth rate**: an increase (decrease) from n to n' moves the economy to a lower (higher) steady state $\hat{k}^{**}$



Comparative dynamics

- **Shocks to the population growth rate**: an increase (decrease) from n to n' moves the economy to a lower (higher) steady state $\hat{k}^{**}$
  - At initial $\hat{k}^*$, investment is too low to keep $\hat{k}^*$ constant, so $\hat{k}$ declines
  - Increase in n lowers the growth rate *temporarily* (along the transition to $\hat{k}^{**}$)



Summary of Solow model with population growth and technological progress

- **With tech. progress, Solow's model can account for sustained growth**
- Key takeaways:
  1. Capital accumulation determined by savings rate (s), depreciation rate (δ), rate of technological progress (g_A), and population growth rate (n)
  2. Richer countries ($\uparrow \tilde{y}$) have higher savings rates ($\uparrow s$), better technology ($\uparrow A_0$), more tech. progress ($\uparrow g_A$), lower depreciation ($\downarrow \delta$), lower pop growth rate ($\downarrow n$)
  3. Gvt policy may have long-run level effects, but no long-run growth effects (eg, increase in savings rate, reduction in pop growth rate, ...)
- **Main problem of Solow model is that all key variables** (savings rate, depreciation rate, pop growth rate, rate of tech. progress) **are exogenous**

Beyond Solow’s Model

Other perspectives on population growth

- According to the Solow model, population growth is bad for econ growth: it reduces output per worker by leading to lower capital-labor ratios

- Other perspectives:
    - Malthus (1798) highlighted interaction of population with natural resources
        - Argued population grows geometrically while means of subsistence linearly
        - Result is scarcity and famine: not enough food to feed people
        - Perceived technological progress as temporary and unsustainable: population would grow in response to it, keeping people in a poverty trap!
        - As later in Solow's model, population growth is bad for economic growth

Other perspectives on population growth

- According to the Solow model, population growth is bad for econ growth: it reduces output per worker by leading to lower capital-labor ratios

- Other perspectives:
  - Romer (1990), Kremer (1993), and others highlighted interaction of population with technology
    - Basic idea is that technological progress depends on technological breakthroughs
    - More people means more potential inventors (ie, 1 Elon Musk per million)
    - The larger the population, the faster technology advances and economy grows
    - Kremer provides evidence in favor of this, using data from 1 million BC to 1990
    - Contrary to Solow's model, population growth is great for economic growth

Augmenting the Solow model

- **Solow model can be augmented in multiple ways**

  (adding human capital, a role for the government, international trade, ...)

- Mankiw, Romer, and Weil (1992) emphasized the role of human capital:
  - Different countries have different levels of education, skills, know-how, ...
  - They extended production function to accommodate this: $Y = F[K, H, AL]$
  - Model highlights human capital investments as growth-enhancing mechanism
  - Augmented model:
    - Fits data much better than original Solow model
    - Suggests 70% of cross-country income differences due to differences in physical and human capital

Insights from modern growth models

New growth theory:

- Endogenizes technological progress and emphasizes innovation
  - **Product-variety models**: innovation causes productivity growth by creating new—not necessarily improved—varieties (eg, Romer 1990, Jones 1995)
  - **Schumpeterian-growth models**: innovation leads to creative destruction and growth (eg, Aghion and Howitt 1992, and co-authors)
- Thinks of countries as parts of a whole rather than isolated units
  - **Technology adoption and skill mismatch** (Acemoglu and Zilibotti 2001):
    - Technologies need to be adapted to local environments
    - Inappropriateness of technology due to climate or skill mismatch
  - **International trade**: recognizes role of FDI and imports/exports
- Studies whether technical change is biased towards particular factors of production, incorporates climate change considerations, ...

Taking Stock

Taking stock

- **Solow's model** is one of the first workhorse models in the growth literature; it **helps us to understand the *mechanics* of growth**
- **Solow's model key features**:
  - **Revolves around neoclassical production function**
  - **Blends Keynesian** (behavioral rules) **& neoclassical ingredients** (optimizing behavior)
  - Allows comparative statics and dynamics
  - **Can account for convergence** (conditional vs. absolute) **and divergence**
  - **Has limited role for government intervention**
  - **Can be augmented to incorporate relevant factors** (eg, human capital)
  - **Allows to bridge theory with empirics** (eg, growth accounting)

Taking stock

- It sheds light on importance of saving/investment rates, population growth, human capital, and technology differences
- Solow model is not entirely satisfactory:
    - Most important variables are exogenous
    - It emphasizes the proximate causes of growth ...
    - ... but to say that a country is poor because it has little capital and inefficient technology is like saying that a person is poor because it has no money!
    - There are factors that make a country to have more physical- and human capital and more efficient technologies—as there are factors that make a person to have more money than others
- Next, we study fundamental causes of growth and take further look at data

Questions?

Thank You!

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Derivation: Way 1

- Taking logs of $\tilde{k}_t = K_t/L_t$:

$$\ln \tilde{k}_t = \ln K_t - \ln L_t$$

- Differentiating both sides with respect to time:

$$\begin{aligned}
& \frac{\mathrm{d} \ln \tilde{k}_t}{\mathrm{d}t} = \frac{\mathrm{d} \ln K_t}{\mathrm{d}t} - \frac{\mathrm{d} \ln L_t}{\mathrm{d}t} \\
\iff \quad & \frac{\mathrm{d} \ln \tilde{k}_t}{\mathrm{d}\tilde{k}_t} \frac{\mathrm{d}\tilde{k}_t}{\mathrm{d}t} = \frac{\mathrm{d} \ln K_t}{\mathrm{d}K_t} \frac{\mathrm{d}K_t}{\mathrm{d}t} - \frac{\mathrm{d} \ln L_t}{\mathrm{d}L_t} \frac{\mathrm{d}L_t}{\mathrm{d}t} \\
\iff \quad & \frac{\dot{\tilde{k}}_t}{\tilde{k}_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{\tilde{L}_t} \qquad \text{(where } \dot{x}_t = \mathrm{d}x/\mathrm{d}t\text{)}
\end{aligned}$$

- Discrete time approximation (ie, $\dot{x}_t \approx \Delta x_t$) of above equation:

$$\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} = \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t}$$

Derivation: Way 2

- An alternative way to reach equation

$$\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} = \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t}$$

  is to learn the “trick” we saw in class and apply it

- This trick is that the percentage change of a ratio is approximately the percentage change of the numerator minus the percentage change of the denominator

- Applying the trick to $\tilde{k}_t = K_t/L_t$, we get

$$\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} = \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t}$$

Back

Derivation

- **Way 1**: Taking logs of $\hat{k}_t = \frac{K_t}{A_t L_t}$, differentiating wrt to time, and approximating works exactly as before (only that we have to take care of an extra term b/c of A_t)

- **Way 2**: With tech. growth also in the denominator of $\hat{k}_t = \frac{K_t}{A_t L_t}$, we use two tricks:
  1. The percentage change of a ratio is approx. the percentage change of the numerator minus the percentage change of the denominator
  2. The percentage change of a product is approx. the sum of percentage changes

- Applying these tricks to $\hat{k}_t = K_t/(A_t L_t)$, we get

$$\begin{aligned}\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} &= \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta (A_{t+1} L_{t+1})}{A_t L_t} \\ &= \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta A_{t+1}}{A_t} - \frac{\Delta L_{t+1}}{L_t}\end{aligned}$$

Back