



ECO 3302 – Intermediate Macroeconomics

Lecture 7: The Solow Model

Luis Pérez

(luisperez@smu.edu)

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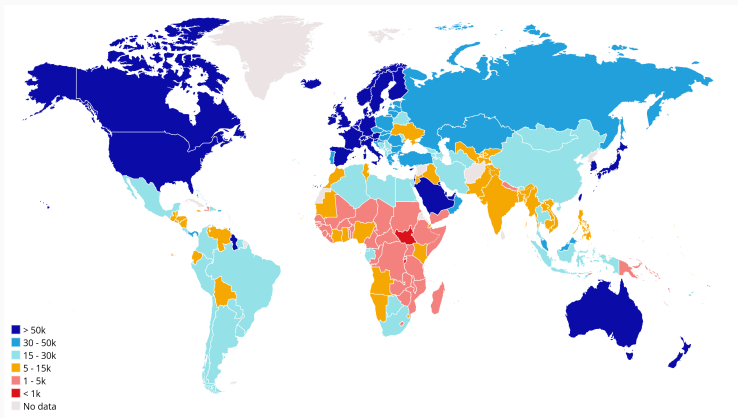
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Introduction to the Solow model

Introduction

- ▶ **Last lecture:** learned to make cross-country income comparisons
- ▶ **Today:** try to understand why some countries much richer than others



► Our background:

- Saw that production factors and technology determine an economy's income:

$$Y = F(K, L; A)$$

- Differences in income across countries and over time come from differences in capital, labor, and technology
- **Static focus:** given technology and factors, study output at given point in time

► Today and next lectures, our focus becomes dynamic: explain why GDP grows over time and why it grows much faster for some countries

- We do this with **Solow growth model**

Environment and Assumptions

Environment: Households

- ▶ Closed economy with single good
- ▶ Discrete time running to an infinite horizon ($t = 0, 1, 2, \dots$)
- ▶ Large number of households that will not be optimizing!
 - ▶ Main difference between Solow and Neoclassical growth models
- ▶ To simplify analysis, assume:
 - All households identical \rightarrow economy admits **representative household**
 - **Households save constant exogenous fraction $s \in (0, 1)$ of disposable income**
 - This is the no-optimizing part on households side
(ie, households don't decide how much to save)
 - It both simplifies and limits analysis
(eg, can't study effects of tax increase on savings and growth)
 - **Households own all factors of production and supply labor inelastically**

- ▶ Assume *all* firms have same production function → **representative firm**

- ▶ **Aggregate production function:**

$$Y_t = F[K_t, L_t, A_t] \quad (1)$$

- Y : final good (think of it as units of real GDP)
- K : capital stock (ie, machines, buildings,...)
- L : labor (eg, population size, labor force size, total hours worked, ...)
- A : total factor productivity (TFP)

- ▶ A is “efficiency” shifter of production function: $\uparrow A \implies \uparrow Y$ with given K, L

A1: Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale

The production function $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is twice continuously differentiable in K and L , and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0$$

Moreover, F exhibits constant returns to scale (CRS) in K and L :

$$F(\lambda K, \lambda L) = \lambda Y, \quad \text{for any } \lambda > 0$$

A2: Inada Conditions

F satisfies:

$$F(0, L, A) = 0 \quad \text{and} \quad F(K, 0, A) = 0 \quad \text{for } K, L, A > 0 \quad (\text{essential inputs})$$

$$\lim_{K \rightarrow 0} F_K(K, L, A) = \infty \quad \text{and} \quad \lim_{K \rightarrow \infty} F_K(K, L, A) = 0, \quad \text{for all } L, A > 0$$

$$\lim_{L \rightarrow 0} F_L(K, L, A) = \infty \quad \text{and} \quad \lim_{L \rightarrow \infty} F_L(K, L, A) = 0, \quad \text{for all } K, A > 0$$

- ▶ Inada (boundary) conditions ensure the existence of *interior equilibria*
- ▶ Assumptions A1 and A2 are “**neoclassical technology assumptions**”
 - Give rise to neoclassical production function

Environment: Market structure, endowments, market clearing

- ▶ Assume **competitive markets and market clearing**

(ie, agents are price-takers and prices clear markets)

- ▶ **Households own all labor and supply it inelastically**

(ie, all labor put to work unless its rental price is 0!)

- ▶ Endowment of labor assumed equal to population size, \bar{L}_t

- ▶ Labor market clearing condition given by

$$L_t = \bar{L}_t, \quad \text{for } t = 0, 1, 2, \dots$$

- ▶ Rental price of labor, the *wage rate*, denoted W_t . Then it must be satisfied,

$$L_t \leq \bar{L}_t, \quad W_t \geq 0, \quad \text{and} \quad [L_t - \bar{L}_t] W_t = 0$$

Employed labor must be lower than population size, wage must be non-negative, and either the labor market clears or the wage is zero

Environment: Market structure, endowments, market clearing

- ▶ Households own all capital and rent it to firms at rental rate R_t

- ▶ Capital market clearing condition given by

$$K_t = \overline{K}_t, \quad \text{for } t = 0, 1, 2, \dots$$

- ▶ We take initial capital endowments $K_0 \geq 0$ as given

- ▶ Capital depreciates at exponential rate $\delta \in (0, 1)$

- 1 unit of capital today is equivalent to $1 - \delta$ units tomorrow

- ▶ Loss of capital affects interest rates (return to savings of households):

$$r_t = R_t - \delta$$

Environment: Firm optimization

- Problem of the representative firm is to choose K, L to maximize profits:

$$\begin{aligned} \max_{K_t \geq 0, L_t \geq 0} \quad & \Pi_t \equiv \underbrace{P_t Y_t}_{\text{revenues}} - \underbrace{W_t L_t + R_t K_t}_{\text{costs}} \\ \text{s.t.} \quad & Y_t = F[K_t, L_t, A_t] \end{aligned}$$

- Important to notice:

1. Maximization problem imposes competitive markets
(firms take factor prices W_t and R_t as given)
2. Can normalize price of final good, $P_t = 1$, in all periods
(final output is the *numeraire*)
3. Can substitute Y into the profits equation Π

Firm optimization

Hence, can write:

$$\max_{K_t \geq 0, L_t \geq 0} \Pi_t \equiv \underbrace{F[K_t, L_t, A_t]}_{\text{revenues}} - \underbrace{W_t L_t - R_t K_t}_{\text{costs}}$$

- Given properties of F (A1–A2), we can take **FOCs to obtain (interior) solution**:

$$\frac{\partial \Pi_t}{\partial K_t} = 0 \quad \implies \quad R_t = F_K[K_t, L_t, A_t] \quad (2)$$

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \quad \implies \quad W_t = F_L[K_t, L_t, A_t] \quad (3)$$

With competitive markets, rental rates equal to marginal (revenue) products

- Solving for K_t and L_t we can derive demand for capital and labor

Fundamental law of motion

- ▶ K depreciates exponentially at rate δ and grows with investment $I \geq 0$.
Capital's law of motion:

$$\underbrace{K_{t+1}}_{\text{capital tomorrow}} = \underbrace{(1 - \delta)K_t}_{\text{today's undepreciated capital}} + \underbrace{I_t}_{\text{today's investment}} \quad (4)$$

- ▶ In closed economy, abstracting from the government:

$$Y_t = C_t + I_t, \quad (5)$$

where C_t denotes consumption.

- ▶ Whatever is not consumed is invested:

$$S_t = I_t = Y_t - C_t$$

Fundamental law of motion

- ▶ Solow's behavioral rule: HHs save constant fraction $s \in (0, 1)$ of income:

$$S_t = I_t = sY_t \quad (6)$$

$$C_t = (1 - s)Y_t \quad (7)$$

- ▶ Equation (4) can be rewritten as

$$\begin{aligned} K_{t+1} &= (1 - \delta)K_t + sY_t \\ &= (1 - \delta)K_t + sF[K_t, L_t, A_t] \end{aligned} \quad (8)$$

- ▶ This equation is the *fundamental law of motion* in Solow's growth model
- ▶ Equation (8) together with the laws of motion of L_t and A_t describe the equilibrium in the Solow growth model

Equilibrium

Definition of Equilibrium

- ▶ Solow model combines features of Keynesian models (ie, behavioral rules) and modern macro approaches (ie, optimization & market clearing)

Definition of Equilibrium

In the basic Solow model for a given sequence of $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial capital stock K_0 , an equilibrium path is a sequence of capital stocks, output levels, consumption levels, rental rates and wages $\{K_t, Y_t, C_t, R_t, W_t\}_{t=0}^{\infty}$ such that K_t satisfies equation (8), Y_t is given by equation (1), C_t is given by equation (7), and R_t and W_t are given by equations (2) and (3), respectively.

- ▶ Equilibrium is an “entire path” of allocations and prices, not a static object!

Theoretical Analysis

National accounting in the Solow model

- ▶ **Remember Euler's theorem?** With it and FOCs from firm's problem, we can now establish our first result!

Simplified version of Euler's Theorem

Suppose A1 holds and markets are competitive. Then, in the equilibrium of the Solow model, **firms make no profits and the following equation holds:**

$$Y_t = w_t L_t + R_t K_t \quad (9)$$

- ▶ **Factor shares** obtained dividing both sides of (9) by Y_t :

$$\Lambda_{Lt} + \Lambda_{Kt} = 1 \implies \Lambda_{Kt} = 1 - \Lambda_{Lt}$$

Equilibrium without population growth and technological progress

► Two simplifying assumptions:

1. No population growth: $L_t = L > 0$ for all $t = 0, 1, 2, \dots$
2. No technological progress: $A_t = A > 0$ for all $t = 0, 1, 2, \dots$

► Study economy in “per capita” terms. Using tildes to denote per-capita vars:

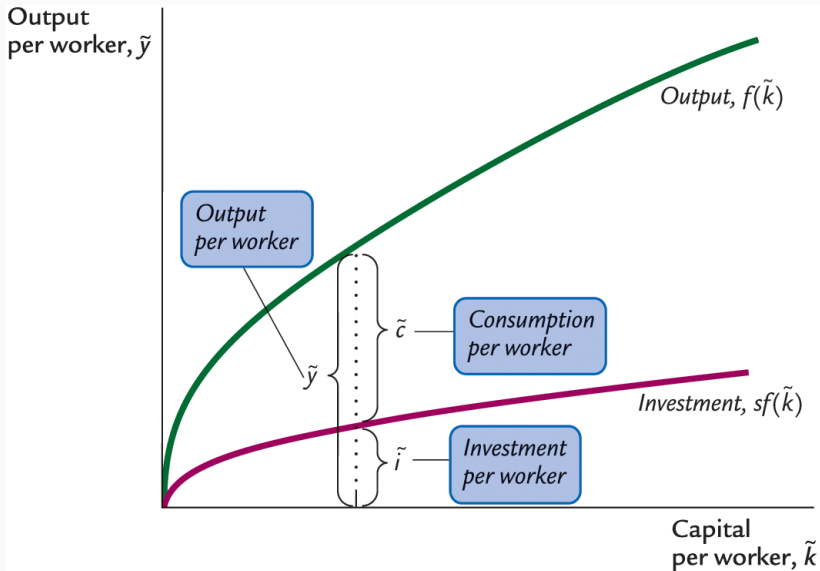
$$\text{Capital-labor ratio : } \tilde{k}_t := \frac{K_t}{L}$$

$$\begin{aligned} \text{Output per capita : } \tilde{y}_t &:= Y_t / L \\ &= F \left[\frac{K_t}{L}, 1, A \right] && \text{(by CRS)} \\ &= f(\tilde{k}_t, A) \end{aligned}$$

Since there is no technological progress, A is constant and can be omitted.

I.e., $\tilde{y}_t = f(\tilde{k}_t)$ (Capital-labor ratio entirely determines output per capita in this economy)

Graphical representation: per-capita production function and investment



Equilibrium without population growth and technological progress

- ▶ Assume Cobb–Douglas production function:

$$Y_t = AK_t^\alpha L^{1-\alpha}, \quad \alpha \in (0, 1)$$

(Clearly satisfies neoclassical technology assumptions A1–A2)

- ▶ Expressing output in per capita terms:

$$\tilde{y}_t = A\tilde{k}_t^\alpha$$

- ▶ FOC wrt \tilde{k} yields rental rate of capital:

$$R_t = \frac{d\tilde{y}_t}{d\tilde{k}_t} = \alpha A\tilde{k}_t^{\alpha-1}$$

- ▶ Wage obtained applying Euler's theorem:

$$\begin{aligned} w_t &= \tilde{y}_t - R_t\tilde{k}_t \\ &= (1 - \alpha)A\tilde{k}_t^\alpha \end{aligned}$$

Equilibrium without population growth and technological progress

- ▶ Same results with original production function
- ▶ Rental rate of capital:

$$\begin{aligned}R_t &= \frac{\partial Y_t}{\partial K_t} = \alpha A K_t^{\alpha-1} L^{1-\alpha} \\&= \alpha A (\tilde{k}_t L)^{\alpha-1} L^{1-\alpha} && \text{(using } K_t = \tilde{k}_t L\text{)} \\&= \alpha A \tilde{k}_t^{\alpha-1}\end{aligned}$$

- ▶ Wage:

$$\begin{aligned}w_t &= \frac{\partial Y_t}{\partial L} = (1 - \alpha) A K_t^{\alpha} L^{-\alpha} \\&= (1 - \alpha) A \tilde{k}_t^{\alpha}\end{aligned}$$

(Can verify Euler's theorem using these w_t and R_t)

Equilibrium without population growth and technological progress

- ▶ Recall fundamental role of capital: $\tilde{y}_t = f(\tilde{k}_t)$
- ▶ Can also rewrite capital's law of motion in per-capita terms:

$$\begin{aligned} K_{t+1} &= (1 - \delta)K_t + sF(K_t, L, A) \\ \iff \frac{K_{t+1}}{L} &= (1 - \delta)\frac{K_t}{L} + s\frac{F(K_t, L, A)}{L} \\ \iff \tilde{k}_{t+1} &= (1 - \delta)\tilde{k}_t + sf(\tilde{k}_t) \end{aligned} \tag{10}$$

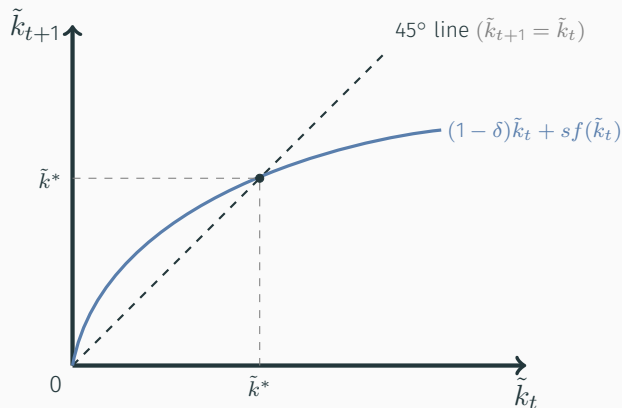
Definition of steady-state equilibrium

A **steady-state equilibrium without technological progress & population growth** is an equilibrium path in which $\tilde{k}_t = \tilde{k}^*$ for all t (ie, constant capital-labor ratio)

- ▶ Economy will approach steady-state (ss) equilibrium over time

Finding the steady-state capital per worker

- ▶ In steady state, capital per worker doesn't grow: $\tilde{k}(t) = \tilde{k}^*$ for all t
- ▶ Can find steady-state capital per worker \tilde{k}^* using capital's law of motion



Finding the steady-state capital per worker

- ▶ In steady state, capital per worker doesn't grow: $\tilde{k}(t) = \tilde{k}^*$ for all t
- ▶ Can find steady-state capital per worker k^* using capital's law of motion:

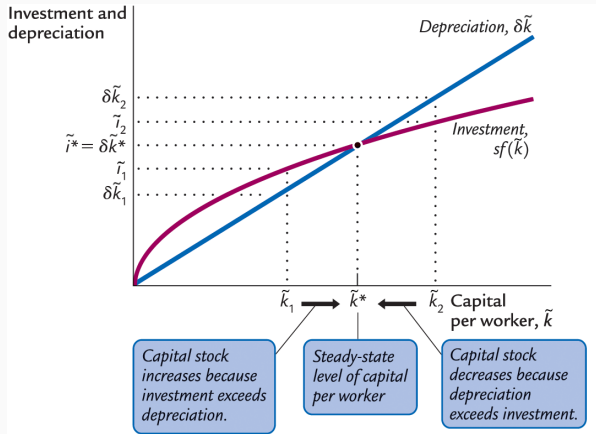
$$\tilde{k}^* = (1 - \delta)\tilde{k}^* + sf(\tilde{k}^*)$$

- ▶ Solution is a fixed point to this equation. **Two candidates:**
 1. $\tilde{k}^* = 0$ has no economic interest (we rule it out by assumption $\tilde{k}_0 > 0$)
 2. $\tilde{k}^* > 0$: positive interesection of $(1 - \delta)\tilde{k} + sf(\tilde{k})$ curve with 45° line
- ▶ Solution characterization:

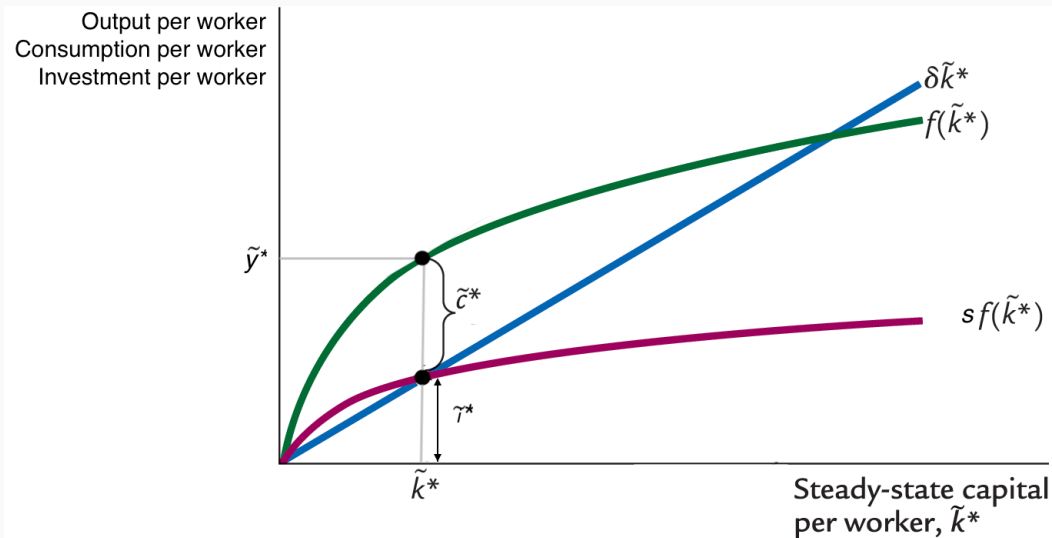
$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta}{s} \quad \Longleftrightarrow \quad \underbrace{sf(\tilde{k}^*)}_{\text{ss investment}} = \underbrace{\delta\tilde{k}^*}_{\text{ss depreciation}}$$

Steady-state capital per worker, investment, and depreciation

In steady state, capital per worker doesn't grow: $\tilde{k}(t) = \tilde{k}^*$ for all t
This only happens when investment ($\tilde{i} = sf(\tilde{k})$) equals depreciation ($\delta\tilde{k}$)



Steady-state equilibrium without technological progress and population growth



Equilibrium characterization

Equilibrium characterization

Under assumptions $A1$ and $A2$, there exists a unique steady-state equilibrium in the Solow model where the capital-labor ratio $\tilde{0} < \tilde{k}^* < \infty$ satisfies

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta}{s}$$

output per capita is

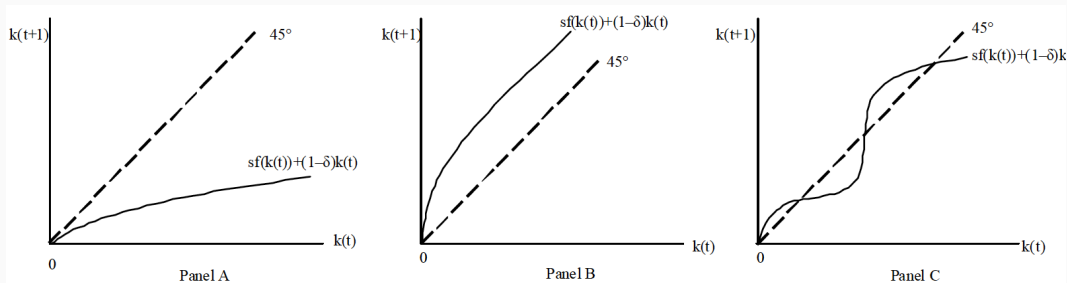
$$\tilde{y}^* = f(\tilde{k}^*) \quad (11)$$

and consumption per capita is

$$\tilde{c}^* = (1 - s)f(\tilde{k}^*) \quad (12)$$

- Existence and uniqueness of equilibrium guaranteed by assumptions $A1$ – $A2$

Examples where existence and uniqueness of equilibrium fails



- ▶ Panels A and B: No equilibrium with $\tilde{k}^* > 0$ (violate A2, Inada conditions)
- ▶ Panel C: Multiple equilibria where $\tilde{k}^* > 0$ (violates A1, decreasing MPs)

- ▶ Countries with higher saving rates ($\uparrow s$) and better technologies ($\uparrow A$) will have higher capital-labor ratios ($\uparrow \tilde{k}^*$) and will be richer ($\uparrow \tilde{y}^*$)
- ▶ Countries with greater technological depreciation ($\uparrow \delta$) will have lower capital-labor ratios ($\downarrow \tilde{k}^*$) and will be poorer ($\downarrow \tilde{y}^*$)
- ▶ The same is true for consumption \tilde{c}^* (since it is a linear function of output per worker)

The golden rule

► There is unique saving rate, s_{gold} , that maximizes steady-state consumption

- This savings rate found by taking the derivative of \tilde{c}^* wrt s
- **Step 1.** Write steady-state \tilde{c}^* as function of s :

$$\begin{aligned}\tilde{c}^* &= (1 - s)f(\tilde{k}^*(s)) \\ &= f(\tilde{k}^*(s)) - \delta\tilde{k}^*(s) \quad (\text{using } sf(\tilde{k}^*) = \delta\tilde{k}^*)\end{aligned}$$

- **Step 2.** Differentiate wrt s :

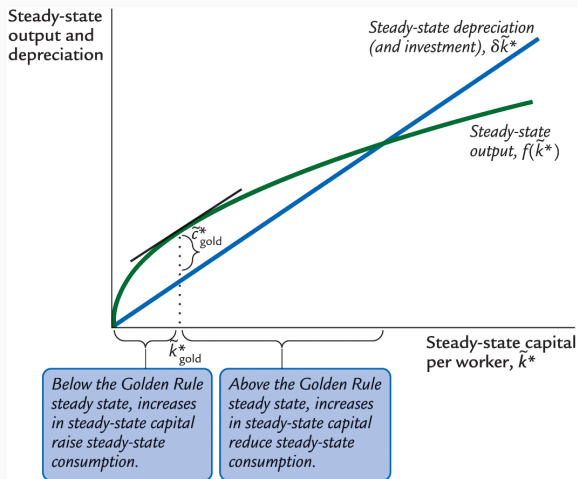
$$\frac{\partial \tilde{c}^*(s)}{\partial s} = \left[f'(\tilde{k}^*(s)) - \delta \right] \frac{\partial \tilde{k}^*}{\partial s}$$

► The golden rule of savings states that the saving rate s_{gold} must be such that

$$\frac{\partial c^*(s_{\text{gold}})}{\partial s} = 0$$

The golden rule

The golden-rule savings rate maximizes steady-state consumption



The golden rule

- ▶ The golden rule savings rate s_{gold} is such that, at that rate, steady-state consumption is maximized. That is,

$$\frac{\partial c^*(s_{\text{gold}})}{\partial s} = 0$$

- ▶ Following result follows:

Result

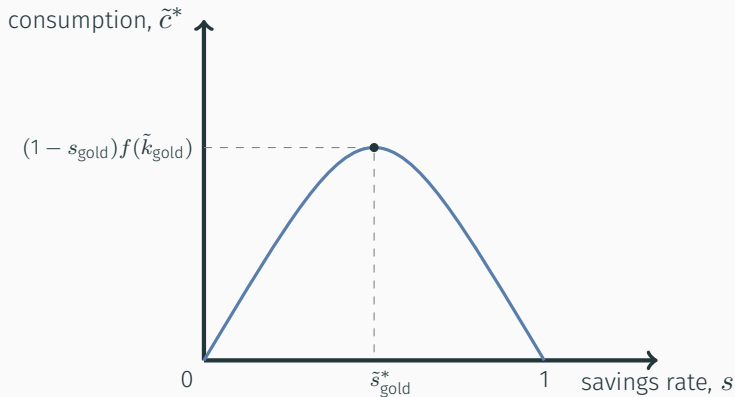
In the basic Solow growth model, the highest level of steady-state consumption is reached at s_{gold} , and the corresponding steady-state capital level $\tilde{k}_{\text{gold}}^*$ such that

$$f'(\tilde{k}_{\text{gold}}^*) = \delta \tag{13}$$

The golden rule

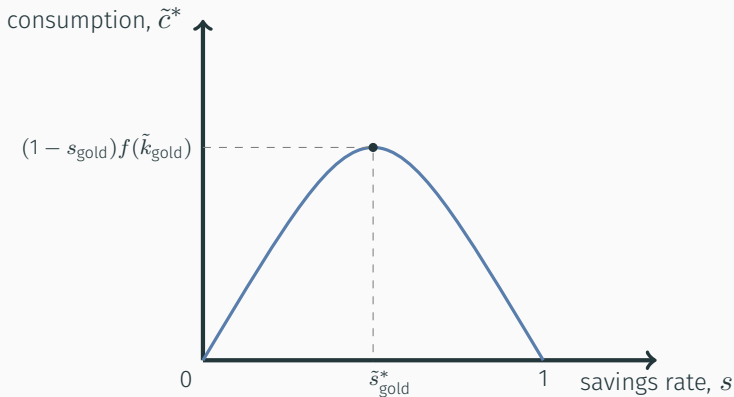
Highest level of steady-state consumption reached at s_{gold} :

$$\tilde{c}^*(s_{\text{gold}}) = (1 - s_{\text{gold}})f(\tilde{k}_{\text{gold}})$$



The golden rule

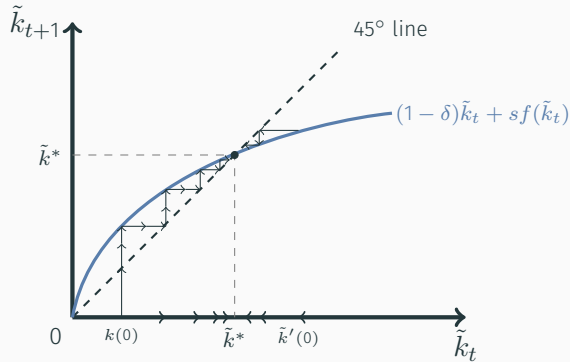
When economy is below $\tilde{k}_{\text{gold}}^*$, a higher saving rate will increase consumption;
Above $\tilde{k}_{\text{gold}}^*$, consumption can be raised by saving less (*dynamic inefficiency*)



Transitional dynamics

- ▶ Equilibrium path refers to entire path of capital stock, output, consumption and factor prices
- ▶ To see how the equilibrium path looks like we need to study the transitional dynamics of equation (10), starting with arbitrary capital-labor ratio, $\tilde{k}_0 > 0$
- ▶ We are often interested only in the steady state equilibrium
- ▶ Let's look at transitional dynamics graphically...

Transitional dynamics



- ▶ **Capital deepening:** Starting at $\tilde{k}_0 < \tilde{k}^*$, economy grows until \tilde{k}^* , so that capital-labor ratio increases (and also income per capita)
- ▶ If economy instead starts at $\tilde{k}'_0 > \tilde{k}^*$, economy decumulates capital until \tilde{k}^* , so capital-labor ratio decreases (and so does income per capita)

An Example: Cobb–Douglas Technology

- ▶ Suppose $Y_t = AK_t^\alpha L^{1-\alpha}$, where $\alpha \in (0, 1)$
- ▶ In steady state:

$$\begin{aligned}\tilde{y}^* &= f(\tilde{k}^*) \\ &= A\tilde{k}^{*\alpha} \\ &= A \left\{ \left(\frac{sA}{\delta} \right)^{\frac{1}{1-\alpha}} \right\}^\alpha \\ &= A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta} \right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

$$\begin{aligned}\tilde{c}^* &= (1-s)\tilde{y}^* \\ &= (1-s)A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta} \right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

Summary of Solow model without population growth and tech. progress

- ▶ Basic Solow model has very nice properties:
 - Unique and stable steady state
 - Simple comparative statics
- ▶ ...but so far has no growth: in steady state, there is no growth in capital-labor ratio (\tilde{k}^*) and no growth in output per capita (\tilde{y}^*)
- ▶ Solow model without technological progress can only explain growth during the transition phase (when $\tilde{k} < \tilde{k}^*$)
 - Growth slows down and eventually comes to a halt!
- ▶ Although not in most desirable manner, Solow's model can account for sustained growth with *exogenous* technological change

Equilibrium with population growth

- ▶ One tweak to basic model: add exogenous population growth

$$L_{t+1} = (1 + n)L_t \quad \Longleftrightarrow \quad \frac{\Delta L_{t+1}}{L_t} = n$$

where $n \geq 0$ is the population growth rate and $L_0 > 0$ given

- ▶ Fundamental law of capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + sY_t \tag{14}$$

- ▶ By definition, capital-labor ratio and output per capita:

$$\tilde{k}_t = \frac{K_t}{L_t} \quad \text{and} \quad \tilde{y}_t = \frac{Y_t}{L_t}$$

Equilibrium with population growth

- Taking logs and differentiating both sides of $\tilde{k}_t = K_t/L_t$ wrt time yields: [details](#)

$$\begin{aligned}\frac{\Delta \tilde{k}_{t+1}}{\tilde{k}_t} &= \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta L_{t+1}}{L_t} \\ &= \frac{sY_t - \delta K_t}{K_t} - n && \text{(using } \Delta K_{t+1} = sY_t - \delta K_t \text{ and } \Delta L_{t+1}/L_t = n\text{)} \\ &= s \frac{Y_t}{K_t} - (\delta + n) \\ &= s \frac{\tilde{y}_t L_t}{\tilde{k}_t L_t} - (\delta + n) && \text{(using } K_t = \tilde{k}_t L_t \text{ and } Y_t = \tilde{y}_t L_t\text{)} \\ &= s \frac{\tilde{y}(t)}{\tilde{k}(t)} - (\delta + n)\end{aligned}\tag{15}$$

Equilibrium with population growth

- ▶ Rearrange equation (15) to get law of motion in per capital terms:

$$\Delta \tilde{k}_t = s\tilde{y}_t - (\delta + n)\tilde{k}_t \quad (16)$$

- ▶ Three remarks:

- Investment per worker ($s\tilde{y}$) increases capital per worker (\tilde{k})
- Depreciation (δ) and population growth (n) reduce capital per worker (\tilde{k})
- Previous version of Solow model nested here for special case $n = 0$

- ▶ Can solve model with these equations and definition of steady state:

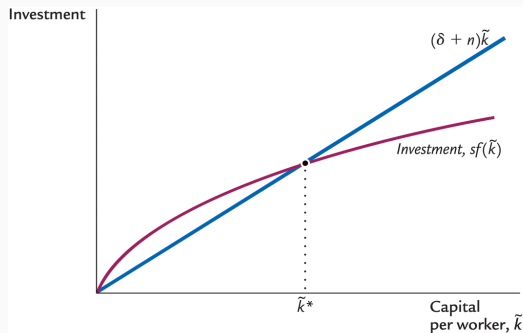
- Graphically
- Analytically

Equilibrium with population growth: Graphical solution

- ▶ In steady state, the capital-labor ratio is constant:

$$\Delta \tilde{k}^* = 0 \implies s\tilde{y}^* = (\delta + n)\tilde{k}^*$$

- ▶ The steady-state level of investment per capita ($s\tilde{y}^*$) is $(\delta + n)\tilde{k}^*$
Steady-state investment makes up for depreciated capital and population growth



Equilibrium with population growth: Analytical solution

- ▶ In steady state, the capital-labor ratio is constant:

$$\Delta \tilde{k}^* = 0 \quad \implies \quad \frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{\delta + n}{s}$$

- ▶ With Cobb–Douglas production function:

$$\begin{aligned}\tilde{y}^* &= A\tilde{k}^{*\alpha} \\ &= A \left(\frac{sA}{\delta + n} \right)^{\frac{\alpha}{1-\alpha}} \\ &= A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta + n} \right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

$$\tilde{c}^* = (1 - s)\tilde{y}^*$$

Summary of Solow model without technological progress

- ▶ Adding population growth hasn't changed basic features of the model:
 - Unique and stable steady state
 - Simple comparative statics
- ▶ New insights:
 - To replenish the capital-labor ratio, investment must be determined by considering both depreciation of physical assets and population growth
 - **Richer countries have** higher savings/investment rates, higher levels of technology, and **lower** depreciation- and **population growth rates**
- ▶ **Same old problems.** Solow model can still not explain sustained growth
 - No per-capita growth once economy reaches steady state: Y grows, \tilde{y} doesn't
 - Growth in per-capita terms slows down and eventually comes to a halt!

The growth slowdown

- ▶ Recall capital accumulation equation (15):

$$\frac{\Delta \tilde{k}_t}{\tilde{k}_t} = s \frac{\tilde{y}_t}{\tilde{k}_t} - (\delta + n)$$

- ▶ With Cobb-Douglas production function:

$$\frac{\Delta \tilde{k}_t}{\tilde{k}_t} = s A \tilde{k}_t^{\alpha-1} - (\delta + n)$$

- ▶ Growth rate of capital-labor ratio \tilde{k} declines over time as \tilde{k} rises since $\alpha < 1$

$$\frac{d}{d\tilde{k}} \left(\frac{\Delta \tilde{k}}{\tilde{k}} \right) = (\alpha - 1) s A \tilde{k}^{\alpha-2} - (\delta + n) < 0$$

- ▶ Given that \tilde{y} is proportional to \tilde{k} , $\tilde{y}_t = f(\tilde{k}_t)$, the same is true for \tilde{y}

The growth slowdown: Graphical representation

The further below (above) is the economy from the steady state \tilde{k}^* ,
the faster the economy grows (de-grows)

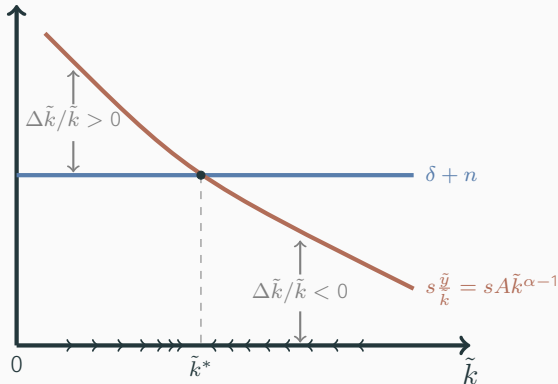


Figure 1: Transitional dynamics with Cobb–Douglas function and no tech. change

Convergence

- ▶ Using a Cobb–Douglas production function, we established that the further below an economy is from its steady state, the faster it grows
- ▶ We now generalize this result and see if it holds in the data
- ▶ Recall that with neoclassical technology assumptions:

$$\begin{aligned}g_{\tilde{k}t} &\equiv \frac{\Delta \tilde{k}_t}{\tilde{k}_t} = s \frac{\tilde{y}_t}{\tilde{k}_t} - (\delta + n) \\&= s f(\tilde{k}_t) \tilde{k}_t^{-1} - (\delta + n)\end{aligned}$$

- ▶ Taking the derivative of $g_{\tilde{k}}$ wrt \tilde{k} :

$$\begin{aligned}\frac{dg_{\tilde{k}}}{d\tilde{k}} &= s f'(\tilde{k}) \tilde{k}^{-1} - s f(\tilde{k}) \tilde{k}^{-2} \\&= \frac{s}{\tilde{k}} \left(f'(\tilde{k}) - \frac{f(\tilde{k})}{\tilde{k}} \right) < 0\end{aligned}$$

Convergence

- **More general result** (neoclassical vs. Cobb-Douglas technology)

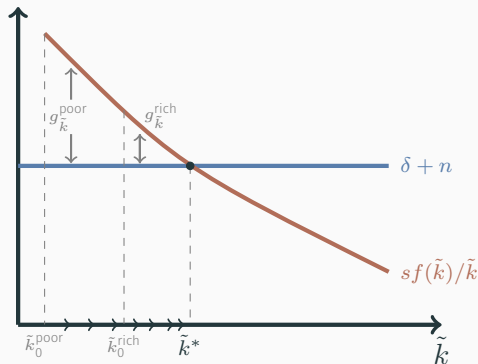
$$\frac{dg_{\tilde{k}}}{d\tilde{k}} = \frac{s}{\tilde{k}} \left(f'(\tilde{k}) - \frac{f(\tilde{k})}{\tilde{k}} \right) < 0$$

implies that smaller values of \tilde{k} associated with larger values of \tilde{g}_k

- Does this result mean that economies with lower capital per worker tend to grow faster? Or, in other words, that there is convergence across countries?
- It depends!
 - If economies structurally similar (ie, same values for s, n, δ, A and same technology f), then yes: they share same steady-state values \tilde{k}^* and \tilde{y}^*
 - If economies not structurally similar, they don't need to converge: they have different steady-state values \tilde{k}^* and \tilde{y}^*

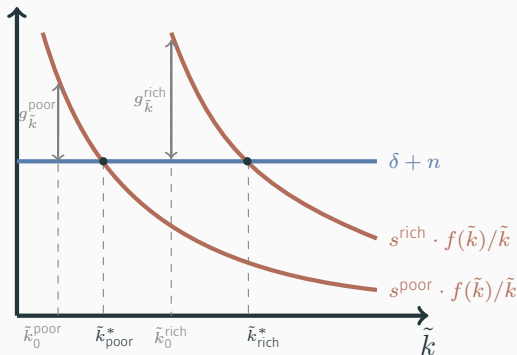
Conditional convergence

- Consider two economies—one rich, one poor—that are structurally similar (ie, same s, n, δ, A, f), but have different initial conditions: $\tilde{k}_0^{\text{rich}} > \tilde{k}_0^{\text{poor}} > 0$
- Model predicts less-advanced economy will exhibit higher growth rate $g_{\tilde{k}}$



Divergence

- Consider two economies—one rich, one poor—that are structurally dissimilar (in that $s^{\text{rich}} > s^{\text{poor}}$) and have different initial conditions: $\tilde{k}_0^{\text{rich}} > \tilde{k}_0^{\text{poor}} > 0$
- Model predicts rich country grows further apart from poor country if rich country further away from its steady state than poor country; ie, $g_{\tilde{k}}^{\text{rich}} > g_{\tilde{k}}^{\text{poor}}$



Conditional vs. absolute convergence

► Two types of convergence:

- **Conditional convergence:** economies with lower starting values of capital-labor ratio will exhibit higher per-capita growth rates and will thereby tend to catch up with initially richer countries when economies are structurally similar
- **Absolute convergence:** poorer countries tend to grow faster than richer ones even without conditioning on steady-state determinants

► Because absolute convergence is a less restrictive form of convergence, it is harder to observe it in practice

Convergence in the data

Empirical evidence supports conditional convergence, not absolute convergence

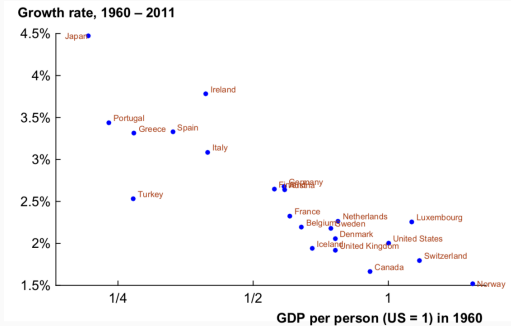
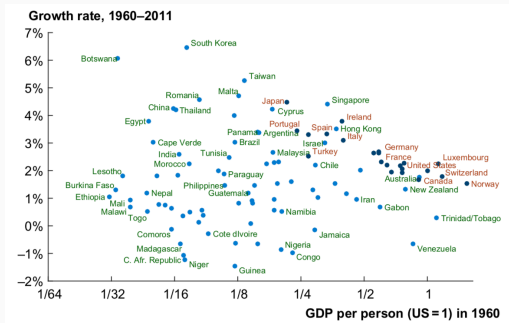


Figure 2: World economies (left) and OECD economies (right)

Convergence in the data

Empirical evidence supports conditional convergence

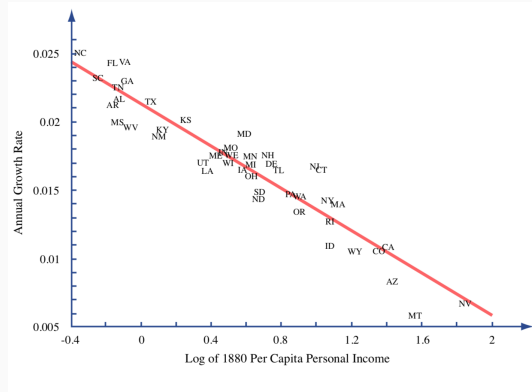


Figure 3: Convergence across US states, 1880–2000

Convergence in the data

Empirical evidence supports conditional convergence

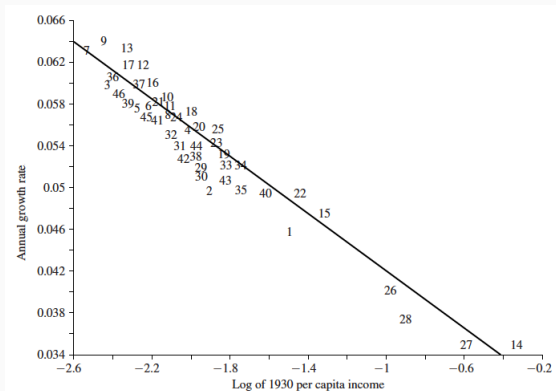


Figure 4: Convergence across Japanese prefectures, 1930–1990

► Adding pop. growth to basic model doesn't solve fundamental problem:

- Model can still not account for sustained growth
 - Once steady state reached, output per capita \tilde{y}^* doesn't grow
- This is at odds with empirical evidence
 - With given inputs, society produces more output per capita today than in past:
 $\tilde{y}(\tilde{k})_{2024} > \tilde{y}(\tilde{k})_{1800}$

► To address this problem, we add technological progress to model

Adding technological progress

► Two tweaks to basic model:

- Population growth: $L_{t+1} = (1 + n)L_t$, where $L_0 > 0$
- Technological progress: $A_{t+1} = (1 + g_A)A_t$, where $A_0 > 0$

► Importantly, we treat tech. progress as exogenous (“manna from heaven”): economic agents cannot influence it!

- Again, this is at odds with reality: technology developed with R&D
- But it is a useful first step
- More advanced models feature endogenous tech. progress
(Romer 1990, Gorssman–Helpman 1991, Aghion–Howitt 1992, Jones 1995, Acemoglu 2002, ...)
- But these models are beyond our scope

Adding technological progress: BGP and Harrod-neutral tech. progress

- ▶ How to introduce technology A into production function F ?
 - **Three options:** Hicks-neutral, Solow-neutral, Harrod-neutral tech. progress
 - Standard approach is to choose one that generates **balanced growth path (BGP)**
 - **BGP:** allocation where output grows at a constant rate and capital-output ratio, interest rate, and factor shares remain constant (in line with Kaldor facts)
- ▶ Important result (due to Uzawa 1961) establishes that only Harrod-neutral or labor-augmenting technological progress can generate balanced growth
- ▶ Thus, we now consider production functions:

$$Y_t = F[K_t, A_t L_t]$$

Equilibrium with population growth and technological progress

- ▶ Capital's fundamental law of motion:

$$\Delta K_{t+1} = sF[K_t, A_t L_t] - \delta K_t$$

- ▶ Convenient to analyze economy in “effective” or “efficiency” units of labor.
Capital-labor ratio in effective units (using hat notation):

$$\hat{k}_t := \frac{K_t}{A_t L_t}$$

- ▶ Taking logs, differentiating wrt time, and approximating:

$$\begin{aligned} \frac{\Delta \hat{k}_{t+1}}{\hat{k}_t} &= \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta A_{t+1}}{A_t} - \frac{\Delta L_{t+1}}{L_t} \\ &= \frac{\Delta K_{t+1}}{K_t} - g_A - n \end{aligned}$$

Equilibrium with population growth and technological progress

- Output per effective unit of labor:

$$\begin{aligned}\hat{y}_t &:= \frac{Y_t}{A_t L_t} \\ &= F\left[\frac{K_t}{A_t L_t}, 1\right] \\ &\equiv f(\hat{k}_t)\end{aligned}$$

- Income per capita:

$$\begin{aligned}\tilde{y}_t &:= \frac{Y_t}{L_t} \\ &= A_t \hat{y}_t \\ &= A_t f(\hat{k}_t)\end{aligned}$$

- Now, even if \hat{y}_t constant, income per capita \tilde{y}_t grows because A_t grows!

Equilibrium with population growth and technological progress

- ▶ With technological progress, we no longer look for steady state but for *balanced growth path*, where income per capita grows at constant rate
 - Transformed variables \hat{y}_t, \hat{k}_t remain constant
 - So BGP can be thought of as steady state of transformed model
 - This explains why in such models economists use “BGP” and “steady state” interchangeably
- ▶ Back to our expression:

$$\begin{aligned}\frac{\Delta \hat{k}_{t+1}}{\hat{k}_t} &= \frac{\Delta K_{t+1}}{K_t} - g_A - n \\ &= \frac{sY_t - \delta K_t}{K_t} - g_A - n && \text{(substituting } \Delta K_t) \\ &= s \frac{Y(t)}{K(t)} - (\delta + g_A + n)\end{aligned}$$

Equilibrium with population growth and technological progress

- We can now use $\hat{k} \equiv K/(AL)$ to write:

$$\begin{aligned}\frac{\Delta \hat{k}_t}{\hat{k}_t} &= s \frac{Y_t}{\hat{k}_t A_t L_t} - (\delta + g_A + n) \\ &= s \frac{\hat{y}(t)}{\hat{k}(t)} - (\delta + g_A + n) \quad (\text{using } \tilde{y} = Y/(AL))\end{aligned}$$

- We can now write the law of motion for capital in effective units:

$$\Delta \hat{k}_t = s \hat{y}_t - (\delta + g_A + n) \hat{k}_t \quad (17)$$

- Three remarks:

- Investment in effective units ($s\hat{y}$) increases effective capital per worker (\hat{k})
- Depreciation (δ), technological progress (g_A), and population growth (n) reduce effective capital per worker (\hat{k})
- Previous model versions nested here when $g_A = 0$ and/or $n = 0$

Equilibrium with population growth and technological progress

- ▶ Law of motion for capital in effective units:

$$\Delta \hat{k}_t = s \hat{y}_t - (\delta + g_A + n) \hat{k}_t$$

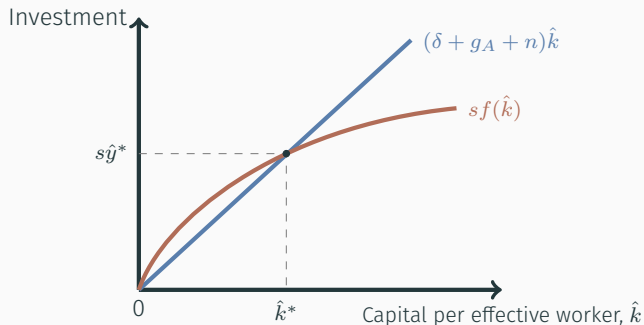
- ▶ A steady state or BGP is now defined as an equilibrium in which the effective capital-labor ratio, \hat{k}_t is constant overtime—that is, $\Delta \hat{k}_t = 0$
- ▶ Let's solve the model:
 - Graphically
 - Analytically

Equilibrium with population growth and technological progress

- In steady state:

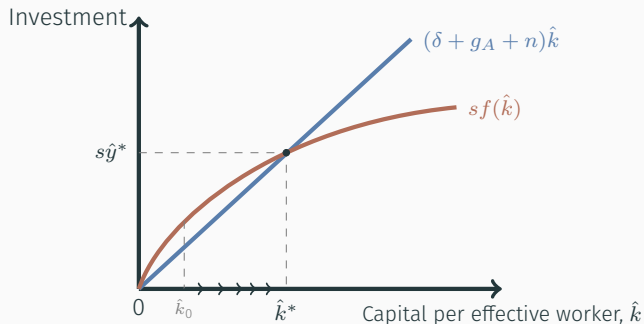
$$\Delta \hat{k}_t^* = 0 \quad \implies \quad s\hat{y}^* = (\delta + g_A + n)\hat{k}^*$$

- The steady-state level of investment in effective units ($s\hat{y}^*$) is $(\delta + g_A + n)\hat{k}^*$
Steady-state investment makes up for depreciated capital, pop. growth, and tech. progress



Equilibrium with population growth and technological progress

- ▶ **Transitional dynamics:** If economy is below (above) its steady state, the effective capital-labor ratio will increase (decline) until \hat{k}^* is reached
- ▶ Once steady-state effective capital-labor ratio \hat{k}^* is reached, economy grows along balanced growth path



Example: Cobb–Douglas production

- ▶ With Cobb–Douglas production $\Delta \hat{k} = 0$ yields:

$$s\hat{y}^* = (\delta + g_A + n)\hat{k}^* \quad \Longleftrightarrow \quad s\hat{k}^\alpha = (\delta + g_A + n)\hat{k}^*$$

- ▶ Solving for \hat{k}^* :

$$\hat{k}^* = \left(\frac{s}{\delta + g_A + n} \right)^{\frac{1}{1-\alpha}}$$

- ▶ Substituting \hat{k}^* into \hat{y}^* :

$$\hat{y}^* = \left(\frac{s}{\delta + g_A + n} \right)^{\frac{\alpha}{1-\alpha}}$$

Example: Cobb–Douglas production

► Output per capita along BGP

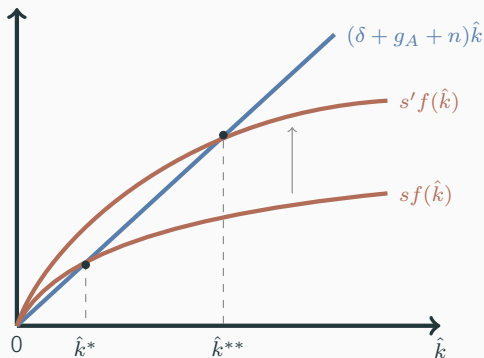
$$\begin{aligned}\tilde{y}_t^* &= A_t \hat{y}^* \\ &= A_t \left(\frac{s}{\delta + g_A + n} \right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$

depends on technology A and time t , as well as on savings rate s , depreciation rate δ , rate of tech. progress g_A , and rate of pop. growth n

- **Level vs. growth effects:** Changes in investment, depreciation, or pop. growth rates affect long-run level of output per capita, not its long-run growth rate

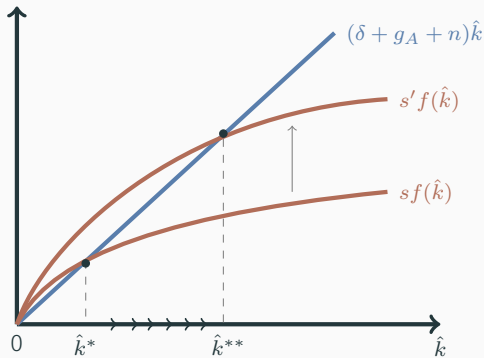
Comparative dynamics

- **Shocks to the investment rate:** an increase (decrease) from s to s' moves the economy to a higher (lower) steady state \hat{k}^{**}



Comparative dynamics

- **Shocks to the investment rate:** an increase (decrease) from s to s' moves the economy to a higher (lower) steady state \hat{k}^{**}
- At initial \hat{k}^* , investment exceeds amount needed to keep \hat{k}^* constant, so \hat{k} rises
 - Increase in s raises the growth rate *temporarily* (along the transition to \hat{k}^{**})



Comparative dynamics

- ▶ Prior to increase in savings rate s , output per worker grew at rate g_A
- ▶ When savings rate increases at t^* , output per capita \tilde{y} grows more rapidly until economy reaches new steady state, when growth rate returns to g_A

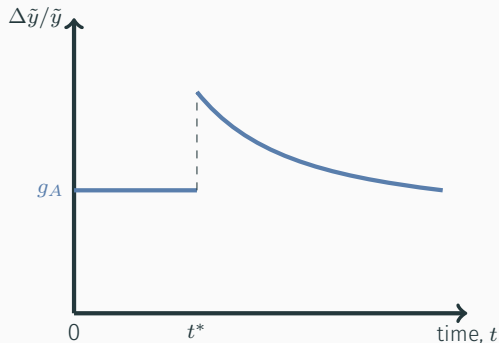


Figure 5: Effect of an increase in the savings rate on growth rate of output per capita

Comparative dynamics

- Policy changes don't have long-run growth effects, only level effects

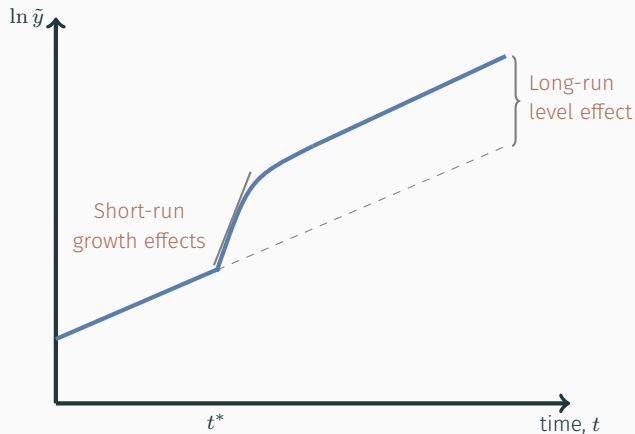
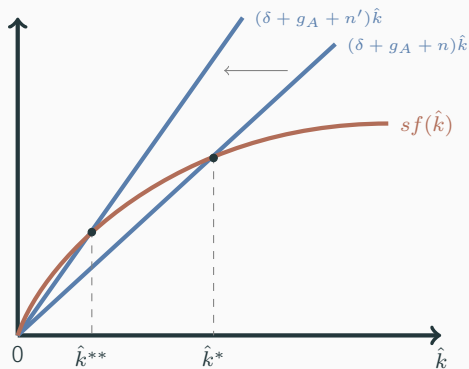


Figure 6: Level effects in the Solow model

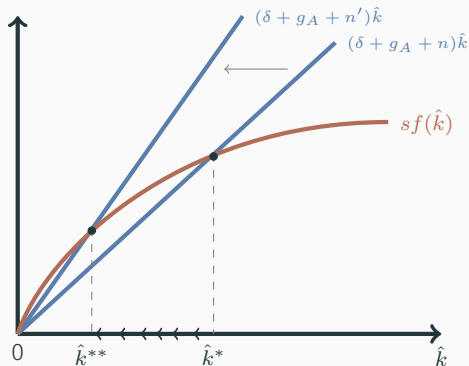
Comparative dynamics

- Shocks to the population growth rate: an increase (decrease) from n to n' moves the economy to a lower (higher) steady state \hat{k}^{**}



Comparative dynamics

- **Shocks to the population growth rate:** an increase (decrease) from n to n' moves the economy to a lower (higher) steady state \hat{k}^{**}
- At initial \hat{k}^* , investment is too low to keep \hat{k}^* constant, so \hat{k} declines
 - Increase in n lowers the growth rate *temporarily* (along the transition to \hat{k}^{**})



Summary of Solow model with population growth and technological progress

- ▶ With tech. progress, Solow's model can account for sustained growth
- ▶ Key takeaways:
 1. Capital accumulation determined by savings rate (s), depreciation rate (δ), rate of technological progress (g_A), and population growth rate (n)
 2. Richer countries ($\uparrow \tilde{y}$) have higher savings rates ($\uparrow s$), better technology ($\uparrow A_0$), more tech. progress ($\uparrow g_A$), lower depreciation ($\downarrow \delta$), lower pop growth rate ($\downarrow n$)
 3. Gvt policy may have long-run level effects, but no long-run growth effects (eg, increase in savings rate, reduction in pop growth rate, ...)
- ▶ **Main problem of Solow model is that all key variables** (savings rate, depreciation rate, pop growth rate, rate of tech. progress) **are exogenous**

Beyond Solow's Model

Other perspectives on population growth

- ▶ According to the Solow model, population growth is bad for econ growth: it reduces output per worker by leading to lower capital-labor ratios
- ▶ Other perspectives:
 - Malthus (1798) highlighted interaction of population with natural resources
 - Argued population grows geometrically while means of subsistence linearly
 - Result is scarcity and famine: not enough food to feed people
 - Perceived technological progress as temporary and unsustainable: population would grow in response to it, keeping people in a poverty trap!
 - As later in Solow's model, population growth is bad for economic growth

Other perspectives on population growth

- ▶ According to the Solow model, population growth is bad for econ growth: it reduces output per worker by leading to lower capital-labor ratios
- ▶ Other perspectives:
 - Romer (1990), Kremer (1993), and others highlighted interaction of population with technology
 - Basic idea is that technological progress depends on technological breakthroughs
 - More people means more potential inventors (ie, 1 Elon Musk per million)
 - The larger the population, the faster technology advances and economy grows
 - Kremer provides evidence in favor of this, using data from 1 million BC to 1990
 - Contrary to Solow's model, population growth is great for economic growth

Augmenting the Solow model

► Solow model can be augmented in multiple ways

(adding human capital, a role for the government, international trade, ...)

► Mankiw, Romer, and Weil (1992) emphasized the role of human capital:

- Different countries have different levels of education, skills, know-how, ...
- They extended production function to accommodate this: $Y = F[K, H, AL]$
- Model highlights human capital investments as growth-enhancing mechanism
- Augmented model:
 - Fits data much better than original Solow model
 - Suggests 70% of cross-country income differences due to differences in physical and human capital

Insights from modern growth models

New growth theory:

- ▶ Endogenizes technological progress and emphasizes innovation
 - **Product-variety models:** innovation causes productivity growth by creating new—not necessarily improved—varieties (eg, Romer 1990, Jones 1995)
 - **Schumpeterian-growth models:** innovation leads to creative destruction and growth (eg, Aghion and Howitt 1992, and co-authors)
- ▶ Thinks of countries as parts of a whole rather than isolated units
 - **Technology adoption and skill mismatch** (Acemoglu and Zilibotti 2001):
 - Technologies need to be adapted to local environments
 - Inappropriateness of technology due to climate or skill mismatch
 - **International trade:** recognizes role of FDI and imports/exports
- ▶ Studies whether technical change is biased towards particular factors of production, incorporates climate change considerations, ...

Taking Stock

- ▶ Solow's model is one of the first workhorse models in the growth literature; it helps us to understand the *mechanics* of growth
- ▶ Solow's model key features:
 - Revolves around neoclassical production function
 - **Blends Keynesian** (behavioral rules) & **neoclassical ingredients** (optimizing behavior)
 - Allows comparative statics and dynamics
 - Can account for **convergence** (conditional vs. absolute) and **divergence**
 - Has limited role for government intervention
 - Can be augmented to incorporate relevant factors (eg, human capital)
 - Allows to bridge theory with empirics (eg, growth accounting)

Taking stock

- ▶ It sheds light on importance of saving/investment rates, population growth, human capital, and technology differences
- ▶ Solow model is not entirely satisfactory:
 - Most important variables are exogenous
 - It emphasizes the proximate causes of growth ...
 - ... but to say that a country is poor because it has little capital and inefficient technology is like saying that a person is poor because it has no money!
 - There are factors that make a country to have more physical- and human capital and more efficient technologies—as there are factors that make a person to have more money than others
- ▶ Next, we study fundamental causes of growth and take further look at data

Questions?

Thank You!

(Email: luisperez@smu.edu)

(Website: <https://luisperezecon.com>)

Derivation

- ▶ Taking logs of $\tilde{k}_t = K_t/L_t$:

$$\ln \tilde{k}_t = \ln K_t - \ln L_t$$

- ▶ Differentiating both sides with respect to time:

$$\begin{aligned} \frac{d \ln \tilde{k}_t}{dt} &= \frac{d \ln K_t}{dt} - \frac{d \ln L_t}{dt} \\ \Leftrightarrow \frac{d \ln \tilde{k}_t}{d \tilde{k}_t} \frac{d \tilde{k}_t}{dt} &= \frac{d \ln K_t}{d K_t} \frac{d K_t}{dt} - \frac{d \ln L_t}{d L_t} \frac{d L_t}{dt} \\ \Leftrightarrow \frac{\dot{\tilde{k}}_t}{\tilde{k}_t} &= \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} \quad (\text{where } \dot{x}_t = dx/dt) \end{aligned}$$

- ▶ Discrete time approximation (ie, $\dot{x}_t \approx \Delta x_t$) of above equation:

$$\frac{\Delta \tilde{k}_t}{\tilde{k}_t} = \frac{\Delta K_t}{K_t} - \frac{\Delta L_t}{L_t}$$