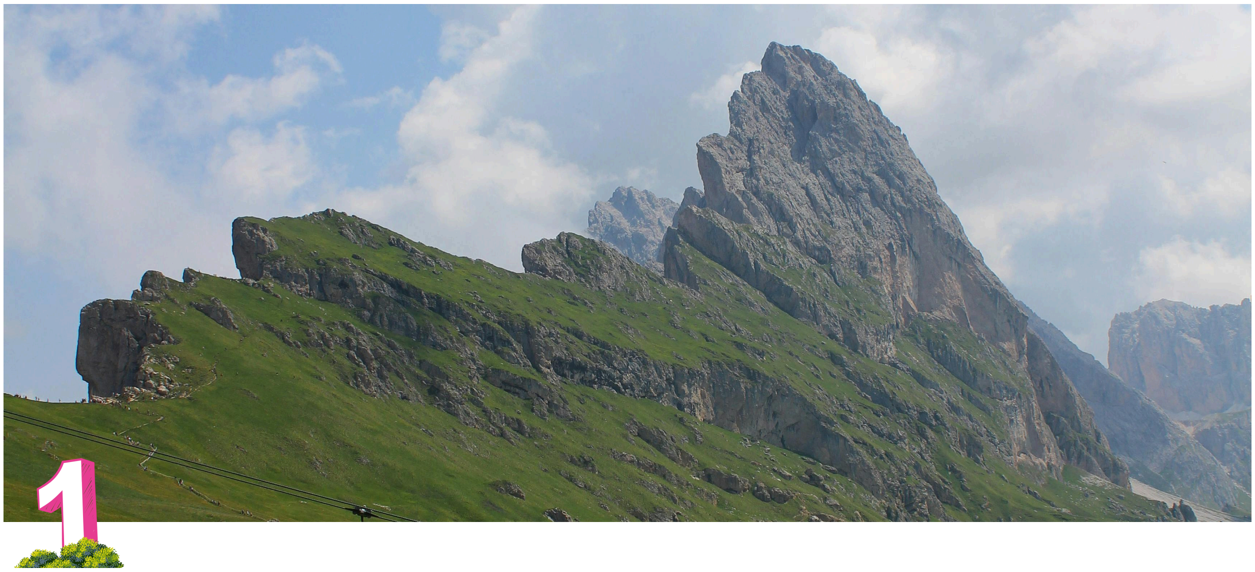

Daily Math Solution - DAY 1

Submitted date: @November 11, 2025

Vector Equality in $\mathbb{R}^n$

I. About

The development of this solution was completed by Theophanie Scholastica Tanzil (@Phanie; https://github.com/lymphoidcell). The Google Slide version is here: [Slide].

Furthermore, I attempted to demonstrate how this topic can be applied in practice, particularly in the context of understanding NumPy. Google Colab: [Code]

- ▼ **Topic**
  - ▼ Vector spaces R^2 and R^3

    Introduction that vectors live in different dimensional spaces (examples: $(2, -5), (7, 9) \in \mathbb{R}^2$; $(0,0,0), (3,4,5) \in \mathbb{R}^3$).
  - ▼ Zero vector

    The concept of the zero vector $(0,0,0)$ in $\mathbb{R}^3$.

▼ Solving a simple linear system

Solving for x, y, z from the provided component equations.

▼ **Category**

Machine Learning Math - Linear Algebra

▼ **Difficulty Level**

Beginner (VERY EASY!!!)

▼ **Time Investment**

- The actual solution took less than five minutes.
- But I have to admit that I love to make my notes as perfect as possible, so I explored about the problem deeply. 😁 *Psst*—it took me 2 hours!

I also created a note related to today's problem: 📓 Chapter 1: Introduction to Vectors in R^n and C^n

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II. Problem Statement

(a) The following are vectors:

$$(2, -5), (7, 9), (0, 0, 0), (3, 4, 5)$$

The first two vectors belong to $\mathbb{R}^2$, whereas the last two belong to $\mathbb{R}^3$.

The third is the zero vector in $\mathbb{R}^3$. (*The first question has been answered in the problem statement, ckckck.* 😂)

(b) Find x, y, z such that $(x - y, x + y, z - 1) = (4, 2, 3)$.

By definition of equality of vectors, corresponding entries must be equal.

$$x - y = 4, x + y = 2, z - 1 = 3$$

Source: 1️⃣ Daily Math Problem - DAY 1

III. Approach and Methodology

There are several approaches to solving the problem above, including **elimination followed by substitution**, **Gaussian elimination**, and **inverse matrix**; then explaining what the system means geometrically using **the Row and Column Picture method —** I just found this topic intriguing, especially after watching the The Geometry of Linear Equations from Prof. Gilbert Strang.

Solution Strategy:

▼ Elimination → Substitution

Solve the 2×2 system for x, y by eliminating one variable, then back-substitute; read z from the third equation.

▼ Gaussian Elimination

Write the augmented matrix for the x, y subsystem and row-reduce to solve; then use the third scalar for z.

▼ Inverse Matrix

- Write the subsystem for x, y as $A \cdot u = b$ with $A = [[1, -1], [1, 1]], u = [[x], [y]]$, $b = [[4], [2]]$.
- Compute A^{-1} and multiply $u = A^{-1}b$; then get z from $z - 1 = 3$.

▼ **What the system means geometrically → Row Picture and Column Picture**

Interpret the two x, y equation as **row picture** and **column picture**. This method will provide the geometric understanding. However, it notes that the systematic algebraic approach is **elimination**.

Preliminary Analysis:

Elimination → Substitution

- Vector equality ⇒ entrywise equations.
- Adding the first two equations eliminates y.
- Uniqueness since coefficient matrix $[[1, -1], [1, 1]]$ has $det = 2 \neq 0$.

Gaussian Elimination

- System: $[[1, -1], [1, 1]]\ [[x], [y]] = [[4], [2]]$.
- Row operations preserve solution; pivot in both columns since $det \neq 0$.

Inverse Matrix

- $det(A) = 1\cdot 1 - (-1\cdot 1) = 2 \neq 0$ ⇒ A is invertible.
- For 2×2, $A^{-1} = (1/det)\cdot [[d, -b], [-c, a]]$.
- Here $a = 1$, $b = -1$, $c = 1$, $d = 1 \Rightarrow A^{-1} = (1/2)\cdot [[1, 1], [-1, 1]]$.

Row Picture and Column Picture

- Columns $c_1 = [[1], [1]]$, $c_2 = [[-1], [1]]$ from a basis of $\mathbb{R}^2$ ($det = 2 \neq 0$); hence a unique combination exists.
- Geometrically, two nonparallel lines intersect at one point.

IV. Detailed Solution

1️⃣ Elimination followed by substitution method

Step 1: Label each equation

- $x - y = 4$ → (1)
- $x + y = 2$ → (2)
- $z - 1 = 3$ → (3), this one is *optional* since it has nothing to do with the x and y 😁

Step 2: Either eliminate they x or y, both ways are valid

The goal is to *eliminate* one variable to find another variable's value. I will go with eliminating y to find the x's value.

$$\begin{array}{rl} x - y = 4 & \quad (1) \\ x + y = 2 & \quad (2) \\ \hline 2x = 6 & + \end{array}$$

Thus, $x = 3$

The value of x then can be substituted to either equation (1) or (2).

If we want to verify whether eliminating x will also work, or we just want to stay with elimination without going further to substitution, let's try:

$$
\begin{array}{ll}
x - y = 4 & (1) \\
x + y = 2 & (2) \\
\hline
-2y = 2 & -
\end{array}
$$

Thus, $y = -1$

The value of y then can be substituted to either equation (1) or (2).

Step 3: Substitution (let's assume that we haven't found the value of the other variable)

Let's substitute $y = -1$ to equation (2)

$x + y = 2$ then $x - 1 = 2$

Thus, $x = 3$

The same answer as we did in the elimination part, right? Math is pretty fun! ⭐⭐⭐⭐⭐

Step 4: Do not forget to find the z value 😄

$$z - 1 = 3$$

Switch the position like usual to get the value of $z = 4$.

Final answer:

$$x = 3, y = -1, z = 4$$

Therefore,

	Conclusion
1	The set of equations $x - y = 4$ and $x + y = 2$ is a linear system.
2	The solution found, $x = 3$ and $y = -1$, can be written as the solution vector $[[3], [-1]]$.
3	Since this solution has two components (x and y), it is a vector in R^2. Meanwhile, z (from $z - 1 = 3$) is a separate equation. If this were part of a system with three variables (for example, if the first equation were $x - y + 0z = 4$), then the complete solution would be $x = 3$, $y = -1$, and $z = 4$. Thus, the solution vector would be $[[3], [-1], [4]]$ in R^3.

2️⃣ Gaussian elimination

This method is also called 'Row Reduction.'

Step 1: Set up the Augmented Matrix

First, let's focus on the x and y equations, as they are a 2×2 system. We'll solve for z at the end (or just skip it, since we already did in the previous section 😂)

- Equation (1): $x - y = 4$
- Equation (2): $x + y = 2$

We can write this system as an "augmented matrix."

Row 1 (R1) comes from Equation (1): $[1 - 1|4]$

Row 2 (R2) comes from Equation (2): $[11|2]$

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 1 & 2 \end{array}\right]$$

Step 2: Perform Row Operations (transform existing form into Row Echelon Form; the most fun part of Gaussian elimination!!!)

The goal is to get a 0 in the bottom-left corner (where the 1 in R2 is). We can do this with the operation: New R2 = R2 - R1.

Let's calculate the new R2:

$$\begin{array}{rr}
[\,1 \quad 1\,|\quad 2\,] & (\mathrm{R2}) \\
-[\,1 \quad \text{-}1\,|\quad 4\,] & (\mathrm{R1}) \\
\hline
[\,0 \quad 2\,|\,\text{-}2\,] \quad & (\mathrm{New\ R2})
\end{array}$$

Now, our matrix looks like this (it's in "Row Echelon Form"):

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 2 & -2 \end{array}\right]$$

Step 3: Back Substitution

Let's turn the matrix back into equations:

- From R2: $0x + 2y = -2$ → $2y = -2$
- From R1: $1x - 1y = 4$ → $x - y = 4$

Now we solve from the bottom up!

From R2:

$$2y = -2$$

$$y = -2 \div 2$$

Thus, $y = -1$

Now substitute $y = -1$ into R1's equation:

$$x - y = 4$$

$$x - (-1) = 4$$

$$x + 1 = 4$$

$$x = 4 - 1$$

Thus, $x = 3$

Final answer is the same as the previous method: $x = 3, y = -1, z = 4$

Therefore,

Conclusion
Through row operations, we transformed the augmented matrix into row echelon form, making it easy to solve by back substitution.

3️⃣ Inverse matrix

This is another powerful method, but it only works if the system has a unique solution. In which, the lines/planes intersect at exactly one point.

To verify, we will check the Determinant. For a 2×2 system with matrix $A = [[a, b], [c, d]]$: $det(A) = ad - bc$.

Rules:

- If $det(A) \neq 0$ → unique solution exists meaning inverse exists
- If $det(A) = 0$ → no unique solution (either no solution or infinitely many)

Let's check our equations:

$A = [[1, -1], [1, 1]]$

$det(A) = (1)(1) - (-1)(1) = 1 + 1 = 2$

which is not equal to 0

Since $determinant = 2 \neq 0$, the system has a unique solution, so we can use the inverse matrix method.

Step 1: Find the Inverse of A (A^{-1})

For a 2×2 matrix $[[a, b], [c, d]]$, the inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Since we already got our determinant which is 2, let's proceed to build the inverse matrix:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Step 2: Solve for $x(X = A^{-1} \cdot B)$

Now we multiply A^{-1} by B:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Let’s do the matrix multiplication (row by column):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (1 \cdot 4) + (1 \cdot 2) \\ (-1 \cdot 4) + (1 \cdot 2) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 + 2 \\ -4 + 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

Finally, multiply by the ½:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \div 2 \\ -2 \div 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

This tells us our values directly: $x = 3$ and $y = -1$

Therefore,

Conclusion
By finding the inverse of matrix A, we can solve for X directly using $X = A^{-1} \cdot B$.

⭐ BONUS: Row Picture & Column Picture ⭐

Lesson learned from this video https://www.youtube.com/watch?v=J7DzL2_Na80&t=430s; this problem involves three equations and three unknowns (x, y, z). Let's remind ourselves what the equations are again:

$$x - y = 4 \qquad x + y = 2 \qquad z - 1 = 3$$

As mentioned before, the approaches that we are using are **row picture** and **column picture**. Detailed explanations are given below:

▼ **Row Picture**

For the x and y variables, the two equations ($x - y = 4$ and $x + y = 2$) can be solved graphically by finding the point where the two **straight lines** meet in the xy-plane. The complete 3×3 system involves the intersection of three planes in three-dimensional space, and the solution (x, y, z) is the single point where those planes meet.

▼ **Column Picture**

This method requires writing the system in matrix form, $Ax = b$, and solving the vector $x = (x, y, z)$ that represents the **linear combination** of the columns of matrix A required to produce the right-hand side vector $b = (4, 2, 3)$.

> ❣️ However, I should have listed this solution before proceeding to the Gaussian elimination. But, it's okay, as long as whoever read this page will understand. (๑>◡<๑)

Step 1: Use 'Row Picture' to find the solution (x, y) by determining where the lines $x - y = 4$ and $x + y = 2$ intersect

Only those two equations first, since they form a 2×2 system. Furthermore, using the Row Picture method visualizes the solution by plotting the lines corresponding to each equation in the xy-plane and finding the single point where they meet.

Row Picture

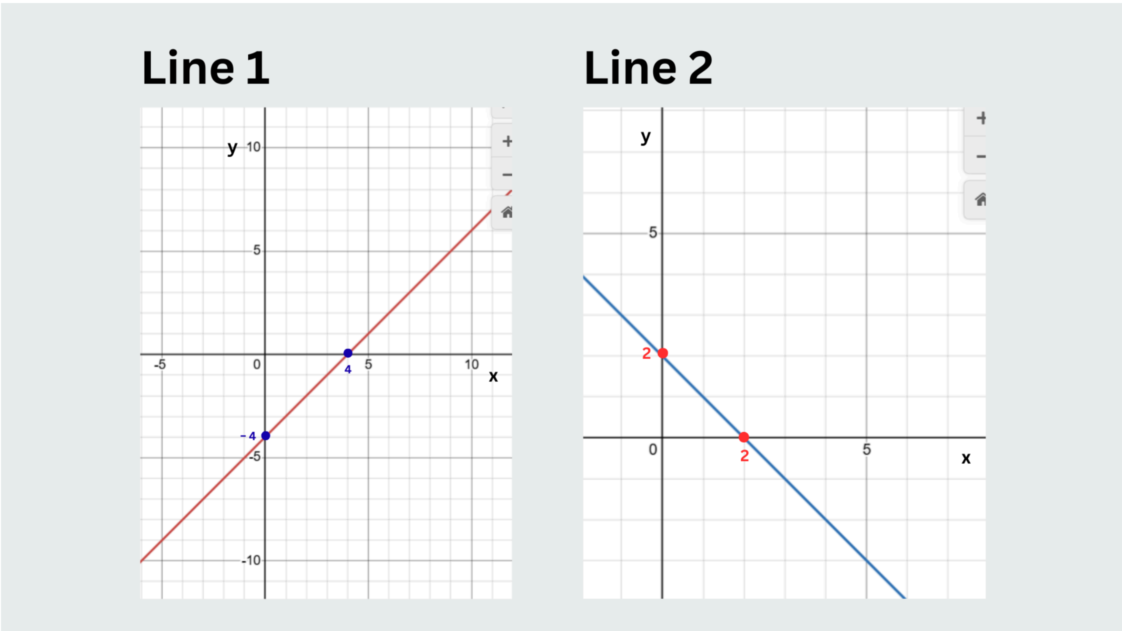
Once again, this means one equation at a time (lines in R^2, planes in R^3); solution = geometric intersection.

We apply this to the 2×2 subsystem:

1. $x - y = 4$ represents a straight line in the xy-plane.
    - If $y = 0$, then $x = 4$. (The point $(4, 0)$).
    - If $x = 0$, then $y = -4$. (The point $(0, -4)$).
2. $x + y = 2$ represents a second straight line.
    - If $y = 0$, then $x = 2$. (The point $(2, 0)$).
    - If $x = 0$, then $y = 2$. (The point $(0, 2)$).

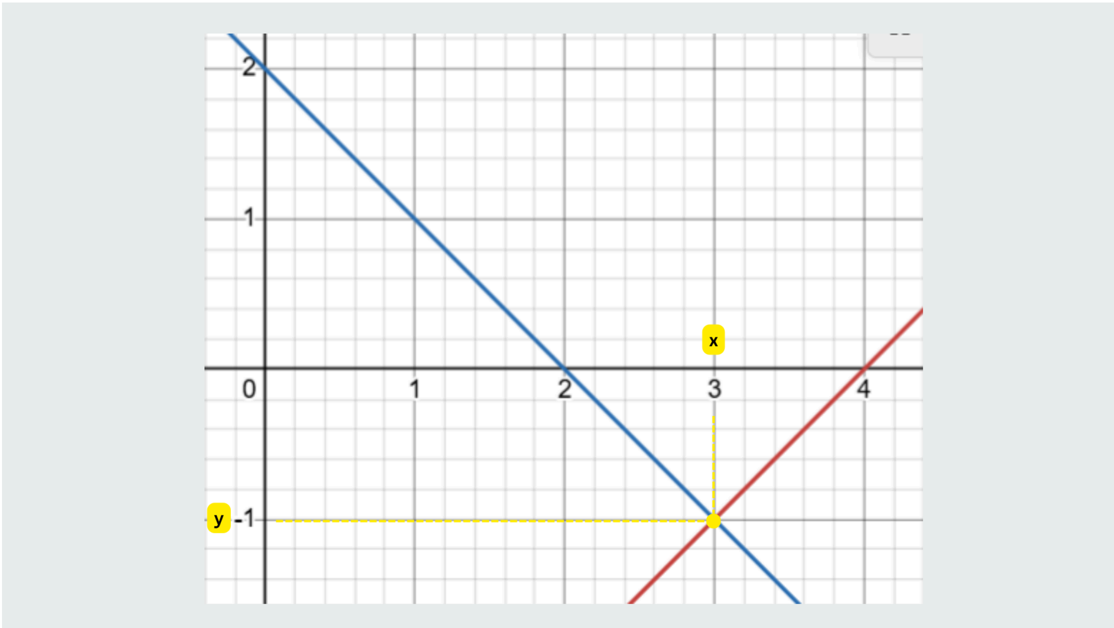
Therefore, we have two lines now:

- **Line 1** → x - y = 4, passing through (4, 0) and (0, -4)
- **Line 2** → x + y = 2, passing through (2, 0) and (0, 2)



The question is, where do these lines intersect?

If we plot these points and draw the lines, we'd see them cross at some point:



Thus, from the graph above we also got that $x = 3$ and $y = -1$.

Step 2: We will now examine the 'Column Picture,' despite having already determined the values of x and y from the previous method.

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A \qquad x \quad = \quad b$$

The Column Picture reinterprets solving the system Ax = b as finding a way to produce the right-hand side vector b by taking a linear combination of the columns of the matrix A. For the 2 × 2 system, we can write this in matrix form, where the columns of the coefficient matrix A and the right-hand side vector are:

Column 1

Column 2

The right-hand side vector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Instead of thinking about lines, think about **combining vectors**. The system can be written as:

$$x \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Or more clearly:

$$x \times (Column1) + y \times (Column2) = b$$

Where:

- **Column 1** = $[[1], [1]]$ (coefficients of x from both equations)
- **Column 2** = $[[-1, [1]]$ (coefficients of y from both equations)
- **b** = $[[4], [2]]$ (right-hand side)

Then, the question becomes, "What values of x and y will make this combination equal to $[[4], [2]]$? Let's try to visualize the vectors, just as we did earlier in the Row Picture's section using the xy-plane. The interesting part about plotting these points is that it leads us to ask, "How much of Column 1 and how much of Column 2 do we need to combine to reach b?"

Thus, we need to observe how the columns combine and solve the system algebraically:

$$x \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Becomes like this:

$$\begin{bmatrix} x & - & y \\ x & + & y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

The combination yields two equations that can be solved through the **elimination method**, giving the values $x = 3$ and $y = 1$. We can now interpret this result in matrix form:

- The solution $x = 3$, $y = -1$ means:

$$3 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

- Let's verify:

$$3 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$(-1) \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Add them:

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Last step, don't forget to find the z value 😂

Conclusion

	Row Picture (Geometric Intersection View)
1	Each equation represents a geometric object (line in 2D, plane in 3D).
2	The solution is where all these geometric objects meet (intersection).
3	Easy to visualize and understand (either for 2×2 or 3×3 systems.
4	This method can also help detect whether equations intersect at one point, are parallel (no solution), or coincide (infinite solutions).
	Column Picture (Linear Combination View)
5	The combining vectors part is an important operation especially when talking about linear algebra.
6	This method works for any size system, even when we can't visualize = in other words, scalable to higher dimensions.
7	This method shows how the columns of matrix A span the space.
8	I think understanding the Column Picture helps lay the groundwork for advanced topics like basis, span, and linear independence.
9	It also helps to understand the practical part for computation since matrix multiplication and solving $Ax = b$ are based on column combinations.
	While both approaches yield the same solution, each highlights different properties of the system:
10	Row Picture asks, "Do these constraints have a common solution?"
11	Column Picture asks, "Can we reach b by combining the columns of A?"
	Let us now turn to the final conclusion regarding the problem set and the insights gained from solving it through both the Row Picture and the Column Picture approaches…
12	The set of equations $x - y = 4$ and $x + y = 2$ is a linear system.
13	The solution found, $x = 3$ and $y = -1$, can be written as the solution vector $[3, -1]$.
14	Since this solution has two components (x and y), it is a vector in R^2. Meanwhile, z (from $z - 1 = 3$) is a separate equation. If this were part of a system with three variables (for example, if the first equation were $x - y + 0z = 4$), then the complete solution would be $x = 3$, $y = -1$, and $z = 4$. Thus, the solution vector would be $[3, -1, 4]$ in R^3.

V. Verification and Validation

❣️ Actually no need, but I just want to fill in the template, hehehe.

Verification Method 1: Direct substitution

- $x - y = 3 - (-1) = 4$ ✔️
- $x + y = 3 + (-1) = 2$ ✔️
- $z - 1 = 4 - 1 = 3$ ✔️

  All constraints satisfied.

Verification Method 2: Linear-algebraic consistency

Write the 2×2 subsystem $AX = b$ with

$A = [[1, -1], [1, 1]]$, $X = [[x\ y]]$, $b = [[4\ 2]]$.

$\det A = 1 \cdot 1 - (-1) \cdot 1 = 2 \neq 0 \Rightarrow A^{-1}$ exists → unique (x, y).

$X = A^{-1}b = ½\ [[1, 1], [-1, 1]]\ [[4\ 2]] = [[3\ -1]]$.

Then z from $z - 1 = 3 \Rightarrow z = 4$.

Residual check: $r = b - AX = 0$.

VI. Analysis and Insights

Key Observations:

- Vector equality enforces componentwise equations; each component is an independent scalar constraint.
- Full-rank 2×2 block gives uniqueness for (x,y); z is decoupled and solved separately.

Critical Insights:

- Check $\det A \neq 0$ first; it tells you solvability/uniqueness instantly.
- Choose elimination that zeroes one variable in one step (add the two equations to kill y).

Alternative Solution Methods:

- Elimination then back-substitution (fewest steps here).
- Gaussian elimination (augmented matrix → REF/RREF).
- Inverse matrix (only because det A ≠ 0); otherwise use elimination.
- Row/column pictures: intersection of two nonparallel lines in R^2; column combination reaches b.

Comparative analysis:

- Elimination: fastest by hand for 2 × 2.
- Gaussian: systematic, scales to larger systems.
- Inverse: computationally heavier; pedagogical for small systems.
- Geometric views: build intuition; not a computation method at scale.

Connections to Broader Theory:

- Rank/uniqueness: rank(A) = 2 ⇒ unique solution; rank < 2 ⇒ none/infinite.
- ML/AI: identical algebra to solving normal equations (X^T X)θ = X^T y; full rank and conditioning of X^T X control identifiability and stability.
- Generalizations: under/over-determined systems, least squares, pseudoinverse, conditioning and numerical stability.

Technical Considerations:

- Common errors: sign errors when adding/subtracting rows; forgetting z is independent; mixing R^2 vs R^3 vectors.
- Assumptions: real scalars; exact arithmetic; A taken from the (x,y) block only.
- Limitations: Inverse method fails/ill-conditioned when det A ≈ 0; geometric intuition doesn't scale to high-dimensional systems.

VII. References

- MIT OpenCourseWare. (2019, September 25). *1. The geometry of linear equations* [Video]. YouTube. https://www.youtube.com/watch?v=J7DzL2_Na80&t=430s

- The Organic Chemistry Tutor. (2018, January 17). *Solving systems of equations by elimination & substitution with 2 variables* [Video]. YouTube. https://www.youtube.com/watch?v=oKqtgz2eo-Y
- Wrath of Math. (2023, May 15). *A guide to Gaussian elimination method (and solving systems of equations) | Linear Algebra* [Video]. YouTube. https://www.youtube.com/watch?v=seet9VyHo3Q

VIII. My Follow-Up Questions

- If det A = 0, classify cases (no solution vs infinitely many) for variants of b.
- Solve the least-squares version when the two (x,y) equations are inconsistent; derive (A^T A)u = A^T b.
- Extend to a coupled 3 × 3 system; analyze rank, null space, and geometric interpretation (intersection of planes).
- Compute condition number $\kappa_2(A)$ and show how small perturbations in b affect (x,y).
- Implement and compare elimination vs $A^{-1}b$ vs `numpy.linalg.solve` timing/accuracy on random n × n systems; report residuals and κ(A).