AULA 6

ESTIMADOR DE
MÁXIMA
VEROSSIMILHANÇA

$$X_1, \dots X_n \sim f(\theta)$$

$$\underline{X} = \underline{x} \qquad f_n(\underline{x} \mid \theta)$$

$$\theta \in \Omega$$

$\mathcal{U} \cdot / L$

MAXIMIZO

$$P_\theta(\underset{\sim}{X} = \underset{\approx}{x})$$

- EMV (EXPONENCIAL)

$\underline{X_i} \sim EXPO(\theta)$ $\boxed{IID}$

$$f(x_i|\theta) = \theta \exp(-\theta x_i)$$

i.i.d. $f_n(\underset{\sim}{x}|\theta) = \prod_{i=1} f(x_i|\theta)$

$$f_n(\underset{\sim}{x}|\theta) = \theta^n \exp(-S\theta)$$

$$S = \sum_{i=1}^{n} x_i \qquad \theta \in \mathbb{R}^+$$

$$\hat{\theta}_{EMV}$$

$$\max_{\theta \in \mathbb{R}^+} f_n(\underset{\sim}{x}|\theta)$$

$$L(\theta) \propto \theta^n e^{-S\theta}$$

$$\underline{\underline{L(\theta)}} = \theta^n e^{-S/\theta}$$

$$\underline{\log L(\theta)} =: \ell(\theta)$$

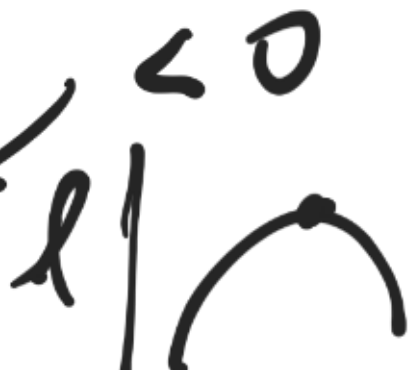
$\hat{\theta}_{max}$

log É MONOTÓNICO

$$\ell(\theta) = n \log\theta - \frac{S}{\theta}$$

$$\frac{d\ell}{d\theta} = \frac{n}{\theta} - S = 0$$

$$\frac{d^2\ell}{d\theta^2} = -\frac{n}{\theta^2} < 0$$



$$\frac{\theta}{\theta}$$

$$\frac{d\ell}{d\theta} = \frac{\theta}{n} - S = 0$$

$$\boxed{\bar{x}_n := \frac{S}{n}} \quad \hat{\theta}_{EMV} = \frac{n}{S} = \frac{1}{\bar{x}_n}$$

$$\hat{\theta}_{EMV} = \frac{1}{\bar{x}_n}$$

BERNOULLI

$$X_i \sim B(\theta), 0<\theta<1$$

$$f_n(\underline{x}|\theta) \propto \theta^{y}(1-\theta)^{n-y}$$

$$f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$$

$$x_i \in \{0,1\}$$

$$y = \sum_{i=1}^{n} x_i.$$

$$l(\theta) = y\log(\theta) + (n-y)\log(1-\theta)$$

$$\frac{dl}{} = n\theta - y$$

$$d\theta \qquad \frac{\qquad}{\theta(1-\theta)}$$

$$\frac{d^2 l}{d\theta^2} = -\frac{n\theta^2 - 2y\theta + y}{[\underbrace{\theta(1-\theta)}]^2}$$

$$\frac{d^2 l}{d\theta^2} < 0$$

$$\frac{dl}{d\theta} = \frac{n\theta - y}{\underbrace{(1-\theta)\theta}_{>0}}$$

$$n\theta - y = 0$$

$$\hat{\theta} = \frac{y}{n} = \bar{x}_n$$

$$\boxed{\hat{\theta}_{EMV} = \bar{x}_n}$$

$\hat{\theta}_{EMV}$ É CONSISTENTE?

LGN $\bar{x}_n \xrightarrow{P} \theta$.

NORMAL

$X_i \sim N(\mu, \sigma^2)$

CASO 1

$\mu = ?$ $\sigma^2 = \sigma^2$ ← [illegible]

$$f(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$f_n(\underline{x} \mid \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i \mid \mu, \sigma$$

$$= \left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]^n \exp\left(-\frac{\sum_{i=1}^{n} (x_i-\mu)^2}{2\sigma^2}\right)$$

$$l(\theta) \propto \log f_n(\underline{x} \mid \theta)$$

$$\theta = (\mu, \sigma^2)$$

$$l(\mu) \propto -\sum (x_i-\mu)^2$$

$$-\sum_{i=1}^{n}\left(x_i^2 - 2\mu x_i + \mu^2\right)$$

$$= -\left(\sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2\right)$$

$$= \underbrace{-\sum x_i^2 + 2\mu\sum x_i - n\mu^2}_{l(\mu)}$$

$$\frac{d\,l(\mu)}{d\mu} = -2n\mu + 2\sum x_i.$$

$$= 0 \Rightarrow \hat{\mu} = \frac{\sum x_i}{n},$$

$$\boxed{\hat{\mu}_{EMV} = \bar{x}_n}$$

$n = 2$

$\sigma^2 = \text{Fixo}$

<u>CASO 2</u> : $\mu = \mu(\text{Fixo})$

$\sigma^2 = ?$

$$\Delta := \sum (x_i - \mu)^2$$

$$\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]^n \exp\left(-\frac{\Delta}{2\sigma^2}\right)$$

$$\propto (\sigma^2)^{-\frac{1}{n}} \exp\left(-\frac{\Delta}{2\sigma^2}\right)$$

$$L(\sigma^2)$$

$$\ell(\sigma^2) = \log L(\sigma^2)$$

$$-\frac{n}{2}\log\sigma^2 - \frac{\Delta}{2\sigma^2}$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{\boxed{\Delta - n\sigma^2}}{2(\sigma^2)^2} \leftarrow \neq 0$$

$$\Delta - n\sigma^2 = 0$$

$$\hat{\sigma}^2_{EMV} = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}$$

EXERCÍCIO 6
NA SEÇÃO 7.5 DG

CASO 3

$$\mu = ?$$

$$\sigma^2 = ?$$

$$L(\mu, \sigma^2)$$

$$\hat{\theta}_{EMV} = (\hat{\mu}_{EMV}, \sigma^2_{EMV})$$

$$(-\infty, \infty) \times (0, \infty)$$

$$\mu, \sigma^2 \in$$

$$\sigma^2_{EMV} = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}$$

(0.802)

$$\hat{\mu}_{EMV} = \bar{x}_n$$

$$\Delta \Big|_{\mu = \bar{x}}$$

$$\hat{\sigma}^2_{EMV} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

$\bar{S}^2$

$\mu = ?$ $\sigma^2 = ?$

<u>CASO 3</u>

$$\hat{\theta}_{EMV} = \left(\bar{x}_n, \frac{\sum_{i=1}^{n} (x_i - \bar{x}_n)^2}{n} \right)$$

$\mu_{max} \rightarrow \sigma^2_{min}$

$f(\mu, \sigma^2) \leftarrow$

UNICIDADE DO EMV

$$f_n(x|\theta) \begin{cases} 1, & \theta \in (m_{y_1}, \\ 0, & c.c. \end{cases}$$

$$\boxed{\hat{\theta} \in [n-1, m]}$$

INVARIANCIA

$X_i \sim B(p)$

$$\omega = \frac{p}{1-p}$$

$$\boxed{\hat{p} = \bar{x}_n}$$

$\hat{\omega}_{EMV}$

$$\hat{\omega}_{EMV} = \frac{\bar{x}_n}{1-\bar{x}_n}$$

$$= \frac{\hat{p}}{1-\hat{p}}$$

$f_n(x \mid \omega)$

$\underset{\omega \in \Omega}{\operatorname{argmax}}$

$$\mathcal{L}^*(\omega) = \sup_{\{\theta : \phi(\theta) = \omega\}} \mathcal{L}(\theta)$$

$$\sup \sup \mathcal{L}(\theta)$$

$$\sup_{w} \quad \{\theta : \phi(\theta) = w\}$$

$$=$$

$$\sup_{w} L^{*}(w)$$

$$= \sup_{\theta} L(\theta)$$

CONSISTÊNCIA EMV

$$E_{\theta_0}[\log f(X|\theta)]$$

$$E_{\theta_0}\left[\log \frac{f(\underline{X}|\theta)}{f(\underline{X}|\theta_0)}\right] \leq 0$$

$$\int_{\underline{x}} \log f(\underline{x}|\theta)\, f(x|\theta_0)\, dx$$

$$\int_{\underline{x}} \log f(\underline{x}|\theta_0)\, f(x|\theta_0)\, dx$$

$$E_{\theta_0}[\log A] - E_{\theta_0}[\log B]$$

$$E_{\theta_0}\left[\log \frac{A}{B}\right]$$

$$\leq \log E_{\theta_0}\left[\frac{A}{B}\right]$$

$A \quad f(\underline{x}|\theta)$

$$\frac{A}{B} = \frac{f(x|\theta)}{\cancel{f(x|\theta_0)}}$$

$$E_{\theta_0}[g] = \int_x \cancel{f(x|\theta_0)}\, g(\theta, x)\, dx$$

$$E_{\theta_0}\left[\frac{A}{B}\right] = \int_x f(x|\theta)\, dx$$

$$\uparrow 1$$

$$\log(1) = 0$$