Grothendieck-Verdier Functor Categories and Deformations of Symmetric Frobenius Algebras

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Chapter 1

Introduction

1.1 Background

Recall that a (lax) monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is composed of a functor F and the data of a natural transformation $\mu : \otimes_{\mathcal{D}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathcal{C}}$. When we talk about deforming a monoidal functor between linear monoidal categories, we mean deforming the natural transformation data, as in [DE; Yet1; Yet2]. These are controlled by Davydov-Yetter cohomology. This has some independent interest, as it gives us families of structures. For instance, we can deform a $\mathcal{C}$-module category $\mathcal{M}$ by deforming monoidal functors $\mathcal{C} \to \mathrm{End}(\mathcal{M})$, giving us families of module categories. If we assume $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ has the structure of a monoidal functor, then deformations are tightly related to deformations of braidings. Finally, deformations of the associator of the underlying monoidal category are controlled by the Davydov-Yetter cohomology of the identity functor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$.

The mathematics of deforming associative algebras is also well-known, as in [Ger]. This gives us families of algebras, controlled by Hochschild cohomology. Following an insight due to Davydov and Street, we can connect the deformation theory of functors to the deformation theory of associative algebras in the following way.

Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear monoidal categories with some reasonable size constraints, and $\mathcal{D}$ copowered over $\mathbf{Vect}_{\mathbb{K}}$. Define the Day convolution $\varoast : \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \times \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ by the coend $F_0 \varoast F_1 := \int^{A,B} \mathrm{Hom}_{\mathcal{C}}(A \otimes B, -) \star (F_0 A \otimes F_1 B)$. Then the category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ together with the Day convolution is a $\mathbb{K}$-linear monoidal category. Lax monoidal functors are then monoids in this monoidal category, and so we can deform them as algebras. The insight of Davydov and Street is that the Davydov-Yetter cohomology of the functor is quasi-isomorphic to the Hochschild cohomology of the functor regarded as an algebra. I provide a proof of this in Section 4.2 as the proof by Davydov and Street is not published at the time of writing. With this insight in hand, we can think about how to handle deformations of other structure on functors via deformations of algebras with additional structure. In particular, we are interested in deforming Frobenius monoidal functors $F : \mathcal{C} \to \mathcal{D}$ between Grothendieck-Verdier categories $\mathcal{C}$ and $\mathcal{D}$.

Given two monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a Frobenius monoidal functor $F : \mathcal{C} \to \mathcal{D}$ comes equipped with two natural transformations, a lax monoidal structure $\mu_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ and an oplax monoidal structure $\Delta_{X,Y} : F(X \otimes Y) \to FX \otimes FY$ that satisfy some coherence conditions. They are also of independent interest. In [DP] they are shown to preserve duals. In [FSY] it is shown that rigid separable Frobenius monoidal functors preserve the graphical calculus of pivotal bicategories. Additional results concerning what is preserved by separable Frobenius monoidal functors is present in [MS].

Motivated by studying (co)completions of symmetric monoidal categories with "dualizing objects" symmetric Grothendieck-Verdier categories were introduced in [Bar1] as *-autonomous categories. Later work, such as [Bar2], expanded the notion of a *-autonomous category to encompass non-symmetric

GV-categories. A GV-category is composed of a category $\mathcal{C}$ and two monoidal products, $\otimes$ and $\parr$. Furthermore the monoidal unit of $\parr$, say K, is called the dualizing object of $\otimes$ and induces a dualizing functor $G : \mathcal{C} \to \mathcal{C}^{op}$. A Frobenius algebra in a GV-category – called a GV-Frobenius algebra – is an algebra in $(\mathcal{C}, \otimes, \mathbb{I})$ and a coalgebra in $(\mathcal{C}, \parr, K)$ together with some compatibility conditions, cf. Definition 5.1.24. Similarly, a Frobenius monoidal functor between two GV-categories $F : \mathcal{C} \to \mathcal{D}$ – called a Frobenius Linearly Distributive-Functor or Frobenius LD-Functor – has the structure of a lax $\otimes$ functor and an oplax $\parr$ functor together with some compatibility conditions, cf. Definition 5.1.21. It is also worth noting that the image of a Frobenius LD-functor from the trivial one-object GV-category into a GV-category is a GV-Frobenius algebra [Egg2, 2.4].

So, to summarize how we want to use the insight from Davydov and Street: We want to deform some Frobenius monoidal functor $F : \mathcal{C} \to \mathcal{D}$. For that we want to see them as Frobenius algebras in the functor category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$. Now, we could define a Frobenius algebra just with respect to the Day convolution. However, since the Day convolution is given by a coend, maps into a product like $F \mathbin{☼} F$ are messy and difficult to work with. Since Frobenius LD-functors and GV-Frobenius algebras use two monoidal products, the structure of a GV-category is much more appealing than merely the structure of a monoidal category. In this thesis, I work out what conditions are required on the GV-categories $\mathcal{C}, \mathcal{D}$ so that $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ inherits the structure of a GV-category. The first monoidal product is the Day convolution, as defined above. I introduce a second monoidal product, named the coDay convolution, defined by the end formula $F_0 \mathbin{◑} F_1 := \int_{A,B} \mathrm{Hom}_{\mathcal{C}}(-, A \parr B)^* \star F_0 A \parr F_1 B$. It is natural to ask if GV-Frobenius algebras in the GV-category $(\mathrm{Funct}(\mathcal{C}, \mathcal{D}), ☼, ◑)$ are the same as Frobenius LD-functors in the functor category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$. In this thesis I prove the answer is in the affirmative. In the course of proving that result, I also develop some significant but preliminary results of Grothendieck-Verdier module categories over this GV-category.

It is now clear that as a first step an understanding of deformations of GV-Frobenius algebras is needed. I make some initial progress in this direction by developing the theory of deformations of associative algebras in monoidal categories that preserve the structure of a symmetric Frobenius algebra. This requires a different cohomology theory than Hochschild cohomology. The appropriate cohomology theory is that of cyclic cohomology, as discussed in [Lod]. Similarly to how Hochschild cohomology uses cosimplicial objects, cyclic cohomology uses cocyclic objects. The typical way to define a cocyclic object assumes the target monoidal category is symmetric. I construct an alternative definition that assumes the target monoidal category is only pivotal. This, combined with some other well-established results, allows me to fully classify deformations of symmetric Frobenius algebras where the underlying algebra is separable as an example of the results.

1.2 Thesis Plan

The chapters should be considered in pairs. The first pair of chapters, 2 and 3, develop the deformation theory of a symmetric Frobenius algebra in a pivotal monoidal category. Towards this end, chapter 2 is a review of the relevant machinery and gives a short exploration of the deformation of an associative algebra in a monoidal category. Chapter 3 then reviews some facts about cyclic cohomology and gives novel definitions of some cocyclic objects, before moving into novel results in Section 3.4.

The second pair of chapters, 4 and 5, develop the relevant context for a deformation theory of Frobenius LD-functors. Chapter 4 gives a review of the machinery in the monoidal case, culminating with a proof of Davydov and Street's insight. Finally, Dhapter 5 gives an exploration of functors between GV-categories and between GV-module categories. This culminates in a proof that Frobenius LD-functors between two appropriate GV-categories are GV-Frobenius algebras in the GV-functor category.

1.3 Summary of Results

This thesis has 5 main results, which to the best of my knowledge are new.

1. The standard way of defining a cocyclic object in a monoidal category presupposes that the monoidal category is symmetric. In Propositions 3.2.5 and 3.2.8 I show that cocyclic objects can be defined in pivotal monoidal categories, and furthermore that when the category is both symmetric and pivotal, the standard definition and my definition agree.

2. When we talk about deforming an algebraic object, attention must be paid to what is held constant and what is changed. In this work, a deformation of a symmetric Frobenius algebra will deform the multiplication and keep the bilinear form constant. This lets us impose the condition that the bilinear form remains invariant, symmetric and non-degenerate with the new multiplication. Theorem 3.4.5 shows that such deformations of a symmetric Frobenius algebra in a monoidal category are controlled by the second cyclic cohomology group.

3. Separable algebras do not have non-trivial deformations as associative algebras. By contrast, separable algebras do have non-trivial deformations when we make sure we respect the structure of the symmetric Frobenius algebra. Theorem 3.4.11 and following lemmata give us a complete classification of deformations of symmetric Frobenius algebras where the underlying algebra is separable, as an example of the machinery I developed in the general case.

4. Section 5.2 develops the theory of the category of functors between GV-categories and the category of functors between GV-module categories. In particular, I show that the category of functors between GV-categories inherits the structure of a GV-category, and the category of functors between GV-module categories inherits the structure of a GV-module category. There are a number of smaller important results discussed, including the sufficient conditions for the functor category between GV-categories to be pivotal.

5. Finally, Section 5.3 proves some relations between the distributors of the GV-categories $\mathcal{C}$,$\mathcal{D}$ and $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$. It finishes with Theorem 5.3.7, which shows that Frobenius LD-functors from $\mathcal{C}$ to $\mathcal{D}$ are GV-Frobenius algebras in $(\mathrm{Funct}(\mathcal{C}, \mathcal{D}), ☼, ◑)$. This theorem is related to a result in [Egg1], but is proven in a new and interesting way.

1.4 Directions for Further Research

The ultimate aim of this research is to develop a deformation theory for Frobenius LD-functors, following the general techniques of generalizing the deformation of algebras to deformations of monoidal functors. The work of this thesis, in particular Chapter 5, indicates that the main method for this is to develop a theory of deformations of GV-Frobenius algebras in pivotal GV-categories.This thesis sets up the required machinery, but further work is required to define deformations of such GV-Frobenius algebras.

There are also some interesting questions raised by the treatment of deformations of symmetric Frobenius algebras in monoidal categories in Section 3.4. I deformed the algebra structure, while holding the non-degenerate bilinear form constant. It is an open question as to if different results can be obtained by deforming the non-degenerate bilinear form while holding the algebra structure constant, or if both the non-degenerate bilinear form and the algebra structure are deformed at the same time. Additionally, I have not developed several important tools for deformations of algebras, such as the taking deformations relative to a subalgebra.

There has been previous research on the deformations of non-symmetric Frobenius algebras in [CCEKS]. It is an open question as to if some modified form of cyclic cohomology could be applied to their results.

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Chapter 2

Deformations of an Associative Algebra

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, will denote the natural numbers, the integers, the real numbers, and the complex numbers respectively. The base field will typically be denoted by $\mathbb{K}$. The notation $\mathbb{K}_p$ for a prime p will denote the finite field $\mathbb{Z}/p\mathbb{Z}$ of characteristic p. We will denote the category of vector spaces over $\mathbb{K}$ by $\mathbf{Vect}_{\mathbb{K}}$, and the category of finite-dimensional vector spaces over $\mathbb{K}$ by $\mathbf{FinVect}_{\mathbb{K}}$. Given a $\mathbb{K}$-linear category $\mathcal{C}$, the hom-spaces of $\mathcal{C}$ will be denoted $\mathrm{Hom}_{\mathcal{C}}$. As a special case, the hom-spaces of $\mathbf{Vect}_{\mathbb{K}}$ and $\mathbf{FinVect}_{\mathbb{K}}$ will be denoted $\mathrm{Hom}_{\mathbb{K}}$. If $V \in \mathbf{FinVect}_{\mathbb{K}}$ is a finite-dimensional vector space, then we denote the dual vector space $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$.

Before we move to the novel results in Chapter 3, we will first review the theory of infinitesimal deformations of an associative algebra in a monoidal category enriched over $\mathbf{Vect}_{\mathbb{K}}$. We will show that the deformations are controlled by Hochschild cohomology theory in a very concrete way.

2.1 The Simplicial Category and Hochschild Cohomology

We will first discuss cochain complexes, simplicial objects, and cohomology theories so we have a good grasp of the notation used for the algebraic constructions that control the deformation theory.

Definition 2.1.1 (Cochain Complex)**.** Let $\mathcal{C}$ be an abelian category. A cochain complex $(C^\bullet, b^\bullet)$ is

- A collection of objects $\{C^n\}_{n\in\mathbb{Z}}$
- A collection of morphisms $b^n : C^n \to C^{n+1}$ such that $b^{n+1} \circ b^n = 0$.

We will typically denote a cochain $(C^\bullet, b^\bullet)$ by just the objects $C^\bullet$. For ease of notation, we will also typically suppress the superscript of the morphisms b. The morphisms b are also typically called the differential of the cochain.

Definition 2.1.2 (Cohomology of a cochain)**.** Let $C^\bullet$ be a cochain complex in an abelian category $\mathcal{C}$. Then the cohomology of $C^\bullet$, denoted by $H^\bullet(C^\bullet)$ is defined by

$$H^n(C^\bullet) := \frac{\ker b^n}{\operatorname{im} b^{n-1}}.$$

We will eventually need the notion of a double complex.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
D^{0,2} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{1,2} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{2,2} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{3,2} \arrow[r, "d^h"] \arrow[u, "d^v"] & \cdots \\
D^{0,1} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{1,1} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{2,1} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{3,1} \arrow[r, "d^h"] \arrow[u, "d^v"] & \cdots \\
D^{0,0} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{1,0} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{2,0} \arrow[r, "d^h"] \arrow[u, "d^v"] & D^{3,0} \arrow[r, "d^h"] \arrow[u, "d^v"] & \cdots
\end{tikzcd}
$$

Figure 2.1: Double Complex Example

Definition 2.1.3 (Double Complex)**.** A double complex $(D^{\bullet\bullet}, d^h, d^v)$, sometimes also called a bicomplex, is a family $\{D^{p,q}\}_{p,q\in\mathbb{Z}}$ of objects of $\mathcal{C}$, combined with maps

$$d^h : D^{p,q} \to D^{p+1,q} \quad \text{and} \quad d^v : D^{p,q} \to D^{p,q+1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$. In this thesis, we work with bounded double complexes, that is, double complexes where $D^{p,q} \cong 0$ for $p < 0$ or $q < 0$.

We will usually visualize a (bounded) double complex as a lattice, as in figure 2.1.

Definition 2.1.4 (Total Complex)**.** Given a bounded bicomplex $D^{\bullet\bullet}$, we define the total complex $\mathrm{Tot}(D^{\bullet\bullet})$ by

$$\mathrm{Tot}^n(D) := \bigoplus_{p+q=n} D^{p,q}$$

with differential $d : d^h + d^v$. The total complex of a double complex is a cochain complex.

One general way to construct cochain complexes is via cosimplicial objects.

Definition 2.1.5 (Simplicial Category, Simplicial and Cosimplicial Objects)**.** The simplicial category Δ has objects the non-empty finite totally ordered sets and morphisms the non-decreasing monotone functions.

A simplicial object X in a category $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$. A cosimplicial object X in a category $\mathcal{C}$ is a functor $X : \Delta \to \mathcal{C}$. Morphisms between (co)simplicial objects are natural transformations.

We will use a complete set of representatives of objects of Δ as given by $[n] := \{0 < 1 < \cdots < n\}$.

The following proposition is well known.

Proposition 2.1.6. *The simplicial category is generated by the so-called face maps ϵ_i and degeneracy maps η_i where*

$$\epsilon_i(j) := \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geqslant i \end{cases} \qquad \eta_i(j) := \begin{cases} j & \text{if } j \leqslant i \\ j-1 & \text{if } j > i \end{cases}$$

These satisfy the identities:

$$\epsilon_j \epsilon_i = \epsilon_i \epsilon_{j-1} \text{ if } i < j$$
$$\eta_j \eta_i = \eta_i \eta_{j+1} \text{ if } i \leqslant j$$
$$\eta_j \epsilon_i = \begin{cases} \epsilon_i \eta_{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ \epsilon_{i-1} \eta_j & \text{if } i > j+1 \end{cases}$$

With some abuse of notation when discussing both simplicial and cosimplicial objects, we will denote $\delta^i := X(\epsilon_i)$ and $\sigma^i := X(\eta_i)$.

Lemma 2.1.7 (The Associated Cohomology). *[Wei, 8.2.1] Let $X : \Delta \to \mathcal{C}$ be a cosimplicial object in an abelian category $\mathcal{C}$. Then the associated cochain complex $C^\bullet(X)$ is defined by $C^n(X) := X(n)$ and $b := \sum_{i=0}^n (-1)^i \delta^i$. The Associated Cohomology of X is $HH^\bullet(X) := H^\bullet(C^\bullet(X))$.*

We here recall the definition of a monoidal category, as they provide a natural setting for certain cohomology theories.

Definition 2.1.8 (Monoidal Category). A monoidal category is a tuple $(\mathcal{C}, \otimes, \mathbb{I}, \alpha, r, \ell)$ where:

- $\mathcal{C}$ is a category.
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, called the monoidal product.
- $\mathbb{I} \in \mathcal{C}$ is an object, called the unit object.
- $\alpha : \otimes \circ (\otimes \times \mathrm{id}_\mathcal{C}) \Rightarrow \otimes \circ (\mathrm{id}_\mathcal{C} \times \otimes)$ is a natural isomorphism, called the associator.
- $r : - \otimes \mathbb{I} \Rightarrow -$ is a natural isomorphism, called the right unitor.
- $\ell : \mathbb{I} \otimes - \Rightarrow -$ is a natural isomorphism, called the left unitor.

These datum must satisfy certain coherency conditions. For these see most textbooks on category theory, such as [Bor2, 6.1].

When working in a $\mathbb{K}$-linear monoidal category, we can define the Hochschild cohomology as follows:

Proposition 2.1.9 (Hochschild Cohomology of an Algebra with Coefficients). *Given a $\mathbb{K}$-linear monoidal category $\mathcal{C}$, an associative, unital algebra $(A, \mu, \eta) \in \mathcal{C}$ and a bimodule $(M, \lambda, \rho) \in {}_A\mathbf{Mod}_A$, where $\lambda : A \otimes M \to M$ is the left action and $\rho : M \otimes A \to M$ is the right action, we have a cosimplicial object $C^\bullet(A; M) : \Delta \to \mathbf{Vect}_\mathbb{K}$ defined by*

$$C^n(A; M) := \mathrm{Hom}_\mathcal{C}(A^{\otimes n}, M)$$

$$(\delta^i f) := \begin{cases} \lambda \circ (\mathrm{id}_A \otimes f) & \text{if } i = 0 \\ f \circ (\mathrm{id}_{A^{\otimes i-1}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes n-i-1}}) & \text{if } 0 < i < n \\ \rho \circ (f \otimes \mathrm{id}_A) & \text{if } i = n \end{cases}$$

$$(\sigma^i f) := f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \eta \otimes \mathrm{id}_{A^{\otimes n-i}}).$$

By Lemma 2.1.7 this gives us an associated chain complex via $b := \sum_{i=0}^n (-1)^i \delta^i$. The Hochschild cohomology of A with coefficients in M is $H^\bullet(C^\bullet(A; M))$ and denoted $H^\bullet(A; M)$.

Remark 2.1.10. Of course, $\mathcal{C}$ could be $\mathbf{Vect}_\mathbb{K}$ or $\mathbf{FinVect}_\mathbb{K}$ itself. We work in more generality because we want to generalize to discussing deformations in functor categories. This has been discussed in even greater generality in [EGNO, 7.22] for Davydov-Yetter cohomology.

2.2 Deformations of an associative algebra

With the objects that will control the deformation now understood, we can move on to discussing deformations, where then we will specialize to infinitesimal deformations.

Suppose we have two commutative rings, say R and S, and an S-module $M \in {}_S\mathbf{Mod}$. Then a ring homomorphism $\theta : R \to S$ defines an R-action on M by $r.m := \theta(r).m$. This induces $\theta^* : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$. Note that as R and S are commutative, this naturally works for bimodules as well. If, instead we have an R-module $M \in {}_R\mathbf{Mod}$, then the same ring homomorphism $\theta : R \to S$ defines an S-action on $S \otimes_R M$ by $r_0.(r, m) := (r_0 r, m)$. This induces another functor $\theta_* : {}_R\mathbf{Mod} \to {}_S\mathbf{Mod}$.

The functor θ^* is called restriction of scalars, and the functor θ_* is called extension of scalars. In fact, they form an adjoint pair $\theta_* \dashv \theta^*$.

This basic machinery is what allows us to define an R-deformation of an S-algebra as follows.

Definition 2.2.1 (Abstract Deformation of an Algebra)**.** [LvdB, 5.1] Let R, S be commutative rings, with a ring homomorphism $\theta : R \to S$. For an S-algebra A, an R-deformation of A is an R-algebra B together with an R-algebra homomorphism $f : B \to \theta^* A$ that induces an isomorphism of algebras $S \otimes_R B \to A$.

Note that S is an R-algebra by applying the functor θ^* to the regular representation of S.

It is necessary to have criteria for when two deformations are equivalent.

Definition 2.2.2 (Equivalence of Abstract Deformations)**.** [LvdB, p. 8.2] Let R, S be commutative rings with a ring homomorphism $\theta : R \to S$. For an S-algebra A, two R-deformations (B_1, f_1) and (B_2, f_2) are equivalent if there is an isomorphism of R-algebras $\varphi : B_2 \to B_1$ such that $f_1 \circ \varphi = f_2$.

Remark 2.2.3. According to [LvdB, 8.2], (flat, small) R-linear deformations of a (flat, small) S-linear category form a groupoid, where morphisms are equivalences of deformations modulo natural isomorphism of functors.

Now, all of this is too general. First of all, we should fix our rings R and S. Let $S = \mathbb{K}$, as we are concerned with deformations of an algebra over a field. Next, consider the ring of dual numbers $\mathbb{D} := {}^{\mathbb{K}[t]}\!/_{(t^2)}$, and set $R = \mathbb{D}$. This comes with an obvious ring homomorphism, often called the augmentation, $\theta : \mathbb{D} \to \mathbb{K}$, $\theta(a + bt) := a$ and another obvious ring homomorphism $\beta : \mathbb{K} \to \mathbb{D}$, $\beta(a) := a + 0t$.

We want to have a notion of a trivial deformation, and this fixes the underlying vector space for all deformations as $\beta_*(A)$ has underlying vector space $\mathbb{D} \otimes_{\mathbb{K}} A$. Specializing all the gadgets above, we get the following:

Definition 2.2.4 (Infinitesimal Deformation)**.** [Ger] Given an associative $\mathbb{K}$-algebra (A, μ), an infinitesimal deformation or deformation is an associative $\mathbb{D}$-algebra (A_t, μ_t) such that $A_t = A \oplus tA$, and μ_t is of the form $\mu_t(a, b) = \mu(a, b) + t\varphi(a, b)$ where $\varphi : A \otimes A \to A$ is a $\mathbb{K}$-bilinear map. Note that for this to be well-defined, we have to set $\mu(xt, y) = \mu(x, yt) = t\mu(x, y)$ and $\varphi(xt, y) = \varphi(x, yt) = t\varphi(x, y)$.

Furthermore, two infinitesimal deformations, (A_t, μ_t), (A_t, μ'_t) are equivalent if there exists a $\mathbb{D}$-linear bijection $\rho_t : A_t \to A_t$ of the form $\rho_t = \mathrm{id} + t\lambda$ where $\lambda : A \to A$ is a $\mathbb{D}$-linear map, such that $\mu'_t = \rho_t^{-1} \circ \mu_t \circ (\rho_t \otimes \rho_t)$. We will say a deformation is trivial if it is equivalent to (A_t, μ_0) where $\mu_0(a, b) = \mu(a, b)$. In that case we will call ρ_t the trivializer of the deformation.

It is worth checking that Definition 2.2.4 of an infinitesimal deformation is a special case of Definition 2.2.1 of an abstract deformation.

Lemma 2.2.5. *Let (A, μ, η) be an associative $\mathbb{K}$-algebra, and (A_t, μ_t) an infinitesimal deformation. Then (A_t, f), $f : A_t \to A$ given by $f(x + yt) = x$ gives an abstract deformation.*

Proof. There are two parts to the proof. First we need to show that f is a $\mathbb{D}$-algebra map. Second we need to show that the induced map is an isomorphism.

First we check that f is a $\mathbb{D}$-linear map. Note first that this is clearly a $\mathbb{K}$-linear map. It remains to check that it commutes with the action of t.

$$f(t(x + yt)) = f(tx) = 0 = 0x = \theta(t)x$$

Next, we check that f is a ring homomorphism.

$$\begin{aligned} f(\mu_t(x_0 + y_0t, x_1 + y_1t)) &= f(x_0x_1 + t[x_0y_1 + x_1y_0 + \varphi(x_0, x_1)]) \\ &= x_0x_1 = \mu(f(x_0 + y_0t), f(x_1 + y_1t)) \end{aligned}$$

So f is an algebra homomorphism. It remains to show that the induced map, $\mathbb{K} \otimes_{\mathbb{D}} (A \oplus tA) \to A$, given by $k \otimes_{\mathbb{D}} (x + yt) \mapsto kx$ is an isomorphism.

By the first isomorphism theorem, it suffices to show that the morphism is surjective and has trivial kernel. Surjectivity is easy; for an element $a \in A$, $1 \otimes_{\mathbb{D}} a \mapsto a$.

Consider now the kernel. $k \otimes_{\mathbb{D}} (x + yt) \mapsto 0$. That is, $kx = 0$ which implies $k = 0$ or $x = 0$. If $k = 0$ then the element is 0. Otherwise, we have an element of the form $k \otimes_{\mathbb{D}} yt$. By $\mathbb{D}$-linearity and the definition of how t acts on $\mathbb{K}$ we get $k \otimes_{\mathbb{D}} yt = k\theta(t) \otimes_{\mathbb{D}} y = 0 \otimes_{\mathbb{D}} y = 0$, or the element is 0. All together this gives that the kernel is trivial. □

The following results make clear what is meant by Hochschild cohomology controlling deformations of associative algebras.

Theorem 2.2.6. *Let (A, μ, η) be an associative unital $\mathbb{K}$-algebra. A $\mathbb{D}$-bilinear map of the form $\mu_t = \mu + t\varphi$ gives an infinitesimal deformation if and only if $\varphi \in \ker(b : C^2(A; A) \to C^3(A; A))$, that is, if φ is a 2-cocycle in the Hochschild complex with coefficients in the algebra.*

Proof. Let μ_t give a deformation. That is, $0 = \mu_t \circ (\mathrm{id} \otimes \mu_t) - \mu_t \circ (\mu_t \otimes \mathrm{id})$. That is,

$$\begin{aligned}
&0 = \mu_t \circ (\mathrm{id} \otimes \mu_t) - \mu_t \circ (\mu_t \otimes \mathrm{id}) \\
&0 = \mu \otimes (\mathrm{id} \otimes \mu) - \mu \circ (\mu \otimes \mathrm{id}) + t[\varphi \circ (\mathrm{id} \otimes \mu) + \mu \circ (\mathrm{id} \otimes \varphi) - \varphi \circ (\mu \otimes \mathrm{id}) - \mu \circ (\varphi \otimes \mathrm{id})] \\
&0 = t[\varphi \circ (\mathrm{id} \otimes \mu) + \mu \circ (\mathrm{id} \otimes \varphi) - \varphi \circ (\mu \otimes \mathrm{id}) - \mu \circ (\varphi \otimes \mathrm{id})]
\end{aligned}$$

This is true if and only if

$$\begin{aligned}
0 &= \varphi \circ (\mathrm{id} \otimes \mu) + \mu \circ (\mathrm{id} \otimes \varphi) - \varphi \circ (\mu \otimes \mathrm{id}) - \mu \circ (\varphi \otimes \mathrm{id}) \\
0 &= \partial_2\varphi + \partial_0\varphi - \partial_1\varphi - \partial_3\varphi \\
0 &= b\varphi
\end{aligned}$$

Which completes the proof. □

Lemma 2.2.7. *Let there be two deformations (A_t, μ_t) and (A_t, μ_t') with $\mu_t = \mu + t\varphi$ and $\mu_t' = \mu + t\psi$. Then (A_t, μ_t) and (A_t, μ_t') are equivalent if and only if $\psi = \varphi + bf$ for some $f \in C^1(A; A)$, that is, if they differ up to a coboundary.*

Proof. First, note that for $\rho_t = \mathrm{id} + t\lambda$ for λ a $\mathbb{D}$-linear map, $\rho_t^{-1} = \mathrm{id} - t\lambda$.

Suppose that $\psi = \varphi + bf$ for some $f \in C^1(A; A)$. Let $\rho_t = \mathrm{id} + tf$. Then

$$\begin{aligned}
\rho_t^{-1} \circ \mu_t \circ (\rho_t \otimes \rho_t) &= (\mathrm{id} - tf) \circ (\mu + t\varphi) \circ ((\mathrm{id} + tf) \otimes (\mathrm{id} + tf)) \\
&= (\mathrm{id} - tf) \circ (\mu + t\varphi) \circ (\mathrm{id} \otimes \mathrm{id} + t[\mathrm{id} \otimes f + f \otimes \mathrm{id}]) \\
&= (\mathrm{id} - tf) \circ (\mu + t[\varphi + \mu \circ (\mathrm{id} \otimes f) + \mu \circ (f \otimes \mathrm{id})]) \\
&= \mu + t[\varphi + \mu \circ (\mathrm{id} \otimes f) + \mu \circ (f \otimes \mathrm{id}) - f \circ \mu] \\
&= \mu + t[\varphi + bf] = \mu_t'
\end{aligned}$$

That is, if the two differ by a coboundary, they are equivalent deformations. Suppose now that they are equivalent deformations. By the same computation above, we find that if the two deformations are equivalent with $\rho_t = \mathrm{id} + tf$ then $\psi = \varphi + bf$, that is, they differ by a coboundary. □

These two lemmas give us a direct corollary.

Corollary 2.2.8. *Given an associative algebra (A, μ) and a deformation (A_t, μ_t) with $\mu_t = \mu + t\varphi$, μ_t determines an element $[\varphi] \in H^2(A; A)$. Furthermore, a second deformation (A_t, μ_t') with $\mu_t' = \mu + t\psi$ is equivalent to (A_t, μ_t) if and only if $[\varphi] = [\psi]$.*

This is how Hochschild cohomology controls the deformations of an associative algebra.

Chapter 3

Cyclic Deformations in Monoidal Categories

This chapter will follow the general approach of the previous chapter. First, I will give a brief overview of the graphical calculus for strict monoidal categories. Then I will lay out the algebraic gadgets required for controlling deformations of a symmetric Frobenius algebra that respect the structure of the symmetric Frobenius algebra. This requires more involved computations. I then give some classifications for symmetric Frobenius algebras in general. Finally I finish by classifying deformations of symmetric Frobenius algebras where the underlying algebra is separable.

Throughout the remainder of the thesis, the descriptions "deformation that preserves the symmetric Frobenius algebra structure" and "deformation of the symmetric Frobenius algebra" should be taken as synonymous.

Remark 3.0.1. In the following sections, let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal category with monoidal product $\otimes$, monoidal unit $\mathbb{I}$, and associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$. Furthermore, recall that:

- A monoidal category is <u>strict</u> if the associator and unitors are identity natural transformations.
- A <u>symmetric</u> monoidal category is endowed with a symmetric braiding $s_{-,-} : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, such that $s^2 = \mathrm{id}$. We will typically denote a symmetric monoidal category by $(\mathcal{C}, \otimes, s)$.
- A <u>rigid</u> monoidal category is endowed with left and right dualizing functors ${}^{\vee}-$ and $-^{\vee}$ and the appropriate left and right evaluation and coevaluation morphisms given below.

$$\mathrm{eval}^r_X : X \otimes X^{\vee} \to \mathbb{I} \qquad \mathrm{eval}^{\ell}_X : {}^{\vee}X \otimes X \to \mathbb{I}$$
$$\mathrm{coeval}^r_X : \mathbb{I} \to X^{\vee} \otimes X \qquad \mathrm{coeval}^{\ell}_X : \mathbb{I} \to X \otimes {}^{\vee}X$$

  These satisfy the four <u>triangle identities</u>:

$$\begin{aligned} \mathrm{id}_X &= (\mathrm{eval}^r_X \otimes \mathrm{id}_X) \circ \alpha^{-1}_{X,X^{\vee},X} \circ (\mathrm{id}_X \otimes \mathrm{coeval}^r_X) \\ \mathrm{id}_{X^{\vee}} &= (\mathrm{id}_{X^{\vee}} \otimes \mathrm{eval}^r_X) \circ \alpha^{-1}_{X^{\vee},X,X^{\vee}} \circ (\mathrm{coeval}^r_X \otimes \mathrm{id}_{X^{\vee}}) \\ \mathrm{id}_X &= (\mathrm{id}_X \otimes \mathrm{eval}^{\ell}_X) \circ \alpha_{X,{}^{\vee}X,X} \circ (\mathrm{coeval}^{\ell}_X \otimes \mathrm{id}_X) \\ \mathrm{id}_{{}^{\vee}X} &= (\mathrm{eval}^{\ell}_X \otimes \mathrm{id}_{{}^{\vee}X}) \circ \alpha_{{}^{\vee}X,X,{}^{\vee}X} \circ (\mathrm{id}_{{}^{\vee}X} \otimes \mathrm{coeval}^{\ell}_X). \end{aligned}$$

- A <u>pivotal</u> monoidal category is a rigid monoidal category endowed with a choice of pivotality natural isomorphism $\omega : -^{\vee\vee} \Rightarrow \mathrm{id}_{\mathcal{C}}$. This natural isomorphism is monoidal. In particular, this pivotality natural isomorphism implies that left and right duals are isomorphic. We will typically denote a pivotal monoidal category by $(\mathcal{C}, \otimes, -^{\vee}, \omega)$.

Remark 3.0.2. Recall that if a symmetric monoidal category is rigid – that is, all objects have left and right duals – then the category has a distinguished pivotal structure. The pivotality natural isomorphism is given by:

$$\omega_X := (\mathrm{id}_X \otimes \mathrm{eval}^r_{X^\vee}) \circ (s_{X^\vee,X} \otimes \mathrm{id}_{X^{\vee\vee}}) \circ (\mathrm{coeval}^r_X \otimes \mathrm{id}_{X^{\vee\vee}}).$$

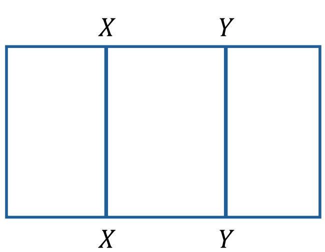
Example 3.0.3. There are a few well-known examples of monoidal categories:

- The category of finite-dimensional $\mathbb{K}$-vector spaces, $\mathbf{FinVect}_{\mathbb{K}}$, is symmetric and pivotal.
- The category of all $\mathbb{K}$-vector spaces, $\mathbf{Vect}_{\mathbb{K}}$, is symmetric but not pivotal. It is not even rigid.
- The category of G-graded vector spaces for a non-abelian group G is pivotal but not symmetric.[EGNO, 2.3.6, 4.7.10]
- Let $S := \{s_1, s_2, \ldots\}$ be a set with more than two elements. Let $\mathfrak{S}$ be the monoidal category where objects are finite sequences of elements of S and the monoidal product is concatenation. The monoidal unit is the empty sequence $\varnothing$. This monoidal category is neither symmetric nor pivotal.

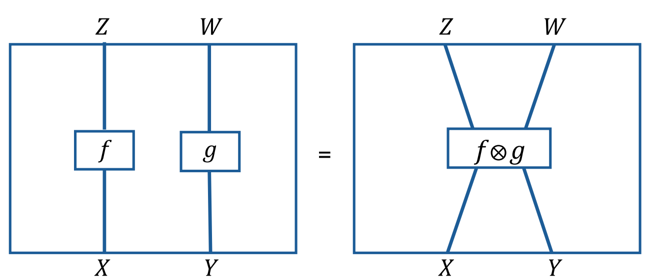
3.1 Graphical Calculus in Monoidal Categories

We refer the reader to [Sel, 4.3–4.4] for a more comprehensive overview of the graphical calculus discussed here.

Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a strict monoidal category. Then we have a graphical calculus for the monoidal category. Our canvas is the closed unit square $[0, 1]^2$. Objects are represented by vertical lines, as in diagram 3.1a and morphisms are represented by boxes, as in diagram 3.2a. We usually do not draw a line for the monoidal unit $\mathbb{I}$. We read taking the monoidal product from left to right, as in diagram 3.1a and diagram 3.1b. We read taking the composition of morphisms from bottom to top, as in diagram 3.2b.



(a) Tensor product $X \otimes Y$



(b) Tensor product of morphisms f and g

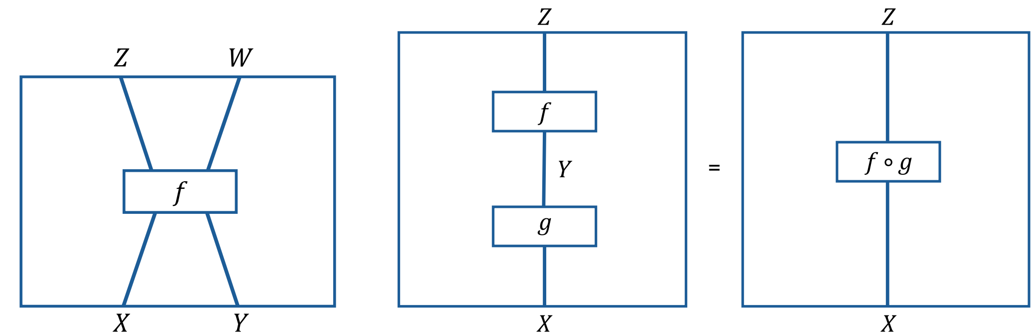
Figure 3.1: Tensor Products

If the monoidal category is rigid, then we represent the coevaluations by diagram 3.4b and diagram 3.4a. We represent the evaluation by diagram 3.5b and diagram 3.5a. If the category is symmetric then we represent the braiding by diagram 3.3. Since a pivotality natural isomorphism in particular entails that left and right duals are isomorphic, we can straighten out loops.

When using this graphical calculus for proofs, we will typically suppress the arrows denoting duality, as well as suppress the labelling of the morphism boxes.

3.2 Cocyclic Objects and Cyclic Cohomology

In the remainder of this chapter, I will prove that deformations of symmetric Frobenius algebras are controlled by cyclic cohomology. Towards this end, and with the aim of having a parallel structure to



(a) Morphism $f : X \otimes Y \to Z \otimes W$ (b) Composition $f \circ g$

Figure 3.2: Morphisms

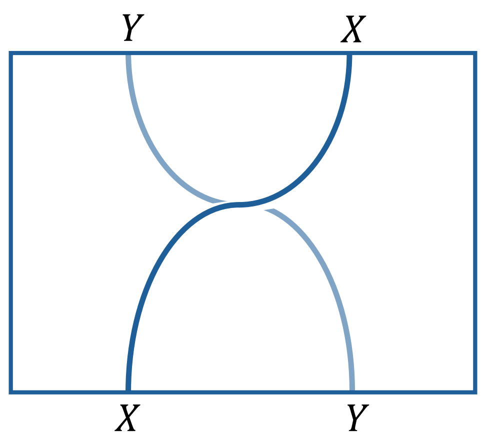


Figure 3.3: Braiding

the previous chapter, I will first discuss cyclic cohomology via cocyclic objects. Previous work on the topic, such as [Lod] and [Wei] define cocyclic objects in the context of a symmetric monoidal category. This is reflected in definition 3.2.2. I generalize cocyclic objects to a pivotal monoidal category in proposition 3.2.4, and provide one more additional definition of a cocyclic object in proposition 3.2.7.

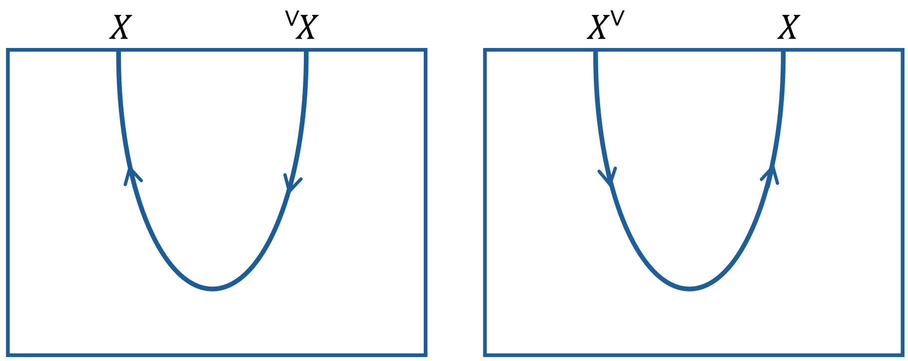
Definition 3.2.1 (Cyclic Category, Cyclic and Cocyclic Objects)**.** The cyclic category ΔC is a subcategory of finite sets. It has objects the non-empty finite totally ordered sets. The morphisms of ΔC are generated by the non-decreasing monotone functions together with the morphisms $t_n : [n] \to [n]$ where

$$t_n(j) := \begin{cases} j-1 & \text{if } j \neq 0 \\ n & \text{if } j = 0 \end{cases} .$$

A cyclic object X in a category $\mathcal{C}$ is a functor $X : \Delta C^{op} \to \mathcal{C}$. A cocyclic object X in a category $\mathcal{C}$ is a functor $X : \Delta C \to \mathcal{C}$. Morphisms of (co)cyclic objects are natural transformations of the functors.

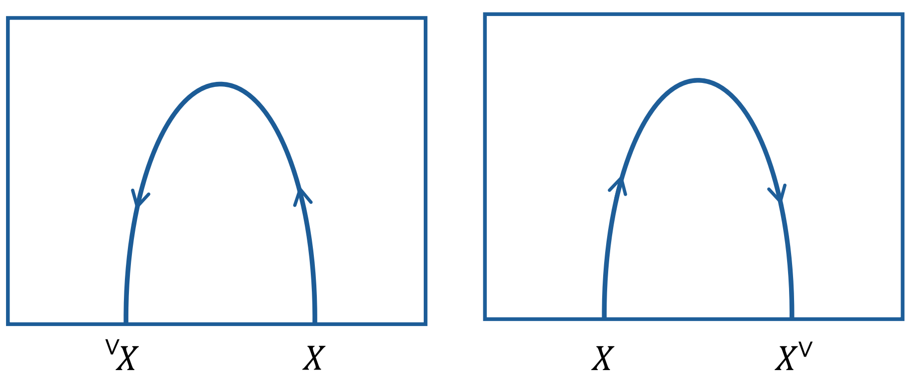
With some abuse of notation we will denote $\partial_i := X(\epsilon_i)$, $\sigma_i := X(\eta_i)$ and $\tau_n := X(t_n)$.

Associative algebras in symmetric monoidal categories and in pivotal monoidal categories provide examples of cocyclic objects. In fact, we will see that the ways to define cocyclic objects in a symmetric monoidal category, as in Definition 3.2.2, and in a pivotal monoidal category, as in Definition 3.2.4, will agree when the category is both symmetric and pivotal with the distinguished pivotal structure from 3.0.2.



(a) Left (b) Right

Figure 3.4: Coevaluations



(a) Left (b) Right

Figure 3.5: Evaluations

Proposition 3.2.2 (Loday Cocyclic Object)**.** *Let $(\mathcal{C}, \otimes, s)$ be a $\mathbb{K}$-linear* <u>*symmetric*</u> *monoidal category, and let $(A, \mu, \eta) \in \mathcal{C}$ be an algebra. Then we have a cocyclic object $\mathrm{LC}^\bullet(A) : \Delta C \to \mathbf{Vect}_{\mathbb{K}}$ given by*

$$\begin{aligned}
[n] &\mapsto \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n+1}, \mathbb{I}) \\
\delta_i(f) &:= \begin{cases} f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes n-i}}) & \text{if } 0 \leqslant i \leqslant n \\ f \circ (\mu \otimes \mathrm{id}_{A^{\otimes n}}) \circ (s_{A^{\otimes n+1}, A}) & \text{if } i = n+1 \end{cases} \\
\sigma_i(f) &:= f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \eta \otimes \mathrm{id}_{A^{\otimes n-i}}) \\
\tau_n(f) &:= f \circ (s_{A^{\otimes n}, A})
\end{aligned}$$

Remark 3.2.3. [Lod] does not explicitly work with cocyclic objects at all, nor does it work with cohomology theories. Instead, it works with homology and uses dualization to extend those results from homology to cohomology. As such, the definition here is extracted from how cyclic objects are set up at [Lod, 2.5.3].

In the following proposition I introduce another definition for cocyclic objects in a pivotal category. The definition of a cocyclic object involves the use of the family of morphisms τ, which naturally appear when working with a pivotal monoidal category.

Proposition 3.2.4 (Pivotal Cocyclic Object)**.** *Let $(\mathcal{C}, \otimes, \mathbb{I}, -^\vee, \omega)$ be a $\mathbb{K}$-linear* <u>*pivotal*</u> *monoidal category, and let $(A, \mu, \eta) \in \mathcal{C}$ be an algebra. Then we have a cocyclic object $\mathrm{PC}^\bullet(A) : \Delta C \to \mathbf{Vect}_{\mathbb{K}}$ given by*

$$\begin{aligned}
[n] &\mapsto \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n+1}, \mathbb{I}) \\
\delta_i(f) &:= \begin{cases} f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes n-i}}) & \text{if } 0 \leqslant i \leqslant n \\ \mathrm{eval}^{\ell}_A \otimes (\omega_{A^\vee} \otimes f \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A^\vee} \otimes \mu \otimes \mathrm{id}_{A^{\otimes n}}) \circ (\mathrm{coeval}^{r}_A \otimes \mathrm{id}_{A^{\otimes n+1}}) & \text{if } i = n+1 \end{cases} \\
\sigma_i(f) &:= f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \eta \otimes \mathrm{id}_{A^{\otimes n-i+1}}) \\
\tau_n(f) &:= \mathrm{eval}^{\ell}_A \circ (\omega_{A^\vee} \otimes f \otimes \mathrm{id}_A) \circ (\mathrm{coeval}^{r}_A \otimes \mathrm{id}_{A^{\otimes n+1}})
\end{aligned}$$

Proof. We have to show that $\mathrm{PC}^\bullet(A)$ is in fact a cocyclic object. We will leave the proof that that $\mathrm{PC}^\bullet(A)$ restricts to a cosimplicial object to the reader, as it just uses the associativity and unitality of A. This means that – following Loday's remarks at [Lod, 6.1.1], we only need to prove:

1. $\tau_n^{n+1} = \mathrm{id}$
2. $\tau_{n+1}\delta_i = \delta_{i-1}\tau_n$ for $1 \leqslant i \leqslant n+1$
3. $\tau_{n+1}\sigma_i = \sigma_{i-1}\tau_{n+2}$ for $1 \leqslant i \leqslant n+1$

We will do so graphically in diagrams 3.6, 3.7, 3.8, and 3.9. We will also explain what we are doing at each step.

1. This is the simplest one. We group all the strings together, and use the monoidality of the pivotality, evaluation and coevaluation morphisms to reduce it from considering $n+1$ strings labelled by A to considering one string labelled by $A^{\otimes n+1}$. Pivotality then lets us straighten out this string, and we can rewrite it as $n+1$ strings labelled by A, as in figure 3.6
2. We have to handle this one in two cases. If $i \neq n+1$ and if $i = n+1$.
   (a) If $i \neq n+1$ then it's a relatively simple counting exercise, as in figure 3.7
   (b) If $i = n+1$ then it comes down to the naturality of the ω, coevaluation and evaluation morphisms, as in figure 3.8.
3. This is again a relatively simple counting exercise, as in figure 3.9.

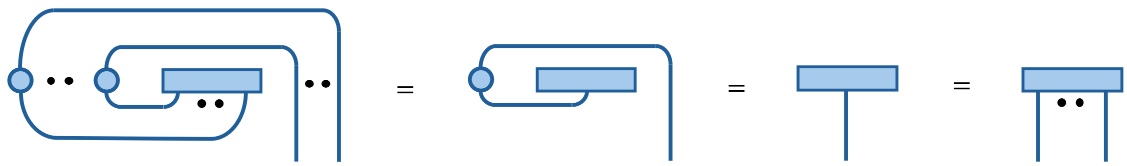


Figure 3.6: $\tau_n^{n+1} = \mathrm{id}$

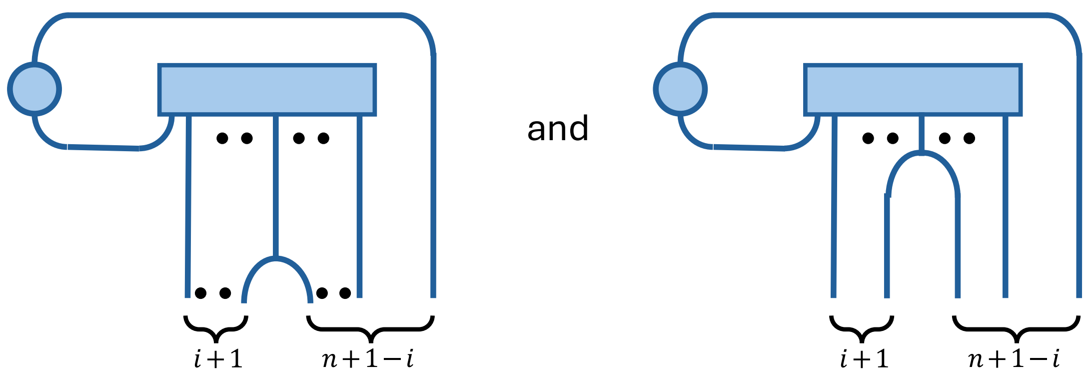


Figure 3.7: $\tau_{n+1}\delta_i = \delta_{i-1}\tau_n$ for $i \neq n+1$

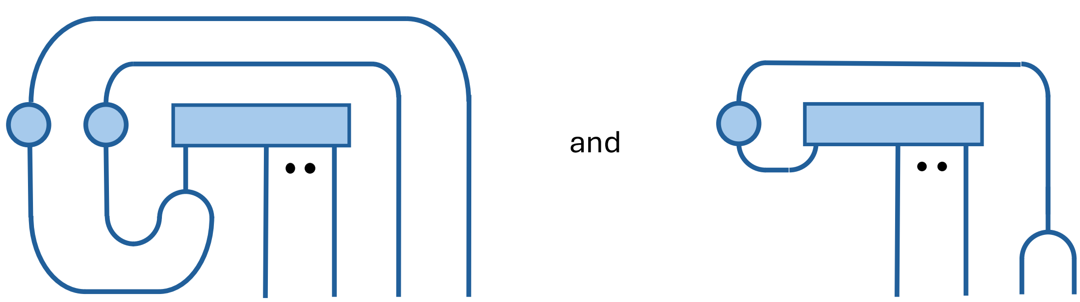


Figure 3.8: $\tau_{n+1}\delta_i = \delta_{i-1}\tau_n$ for $i = n+1$

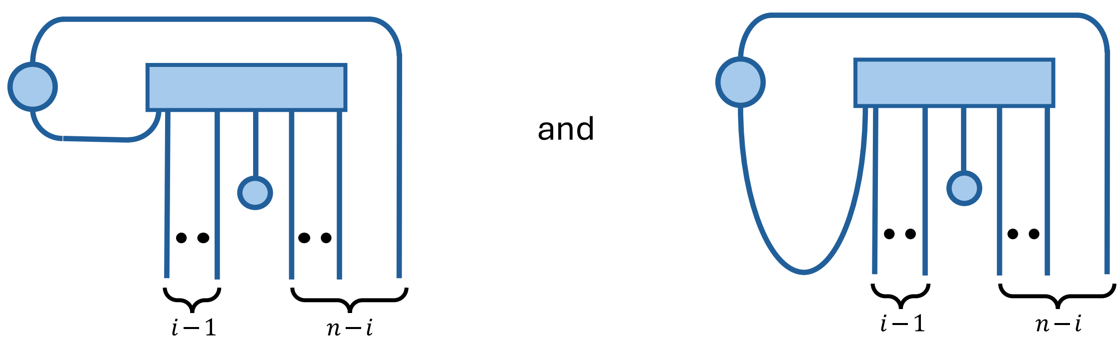


Figure 3.9: $\tau_{n+1}\sigma_i = \sigma_{i-1}\tau_{n+2}$

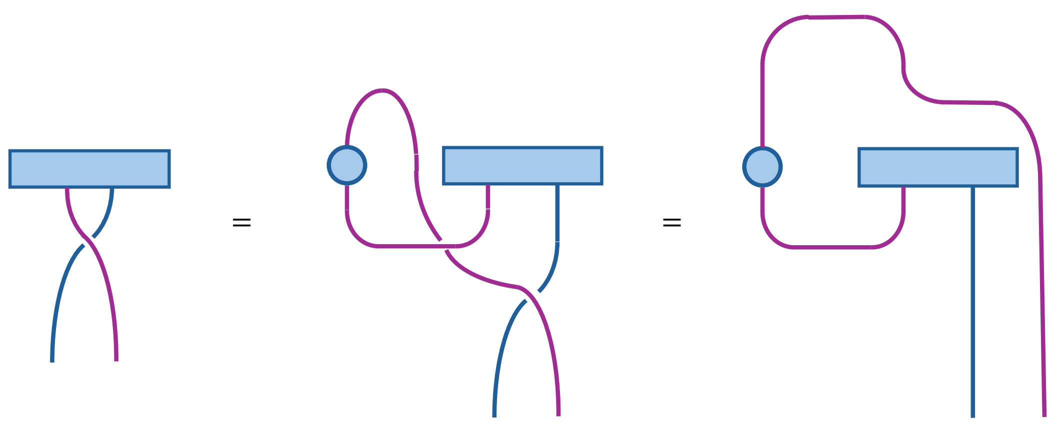


Figure 3.10: Equivalence of Cyclic actions

□

The following lemma proves that this definition of a cocyclic object in a pivotal monoidal category agrees with the definition of a cocyclic object in a symmetric monoidal category.

Lemma 3.2.5. *Let $(\mathcal{C}, \otimes, -^\vee, \omega, s)$ be a $\mathbb{K}$-linear symmetric, rigid monoidal category and endowed with its distinguished pivotal structure. Then for any algebra $A \in \mathcal{C}$, we have $\mathrm{LC}^\bullet(A) \cong \mathrm{PC}^\bullet(A)$, that is, the Loday cocyclic object and the pivotal cocyclic object coincide.*

Proof. Note that the definitions agree on everything except for the cyclic permutation of the inputs. We just need to show that the two cyclic permutations are the same. For that, it suffices to show that for any $f \in \mathrm{Hom}_\mathcal{C}(X \otimes Y, \mathbb{I})$, we have

$$f \circ s_{Y,X} = \mathrm{eval}^\ell_X \circ (\omega_{X^\vee} \otimes f \otimes \mathrm{id}_X) \circ \mathrm{coeval}^r_X \otimes \mathrm{id}_{Y \otimes X}\,.$$

This is easy to see with the graphical calculus in figure 3.10. We simply apply the result from remark 3.0.2, and then apply naturality of the braiding. □

Remark 3.2.6. Let $A := \mathbb{K}[x]$ be the algebra of polynomials in $\mathbb{K}$. As $A \in \mathbf{Vect}_\mathbb{K}$, and $\mathbf{Vect}_\mathbb{K}$ is symmetric, we get a Loday cocyclic object from the construction in 3.2.2. Could we construct a pivotal cocyclic object for $\mathbb{K}[x]$? The dimension of $\mathbb{K}[x]$ is $\aleph_0$, and the dimension of $\mathbb{K}[x]^*$ is $2^{\aleph_0}$. Since taking the dual of a vector space cannot drop the dimension, we get that the dimension of $\mathbb{K}[x]^{**}$ is strictly larger than the dimension of $\mathbb{K}[x]$. So, the algebra doesn't have a categorical dual, let alone a natural isomorphism to give us a pivotal structure.

Recall that there are a few equivalent definitions of a symmetric Frobenius algebra. For an overview, see [FS]. In particular, there are three definitions we care about.

Let $\mathcal{C}$ be a $\mathbb{K}$-linear pivotal monoidal category. First, an associative unital algebra $(A, \mu, \eta) \in \mathcal{C}$ with a bilinear form κ is a symmetric Frobenius algebra if κ is invariant with respect to μ, non-degenerate, and symmetric. Taking each in turn, κ is invariant with respect to μ if $\kappa \circ (\mu \otimes \mathrm{id}) = \kappa \circ (\mathrm{id} \otimes \mu)$. It is non-degenerate if $(\kappa \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coeval}^\ell_A)$ is invertible. Finally, it is symmetric if $\kappa = \mathrm{eval}^\ell_A \circ (\omega_{{}^\vee A} \otimes \kappa \otimes \mathrm{id}) \circ (\mathrm{coeval}^\ell_A \otimes \mathrm{id} \otimes \mathrm{id}))$. We say such a κ is a symmetric Frobenius structure.

Equivalently, an associative unital algebra (A, μ, η) which is also a coassociative counital coalgebra (A, Δ, ε) is a symmetric Frobenius algebra if it satisfies the identities

$$(\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ \mu = (\mathrm{id}_A \otimes \mu) \circ (\Delta \otimes \mathrm{id}_A),$$

and, crucially, for all $x, y \in A$, $\varepsilon(\mu(x, y)) = \varepsilon(\mu(y, x))$.

Finally, for an associative unital algebra (A, μ, η), we can regard the right dual $A^\vee$ as a coassociative counital coalgebra by dualizing the multiplication and unit morphisms. This lets us give $A^\vee$ the structure of an A bimodule via

$$\lambda := (\mathrm{id}_{A^\vee} \otimes \mathrm{eval}^r_A) \circ (\mu^* \otimes \mathrm{id}_A)$$
$$\rho := (\mathrm{eval}^\ell_A \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_A \otimes \omega_{A^\vee} \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_A \otimes \mu^*).$$

Then symmetric Frobenius algebra structures on A are in bijection with A-bimodule isomorphisms $\Phi : A \to A^\vee$. Importantly, note that by a similar argument these are also in bijection with A-bimodule isomorphisms $\Psi : A \to {}^\vee A$. To make these bijections explicit, $\Phi = (\mathrm{id}_{A^\vee} \otimes (\varepsilon \circ \mu)) \circ (\mathrm{coeval}^r \otimes \mathrm{id}_A)$ and $\Psi = ((\varepsilon \circ \mu) \otimes \mathrm{id}_{{}^\vee A}) \circ (\mathrm{id}_A \otimes \mathrm{coeval}^\ell)$.

Note that to formulate the notion of a symmetric Frobenius algebra, the ambient category has to be pivotal.[FS, 4] As such, suppose that $\mathcal{C}$ is pivotal and furthermore that A has the structure of a symmetric Frobenius algebra. With that, I define a cocyclic object analogous to the pivotal cocyclic object of 3.2.4. I use $\varepsilon \circ \mu$ as the evaluation morphism and $\Delta \circ \eta$ as the coevaluation morphism.

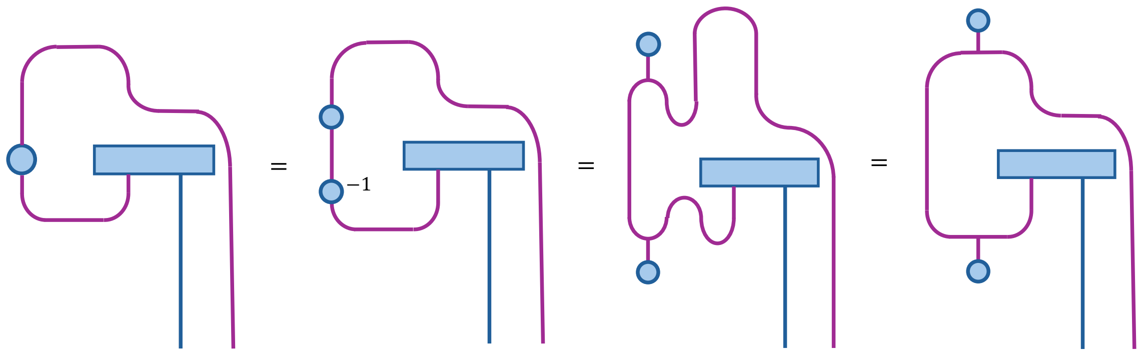


Figure 3.11: Definition of sFC.

Proposition 3.2.7 (Symmetric Frobenius Cocyclic Object)**.** *Let $(\mathcal{C}, \otimes, -^{\vee}, \omega)$ be a $\mathbb{K}$-linear* *pivotal* *monoidal category, and let $(A, \mu, \eta, \Delta, \epsilon) \in \mathcal{C}$ be a symmetric Frobenius algebra. Then we have a cocyclic object $sFC^{\bullet}(A) : \Delta C \to \mathbf{Vect}_{\mathbb{K}}$ given by*

$$[n] \mapsto \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n+1}, \mathbb{I})$$

$$\delta_i(f) := \begin{cases} f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes n-i}}) & \text{if } 0 \leqslant i \leqslant n \\ \epsilon \circ \mu \circ (\mathrm{id}_A \otimes f \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \mu \otimes \mathrm{id}_{A^{\otimes n}}) \circ ((\Delta \circ \eta) \otimes \mathrm{id}_{A^{\otimes n+1}}) & \text{if } i = n+1 \end{cases}$$

$$\sigma_i(f) := f \circ (\mathrm{id}_{A^{\otimes i}} \otimes \eta \otimes \mathrm{id}_{A^{\otimes n-i}})$$

$$\tau_n(f) := \epsilon \circ \mu \circ (\mathrm{id}_A \otimes f \otimes \mathrm{id}_A) \circ ((\Delta \circ \eta) \otimes \mathrm{id}_{A^{\otimes n+1}})$$

Note that in this definition, the Frobenius structure enters in the morphisms δ_{n+1} and τ_n.

I will skip the direct proof that $sFC^{\bullet}(A)$ is a cocyclic object by instead proving that it coincides with another cocyclic object.

Lemma 3.2.8. *Let $(\mathcal{C}, \otimes, -^{\vee}, \omega)$ be a $\mathbb{K}$-linear pivotal category. Then for any symmetric Frobenius algebra $A \in \mathcal{C}$, we have $sFC^{\bullet}(A) \cong PC^{\bullet}(A)$, that is, the pivotal cocyclic object and the symmetric Frobenius algebra cocyclic object coincide. Note that this also implies that the symmetric Frobenius algebra cocyclic object is independent of the specific Frobenius structure.*

Proof. Again, the only thing we need to check is the cyclic permutation of the inputs. Here we just need to check that for any $f \in \mathrm{Hom}_{\mathcal{C}}(A \otimes X, \mathbb{I})$, we have

$$\mathrm{eval}^{\ell}_A \circ (\omega_{A^*} \otimes f \otimes \mathrm{id}_A) \circ \mathrm{coeval}^r_A \otimes \mathrm{id}_{X \otimes A} = \epsilon \circ \mu \circ (\mathrm{id}_A \otimes f \otimes \mathrm{id}_A) \circ ((\Delta \circ \eta) \otimes \mathrm{id}_{X \otimes A}).$$

This is even easier to see with the graphical calculus than the proof for lemma 3.2.5, as seen in figure 3.11. In the first move, we replace $\omega_{A^{\vee}} : A^{\vee} \to {}^{\vee}A$ by the composition $A^{\vee} \xrightarrow{\Phi^{-1}} A \xrightarrow{\Psi} {}^{\vee}A$. From there, it is simply expressing the bimodule isomorphisms in the graphical calculus and applying relations.

There are a few things we should take note of in the proof. First, we need that A is symmetric, in particular so we can decompose $\omega_{A^{\vee}}$ into two bimodule isomorphisms. Second, consider two symmetric Frobenius algebras A and A' that are isomorphic as algebras but not isomorphic as Frobenius algebras. As they are isomorphic as algebras, they will have isomorphic pivotal cocyclic objects, $PC^{\bullet}(A) \cong PC^{\bullet}(A')$. This means that $sFC(A) \cong PC^{\bullet}(A) \cong PC^{\bullet}(A') \cong sFC(A')$, or they have isomorphic symmetric Frobenius cocyclic objects. In particular, if one endows the same underlying algebra with two distinct symmetric Frobenius algebras, the symmetric Frobenius cocyclic objects will be isomorphic. □

Remark 3.2.9. The above proof may seem to make Proposition 3.2.7 redundant, but it actually serves two main goals: first, it provides a glimpse into how tightly cyclic cohomology and symmetric Frobenius algebras are linked. Second, in later work, where we plan to generalize from rigid monoidal categories to pivotal GV-categories, generalizing Definition 3.2.7 will help us make some computations.

3.3 Cyclic Cohomology

The notion of cyclic cohomology is built on the notion of Hochschild cohomology, much in the same way that cocyclic objects are built via the structure of cosimplicial objects. We begin with deriving a cyclic cohomology from a cocyclic object in Definition 3.3.1. This requires a few extra bits of machinery, but with those defined the construction of cyclic cohomology is clear. However, the definition of cyclic cohomology in Definition 3.3.1 is often difficult to compute with, and is not clearly directly connected to symmetric Frobenius algebras. To help with this, we introduce another notion of cohomology in Proposition 3.3.8, with a clearer relation to symmetric Frobenius algebras, and show in Theorem 3.3.10 that these two notions of cohomology agree when the base field is of characteristic 0.

The cocyclic category has more morphisms than the simplicial category, and so we will introduce more operators on a cocyclic object in a linear category than in the simplicial case from Lemma 2.1.7. First is the norm operator, $N := \sum_{i=0}^{n}(-1)^i\tau_n^i$ – note that here the superscript i denotes iterating the operator τ_n i times – and second is the reduced differential operator $b' := \sum_{i=0}^{n-1}(-1)^i\delta_i$. Note that b' has one less summand than b.

Definition 3.3.1 (Tsygan's Double Complex, Cyclic Cohomology)**.** [Wei, 9.6.6] Let $\mathcal{B}$ be a $\mathbb{K}$-linear abelian category and $X^\bullet : \Delta C \to \mathcal{B}$ a cocyclic object. Then there is a bounded double complex, called Tsygan's Double Complex, say $\mathrm{Tsy}^{\bullet\bullet}(X)$, as shown in 3.1. Then the cyclic cohomology of X is defined as $HC^\bullet(X) := H^\bullet(\mathrm{Tot}(\mathrm{Tsy}^{\bullet\bullet}(X)))$.

Consider the first column. This is the same as the cochain complex we get from forgetting the morphism τ and applying Lemma 2.1.7 to the resultant simplicial object X.

Using this, we also will define the simplicial cohomology associated to X by $HH^\bullet(X) := H^\bullet(X, b^\bullet) \cong H^\bullet(\mathrm{Tsy}^{0,\bullet}(X))$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
X^2 \arrow[u, "b"] \arrow[r, "1-\tau"] & X^2 \arrow[u, "-b'"'] \arrow[r, "N"] & X^2 \arrow[u, "b"] \arrow[r, "1-\tau"] & X^2 \arrow[u, "-b'"'] \arrow[r, "N"] & \cdots \\
X^1 \arrow[u, "b"] \arrow[r, "1+\tau"] & X^1 \arrow[u, "-b'"'] \arrow[r, "N"] & X^1 \arrow[u, "b"] \arrow[r, "1+\tau"] & X^1 \arrow[u, "-b'"'] \arrow[r, "N"] & \cdots \\
X^0 \arrow[u, "b"] \arrow[r, "1-\tau"] & X^0 \arrow[u, "-b'"'] \arrow[r, "N"] & X^0 \arrow[u, "b"] \arrow[r, "1-\tau"] & X^0 \arrow[u, "-b'"'] \arrow[r, "N"] & \cdots
\end{tikzcd} \tag{3.1}$$

Remark 3.3.2. When A is an algebra, $\mathcal{C}$ is a $\mathbb{K}$-linear symmetric monoidal category, then we use the Loday cocyclic object, $LC^\bullet(A)$, from definition 3.2.2 to write $HC^\bullet(A) := HC^\bullet(LC^\bullet(A))$ and $HH^\bullet(A) := HH^\bullet(LC^\bullet(A))$. If instead $\mathcal{C}$ is a $\mathbb{K}$-linear pivotal monoidal category we use the pivotal cocyclic object, $PC^\bullet(A)$, from definition 3.2.4 to write $HC^\bullet(A) := HC^\bullet(PC^\bullet(A))$ and $HH^\bullet(A) := HH^\bullet(PC^\bullet(A))$.

One of the most significant results related to Tsygan's double complex is the construction of a long exact sequence of period 2 in cohomology. Before we prove Theorem 3.3.4, we first want to prove a small lemma.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
* \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
* \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
* \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
* \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

(a) $U(X)$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

(b) $V(X)$

Figure 3.12: The decomposition of Tsy(X)

Lemma 3.3.3. *Let X, Y, Z be double complexes in an abelian category $\mathcal{B}$. If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence of bounded double complexes, then there is a short exact sequence $0 \to \mathrm{Tot}(X) \to \mathrm{Tot}(Y) \to \mathrm{Tot}(Z) \to 0$ of cochain complexes.*

Proof. First, $f : X \to Y$ is a family of morphisms $f^{i,j} : X^{i,j} \to Y^{i,j}$ that respect the differentials, that is,

$$f^{i,j+1} \circ d^h = d^h \circ f^{i,j}$$

and

$$f^{i+1,j} \circ d^v = d^v \circ f^{i,j}.$$

This determines a map

$$\bar{f}^n : \bigoplus_{p+q=n} X^{p,q} \to \bigoplus_{p+q=n} Y^{p,q}$$

such that $\bar{f} : \mathrm{Tot}(X) \to \mathrm{Tot}(Y)$ is a morphism of chain complexes. Similarly we get $\bar{g} : \mathrm{Tot}(Y) \to \mathrm{Tot}(Z)$. It remains to show that $\mathrm{im}(\bar{f}) = \ker(\bar{g})$. Say $(y_{(p,q)})_{p+q=n} \in \mathrm{Tot}^n(Y)$ is in the kernel of $\bar{g}$. That is, $g^{p,q}(y_{(p,q)}) = 0$. As $\mathrm{im}\, f = \ker g$, there must exist some $(x_{(p,q)})_{p+q=n} \in \mathrm{Tot}^n(X)$ such that $f^{p,q}(x_{(p,q)}) = y_{(p,q)}$. That is, $\ker \bar{g} \supset \mathrm{im}\, \bar{f}$. For the other direction, simply note that $g^{p,q} \circ f^{i,j} = 0$ for all p, q, i, j. ☐

Theorem 3.3.4 (Connes' Long Exact Sequence of Periodicity Two)**.** *[Lod, 2.2.1, 2.4.4] Let X be a cocyclic object in a $\mathbb{K}$-linear abelian category $\mathcal{C}$. Then there is a natural long exact sequence*

$$\cdots \to HH^n(X) \xrightarrow{B} HC^{n-1}(X) \xrightarrow{S} HC^{n+1}(X) \xrightarrow{I} HH^{n+1}(X) \to \cdots \tag{3.2}$$

This is called the Connes long exact sequence in cohomology.

Proof. Consider $\mathrm{Tsy}^{\bullet\bullet}(X)$. We decompose it into two bicomplexes; $U(X) := \mathrm{Tsy}^{i\bullet}(X)$ with $i \in \{0, 1\}$ and zero elsewhere, and $V(X) := \mathrm{Tsy}^{j\bullet}(X)$ with $j \geqslant 2$ and zero elsewhere. These are illustrated in diagram 3.12. It is clear from the direct sum decomposition that we get an exact sequence of bicomplexes:

$$0 \to V(X) \to \mathrm{Tsy}(X) \to U(X) \to 0. \tag{3.3}$$

By Lemma 3.3.3 this gives us a short exact sequence of the total complexes

$$0 \to \mathrm{Tot}(V(X)) \to \mathrm{Tot}(\mathrm{Tsy}(X)) \to \mathrm{Tot}(U(X)) \to 0 \tag{3.4}$$

This gives us a long exact sequence in cohomology

$$\cdots \to H^n(\mathrm{Tot}(U(X))) \to H^{n+1}(\mathrm{Tot}(V(X))) \to H^{n+1}(\mathrm{Tot}(\mathrm{Tsy}(X))) \to H^{n+1}(\mathrm{Tot}(U(X))) \to \cdots \quad (3.5)$$

Now, by definition $H^n(\mathrm{Tot}(\mathrm{Tsy}(X))) \cong HC^n(X)$. Note that by the periodicity of the Tsygan double complex, $V(X)$ is the same as $\mathrm{Tsy}(X)$ just shifted by two. This means that $H^n(\mathrm{Tot}(V(X))) \cong HC^{n-2}(X)$. Finally, $U(X)$ is made up of two columns. The second column, $\mathrm{Tsy}^{1,\bullet}(X)$ is a contractable complex by [Lod, 1.1.12] which means that $H^n(\mathrm{Tot}(U(X))) \cong H^n(\mathrm{Tsy}^{0,\bullet}(X))$ by lemma [Lod, 2.1.6]. By definition this gives $H^n(\mathrm{Tot}(U(X))) \cong HH^n(X)$. Putting this all together gives us

$$\cdots \to HH^n(X) \to HC^{n-1}(X) \to HC^{n+1}(X))) \to HH^{n+1}(X)) \to \cdots \quad (3.6)$$

Which is exactly the long exact sequence from Equation (3.2). □

As mentioned at the start of Section 3.3, there are two main problems with this definition of cyclic cohomology. First of all, it is not clear how to give an interpretation of HC in terms of a symmetric Frobenius algebra. Secondly, this definition of cyclic cohomology can be difficult to compute in certain cases. We can see this just by considering the dimension of $\mathrm{Tot}^n(\mathrm{Tsy}(A))$ for a $\mathbb{K}$-algebra A. This can quickly grow out of control if the dimension of A as a $\mathbb{K}$-algebra is sufficiently large. We will resolve both of these problems by introducing another cohomology theory. We will begin with a few preliminaries.

Definition 3.3.5. Let X be a cocyclic object in a $\mathbb{K}$-linear abelian category $\mathcal{B}$. Define the nth cyclic subobject C^n_λ as the kernel of $1-(-1)^n\tau_n : X^n \to X^n$. If objects in $\mathcal{B}$ have elements, then a cochain $f \in X^n$ is called cyclic if it satisfies $f-(-1)^n\tau_n(f) = 0$. We will often call C^n_λ the cyclic cochains even if objects in $\mathcal{B}$ are not assumed to have elements.

Example 3.3.6. [PS] Let A be an associative and unital algebra in $\mathbf{Vect}_{\mathbb{K}}$. Then the cyclic cochains in the Loday cocyclic object $LC^\bullet(A)$, introduced in 3.2.2, are the maps $f : A^{\otimes n+1} \to \mathbb{K}$ such that

$$f(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1}), \quad a_i \in A. \quad (3.7)$$

Lemma 3.3.7. *Let X be a cocyclic object in a $\mathbb{K}$-linear abelian category $\mathcal{B}$, and assume that objects in $\mathcal{B}$ have elements. Then there is a cochain complex $C^\bullet_\lambda(X)$, where the cochain groups are composed of the cyclic cochains, that is, $C^n_\lambda(X) := \{f \in X^n : f-(-1)^n\tau_n(f) = 0\}$, and the differential is the Hochschild differential. This is inherited from regarding X as a simplicial object by forgetting τ, or by taking the first column of Tsygan's double complex.*

Proof. It suffices to prove that if f is cyclic then bf is cyclic.

$$\begin{aligned}
bf-(-1)^{n+1}\tau_{n+1}(bf) &= \sum_{i=0}^{n+1}(-1)^i\partial_i f-(-1)^{n+1}\sum_{i=0}^{n+1}(-1)^i\tau_{n+1}\partial_i f \\
&= \sum_{i=0}^{n+1}(-1)^i\partial_i f-(-1)^{n+1}\left[\partial_{n+1}f+\sum_{i=0}^{n}(-1)^{i+1}\partial_i\tau_n f\right] \\
&= (-1)^{n+1}\partial_{n+1}f-(-1)^{n+1}\partial_{n+1}f+\sum_{i=0}^{n}\left[(-1)^i\partial_i f-(-1)^{i+n+2}\partial_i\tau_n f\right] \\
&= \sum_{i=0}^{n}\left[(-1)^i\partial_i(f-(-1)^n\tau_n f)\right] \\
&= 0
\end{aligned}$$

□

Definition 3.3.8 (Connes' Definition of Cyclic Cohomology)**.** [Lod, 1.5.1, 2.4.2] Let $\mathcal{B}$ be a $\mathbb{K}$-linear abelian category and $X^\bullet : \Delta C \to \mathcal{B}$ a cocyclic object. Define the Connes cohomology by

$$H^n_\lambda(X) := H^n(C^\bullet_\lambda(X)). \tag{3.8}$$

Note here that the subscript λ is just notation and not a parameter.

With the following results, we can show that when the base field is of characteristic 0, the two notions of cohomology given in Definition 3.3.1 and Definition 3.3.8 are the same. First of all, note that for a cocyclic object X, we have a canonical injective map of cochain complexes $C^\bullet_\lambda(X) \hookrightarrow \mathrm{Tot}(\mathrm{Tsy}^{\bullet\bullet}(X))$.

Lemma 3.3.9. *Let X be a cocyclic object in a $\mathbb{K}$-linear abelian category $\mathcal{B}$. There exists an injective map of cochain complexes $\iota_X : C^\bullet_\lambda(X) \hookrightarrow \mathrm{Tot}(\mathrm{Tsy}^{\bullet\bullet}(X))$.*

Proof. We will define ι_X term-wise. First, note that by as $C^n_\lambda(X)$ is defined as a kernel, we have a natural inclusion map $C^n_\lambda(X) \hookrightarrow X^n = \mathrm{Tsy}^{0,n}(X)$ which is automatically injective. This is depicted in diagram (3.9).

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^2_\lambda(X) \arrow[u, "b"] \arrow[r, hook] & X^2 \arrow[u, "b"] \arrow[r, "1-\tau"] & X^2 \arrow[u, "-b'"] \arrow[r, "N"] & \cdots \\
C^1_\lambda(X) \arrow[u, "b"] \arrow[r, hook] & X^1 \arrow[u, "b"] \arrow[r, "1+\tau"] & X^1 \arrow[u, "-b'"] \arrow[r, "N"] & \cdots \\
C^0_\lambda(X) \arrow[u, "b"] \arrow[r, hook] & X^0 \arrow[u, "b"] \arrow[r, "1-\tau"] & X^0 \arrow[u, "-b'"] \arrow[r, "N"] & \cdots
\end{tikzcd} \tag{3.9}$$

There is then another canonical map $\mathrm{Tsy}^{\bullet\bullet}(X) \to \mathrm{Tot}(\mathrm{Tsy}^{\bullet\bullet}(X))$ as we have defined the double complex using the direct sum. Define ι^n_X as the composition of these two canonical maps. It remains to show that ι_X as defined is injective and a map of cochains.

We will start with injectivity. Note that the surjection $\mathrm{Tsy}^{\bullet\bullet}(X) \to \mathrm{Tot}(\mathrm{Tsy}^{\bullet\bullet}(X))$ sends terms of degree (p, q) to terms of degree $p+q$. So, if $n_0 \neq n_1$, the surjection will send the image of $C^{n_0}_\lambda(X)$ and $C^{n_1}_\lambda(X)$ to components of different degrees. This, combined with the injectivity of the inclusion, entails that ι_X must be injective.

To see that it is a map of cochain complexes, it suffices to notice that $C^\bullet_\lambda(X)$ is defined as the kernel of the first horizontal differentials. □

For the next theorem, recall that for an abelian category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, A)$ canonically has the structure of an abelian group.

Theorem 3.3.10. *[Lod, 2.4.2] Let X be a cocyclic object in a $\mathbb{K}$-linear abelian category $\mathcal{C}$. Consider the action of the integers $\mathbb{Z}$ on the abelian groups $\mathrm{Hom}_{\mathcal{C}}(X^{m-1}, X^{m-1})$. Let k be the maximal integer such that for all $m < k$, the action of m on $\mathrm{Hom}_{\mathcal{C}}(X^{m-1}, X^{m-1})$ is invertible. Then the injection from Lemma 3.3.9 induces a collection of isomorphisms for all $n \leqslant k$, or just on all n if there is no such maximal k:*

$$H^n_\lambda(X) \xrightarrow{\cong} HC^n(X), \quad n \geqslant 0. \tag{3.10}$$

For a proof of this theorem, see Appendix A.

Corollary 3.3.11. *Let A be an associative algebra over a field $\mathbb{K}$ of characteristic k. Then we have a sequence of isomorphisms for $m < k$,*

$$H^m_\lambda(LC(A)) \xrightarrow{\cong} HC^m(LC(A))$$

Lemma 3.3.12. *Suppose A is a symmetric Frobenius algebra in $\mathbf{FinVect}_{\mathbb{K}}$. Then the associated cohomology of the simplicial object we get from the algebra is isomorphic to the Hochschild cohomology of the algebra with coefficients in the regular bimodule ${}_AA_A$ of A. That is, $HH^\bullet(A) \cong H^\bullet(A;A)$.*

Proof. From above we have that one definition of a symmetric Frobenius algebra over $\mathbf{FinVect}_{\mathbb{K}}$ is that there is a bimodule isomorphism $\Phi : A \to A^*$, and $A^* \cong \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$.

The chain groups we work with when considering $HH^\bullet(A)$ are $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes n+1}, \mathbb{K})$, and the chain groups we work with when considering $H^\bullet(A;A)$ are $\mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, A)$. Consider now the sequence of maps

$$\mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, A) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, A^*) \to \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n} \otimes A, \mathbb{K}).$$

The first map is given by post-composition with Φ, and the second is given by the tensor-hom adjunction. As Φ is a bimodule isomorphism and the tensor-hom adjunction is indeed an adjunction, this will give us an isomorphism.

Note, importantly, that the isomorphism takes the form $f \mapsto ((a_0, \dots, a_n) \mapsto \kappa(f(a_0, \dots, a_{n-1}), a_n))$.

It is straightforward to check that these isomorphisms commute with the differentials, which means that it is a quasi-isomorphism and induces an isomorphism on the level of cohomology. ☐

Remark 3.3.13. Given a map $f \in \mathrm{Hom}_{\mathbb{K}}(A^{\otimes n}, \mathbb{K})$, we will write $\tilde{f}$ for the map

$$(a_0, \dots, a_n) \mapsto \kappa(f(a_0, \dots, a_{n-1}), a_n).$$

3.4 Deformations of a Symmetric Frobenius Algebra

The following section, while mostly novel results, builds heavily on the results from Section 2.2. It also builds on [PS].

Suppose V is a $\mathbb{K}$-vector space with an inner product $\beta : V \otimes V \to \mathbb{K}$. Define a $\mathbb{D}$-vector space V_t by $V_t := V \oplus tV$. Then we can lift the inner product β to a $\mathbb{D}$-bilinear inner product $\hat{\beta}$ by the formula

$$\hat{\beta}(v + t\bar{v}, w + t\bar{w}) := \beta(v, w) + t[\beta(v, \bar{w}) + \beta(\bar{v}, w)].$$

It is trivial to check that this defines a $\mathbb{D}$-bilinear form $\hat{\beta} : V_t \otimes V_t \to \mathbb{D}$.

Since a symmetric Frobenius algebra can be considered as an associative algebra with a symmetric, non-degenerate bilinear form that is invariant with respect to the multiplication, we can now define another type of deformation.

Definition 3.4.1 (Symmetric Frobenius Deformation)**.** Let (A, μ, β) be a symmetric Frobenius algebra, that is an associative algebra with a symmetric, non-degenerate, inner product β that is invariant with respect to μ. We say a deformation (A_t, μ_t) preserves the inner product if $\hat{\beta}$ is symmetric, non-degenerate and invariant with respect to μ_t. We will write such a deformation as $(A_t, \mu_t, \hat{\beta})$.

Furthermore, two deformations that preserve the inner product $(A_t, \mu_t, \hat{\beta})$ and $(A_t, \mu'_t, \hat{\beta})$ are equivalent as deformations that preserve the inner product if there exists a linear bijection $\rho_t : A_t \to A_t$ of the form $\rho_t = \mathrm{id} + t\lambda$ such that $\mu'_t = \rho_t^{-1} \circ \mu_t \circ (\rho_t \otimes \rho_t)$ and $\hat{\beta} = \hat{\beta} \circ (\rho_t \otimes \rho_t)$. We will say a deformation is trivial if it is equivalent to $(A_t, \mu_0, \hat{\beta})$. In that case we will call ρ_t the trivializer of the deformation.

Note that often we will, by abuse of notation, write $\hat{\beta}$ as β.

Lemma 3.4.2. *Let (A, μ, κ) be a symmetric Frobenius algebra, with $(A_t, \mu_t, \hat{\kappa})$ and $(A_t, \mu'_t, \hat{\kappa})$ two equivalent deformations with $\mu_t = \mu + t\varphi$ and $\mu'_t = \mu + t\psi$ and linear bijection $\rho_t := \mathrm{id} + t\lambda$. Then $\tilde{\lambda}$, as discussed in Remark 3.3.13 is cyclic.*

Proof. By the fact ρ_t preserves the bilinear form, we have that

$$\begin{aligned}\hat{\kappa}(x,y) &= \hat{\kappa}(\rho_t(x),\rho_t(y))\\ &= \hat{\kappa}(x+t\lambda(x), y+t\lambda(y))\\ &= \hat{\kappa}(x,y)+\hat{\kappa}(t\lambda(x),y)+\hat{\kappa}(x,t\lambda(y))\end{aligned}$$

This in particular implies that

$$\tilde{\lambda}(x,y)=\kappa(\lambda(x),y)=-\kappa(x,\lambda(y))=-\kappa(\lambda(y),x)=-\tilde{\lambda}(y,x).$$

□

Lemma 3.4.3. *Given (A,μ,β) an associative algebra with a symmetric, non-degenerate, inner product β that in invariant with respect to μ, a deformation (A_t,μ_t) preserves the inner product if and only if $\tilde{\varphi}(a,b,c):=\hat{\beta}(\varphi(a,b),c)$ is cyclic as in Definition 3.3.8.*

Proof. In order for (A_t,μ_t) to preserve the inner form, we need three conditions. Two of these concern only $\hat{\beta}$, as it must be symmetric and non-degenerate. Both of these conditions are left as an exercise for the reader as they are always preserved regardless of if $\tilde{\varphi}$ is cyclic or not. This leaves us with showing that $\tilde{\varphi}$ is cyclic if and only if $\hat{\beta}$ is invariant with respect to μ_t.

Expanding the invariance of $\hat{\beta}\circ(\mu_t\otimes \mathrm{id}_{A_t})=\hat{\beta}\circ(\mathrm{id}_{A_t}\otimes\mu_t)$ on the left and then on the right we first get on the left:

$$\begin{aligned}&\hat{\beta}(\mu_t(x+t\bar{x},y+t\bar{y}),z+t\bar{z})\\ &=\hat{\beta}(xy+t\varphi(x,y)+t\bar{x}y+tx\bar{y},z+t\bar{z})\\ &=\beta(xy,z)+t(\beta(\varphi(x,y),z)+\beta(\bar{x}y,z)+\beta(x\bar{y},z)+\beta(xy,\bar{z}))\end{aligned}$$

Then on the right:

$$\begin{aligned}&\hat{\beta}(x+t\bar{x},\mu_t(y+t\bar{y},z+t\bar{z}))\\ &=\hat{\beta}(x+t\bar{x},yz+t\varphi(y,z)+t\bar{y}z+ty\bar{z})\\ &=\beta(x,yz)+t(\beta(x,\varphi(y,z))+\beta(\bar{x},yz)+\beta(x,\bar{y}z)+\beta(x,y\bar{z}))\end{aligned}$$

Using the invariance of β, we find that $\hat{\beta}$ is invariant if and only if $\beta(\varphi(x,y),z)=\beta(x,\varphi(y,z))$.

Suppose first that $\hat{\beta}$ is invariant. By the symmetry of β, this implies that

$$\tilde{\varphi}(x\otimes y\otimes z)=\beta(\varphi(x,y),z)=\beta(x,\varphi(y,z))=\beta(\varphi(y,z),x)=\tilde{\varphi}(y\otimes z\otimes x).$$

Now suppose that $\tilde{\varphi}$ is cyclic. Then

$$\beta(\varphi(x,y),z)=\tilde{\varphi}(x\otimes y\otimes z)=\tilde{\varphi}(y\otimes z\otimes x)=\beta(\varphi(y,z),x)=\beta(x,\varphi(y,z)).$$

This proves what we need to prove. □

Remark 3.4.4. This is rephrasing a statement in [PS, §2] in more familiar notation.

This lemma lets us prove the following theorem; that deformations of symmetric Frobenius algebras are controlled by the second Connes cohomology group.

Theorem 3.4.5. *Given a symmetric Frobenius algebra (A,μ,κ) and a deformation that preserves the inner product $(A_t,\mu_t,\hat{\kappa})$ with $\mu_t=\mu+t\varphi$, the μ_t determines an element $[\tilde{\varphi}]\in H^2_\lambda(A)$. Furthermore, a second deformation $(A_t,\mu_t',\hat{\kappa})$ with $\mu_t'=\mu+t\psi$ is equivalent to $(A_t,\mu_t,\hat{\kappa})$ if and only if $[\tilde{\varphi}]=[\tilde{\psi}]$.*

Proof. First note that $\tilde{\varphi}$ is cyclic in the sense of (3.7) by Lemma 3.4.3. Next, note that by Lemma 2.2.6, φ is a 2-cocycle in the Hochschild cochain complex $C^\bullet(A;A)$. Combining this with Lemma 3.3.12, we see that $\tilde{\varphi}$ as defined above is a 2-cocycle of $C^\bullet_\lambda(A)$. This necessarily determines an element of $H^2_\lambda(A)$.

Now assume that $(A_t, \mu_t, \hat{\kappa})$ is equivalent to $(A_t, \mu'_t, \hat{\kappa})$. That is, there is some $\rho_t = \mathrm{id} + t\lambda$ such that $\mu'_t = \rho_t^{-1} \circ \mu_t \circ (\rho_t \otimes \rho_t)$ and $\hat{\kappa} = \hat{\kappa} \circ (\rho_t \otimes \rho_t)$. By Lemma 3.4.2, we know that $\tilde{\lambda}$ is cyclic.

Expanding out $\mu'_t = \rho_t^{-1} \circ \mu_t \circ (\rho_t \otimes \rho_t)$ in functional notation gives us the unwieldy

$$\mu + t\psi = (\mathrm{id} - t\lambda) \circ (\mu + t\varphi) \circ [(\mathrm{id} + t\lambda) \otimes (\mathrm{id} + t\lambda)].$$

Expanding it further via linearity and considering the non-trivial t component, we get the two equations

$$\psi = \varphi + \mu \circ (\lambda \otimes \mathrm{id}) - \lambda \circ \mu + \mu \circ (\mathrm{id} \otimes \lambda)$$
$$\psi = \varphi + b\lambda$$

This implies that $[\psi] = [\varphi]$ in $H^2(A;A)$. By Lemma 3.3.12 and the fact $\tilde{\lambda}$ is cyclic, this implies that $[\tilde{\psi}] = [\tilde{\varphi}]$. For the converse, it suffices to note that if $\tilde{\psi} = \tilde{\varphi} + b\tilde{f}$ then $\rho_t = \mathrm{id} + tf$ works as an invertible linear map that renders the two deformations equivalent.

□

By Lemma 3.3.10 we get the following corollary.

Corollary 3.4.6. *Given a symmetric Frobenius algebra (A, μ, κ) over $\mathbb{K}$ where $\mathrm{char}(\mathbb{K}) = 0$, deformations that preserve the inner form are controlled by the second cyclic cohomology group $HC^2(A)$.*

Remark 3.4.7. Note that it also suffices that $\mathrm{char}(\mathbb{K}) > 3$. For some examples of where this theorem fails, see Appendix B.

Remark 3.4.8. As discussed at [GS, p. 25], all deformations of an associative unital algebra are equivalent to a deformation that preserves the unit. It is unclear if a similar result holds for deformations of symmetric Frobenius algebras.

It is worth spending some time working through an example of this approach to deformations of symmetric Frobenius algebras. Through Theorem 3.3.4, it is clear that this will be easiest to calculate for an algebra A when $HH^*(A) = 0$ for $* \neq 0$, that is, if the algebra A is separable. In contrast to Hochschild cohomology where separability of the algebra implies a trivial cohomological structure, we find that separable algebras can have non-trivial cyclic cohomology. The central goal will be Theorem 3.4.9.

Theorem 3.4.9. *Suppose A is a separable, associative, unital algebra over $\mathbb{K}$. Then the cyclic cohomology $HC^\bullet(A)$ is completely determined by the zeroth simplicial cohomology group via:*

$$HC^{2n}(A) \cong HH^0(A)$$
$$HC^{2n-1}(A) \cong 0.$$

Proof. From [Wei, 9.2.11], we get that A is separable if and only if $HH^*(A) = 0$ for all $* \neq 0$. Via Theorem 3.3.4, we get that the long exact sequence breaks at $HH^*(A)$ for $* \neq 0$ into a long exact sequence of length 5 and a family of isomorphisms:

$$0 \to HC^{-2}(A) \to HC^0(A) \to HH^0(A) \to HC^{-1}(A) \to HC^1(A) \to 0$$
$$0 \to HC^n(A) \to HC^{n+2}(A) \to 0 \quad \text{for } n \neq -1, -2.$$

Now all we need to do is figure out what $HC^{-2}(A), HC^{-1}(A), HC^0(A)$ and $HC^1(A)$ are.

From [Lod, 2.4.5] we get that $HH^0(A) = \{f : A \to \mathbb{K} \mid f(aa') = f(a'a)\}$. Noting that for cyclic cohomology, $\mathrm{im}(D_{-1}) \cong 0$ and $\ker(D_1) = \ker(f_0 \mapsto (b^* f_0, (1-\tau)^* f_0)) = \{f_0 : A \to \mathbb{K} \mid f_0(a_0 a_1) = f_0(a_1 a_0)\}$, we get that $HC^0(A) \cong HH^0(A)$. This forces $HC^{-1}(A) \cong HC^{-2}(A) \cong 0$, so $HC^1(A) \cong 0$. □

Remark 3.4.10. This result is obvious via the construction of the long exact sequence from Theorem 3.3.4. However, it has not been explicitly computed in any sources we have found. In particular, it has not been referenced in [PS], presumably because they don't try to generalize the long exact sequence to the case of A_∞-algebras.

Using [PS] along with the sequence from Theorem 3.3.4, we can work out all of the infinitesimal deformations of a separable, symmetric Frobenius algebra over a field of characteristic 0.

Theorem 3.4.11. *Let* A *be a separable, symmetric Frobenius algebra over* $\mathbb{K}$ *with* $\mathrm{char}(\mathbb{K}) = 0$ *and Frobenius form* $\kappa : A^2 \to \mathbb{K}$*. Then all infinitesimal deformations that preserve the Frobenius form are of the form* $\mu_t = \mu + t\varphi$ *where* φ *is determined by the condition:*

$$\kappa(\varphi(a_0, a_1), a_2) = \tilde{\varphi}(a_0, a_1, a_2) = \frac{1}{2} f(a_0 a_1 a_2)$$

where $f : A \to \mathbb{K}$ *is in* $HH^0(A) = \{g \mid g(a_0 a_1) = g(a_1 a_0)\}$.

Proof. From Theorem 3.4.9 we get that

$$HC^n(A) \cong \begin{cases} HH^0(A) & \text{if } n \in 2\mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 3.3.10 we get that $HC^n(A) \cong H^n_\lambda(A)$ for all n as $\mathrm{char}(\mathbb{K}) = 0$.Already that means that $H^3_\lambda(A) \cong 0$.

In the case of $\mathrm{char}(\mathbb{K}) = 0$, we can even go further and compute it explicitly. Following [Lod, 2.2.5] onwards and dualizing we get that the $S : H^0_\lambda(A) \to H^2_\lambda(A)$ is given by:

$$\begin{aligned} Sf(a_0 \otimes a_1 \otimes a_2) &= f(S(a_0 \otimes a_1 \otimes a_2)) \\ &= f\left(-\frac{1}{2}(-a_0 a_1 a_2 + a_2 a_0 a_1 - a_1 a_2 a_0)\right) \\ &= \frac{1}{2}(f(a_0 a_1 a_2) - f(a_2 a_0 a_1) + f(a_1 a_2 a_0)) \end{aligned}$$

Noting that as $H^0_\lambda(A) = HH^0(A)$ so $f(xy) = f(yx)$, we get that $f((a_2 a_0) a_1) = f(a_1(a_2 a_0))$. That simplifies our equation to $\frac{1}{2} f(a_0 a_1 a_2)$. This is living in $\mathrm{Hom}_\mathbb{K}(A^3, \mathbb{K})$; we need to bring it into $\mathrm{Hom}_\mathbb{K}(A^2, A)$, which we do via the identity

$$\kappa(\varphi(a_0, a_1), a_2) = \frac{1}{2} f(a_0 a_1 a_2)$$

□

It is worth taking a moment to compute this for a low-dimensional example. In particular, we will see how $HH^2(A; A) \cong 0$ while $HC^2(A) \ncong 0$.

Example 3.4.12. Recall that $\mathbb{C}$ is a semisimple algebra over $\mathbb{R}$, and as $\mathbb{R}$ is a perfect field, this means $\mathbb{C}$ is separable. We can use Theorem 3.4.11 to classify all possible infinitesimal deformations that preserve some given Frobenius form. Let κ be our Frobenius form, defined by $\kappa(x, y) := \mathfrak{R}(x \cdot y)$ where $\mathfrak{R} : \mathbb{C} \to \mathbb{R}$, $(a + bi) \mapsto a$ is the real component of the complex number. This is known in the literature as Sweedler's trigonometric coalgebra [Swe, p. 140]. As $\mathbb{C}$ is commutative, all linear forms $f : \mathbb{C} \to \mathbb{R}$ are elements of $HH^0(\mathbb{C})$. We know that $\mathbb{C}^*$ is spanned by $g_1(a + bi) = a$ and $g_2(a + bi) = b$.

Consider a deformation $\mu_t = \mu + t\varphi$, and assume the deformation preserves the bilinear form. Letting a general element of $HH^0(\mathbb{C})$ be written $f = \lambda_1 g_1 + \lambda_2 g_2$, we want to determine the corresponding

$\varphi : A^2 \to A$. By Theorem 3.4.11 and evaluate on all pairs. Now, as $\mathbb{C}$ is commutative this gives us 6 equations:

$$\begin{aligned}
&\mathfrak{R}(\varphi(1,1)) = \frac{\lambda_1}{2} && \mathfrak{R}(\varphi(1,i)) = \mathfrak{R}(\varphi(i,1)) = \frac{\lambda_2}{2} && \mathfrak{R}(\varphi(i,i)) = -\frac{\lambda_1}{2} \\
&\mathfrak{I}(\varphi(1,1)) = -\frac{\lambda_2}{2} && \mathfrak{I}(\varphi(1,i)) = \mathfrak{I}(\varphi(i,1)) = \frac{\lambda_1}{2} && \mathfrak{I}(\varphi(i,i)) = \frac{\lambda_2}{2}
\end{aligned}$$

Note that these arise from $\tilde{\varphi}(a,b,c) = \mathfrak{R}(\varphi(a,b)\cdot c) = \frac{1}{2}(\lambda_1 g_1(abc) + \lambda_2 g_2(abc))$.

We should now check that in fact, the new $\mu_t = \mu + t\varphi$ works as an infinitesimal deformation preserving the appropriate inner form. We then get that:

$$\begin{aligned}
&\mu_t(a + a'i, b + b'i) = (ab - a'b') + i(ab' + a'b) \\
&+ \frac{1}{2}t[\lambda_1(ab - a'b') + \lambda_2(ab' + a'b) + i(\lambda_2(a'b' - ab) + \lambda_1(ab' + a'b))]
\end{aligned}$$

We can explicitly check that this definition of μ_t is invariant under κ and associative, but this is unnecessary by Lemmas 3.4.3 and 2.2.6.

We also want to understand why these deformations become trivial when considering ordinary deformations instead of cyclic deformations. Let $\rho_t : A_t \to A_t$ be our trivializer, of the form $\rho_t = \mathrm{id} + th$ where $h : A \to A$ is just an $\mathbb{R}$-linear map. The deformation is trivial if h is bijective and $\mu = \rho_t^{-1} \circ \mu_t \circ (\rho_t \otimes \rho_t)$. Expanding this out and simplifying gives us the following equation on h.

$$\mu = \mu + t[\varphi + \mu \circ (\mathrm{id} \otimes h) + \mu \circ (h \otimes \mathrm{id}) - h \circ \mu]$$

As this is graded by powers of t, we only need to consider the degree 1 component.

$$0 = \mu \circ (\mathrm{id} \otimes h) + \mu \circ (h \otimes \mathrm{id}) - h \circ \mu + \varphi$$

Then we can evaluate this equation on all pairs of the basis elements $1, i$. Since the form is symmetric, this gives us three equations on f.

$$\begin{aligned}
0 &= h(1) + h(1) - h(1) + \varphi(1,1) \\
0 &= h(i) + ih(1) - h(i) + \varphi(1,i) \\
0 &= ih(i) + ih(i) - h(-1) + \varphi(i,i)
\end{aligned}$$

These rearrange and reduce to:

$$\begin{aligned}
h(1) &= -\varphi(1,1) \\
h(i) &= \frac{1}{2}i(\varphi(1,1) + \varphi(i,i))
\end{aligned}$$

This gives us that

$$h(1) = \frac{1}{2}(-\lambda_1 + i\lambda_2) \qquad\qquad h(i) = \frac{1}{2}(-\lambda_2 - i\lambda_1).$$

That is, $h(x) = \frac{1}{2}(-\lambda_1 + i\lambda_2)x$. This is bijective so long as $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. This makes the deformation trivial in the sense of Definition 2.2.4.

Following Definition 3.4.1, we now check if $(\mathfrak{R} \circ \mu) \circ (\rho_t \otimes \rho_t) = \mathfrak{R} \circ \mu$. Expanding and simplifying, we get that $\mathfrak{R}(\mu \circ (\mathrm{id} \otimes h) + \mu \circ (h \otimes \mathrm{id})) = 0$. As $h(x) = \frac{1}{2}(-\lambda_1 + i\lambda_2)x$, we get that for all $x, y \in \mathbb{C}$,

$$0 = \mathfrak{R}((\lambda_1 - i\lambda_2)xy).$$

First, take $x = 1$, $y = 1$. This implies that $\lambda_1 = 0$. Then take $x = -i$, $y = 1$. This implies $\lambda_2 = 0$, which all together implies that $h = 0$. That is, ρ_t only preserves the inner form if $h = 0$, that is all deformations that preserve the inner form are non-trivial.

As mentioned in Section 1.4, this example points to an interesting avenue of further study that we do not look at in this thesis. Instead of keeping the bilinear form κ constant and deforming μ, we could hold μ constant and deform κ. This would give us another sensible notion of what a "deformation of a symmetric Frobenius algebra" could be. In the above example, the trivializer appears to deform the bilinear form as it brings the deformed multiplication back to the original multiplication.

With all these results in hand, we can fully classify the deformations of separable symmetric Frobenius algebras. This reveals a tight connection between cyclic deformations and Frobenius forms. It is worth recalling that all separable algebras are semisimple.

Lemma 3.4.13. *If A is a simple unital algebra, then elements of $HH^0(A)$ are in a bijection with symmetric Frobenius forms on A.*

Proof. Note that for an invariant inner form, say β, on a simple algebra, either β is non-degenerate or $\beta = 0$.

Suppose not, so there is some $0 \neq \beta : A^2 \to \mathbb{K}$ and some $0 \neq x \in A$ such that $\beta(x, y) = 0$ for all $y \in A$. By invariance of β we get that

$$\beta(xy, 1) = \beta(x, y) = 0 = \beta(x, y) = \beta(y, x) = \beta(yx, 1)$$

The left hand entries of β generate the ideal $I = \langle x \rangle$. As A is simple, we know that the only two sided ideals are $\{0\}$ and A. If $I = \{0\}$ then $x = 0$ a contradiction. If $I = A$ then $\beta = 0$, again a contradiction.

Let Invar(A) be the set of invariant symmetric inner forms on A. Then consider the two assignments $\Phi : HH^0(A) \to \text{Invar}(A)$ with $\Phi : f \mapsto (\beta_f : (x, y) \mapsto f(xy))$ and $\Psi : \text{Invar}(A) \to HH^0(A)$ with $\kappa \mapsto (f_\kappa : x \mapsto \kappa(x, 1))$. Both Φ and Ψ are clearly injective. Furthermore, they are inverses of each other. Now, from above we have that all non-zero invariant symmetric inner forms are non-degenerate and thus Frobenius forms, which gives us our bijection. □

Lemma 3.4.14. *If A is a semisimple unital algebra, then $HH^0(A)$ is spanned by $\{f_\kappa \mid \kappa \text{ is a symmetric Frobenius form}\}$.*

Proof. As A is semisimple, we have by Artin-Wedderburn that

$$A \cong \bigoplus_{i=1}^{N} M_{n_i}(\mathcal{D}_i)$$

where $\mathcal{D}_i$ are division algebras over $\mathbb{K}$. Furthermore, it's clear that

$$HH^0(A) \cong \bigoplus_{i=1}^{N} HH^0(M_{n_i}(\mathcal{D}_i))$$

just from writing out definitions. By inspection and Lemma 3.4.13 the image of symmetric Frobenius form on A is of the form $(f_i)_{i=1}^N$ where $f_i \in HH^0(M_{n_i}(\mathcal{D}_i))$ and $f_i \neq 0$. Clearly these span $HH^0(A)$, and we have our result. □

Remark 3.4.15. Theorem 3.4.11 deserves some interpretation. Firstly, $HH^0(A)$ has a clean interpretation; [Lod, 2.4.5] calls an element of $HH^0(A)$ a trace on A. Deformations of Frobenius algebras over A are completely controlled by the traces on A, no matter the particular Frobenius form. The concrete φ does depend on the Frobenius form via the isomorphism it induces $\text{Hom}_{\mathbb{K}}(A^3, \mathbb{K}) \to \text{Hom}_{\mathbb{K}}(A^2, A)$.

As well, the precise relation between the cyclic cohomology and deformations preserving a symmetric Frobenius form is clearer. From Lemma 3.4.14 and that all separable algebras are semisimple, we know that all traces on A are determined by the Frobenius forms on A. So long as $\text{char}(\mathbb{K}) \neq 2, 3$ we get a map $HH^0(A) \to H^2_\lambda(A)$ by composing a few isomorphisms and S. This means that in some sense we are deforming the algebra using a Frobenius form.

Chapter 4

Deformations of a Monoidal Functor

4.1 A Review of Kan Extensions

In order to work with functor categories in the following sections, we will require some basic results involving Kan extensions. We also need to briefly review the two notions of composition of natural transformations.

Definition 4.1.1 (Horizontal and Vertical Composition)**.** [Mac, II.5]

- Given functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and natural transformations $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$, we define the vertical composition $\beta \circ \alpha : F \Rightarrow H$, by $(\beta \circ \alpha)_x := \beta_x \circ \alpha_x$. This is depicted in Diagram (4.1)

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{D}
\arrow[from=F, to=G, Rightarrow, "\alpha"]
\arrow[from=G, to=H, Rightarrow, "\beta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "H"'{name=H}] & \mathcal{D}
\arrow[from=F, to=H, Rightarrow, "\beta\circ\alpha"]
\end{tikzcd}
\tag{4.1}
$$

- Given functors $F_0, G_0 : \mathcal{C} \to \mathcal{D}$, $F_1, G_1 : \mathcal{D} \to \mathcal{E}$ and natural transformations $\alpha : F_0 \Rightarrow G_0$, $\alpha : F_1 \Rightarrow G_1$, we define the horizontal composition $\beta \bullet \alpha : F_1 F_1 \Rightarrow G_1 G_0$ by $(\beta \bullet \alpha)_x := \beta_{G_0 x} \circ F_1(\alpha_x)$. This is depicted in Diagram (4.2).

$$
\begin{gathered}
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_0"{name=A}] \arrow[r, bend right=50, "G_0"'{name=B}] & \mathcal{D} \arrow[r, bend left=50, "F_1"{name=C}] \arrow[r, bend right=50, "G_1"'{name=D}] & \mathcal{E}
\arrow[from=A, to=B, Rightarrow, "\alpha"]
\arrow[from=C, to=D, Rightarrow, "\beta"]
\end{tikzcd}
\\
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1F_0"{name=A}] \arrow[r, bend right=50, "G_1G_0"'{name=B}] & \mathcal{E}
\arrow[from=A, to=B, Rightarrow, "\beta\bullet\alpha"]
\end{tikzcd}
\end{gathered}
\tag{4.2}
$$

We will, by abuse of notation, write the functor F for the natural transformation id_F when calculating with horizontal and vertical composition.

Remark 4.1.2. It is worth mentioning two basic facts about horizontal and vertical composition without proof. We have the following identities:

- For functors $F : \mathcal{C} \to \mathcal{D}$, $G_0, G_1 : \mathcal{E} \to \mathcal{C}$ and a natural transformation $\alpha : G_0 \Rightarrow G_1$, we have:

$$F(\alpha) = F \bullet \alpha.$$

- For functors $F_0, F_1, F_2 : \mathcal{C} \to \mathcal{D}$, $G_0, G_1, G_2 : \mathcal{D} \to \mathcal{E}$ and natural transformations $\alpha : G_1 \Rightarrow G_2$, $\beta : G_0 \Rightarrow G_1$, $\gamma : F_1 \Rightarrow F_2$, $\delta : F_0 \Rightarrow F_1$, we have:

$$(\alpha \circ \beta) \bullet (\gamma \circ \delta) = (\alpha \bullet \gamma) \circ (\beta \bullet \delta).$$

  In particular, this is the interchange law of the 2-category of (small) categories.

For a deeper account of 2-categories, we direct the reader to [Mac, XII.3-4].

Definition 4.1.3 (Kan Extension)**.** [Bor1, 3.7.1][Mac, X.3]Consider two functors $F : \mathcal{B} \to \mathcal{D}$, $p : \mathcal{B} \to \mathcal{C}$. Then

- The <u>left Kan extension of F along p</u>, if it exists, is a pair (K, λ) with
  - $K : \mathcal{C} \to \mathcal{D}$ is a functor,
  - $\lambda : F \Rightarrow K \circ p$ is a natural transformation,

  together with the following universal property.

  If (H, γ) is another pair with
  - $H : \mathcal{C} \to \mathcal{D}$ is a functor,
  - $\gamma : F \Rightarrow H \circ p$ is a natural transformation,

  then there exists a unique natural transformation $\xi : K \Rightarrow H$ satisfying the equality $(\xi \bullet p) \circ \lambda = \gamma$. We will denote the left Kan extension of F along p by $\mathrm{Lan}_p(F)$.

- The <u>right Kan extension of F along p</u>, if it exists, is a pair (K, ρ) with
  - $K : \mathcal{C} \to \mathcal{D}$ is a functor,
  - $\rho : K \circ p \Rightarrow F$ is a natural transformation,

  together with the following universal property.

  If (H, γ) is another pair with
  - $H : \mathcal{C} \to \mathcal{D}$ is a functor,
  - $\gamma : H \circ p \Rightarrow F$ is a natural transformation,

  then there exists a unique natural transformation $\xi : H \Rightarrow K$ satisfying the equality $\lambda \circ (\xi \bullet p) = \gamma$. We will denote the right Kan extension of F along p by $\mathrm{Ran}_p(F)$.

Lemma 4.1.4 (Kan Extension as adjoints)**.** *Fix some functor $p : \mathcal{B} \to \mathcal{C}$ and some other category $\mathcal{D}$. Then if the left Kan extension $\mathrm{Lan}_p(F)$ exists for all $F : \mathcal{B} \to \mathcal{D}$, then the functor*

$$\mathrm{Lan}_p(-) : \mathrm{Funct}(\mathcal{B}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{D})$$

is left adjoint to $p^* : \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{B}, \mathcal{D})$.

Similarly, if the right Kan extension $\mathrm{Ran}_p(F)$ exists for all $F : \mathcal{B} \to \mathcal{D}$, then the functor

$$\mathrm{Ran}_p(-) : \mathrm{Funct}(\mathcal{B}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{D})$$

is right adjoint to $p^* : \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{B}, \mathcal{D})$.

Proof. We will prove this for the left Kan extension, as the case for the right Kan extension is similar.

We first need candidates for the unit and counit of the adjunction. Consider the structure morphism of the left Kan extension, $\lambda_F : F \Rightarrow \mathrm{Lan}_p(F) \circ p$. Let $\eta_F := \lambda_F$. By the universal property of the left Kan extension, we also have $\varepsilon_F : \mathrm{Lan}_p(F \circ p) \Rightarrow F$. The triangle identities come down to Diagrams 4.1. Each one of these exhibits the correct universal properties. ☐

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "p"] \arrow[d, "p"'] \arrow[rr, bend left=60, "F"] \arrow[r, phantom, bend left=40, "\eta\Downarrow"] & \mathcal{C} \arrow[r, "\mathrm{Lan}_p(F)"] \arrow[d, phantom, "\mathrm{Lan}_p\bullet\eta\Downarrow"] & \mathcal{D} \\
\mathcal{C} \arrow[urr, "\mathrm{Lan}_p(\mathrm{Lan}_p(F)\circ p)" description] \arrow[urr, bend right=60, "\mathrm{Lan}_p(F)"'] \arrow[r, phantom, "\varepsilon\bullet\mathrm{Lan}_p\Downarrow"] & {} &
\end{tikzcd}$$

(a) Left Triangle Identity

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "p"] \arrow[dr, "p"'] & \mathcal{C} \arrow[d, equal] \arrow[d, phantom, "\eta\bullet p^*\Downarrow"'] \arrow[dr, "G"] & \\
 & \mathcal{C} \arrow[r, "\mathrm{Lan}_p(G\circ p)"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, bend right=30, "\Downarrow\varepsilon"] & \mathcal{D}
\end{tikzcd}$$

(b) Right Triangle Identity

Figure 4.1: Triangle Identities

This expression in terms of adjoints gives us some natural isomorphisms.

Corollary 4.1.5 (Kan Extensions give natural isomorphisms)**.** *If* $\mathrm{Lan}_p(F)$ *exists, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Funct}(\mathcal{B},\mathcal{D})}(F, p^*(-)) \cong \mathrm{Hom}_{\mathrm{Funct}(\mathcal{C},\mathcal{D})}(\mathrm{Lan}_p(F), -).$$

Similarly, if $\mathrm{Ran}_p(F)$ *exists, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Funct}(\mathcal{B},\mathcal{D})}(p^*(-), F) \cong \mathrm{Hom}_{\mathrm{Funct}(\mathcal{C},\mathcal{D})}(-, \mathrm{Ran}_p(F)).$$

To express Kan extensions as (co)ends we need to have the notion of a copowered category. The notion of a powered category will be useful for some later computations.

Definition 4.1.6 (Powered and Copowered Categories)**.** [Bor2, 6.5.1] Let $\mathcal{C}$ be a $\mathbb{K}$-linear category. Let $C \in \mathcal{C}$ and $V \in \mathbf{Vect}_{\mathbb{K}}$ be two objects.

- The powering of V and C exists if there is an object $V \pitchfork C \in \mathcal{C}$ together with the isomorphisms of vector spaces
$$\mathrm{Hom}_{\mathcal{C}}(C_0, V \pitchfork C) \cong \mathrm{Hom}_{\mathbb{K}}(V, \mathrm{Hom}_{\mathcal{C}}(C_0, C))$$
that are natural in $C_0 \in \mathcal{C}$. We say $\mathcal{C}$ is powered when $V \pitchfork C$ exists for all $V \in \mathbf{Vect}_{\mathbb{K}}$ and $C \in \mathcal{C}$. We say $\mathcal{C}$ is finitely powered if $V \pitchfork C$ exists for all $V \in \mathbf{FinVect}_{\mathbb{K}}$ and $C \in \mathcal{C}$.
- The copowering of V and C exists if there is an object $V \star C$ together with the isomorphisms of vector spaces
$$\mathrm{Hom}_{\mathcal{C}}(V \star C, C_0) \cong \mathrm{Hom}_{\mathbb{K}}(V, \mathrm{Hom}_{\mathcal{C}}(C, C_0))$$
that are natural in $C_0 \in \mathcal{C}$. We say $\mathcal{C}$ is copowered when $V \star C$ exists for all $V \in \mathbf{Vect}_{\mathbb{K}}$ and $C \in \mathcal{C}$. We say $\mathcal{C}$ is finitely copowered if $V \star C$ exists for all $V \in \mathbf{FinVect}_{\mathbb{K}}$ and $C \in \mathcal{C}$.

When doing computations with (co)powered categories, the following lemmata will be useful.

Lemma 4.1.7. *[Bor2, 6.5.4] Let $\mathcal{C}$ be a copowered $\mathbb{K}$-linear category. Then $V \star (U \star C) \cong (V \otimes U) \star C$ for all $V, U \in \mathbf{Vect}_{\mathbb{K}}$, $C \in \mathcal{C}$.*

Proof. The proof follows directly from the natural isomorphisms and the Yoneda lemma. Let $C_0 \in \mathcal{C}$ be an arbitrary object.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(V \star (U \star C), C_0) &\cong \mathrm{Hom}_{\mathbb{K}}(V, \mathrm{Hom}_{\mathcal{C}}(U \star C, C_0)) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(V, \mathrm{Hom}_{\mathbb{K}}(U, \mathrm{Hom}_{\mathcal{C}}(C, C_0))) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(V \otimes U, \mathrm{Hom}_{\mathcal{C}}(C, C_0)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}((V \otimes U) \star C, C_0)
\end{aligned}$$

By the Yoneda lemma, this implies that $V \star (U \star C) \cong (V \otimes U) \star C$. □

Lemma 4.1.8. *If a $\mathbb{K}$-linear category $\mathcal{C}$ is finitely copowered, then it is finitely powered.*

Proof. First, note that for $V \in \mathbf{FinVect}_{\mathbb{K}}$ the functor $V \star - : \mathcal{C} \to \mathcal{C}$ has a right adjoint, given by $V^* \star - : \mathcal{C} \to \mathcal{C}$. The unit of the adjunction is the coevaluation $\mathrm{coeval}^r \bullet \mathrm{id}_{\mathcal{C}}$, and the counit of the adjunction is the evaluation $\mathrm{eval}^r \bullet \mathrm{id}_{\mathcal{C}}$. This implies that there is a natural isomorphism of hom-spaces

$$\mathrm{Hom}_{\mathcal{C}}(V \star C_0, C_1) \cong \mathrm{Hom}_{\mathcal{C}}(C_0, V^* \star C_1).$$

Composing this natural isomorphism together with the defining natural isomorphism of the copowering, $\mathrm{Hom}_{\mathcal{C}}(V \star C_0, C_1) \cong \mathrm{Hom}_{\mathbb{K}}(V, \mathrm{Hom}_{\mathcal{C}}(C_0, C_1))$, gives us a natural isomorphism of the form:

$$\mathrm{Hom}_{\mathcal{C}}(C_0, V^* \star C_1) \cong \mathrm{Hom}_{\mathbb{K}}(V, \mathrm{Hom}_{\mathcal{C}}(C_0, C_1)).$$

This is exactly the defining natural isomorphism of a powering. This exists for all finite-dimensional vector spaces, which implies $\mathcal{C}$ is finitely powered. □

Lemma 4.1.9 (Kan Extensions as (co)ends)**.** *Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear categories with finite-dimensional hom-spaces, and let $\mathcal{D}$ be finitely copowered. Then we can express the left Kan extension of $F : \mathcal{B} \to \mathcal{D}$ along $p : \mathcal{B} \to \mathcal{C}$ by*

$$\mathrm{Lan}_p(F)(x) \cong \int^{t \in \mathcal{B}} \mathrm{Hom}_{\mathcal{C}}(p(t), x) \star F(t).$$

Similarly, we can express the right Kan extension of F along p by

$$\mathrm{Ran}_p(F)(x) \cong \int_{t \in \mathcal{B}} (\mathrm{Hom}_{\mathcal{C}}(x, p(t)))^* \star F(t).$$

Proof. We will prove this for left Kan extensions, as right Kan extensions are similar. The following proof is drawn from [Lor, 2.3.6.], using coend calculus:

$$\begin{aligned}
\mathrm{Funct}(\mathcal{C}, \mathcal{D}) \left(\int^t \mathrm{Hom}_{\mathcal{C}}(p(t), -) \star Ft, H \right) &\cong \int_x \mathrm{Hom}_{\mathcal{D}} \left(\int^t \mathrm{Hom}_{\mathcal{C}}(p(t), x) \star Ft, Hx \right) \\
&\cong \int_{xt} \mathrm{Hom}_{\mathcal{D}}(\mathrm{Hom}_{\mathcal{C}}(p(t), x) \star Ft, Hx) \\
&\cong \int_{xt} \mathrm{Hom}_{\mathbb{K}}(\mathrm{Hom}_{\mathcal{C}}(p(t), x), \mathrm{Hom}_{\mathcal{D}}(Ft, Hx)) \\
&\cong \int_t \mathrm{Hom}_{\mathcal{D}}(Ft, Hp(t)) \cong \mathrm{Funct}(\mathcal{B}, \mathcal{D})(F, H \circ p)
\end{aligned}$$

By the universal property of Corollary 4.1.5, this means that the coend exhibits the relevant Kan extension. □

Lemma 4.1.10. *Let $\mathcal{C}, \mathcal{D}$ be categories. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ such that $L \dashv R$ be adjoint functors with counit $\varepsilon : LR \Rightarrow \mathrm{id}$ and unit $\eta : \mathrm{id} \Rightarrow RL$. Then the following results hold:*

1. *For any category $\mathcal{E}$, the precomposition functor $L \circ - : \mathrm{Funct}(\mathcal{E}, \mathcal{C}) \to \mathrm{Funct}(\mathcal{E}, \mathcal{D})$ has the <u>right</u> adjoint $R \circ - : \mathrm{Funct}(\mathcal{E}, \mathcal{D}) \to \mathrm{Funct}(\mathcal{E}, \mathcal{C})$.*

2. *For any category $\mathcal{E}$, the postcomposition functor $- \circ L : \mathrm{Funct}(\mathcal{D}, \mathcal{E}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{E})$ has the <u>left</u> adjoint $- \circ R : \mathrm{Funct}(\mathcal{C}, \mathcal{E}) \to \mathrm{Funct}(\mathcal{D}, \mathcal{E})$.*

3. *For a functor $F : \mathcal{C} \to \mathcal{E}$, the left Kan extension $\mathrm{Lan}_L(F)$ is given by $F \circ R$, and the universal natural transformation is given by $F \bullet \eta$.*

4. *For a functor $G : \mathcal{D} \to \mathcal{E}$the right Kan extension $\mathrm{Ran}_R(G)$ is given by $F \circ L$ and the universal natural transformation is given by $G \bullet \varepsilon$.*

Proof. We will prove each in turn.

1. The counit of the adjunction should be of the form $\tilde{\varepsilon} : LRF \Rightarrow F$, and the unit of the adjunction should be of the form $\tilde{\eta} : F \Rightarrow RLF$. It is immediate that $\tilde{\varepsilon} := \varepsilon \bullet F$ and $\tilde{\eta} := \eta \bullet F$ satisfy the triangle identities.

2. The counit of the adjunction should be of the form $\tilde{\varepsilon} : FLR \Rightarrow F$, and the unit of the adjunction should be of the form $\tilde{\eta} : F \Rightarrow FRL$. It is immediate that $\tilde{\varepsilon} := F \bullet \varepsilon$ and $\tilde{\eta} := F \bullet \eta$ satisfy our triangle identities.

3. By Corollary 4.1.5, we know that a left Kan extension $\mathrm{Lan}_L F$ is equipped with natural isomorphisms for every functor of the form $G : \mathcal{D} \to \mathcal{E}$:

$$\Phi_G : \mathrm{Hom}_{\mathrm{Funct}(\mathcal{C},\mathcal{E})}(F, G \circ L) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{D},\mathcal{E})}(\mathrm{Lan}_L F, G).$$

   By above, we have that there is a natural isomorphism

$$\Psi_G : \mathrm{Hom}_{\mathrm{Funct}(\mathcal{C},\mathcal{E})}(F, G \circ L) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{D},\mathcal{E})}(F \circ R, G).$$

   By the universal properties of adjoints and Kan extensions, this exhibits the Kan extension $\mathrm{Lan}_L F$ as $F \circ R$. This implies that the natural transformation $\lambda_F : F \Rightarrow (\mathrm{Lan}_L F) \circ L \cong F \circ R \circ L$ will be the adjunct of the identity natural transformation $\mathrm{id}_{F \circ R} : F \circ R \Rightarrow F \circ R$ under $\Psi^{-1}_{F \circ R}$, that is, $F \bullet \eta$.

4. By Corollary 4.1.5, we know that a right Kan extension $\mathrm{Ran}_R G$ is equipped with natural isomorphisms for every functor of the form $F : \mathcal{C} \to \mathcal{E}$:

$$\Phi_F : \mathrm{Hom}_{\mathrm{Funct}(\mathcal{D},\mathcal{E})}(F \circ R, G) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{C},\mathcal{E})}(F, \mathrm{Ran}_R G)$$

   By above, we have that there is a natural isomorphism

$$\Psi_F : \mathrm{Hom}_{\mathrm{Funct}(\mathcal{D},\mathcal{E})}(F \circ R, G) \to \mathrm{Hom}_{\mathrm{Funct}(\mathcal{C},\mathcal{E})}(F, G \circ L)$$

   By the universal properties of adjoints and Kan extensions, this exhibits the Kan extension $\mathrm{Ran}_R G$ as $G \circ L$. The natural transformation $\rho_G : (\mathrm{Ran}_R G) \circ R \cong G \circ L \circ R \Rightarrow G$ will be the adjunct of the identity natural transformation $\mathrm{id}_{G \circ L} : G \circ L \Rightarrow G \circ L$ under $\Psi^{-1}_{G \circ L}$, that is, $G \bullet \varepsilon$.

This completes the proof. □

Lemma 4.1.11. *Let $\mathcal{C}_0, \mathcal{D}_0, \mathcal{C}_1, \mathcal{D}_1$ be categories together with functors $L_i : \mathcal{C}_i \to \mathcal{D}_i$ and $R_i : \mathcal{D}_i \to \mathcal{C}_i$ such that $L_i \dashv R_i$ are adjoints with counit and unit ε_i and η_i. Then the following results hold:*

1. *For all $F : \mathcal{C}_0 \to \mathcal{C}_1$, the Kan extension $\mathrm{Lan}_{L_0}(L_1 \circ F)$ exists, is isomorphic to $L_1 \circ F \circ R_0$ and has structure morphism $\lambda_F = L_1 F \bullet \eta_0$. We define a functor $\tilde{L} : \mathrm{Funct}(\mathcal{C}_0, \mathcal{C}_1) \to \mathrm{Funct}(\mathcal{D}_0, \mathcal{D}_1)$ by $\tilde{L}(F) := \mathrm{Lan}_{L_0}(L_1 \circ F)$.*

2. *For all $G : \mathcal{D}_0 \to \mathcal{D}_1$ the Kan extension $\mathrm{Ran}_{R_0}(R_1 \circ G)$ exists, is isomorphic to $R_1 \circ G \circ L_0$, and has structure morphism $\rho_G = R_1 G \bullet \varepsilon_0$. We define a functor $\tilde{R} : \mathrm{Funct}(\mathcal{D}_0, \mathcal{D}_1) \to \mathrm{Funct}(\mathcal{C}_0, \mathcal{C}_1)$ by $\tilde{R}(G) := \mathrm{Ran}_{R_0}(R_1 \circ G)$.*

3. *These Kan extensions form an adjoint pair $\tilde{L} \dashv \tilde{R}$ with the counit $\tilde{\varepsilon}_F := \varepsilon_1 \bullet F \bullet \varepsilon_0$ and unit $\tilde{\eta}_G := \eta_1 \bullet G \bullet \eta_0$.*

4. *We have the equality*

$$\rho_G = (R_1 \bullet \tilde{\varepsilon}_G \bullet \varepsilon_0) \circ (R_1 \bullet \lambda_{\tilde{R}(G)} \bullet R_0) \circ (\eta_1 \bullet R_1 G L_0 R_0)$$

Proof. We will prove each in turn.

1. This follows directly from part 3 of Lemma 4.1.10.

2. This follows directly from part 4 of Lemma 4.1.10.

3. We will simply prove the triangle identities directly, using the interchange law of Remark 4.1.2. First:

$$\begin{aligned}\tilde{\varepsilon}_{\tilde{L}F} \circ \tilde{L}(\tilde{\eta}_F) &= (\varepsilon_1 \bullet L_1 \bullet FR_0 \bullet \varepsilon_0) \circ (L_1 \bullet \eta_1 \bullet F \bullet \eta_0 \bullet R_0) \\ &= ((\varepsilon_1 \bullet L_1) \circ (L_1 \bullet \eta_1)) \bullet (F) \bullet ((\varepsilon_0 \bullet R_0) \circ (R_0 \bullet \eta_0)) \\ &= L_1 F R_0 = \tilde{L}(F)\end{aligned}$$

Then:

$$\begin{aligned}\tilde{R}(\tilde{\varepsilon}_G) \circ \tilde{\eta}_{\tilde{R}(G)} &= (R_1 \bullet \varepsilon_1 \bullet G \bullet \varepsilon_0 \bullet L_0) \circ (\eta_1 \bullet R_1 \bullet G \bullet L_0 \bullet \eta_0) \\ &= ((R_1 \bullet \varepsilon_1) \circ (\eta_1 \bullet R_1)) \bullet (G) \bullet ((L_0 \bullet \varepsilon_0) \circ (\eta_0 \bullet L_0)) \\ &= R_1 \bullet G \bullet L_0 = \tilde{R}(G)\end{aligned}$$

4. This will follow from direct computation.

$$\begin{aligned}&(R_1 \bullet \tilde{\varepsilon}_G \bullet \varepsilon_0) \circ (R_1 \bullet \lambda_{\tilde{R}(G)} \bullet R_0) \circ (\eta_1 \bullet R_1 G L_0 R_0) \\ &\quad = (R_1 \bullet \varepsilon_1 \bullet G \bullet \varepsilon_0 \bullet \varepsilon_0) \circ (R_1 L_1 R_1 G L_0 \bullet \eta_0 \bullet R_0) \circ (\eta_1 \bullet R_1 G L_0 R_0) \\ &\quad = ((R_1 \bullet \varepsilon_1) \circ (\eta_1 \bullet R_1)) \bullet (G) \bullet (\varepsilon_0 \circ (\varepsilon_0 \bullet L_0 R_0) \circ (L_0 \bullet \eta_0 \bullet R_0)) \\ &\quad = R_1 G \bullet \varepsilon_0 = \rho_G\end{aligned}$$

□

Lemma 4.1.12. *Consider a left Kan extension* $(\mathrm{Lan}_K F, \lambda)$*. If there are two natural transformations* $\gamma, \gamma' : \mathrm{Lan}_K F \Rightarrow G$ *such that* $\gamma \circ \lambda = \gamma' \circ \lambda$ *then* $\gamma = \gamma'$*. Similarly, if we have a right Kan extension,* $(\mathrm{Ran}_K F, \rho)$ *and two natural transformations* $\gamma, \gamma' : G \Rightarrow \mathrm{Ran}_K F$ *such that* $\rho \circ \gamma = \rho \circ \gamma'$ *then* $\gamma = \gamma'$*.*

Proof. By the universal property of the left Kan extension, we know that for a natural transformation $\alpha : F \Rightarrow GK$, there exists a unique natural transformation $\beta : \mathrm{Lan}_K F \Rightarrow G$ such that $\alpha = \beta \circ \lambda$. If $\alpha = \gamma \circ \lambda = \gamma' \circ \lambda$, the uniqueness of the decomposition gives us that $\gamma = \gamma'$. A similar proof applies for the right Kan extension. □

4.2 Deformations of a Monoidal Functor

Throughout the remainder of the thesis, we will use (co)end calculus. For an extended discussion of (co)end calculus, the reader is encouraged to consult [Lor]. In the interest of making some computations more compact, for a category $\mathcal{C}$ and objects $A, B \in \mathcal{C}$ the hom-space $\mathrm{Hom}_{\mathcal{C}}(A, B)$ will be abbreviated as $\mathcal{C}_B^A$ when compactness is needed. It will be avoided if it would reduce legibility.

Definition 4.2.1 (The Day convolution)**.** [Day1; Day2] Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear monoidal categories with finite-dimensional hom-spaces, $\mathcal{C}$ small and $\mathcal{D}$ being finitely copowered and having all colimits of size $\mathcal{C}$. We then define the Day convolution on $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ by

$$F_1 \mathbin{☼} F_2 := \int^{XY} \mathrm{Hom}_{\mathcal{C}}(X \otimes_{\mathcal{C}} Y, -) \star F_1 X \otimes_{\mathcal{D}} F_2 Y \tag{4.3}$$

Theorem 4.2.2. *[Day1; Day2] Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. Then* $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *is a monoidal category with the Day convolution and unit object* $\tilde{\mathbb{I}} := \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}_{\mathcal{C}}, -) \star \mathbb{I}_{\mathcal{D}}$.

Proof. It suffices to show associativity and unitality. The following proof is adapted from [Lor, 6.2.1].

First, for associativity,

$$\begin{aligned}
(F \,\text{☼}\, (G \,\text{☼}\, H))(x) &\cong \int^{AB} \mathcal{C}_x^{A \otimes_{\mathcal{C}} B} \star (FA \otimes_{\mathcal{D}} (G \,\text{☼}\, H)(B)) && \text{def. of conv.}\\
&\cong \int^{AB} \mathcal{C}_x^{A \otimes_{\mathcal{C}} B} \star \left(FA \otimes_{\mathcal{D}} \left(\int^{CD} \mathcal{C}_B^{C \otimes_{\mathcal{C}} D} \star (GC \otimes_{\mathcal{D}} HD) \right) \right) && \text{def. of conv.}\\
&\cong \int^{ABCD} (\mathcal{C}_x^{A \otimes_{\mathcal{C}} B} \otimes_{\mathbb{K}} \mathcal{C}_B^{C \otimes_{\mathcal{C}} D}) \star (FA \otimes_{\mathcal{D}} (GC \otimes_{\mathcal{D}} HD)) && \text{rearrange}\\
&\cong \int^{ACD} \mathcal{C}_x^{A \otimes_{\mathcal{C}} (C \otimes_{\mathcal{C}} D)} \star (FA \otimes_{\mathcal{D}} (GC \otimes_{\mathcal{D}} HD)) && \text{Yoneda lemma}
\end{aligned}$$

By a similar computation we get that

$$((F \,\text{☼}\, G) \,\text{☼}\, H)(x) \cong \int^{ABD} \mathcal{C}_x^{(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} D} \star ((FA \otimes_{\mathcal{D}} GB) \otimes_{\mathcal{D}} HD)$$

Applying some relabelling and associators from the underlying categories we have associativity.

For unitality, we will show right unitality; left unitality is sufficiently similar.

$$\begin{aligned}
(F \,\text{☼}\, \tilde{\mathbb{I}})(x) &\cong \int^{AB} \mathcal{C}_x^{A \otimes_{\mathcal{C}} B} \star (FA \otimes_{\mathcal{D}} (\mathcal{C}_B^{\mathbb{I}_{\mathcal{C}}} \star \mathbb{I}_{\mathcal{D}})) && \text{def. of conv.}\\
&\cong \int^{AB} (\mathcal{C}_x^{A \otimes_{\mathcal{C}} B} \otimes_{\mathbb{K}} \mathcal{C}_B^{\mathbb{I}_{\mathcal{C}}}) \star (FA \otimes_{\mathcal{D}} \mathbb{I}_{\mathcal{D}}) && \text{rearrange}\\
&\cong \int^{A} \mathcal{C}_x^{A \otimes_{\mathcal{C}} \mathbb{I}_{\mathcal{C}}} \star (FA \otimes_{\mathcal{D}} \mathbb{I}_{\mathcal{D}}) && \text{Yoneda lemma}\\
&\cong \int^{A} \mathrm{Hom}_{\mathcal{C}}(A, x) \star FA && \text{units}\\
&\cong Fx && \text{Yoneda lemma}
\end{aligned}$$

□

Lemma 4.2.3. *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear monoidal categories, then we have that* $F_1 \,\text{☼}\, F_2 = \mathrm{Lan}_{\otimes_{\mathcal{C}}}(\otimes_{\mathcal{D}} \circ (F_1 \times F_2))$.

Proof. This is the pointwise expression of a left Kan extension as a coend, as discussed above at Lemma 4.1.9. □

Corollary 4.2.4. *The operations of the Day convolution is universally characterized by the property that there are natural isomorphisms:*

$$\mathrm{Funct}(\mathcal{C}, \mathcal{D})(F_1 \,\text{☼}\, F_2, F_3) \cong \mathrm{Funct}(\mathcal{C} \times \mathcal{C}, \mathcal{D})(\otimes_{\mathcal{D}} \circ (F_1 \times F_2), F_3 \circ \otimes_{\mathcal{C}}).$$

Corollary 4.2.5. *We have a natural isomorphism*

$$\mathrm{Nat}(F_1 \,\text{☼}\, \dots \,\text{☼}\, F_n, F) \cong \mathrm{Nat}(\otimes_{\mathcal{D}}^n \circ (F_i)_{i=1}^n, F \circ (\otimes_{\mathcal{C}})^n)$$

where we chose the standard left bracketing.

Proof. Consider the left Kan extension $\mathrm{Lan}_{\otimes_{\mathcal{C}}^n}(\otimes_{\mathcal{D}}^n \circ (F_i)_{i=1}^n)$, and write it as a coend.

$$\int^{A_i} \mathrm{Hom}_{\mathcal{C}}(\otimes_i A_i, -) \star \otimes_{\mathcal{D}}^n \circ (F_i)_{i=1}^n$$

By the associativity calculation above, we see that $F_1 \varoast \cdots \varoast F_n \cong \mathrm{Lan}_{\otimes_{\mathcal{C}}^n}(\otimes_{\mathcal{D}}^n \circ (F_i)_{i=1}^n)$. This, by the definition of the Kan extension, gives us the natural isomorphism we want to consider. □

Theorem 4.2.6. *Let* $F \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *be a functor between monoidal categories. Then lax monoidal structures* $\hat{\mu} : \otimes_{\mathcal{D}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathcal{C}}$ *are in bijection with natural transformations* $\mu : F \varoast F \Rightarrow F$ *that make* F *into an algebra in* $(\mathrm{Funct}(\mathcal{C}, \mathcal{D}), \varoast)$.

Proof. By Definition 4.2.1 and Lemma 4.1.5, we get that a natural transformation $\mu : F \varoast F \Rightarrow F$ gives us a unique natural transformation $\hat{\mu} : \otimes_{\mathcal{D}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathcal{C}}$, and vice versa. □

For a lax monoidal functor (F, μ) we will write $\hat{\mu} : F \varoast F \to F$ as the structure morphism induced by the isomorphism.

We now have two notions of cohomology. We can regard $(F, \hat{\mu})$ as an algebra in $(\mathrm{Funct}(\mathcal{C}, \mathcal{D}), \varoast)$ and work out the Hochschild cohomology. Or, we can take the notion of Davydov-Yetter cohomology as expressed in [Yet1; Yet2; DE; GHS]. The two are actually tightly related. To see this best, it is worth taking a detour through the world of multicategories. We pull primarily from [Lei].

Definition 4.2.7 (Multicategory)**.** [Lei, 2.1.1] A multicategory $\mathcal{C}$ consists of

- A class of objects $\mathrm{Ob}(\mathcal{C})$.
- For each $n \in \mathbb{N}$ and $a_1, \ldots, a_n, a \in \mathrm{Ob}(\mathcal{C})$, a class $\mathrm{Map}(a_1, \ldots, a_n; a)$ of arrows or maps, where a map θ is depicted as:
$$a_1, \ldots, a_n \xrightarrow{\theta} a.$$
- For each $n, k_1, \ldots, k_n \in \mathbb{N}$ and $a, a_i, a_i^j \in \mathrm{Ob}(\mathcal{C})$, a function
$$\begin{aligned}\mathrm{Map}(a_1, \ldots, a_n; a) \times \mathrm{Map}(a_1^1, \ldots, a_1^{k_1}; a_1) \times \cdots \times \mathrm{Map}(a_n^1, \ldots, a_n^{k_n}; a_n)& \\ \to \mathrm{Map}(a_1^1, \ldots, a_1^{k_1}, \ldots, a_n^1, \ldots, a_n^{k_n}; a)&.\end{aligned}$$
This is called composition, and is written:
$$(\theta, \theta_1, \ldots, \theta_n) \mapsto \theta \circ (\theta_1, \ldots, \theta_n)$$
- For each $a \in \mathrm{Ob}(\mathcal{C})$, an element $1_a \in \mathrm{Map}(a; a)$ called the identity on a.

These satisfy:

- Associativity:
$$\begin{aligned}&\theta \circ \Big(\theta_1 \circ (\theta_1^1, \ldots, \theta_1^{k_1}), \ldots, \theta_n \circ (\theta_n^1, \ldots, \theta_n^{k_n})\Big) \\ &\quad = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_1^1, \ldots, \theta_1^{k_1}, \ldots, \theta_n^1, \ldots, \theta_n^{k_n})\end{aligned}$$
where $\theta, \theta_i, \theta_i^j$ are such that the compositions make sense.
- Identity:
$$\theta \circ (1_{a_1}, \ldots, 1_{a_n}) = \theta = 1_a \circ (\theta)$$
where $\theta : a_1, \ldots, a_n \to a$ is a map.

Proposition 4.2.8. *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear monoidal categories. We define the* <u>*functor multicategory*</u> *$\mathcal{M}(\mathcal{C}, \mathcal{D})$, following [Lei, 2.2.1], consisting of:*

- *The class of objects of $\mathcal{M}(\mathcal{C}, \mathcal{D})$ are $\mathbb{K}$-linear functors $F : \mathcal{C} \to \mathcal{D}$.*
- *For each $(n+1)$-tuple of functors $F_1, \dots, F_n, F$, we define the maps as the multinatural transformations:*
$$\mathsf{Map}(F_1, \dots, F_n; F) := \mathsf{Nat}_{\mathbb{K}}\left(\otimes^n_{\mathcal{D}} \circ (F_i)^n_{i=1}, F \circ \otimes^n_{\mathcal{C}}\right)$$
- *For any $n, k_1, \dots, k_n \in \mathbb{N}$ and any functors F, F_i, F_i^j composition is defined by*
$$\begin{aligned} \mathsf{Map}(F_1, \dots, F_n; F) \times \mathsf{Map}(F_1^1, \dots, F_1^{k_1}; F_1) \times \cdots \times \mathsf{Map}(F_n^1, \dots, F_n^{k_n}; F_n) \\ \to \mathsf{Map}(F_1^1, \dots, F_1^{k_1}, \dots, F_n^1, \dots, F_n^{k_n}; F) \\ (\eta, \eta_1, \dots, \eta_n) \mapsto \eta \circ \otimes^n_{\mathcal{D}}(\eta_1, \dots, \eta_n) \end{aligned}$$
- *The identity map $1_F \in \mathsf{Map}(F; F)$ is just the identity natural transformation in $\mathsf{Nat}_{\mathbb{K}}(F, F)$.*

It is easy, following [Lei] to see that this forms a multicategory.

We now need a small proposition:

Proposition 4.2.9. *[Lei, 2.1.3] There is a functor $V : \mathbf{MonCat} \to \mathbf{MultiCat}$ from monoidal categories to multicategories, where*

- *Objects of $V(\mathcal{C})$ are objects of $\mathcal{C}$.*
- *The maps $\mathsf{Map}(A_1, \dots, A_n; B)$ are $\mathsf{Hom}_{\mathcal{C}}(A_1 \otimes \cdots \otimes A_n, B)$ where $A_1 \otimes \cdots \otimes A_n$ has the standard left bracketing.*
- *The identity map in $\mathsf{Map}(A; A)$ is just the identity morphism in $\mathsf{Hom}_{\mathcal{C}}(A, A)$.*

We say that $V(\mathcal{C})$ is the <u>underlying multicategory of $\mathcal{C}$</u>, and if for a multicategory $\mathcal{E} \cong V(\mathcal{C})$ for some monoidal category $\mathcal{C}$, we say $\mathcal{E}$ is <u>represented by $\mathcal{C}$</u>. We want that $\mathcal{M}(\mathcal{C}, \mathcal{D})$ is represented by $(\mathsf{Funct}(\mathcal{C}, \mathcal{D}), \sun)$.

In particular, we have a way to describe the representation of a multicategory.

Definition 4.2.10 (Representation of a multicategory)**.** [Lei, 3.3.1] A <u>representation</u> of a multicategory $\mathcal{M}$ is an object $\otimes(c_1, \dots, c_n)$ and a map $u(c_1, \dots, c_n) : c_1 \dots, c_n \to \otimes(c_1, \dots, c_n)$ for each natural number n and list of objects $c_1, \dots, c_n \in \mathcal{M}$ such that we have a universal factorization property. That is, for any objects $c_1^1, \dots, c_1^{k_1}, \dots, c_n^1 \dots, c_n^{k_n}, c \in \mathcal{M}$ and any map $f : c_1^1, \dots, c_n^{k_n} \to c$ there is a unique map
$$\bar{f} : \otimes(c_1^1, \dots, c_1^{k_1}), \dots, \otimes(c_n^1, \dots, c_n^{k_n}) \to c$$
such that
$$\bar{f} \circ (u(c_1^1, \dots, c_1^{k_1}), \dots, u(c_n^1, \dots, c_n^{k_n})) = f.$$
If $\mathcal{M}$ admits a representation, we say it is <u>representable</u>.

With this in hand, we can prove the following:

Theorem 4.2.11. *$\mathcal{M}(\mathcal{C}, \mathcal{D})$ is a representable multicategory with $\sun$ as a representation.*

Proof. First we need to define our objects $\sun(F_1, \dots, F_n)$ and our universal maps $u(F_1, \dots, F_n)$. Define $\sun(F_1, \dots, F_n) := (\cdots(F_1 \sun F_2) \sun \cdots \sun F_{n-1}) \sun F_n$ using the standard left bracketing.

To construct u, consider Corollary 4.2.5. That gives us a natural isomorphism
$$\Phi : \mathsf{Nat}(F_1 \sun \cdots \sun F_n, F_1 \sun \cdots \sun F_n) \to \mathsf{Nat}(\otimes^n_{\mathcal{D}} \circ (F_i)^n_{i=1}, (F_1 \sun \cdots \sun F_n) \circ \otimes^n_{\mathcal{C}})$$

Define $u := \Phi(\mathrm{id})$.

It remains to prove that for a sequence of functors $F_1^1, \ldots, F_1^{k_1}, \ldots, F_n^1, \ldots, F_n^{k_n}$, with $k_1 + \cdots + k_n = m$, for any $f \in \mathrm{Nat}(\otimes_{i,j} F_i^j, F \circ \otimes_{\mathcal{C}}^m)$ we have $\bar{f} : \mathrm{Nat}(\otimes_j \star_i F_i^j, F \circ \otimes_{\mathcal{C}}^n)$ with the proper factorization constraint. This is clear from the universal property of the left Kan extension and Corollary 4.2.5. □

This allows us to define a cochain complex of F considered as an algebra.

Definition 4.2.12. Let (F, μ) be a lax monoidal functor between $\mathbb{K}$-linear monoidal categories where we can define the Day convolution. Then we define a cochain complex $C^\bullet_{alg}(F)$ in $\mathbf{Vect}_{\mathbb{K}}$ with $[n] \mapsto \mathrm{Nat}(F^{\star n}, F)$ and for $\phi : F^{\star n} \to F$:

$$\begin{aligned} \partial_0(\phi) &:= \mu \circ (\mathrm{id}_F \star \phi) \\ \partial_i(\phi) &:= \phi \circ (\mathrm{id}_{F^{\star i-1}} \star \mu \star \mathrm{id}_{F^{\star n-i-1}}) \\ \partial_{n+1}(\phi) &:= \mu \circ (\phi \star \mathrm{id}_F) \end{aligned}$$

Then this gives us a cochain complex by $b^{alg}(\phi) := \sum_{i=0}^{n+1} (-1)^i \partial_i(\phi)$. We define the algebraic Day cohomology $H^n_{alg}(F)$ in the standard way.

To control deformations of monoidal functors we introduce Davydov-Yetter cohomology. For additional treatments of Davydov-Yetter cohomology, we direct the reader to [Yet2; Yet1; DE; GHS] and [EGNO, 7.22]

Definition 4.2.13. Let $(F, \hat{\mu})$ be a lax monoidal functor between two $\mathbb{K}$-linear monoidal categories $\mathcal{C}$ and $\mathcal{D}$. Define a cochain complex $C^\bullet_{DY}(F)$ in $\mathbf{Vect}_{\mathbb{K}}$ with $[n] \mapsto \mathrm{Nat}({}^n\otimes \circ (F^n), F \circ \otimes^n) = \mathrm{Map}(F, \ldots, F; F)$ and for $\phi : \underbrace{F, F, \ldots, F}_{n \text{ times}} \to F$,

$$\begin{aligned} \partial_0^{DY}(\phi)_{X_0,\ldots,X_n} &:= \hat{\mu} \circ (\mathrm{id}_{FX_0} \otimes \phi_{X_1,\ldots,X_n}) \\ \partial_i^{DY}(\phi)_{X_0,\ldots,X_n} &:= \phi_{X_0,\ldots,X_i \otimes X_{i+1},\ldots,X_n} \circ (\mathrm{id}_{FX_0 \otimes \cdots \otimes FX_{i-1}} \otimes \hat{\mu} \otimes \mathrm{id}_{FX_{i+2} \otimes \cdots \otimes FX_n}) \text{ for } 0 < i < n \\ \partial_{n+1}^{DY}(\phi)_{X_0,\ldots,X_n} &:= \hat{\mu} \circ (\phi_{X_0,\ldots,X_n} \otimes \mathrm{id}_{FX_n}) \end{aligned}$$

Then this gives us a cochain complex by $b^{DY}(\phi) := \sum_{i=0}^{n+1} (-1)^i \partial_i^{DY}(\phi)$. We define the Davydov-Yetter cohomology $H^n_{DY}(F)$ in the standard way.

The following result is a generalization of [EGNO, 7.22.7].

Lemma 4.2.14. *For a lax monoidal functor F there is a quasi-isomorphism between the two cochain complexes $C^\bullet_{DY}(F)$ and $C^\bullet_{alg}(F)$.*

Proof. It is clear that $H^\bullet(V(C^\bullet_{alg}(F))) \cong H^\bullet(C^\bullet_{alg}(F))$. So, it only remains to prove that $H^\bullet(C^\bullet_{DY}(F)) \cong H^\bullet(V(C^\bullet_{alg}(F)))$. It will be sufficient to show that $b^{DY}V(\phi) = V(b^{alg}(\phi))$ for $\phi : F^{\star n} \to F$. We will do this on the ∂_0 term, as the rest are similar. Recall that $\hat{\mu} = \mu \circ \star$. This gives us

$$\begin{aligned} \partial_0^{DY}(V(\phi))_{X_0,\ldots,X_n} &= \hat{\mu} \circ (\mathrm{id}_{FX_0} \otimes (\phi \circ \star^n)_{X_1,\ldots,X_n}) \\ &= (\mu \circ (\mathrm{id}_F \star \phi)) \circ \star^{n+1} \end{aligned}$$

Essentially, the universal factorization property of a representation of a multicategory lets us pull all the universal objects to the start of the composition. This is the same for every other ∂_i, which proves that the two cochain complexes are quasi-isomorphic. □

This means that we should regard an infinitesimal deformation of a lax monoidal functor (F, μ) in $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ as a lax monoidal functor (F_t, μ_t) in $\mathrm{Funct}(\mathbb{D} \otimes \mathcal{C}, \mathbb{D} \otimes \mathcal{D})$ such that (F_t, μ_t) is (F, μ) when regarded as a $\mathbb{K}$-linear functor. This implies that $F_t = F$, $\mu_t = \mu + t\varphi$.

Definition 4.2.15. Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear monoidal categories such that $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ has finite-dimensional hom spaces, $\mathcal{C}$ small, and $\mathcal{D}$ has all $\mathcal{C}$ colimits. Then let (F, μ) be a lax monoidal functor in $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$. A $\mathbb{D}$-deformation of F is a lax monoidal functor (F_t, μ_t) in $\mathrm{Funct}(\mathbb{D} \otimes_{\mathbb{K}} \mathcal{C}, \mathbb{D} \otimes_{\mathbb{K}} \mathcal{D})$ such that $\mathbb{K} \otimes_{\mathbb{D}} F_t$ is isomorphic as a lax monoidal functor to F.

Two deformations, (F_t, μ_t), (F_t, μ_t') are equivalent if there exists a monoidal natural isomorphism $\eta : F_t \Rightarrow F_t$ such that $\mathbb{K} \otimes_{\mathbb{D}} \eta : F \Rightarrow F$ is the identity. Equivalently, $\nu_1 = \eta^{-1} \circ \nu_0 \circ (\eta \otimes \eta)$ and $\mathbb{K} \otimes_{\mathbb{D}} \eta : F \Rightarrow F$ is the identity.

Lemma 4.2.16. *A lax monoidal functor* (G, ν) *in* $\mathrm{Funct}(\mathbb{D}\otimes_{\mathbb{K}}\mathcal{C}, \mathbb{D}\otimes_{\mathbb{K}}\mathcal{D})$ *with* $\nu := \mu + t\varphi$ *is a deformation of* (F, μ) *if and only if* φ *is a 2-coboundary in the Davydov-Yetter cohomology.*

Proof. Consider the associativity of ν; working through the commutative diagram and bilinearity of composition, we get that:

$$\begin{aligned}
&0 = \nu \circ (\nu \otimes \mathrm{id}) - \nu \circ (\mathrm{id} \otimes \nu)\\
&0 = \mu \circ (\mu \otimes \mathrm{id}) + t[\varphi \circ (\mu \otimes \mathrm{id}) + \mu \circ (\varphi \otimes \mathrm{id})] - \mu \circ (\mathrm{id} \otimes \mu) - t[\varphi \circ (\mathrm{id} \otimes \mu) + \mu \circ (\mathrm{id} \otimes \varphi)]\\
&0 = [\mu \circ (\mu \otimes \mathrm{id}) - \mu \circ (\mathrm{id} \otimes \mu)] + t[\mu \circ (\varphi \otimes \mathrm{id}) - \varphi \circ (\mu \otimes \mathrm{id}) + \varphi \circ (\mathrm{id} \otimes \mu) - \mu \circ (\mathrm{id} \otimes \varphi)]\\
&0 = [\mu \circ (\mu \otimes \mathrm{id}) - \mu \circ (\mathrm{id} \otimes \mu)] + t b^{DY}(\varphi)
\end{aligned}$$

Note that $\mu \circ (\mu \otimes \mathrm{id}) - \mu \circ (\mathrm{id} \otimes \mu) = 0$ by (F, μ) being a lax monoidal functor. ☐

Lemma 4.2.17. *Two deformations of* (F, μ)*, say* (G_0, ν_0)*,* (G_1, ν_1) *with* $\nu_0 := \mu + t\varphi_0$ *and* $\nu_1 := \mu + t\varphi_1$ *are equivalent if and only if* $\varphi_0 = \varphi_1 + b^{DY} f$ *for* f *a 1-cocycle in the Davydov-Yetter cohomology.*

Proof. Say first that $\varphi_1 = \varphi_0 + b^{DY} f$. Then define $\eta := \mathrm{id} + tf$. Note that $\eta^{-1} = \mathrm{id} - tf$. This gives us:

$$\begin{aligned}
\eta^{-1} \circ \nu_0 \circ \eta \otimes \eta &= (\mathrm{id} - tf) \circ (\mu + t\varphi_0) \circ ((\mathrm{id} + tf) \otimes (\mathrm{id} + tf))\\
&= (\mathrm{id} - tf) \circ (\mu + t\varphi) \circ (\mathrm{id} \otimes \mathrm{id} + t[\mathrm{id} \otimes f + f \otimes \mathrm{id}])\\
&= (\mathrm{id} - tf) \circ (\mu + t[\varphi + \mu \circ (\mathrm{id} \otimes f) + \mu \circ (f \otimes \mathrm{id})])\\
&= \mu + t[\varphi + \mu \circ (\mathrm{id} \otimes f) + \mu \circ (f \otimes \mathrm{id}) - f \circ \mu]\\
&= \mu + t[\varphi_0 + b^{DY} f] = \nu_1
\end{aligned}$$

To see it the other way, let $\eta : F_t \Rightarrow F_t$ be a monoidal natural isomorphism. We just need that $\eta = \mathrm{id} + tf$ for f a natural transformation. Then the above computation gives us the identity we need. ☐

Chapter 5

Grothendieck-Verdier Categories

The purpose of this chapter is to work out the structure of the functor category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}, \mathcal{D}$ are $\mathbb{K}$-linear GV-categories. I prove that given some sensible assumptions, $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ has the structure of a $\mathbb{K}$-linear GV-category itself. The capstone of this chapter is Theorem 5.3.7, that under the appropriate conditions, Frobenius Linearly Distributive Functors are GV-Frobenius Algebras in the functor category. Along the way, I construct a class of GV-module categories for the GV-category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ and develop a few computational tools.

5.1 General Grothendieck-Verdier Categories

Here we will define a general Grothendieck-Verdier category. We will prove a couple small results that make the most sense in the case of $\mathbb{K}$-linear GV-categories.

Definition 5.1.1 (Grothendieck-Verdier Category)**.** [BD, 1.1][Bar1]

- Let $(\mathcal{C}, \otimes)$ be a monoidal category. An object $\mathsf{K} \in \mathcal{C}$ is a dualizing object if for all $\mathsf{Y} \in \mathcal{C}$, the functor $\mathsf{X} \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathsf{X} \otimes \mathsf{Y}, \mathsf{K})$ is representable by some object GY, and the contravariant functor $\mathsf{G} : \mathcal{C} \to \mathcal{C}$ is an anti-equivalence of categories. We call G the duality functor with respect to K.
- A choice of dualizing object K for a category $\mathcal{C}$ is a Grothendieck-Verdier structure, and a monoidal category with a Grothendieck-Verdier structure is a Grothendieck-Verdier category. We may also call these a GV-structure and a GV-category.

Remark 5.1.2. Recall that a *-autonomous category in the sense of [Bar1] is a symmetric GV-category in our terminology. Later work, such as [Bar2], expanded the notion of a *-autonomous category to non-symmetric GV-categories. Care should be taken when reading the literature to avoid confusion.

Lemma 5.1.3. *If $\mathcal{C}$ is a GV-category with dualizing object K and dualizing functor G, then we have the isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathsf{X} \otimes \mathsf{Y}, \mathsf{K}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathsf{Y}, \mathsf{G}^{-1}\mathsf{X})$*

Proof. As G is an anti-equivalence of categories, we get:

$$\mathrm{Hom}_{\mathcal{C}}(\mathsf{X} \otimes \mathsf{Y}, \mathsf{K}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{GY}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathsf{G}^{-1}\mathsf{GY}, \mathsf{G}^{-1}\mathsf{X}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathsf{Y}, \mathsf{G}^{-1}\mathsf{X}).$$

□

Lemma 5.1.4. *Given a GV-category $(\mathcal{C}, \otimes, \mathsf{K})$, there is a second monoidal structure on $\mathcal{C}$, denoted by $⅋$ and defined by $\mathsf{X} ⅋ \mathsf{Y} := \mathsf{G}^{-1}(\mathsf{GY} \otimes \mathsf{GX})$. The monoidal unit of $(\mathcal{C}, ⅋)$ is K.*

Proof. We only need to prove a few things.

First is associativity.

$$\begin{aligned} X \parr (Y \parr Z) &\cong G^{-1}(G(Y \parr Z) \otimes GX) \\ &\cong G^{-1}(GG^{-1}(GZ \otimes GY) \otimes GX) \\ &\cong G^{-1}((GZ \otimes GY) \otimes GX) \\ &\cong G^{-1}(GZ \otimes (GY \otimes GX)) \\ &\cong G^{-1}(GZ \otimes GG^{-1}(GY \otimes GX)) \\ &\cong G^{-1}(GZ \otimes G(X \parr Y)) \\ &\cong (X \parr Y) \parr Z \end{aligned}$$

The isomorphism is $\alpha^{\parr}_{X,Y,Z} = G^{-1}(\alpha^{\otimes}_{GZ,GY,GX})$. It obviously obeys the pentagon axiom so we can take it as the associator for $\parr$. We will always consider $\parr$ with this associator.

Next we need to show that K behaves as the monoidal unit with respect to $\parr$. By Lemma 5.1.3, we get:

$$\mathrm{Hom}_{\mathcal{C}}(X, K) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I} \otimes X, K) \cong \mathrm{Hom}_{\mathcal{C}}(X, G^{-1}\mathbb{I})$$

By the Yoneda lemma, this implies that $G^{-1}(\mathbb{I}) \cong K$. Note this also implies that $GK \cong \mathbb{I}$, and a similar computation shows $G\mathbb{I} \cong K$ and $G^{-1}K \cong \mathbb{I}$.

$$X \parr K \cong G^{-1}(GK \otimes GX) \cong G^{-1}(\mathbb{I} \otimes GX) \cong G^{-1}GX \cong X.$$

Proof of the coherence conditions is left as an exercise for the reader. □

Following [FSSW1, p. 7], we get the two isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(x \otimes y, z) &\cong \mathrm{Hom}_{\mathcal{C}}(x, z \parr Gy) \\ \mathrm{Hom}_{\mathcal{C}}(x \otimes y, z) &\cong \mathrm{Hom}_{\mathcal{C}}(y, G^{-1}x \parr z) \end{aligned}$$

This gives the impression of G acting like a right duality and G^{-1} acting like a left duality in some sense. We actually want to define evaluation and coevaluation in a GV-category based on that impression.

Remark 5.1.5. [FSSW1, 3.33] By definition, we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Gy \otimes y, K) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(Gy, Gy) \\ \mathrm{Hom}_{\mathcal{C}}(y \otimes G^{-1}y, K) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(y, y) \\ \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, y \parr Gy) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(Gy, Gy) \\ \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, G^{-1}y \parr y) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(y, y) \end{aligned}$$

Taking the pre-image of the identity morphisms under each of these isomorphisms we get the following morphisms:

$$\begin{aligned} \mathrm{eval}^{\ell}_y &: Gy \otimes y \to K \\ \mathrm{eval}^{r}_y &: y \otimes G^{-1}y \to K \\ \mathrm{coeval}^{\ell}_y &: \mathbb{I} \to y \parr Gy \\ \mathrm{coeval}^{r}_y &: \mathbb{I} \to G^{-1}y \parr y \end{aligned}$$

The proof that these behave like our familiar evaluations and coevaluations – as discussed in Remark 3.0.1 – is too long for this thesis, and so I refer the reader to [FSSW1, 3.33-3.35]. However, we will see below that there are some interesting nuances to this fact.

Lemma 5.1.6. *[BD, 4.2]For a GV-category $\mathcal{C}$, the square of the dualizing functor $G^2 : \mathcal{C} \to \mathcal{C}$ has the structure of a strong monoidal functor.*

Following [BD], we take the notion of a pivotal GV-category.

Definition 5.1.7 (Pivotal GV-category)**.** [BD, 5.1][FSSW1, 4.16] A pivotal structure on a GV-category $\mathcal{C}$ is a natural family of isomorphisms

$$\psi_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x \otimes y, K) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(y \otimes x, K)$$

for $x, y \in \mathcal{C}$ such that

$$\psi_{x,y} \circ \psi_{y,x} = \mathrm{id} \quad \text{and} \quad \psi_{x\otimes y,z} \circ \psi_{y\otimes z,x} \circ \psi_{z\otimes x,y} = \mathrm{id}$$

for $x, y, z \in \mathcal{C}$. A pivotal GV-category is a GV-category together with a choice of pivotal structure.

When establishing a further result in the thesis, an equivalent characterization of pivotality will be more useful. The following two lemmas establish that characterization.

Lemma 5.1.8. *[BD, 5.6] There is a bijection between functoral isomorphisms $\psi : \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, K) \to \mathrm{Hom}_{\mathcal{C}}(Y \otimes X, K)$ on a GV-category $\mathcal{C}$ and natural isomorphisms $f : \mathrm{id} \Rightarrow G^2$. In particular, in the diagram* (5.1)*, given a fixed ψ there is a unique ϖ and vice-versa.*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, K) \arrow[d, "\cong"] \arrow[dr, "\psi_{X,Y}"] & \\ \mathrm{Hom}_{\mathcal{C}}(G^2Y \otimes X, K) \arrow[r, "(f_Y\otimes \mathrm{id})^*"] & \mathrm{Hom}_{\mathcal{C}}(Y \otimes X, K) \end{tikzcd} \tag{5.1}$$

Proof. This holds by Yoneda's lemma. □

Proposition 5.1.9. *[BD, 5.7] A natural isomorphism $f : \mathrm{id} \to G^2$ corresponds to a pivotal structure if and only if it satisfies:*

- f *is monoidal.*
- $f_K : K \to G^2K$ *equals the composition*

$$K \xrightarrow{\cong} G\mathbb{I} \xrightarrow{\cong} G^2G^{-1}\mathbb{I} \xrightarrow{\cong} G^2K.$$

In this case, we will denote the natural isomorphism by ϖ.

For a proof of this, see [BD, 13].

Following [FSSW1], we define the notion of a GV-module category. First, we need to define a module category of a monoidal category.

Definition 5.1.10 (Module Category)**.** [EGNO, 7.1.1] Let $(\mathcal{C}, \otimes)$ be a monoidal category. A right module category $\mathcal{M}$ over $(\mathcal{C}, \otimes)$ is a category together with a bifunctor $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$, called the action, and two natural isomorphisms, $s^{\triangleleft}_{M,X,Y} : M \triangleleft (X \otimes Y) \to (M \triangleleft X) \triangleleft Y$, called the module associativity constraint and $u^{\triangleleft}_{M} : M \to M \triangleleft \mathbb{I}$, called the module unitor constraint. The module associativity constraint satisfies a version of the pentagon axiom:

$$
\begin{tikzcd}
M \triangleleft ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_M \triangleleft \alpha_{X,Y,Z}"] \arrow[d, "s^{\triangleleft}_{M,X\otimes Y,Z}"] & & M \triangleleft (X \otimes (Y \otimes Z)) \arrow[d, "s^{\triangleleft}_{M,X,Y\otimes Z}"] \\
(M \triangleleft (X \otimes Y)) \triangleleft Z \arrow[dr, "s^{\triangleleft}_{M,X,Y} \triangleleft \mathrm{id}_Z"'] & & (M \triangleleft X) \triangleleft (Y \otimes Z) \arrow[dl, "s^{\triangleleft}_{M\triangleleft X,Y,Z}"] \\
& ((M \triangleleft X) \triangleleft Y) \triangleleft Z &
\end{tikzcd}
\tag{5.2}
$$

And the module unitor constraint satisfies the triangle diagram:

$$
\begin{tikzcd}
M \triangleleft (\mathbb{I} \otimes X) \arrow[rr, "s^{\triangleleft}_{M,\mathbb{I},X}"] \arrow[dr, "\mathrm{id}_M \triangleleft \ell_X"'] & & (M \triangleleft \mathbb{I}) \triangleleft X \arrow[dl, "u^{\triangleleft}_M \triangleleft X"] \\
& M \triangleleft X &
\end{tikzcd}
\tag{5.3}
$$

With that in hand, we can define a GV-module category.

Definition 5.1.11 (GV-module category)**.** [FSSW1, 3.1] Let $(\mathcal{C}, \otimes, \invamp)$ be a GV category. A right GV-module category of $\mathcal{C}$ is a right module category $\mathcal{M}$ over the monoidal category $(\mathcal{C}, \otimes)$ together with a right $\otimes$-action, $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ such that the functors

$$- \triangleleft c : \mathcal{M} \to \mathcal{M} \quad \text{and} \quad m \triangleleft - : \mathcal{C} \to \mathcal{M}$$

have a right adjoint for all objects $c \in \mathcal{C}$ and $m \in \mathcal{M}$.

Remark 5.1.12. We denote these right adjoints by:

$$(- \triangleleft c) \dashv \Gamma_c \quad \text{and} \quad (m \triangleleft -) \dashv \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m, -).$$

In particular, the right adjoint to $m \triangleleft -$ gives us the bifunctor $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, called the left inner hom.

Remark 5.1.13. The definition for a left module category and a left GV-module category is similar. We prioritize the right in this work because it makes the graphical calculus easier to read. All proofs regarding the right GV-module categories can be done, in exactly the same way, for left GV-module categories.

Lemma 5.1.14. *Let* $(\mathcal{C}, \otimes, K)$ *be a GV-category. Then* $\mathcal{C}$ *forms a right GV-module category over itself with* $- \triangleleft C := - \otimes C$*. The adjoints are then given by* $H_c(x) := x \invamp Gc$ *and* $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m, x) := G^{-1}m \invamp x$.

Proof. The proof is left as an informative exercise for the reader. □

Definition 5.1.15 (Module Functors)**.** [Shi, Section 2.4][EGNO, 7.2.1] Let $\mathcal{M}$, $\mathcal{N}$ be right $\mathcal{C}$-module categories. A lax <u>module functor</u> is a pair (F, σ) consisting of a functor $F : \mathcal{M} \to \mathcal{N}$ and a natural transformation:

$$\sigma : \triangleleft \circ (F \times \mathrm{id}_{\mathcal{C}}) \Rightarrow F \circ \triangleleft$$

such that for all $M \in \mathcal{M}$, $X, Y \in \mathcal{C}$ we have

$$\sigma_{M,\mathbb{I}} = \mathrm{id}_{F(M)} \quad \text{and} \quad \sigma_{M,X\otimes Y} = \sigma_{M\triangleleft X,Y} \circ (\sigma_{M,X} \times \mathrm{id}_Y).$$

An op-lax module functor has instead $\sigma : F \circ \triangleleft \Rightarrow \triangleleft \circ (F \times \mathrm{id}_{\mathcal{C}})$ with similar conditions as above. Finally, a module functor is called <u>strong</u> if σ is an isomorphism.

Definition 5.1.16 (Morphism of Module Functors)**.** [EGNO, 7.2.2] A morphism of lax $\mathcal{C}$-module functors from (F, s) to (G, t) is a natural transformation $\gamma : F \Rightarrow G$ such that for all $C \in \mathcal{C}$, $M \in \mathcal{M}$ we have the commuting square (5.4). Morphisms of op-lax or strong $\mathcal{C}$-module functors are similarly defined.

$$\begin{tikzcd} F(M) \triangleleft C \arrow[r, "s_{M,C}"] \arrow[d, "\beta_M \triangleleft \mathrm{id}_C"'] & F(M \triangleleft C) \arrow[d, "\beta_{M \triangleleft C}"] \\ G(M) \triangleleft C \arrow[r, "t_{M,C}"] & G(M \triangleleft C) \end{tikzcd} \tag{5.4}$$

Note that for a GV-category $\mathcal{C}$ and a right $\mathcal{C}$ GV-module category $\mathcal{M}$, when we regard $\mathcal{C}$ as the right regular module, the functor $m \triangleleft - : \mathcal{C} \to \mathcal{M}$ is a strong module functor. The following result shows that since the left inner hom is a right adjoint to $m \triangleleft -$, it has the structure of a lax module functor. The proof of the theorem requires significant discussion of module profunctors, a topic beyond the scope of this thesis. As such, the reader is referred to [Shi] for the proof.

Lemma 5.1.17. *[Shi, Lemma 2.5] Let $F : \mathcal{M} \to \mathcal{N}$ be a functor with a right adjoint G between right $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$. If (F, s) is a strong right module functor then G has the unique structure of a lax right module functor such that the counit and unit of the adjoint pair $F \dashv G$ are morphisms of lax right module functors.*

Corollary 5.1.18 (Inner Homs are Lax Module Functors)**.** *Let $\mathcal{C}$ be a GV-category and $\mathcal{M}$ be a right $\mathcal{C}$ GV-module category. Then for all $m \in \mathcal{M}$, $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m, -) : \mathcal{M} \to \mathcal{C}$ is a lax right module functor.*

Proof. We know that we have the adjoint pair $(m \triangleleft -) \dashv \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m, -)$. Now, to use Lemma 5.1.17, we need that $m \triangleleft -$ is a strong right module functor. That is, we have a natural isomorphism $\sigma_{x,y} : m \triangleleft (x \otimes y) \to (m \triangleleft x) \triangleleft y$. Such a natural isomorphism exists, namely, set $\sigma_{x,y} := s^{\triangleleft}_{m,x,y}$. So the lemma applies. □

A special case of this corollary is sufficiently important that we should consider it a proposition.

Proposition 5.1.19 (Distributors)**.** *[FSSW2, 4.11][CS1] Let $\mathcal{C}$ be a GV-category. Then there exist natural transformations δ^{ℓ}, δ^{r} called the distributors.*

$$\delta^{\ell} : \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \invamp) \Rightarrow \invamp \circ (\otimes \times \mathrm{id}_{\mathcal{C}})$$
$$\delta^{r} : \otimes \circ (\invamp \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \invamp \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$$

These satisfy ten coherence conditions in Appendix C.

Proof. Consider the right regular module of $\mathcal{C}$. Then for $c \in \mathcal{C}$, define a functor $Q_{Gc} : \mathcal{C} \to \mathcal{C}$, $x \mapsto Gc \otimes x$. This has the right adjoint $H_c : \mathcal{C} \to \mathcal{C}$, $x \mapsto c \invamp x$. Now, Q_{Gc} is a strong module functor, which means by Lemma 5.1.17, we know that H_c has the unique structure of a lax module functor. That is, there is some natural transformation d such that $d_{x,y} : (c \invamp x) \otimes y \to c \invamp (x \otimes y)$. This is our right distributors δ^{r}. Our left distributor δ^{ℓ} is given by looking at the left regular module of $\mathcal{C}$. What remains is the coherency conditions. For those we refer the reader to [FSSW2, 4.11]. □

Remark 5.1.20. With the distributors established we can now clarify how exactly the morphisms

$$\begin{aligned} \mathrm{eval}^{\ell}_{y} &: Gy \otimes y \to K \\ \mathrm{eval}^{r}_{y} &: y \otimes G^{-1}y \to K \\ \mathrm{coeval}^{\ell}_{y} &: \mathbb{I} \to y \invamp Gy \\ \mathrm{coeval}^{r}_{y} &: \mathbb{I} \to G^{-1}y \invamp y \end{aligned}$$

behave like our familiar evaluations and coevaluations. They satisfy the same triangle identities, but instead of using the associator we instead need to use the distributor. These are:

$$\begin{aligned}
\mathrm{id}_X &= (\mathrm{eval}^r_X \parr \mathrm{id}_X) \circ \delta^\ell_{X,G^{-1}X,X} \circ (\mathrm{id}_X \otimes \mathrm{coeval}^r_X) \\
\mathrm{id}_{G^{-1}X} &= (\mathrm{id}_{G^{-1}X} \parr \mathrm{eval}^r_X) \circ \delta^r_{G^{-1}X,X,G^{-1}X} \circ (\mathrm{coeval}^r_X \otimes \mathrm{id}_{G^{-1}X}) \\
\mathrm{id}_X &= (\mathrm{id}_X \parr \mathrm{eval}^\ell_X) \circ \delta^r_{X,GX,X} \circ (\mathrm{coeval}^\ell_X \otimes \mathrm{id}_X) \\
\mathrm{id}_{GX} &= (\mathrm{eval}^\ell_X \parr \mathrm{id}_{GX}) \circ \delta^\ell_{GX,X,GX} \circ (\mathrm{id}_{GX} \otimes \mathrm{coeval}^\ell_X).
\end{aligned}$$

With the distributors established, we can define both Frobenius LD-functors and GV-Frobenius algebras.

Definition 5.1.21 (Frobenius Linearly Distributive Functor)**.** [CS2, 1] Let $\mathcal{C}$, $\mathcal{D}$ be GV-categories. Then a Frobenius Linear Distributive Functor or Frobenius LD Functor, is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it has a lax $\otimes$-monoidal structure μ, an op-lax $\parr$-monoidal structure Δ, and it satisfies the diagrams (5.5).

$$\begin{tikzcd}
FX \otimes F(Y \parr Z) \arrow[r, "\mathrm{id}\otimes\Delta"] \arrow[d, "\mu"] & FX \otimes (FY \parr FZ) \arrow[d, "\delta^\ell"] & F(X \parr Y) \otimes FZ \arrow[r, "\Delta\otimes\mathrm{id}"] \arrow[d, "\mu"] & (FX \parr FY) \otimes FZ \arrow[d, "\delta^r"] \\
F(X \otimes (Y \parr Z)) \arrow[d, "F\bullet\delta^\ell"] & (FX \otimes FY) \parr FZ \arrow[d, "\mu\parr\mathrm{id}"] & F((X \parr Y) \otimes Z) \arrow[d, "F\bullet\delta^r"] & FX \parr (FY \otimes FZ) \arrow[d, "\mathrm{id}\parr\mu"] \\
F((X \otimes Y) \parr Z) \arrow[r, "\Delta"] & F(X \otimes Y) \parr FZ & F(X \parr (Y \otimes Z)) \arrow[r, "\Delta"] & FX \parr F(Y \otimes Z)
\end{tikzcd} \tag{5.5}$$

Remark 5.1.22. These are a generalization of Frobenius monoidal functors. Those are discussed in [DP].

Remark 5.1.23. If we take the one-object GV-category $\mathcal{I}$, then the image of a Frobenius LD-functor between GV-categories $F : \mathcal{I} \to \mathcal{C}$ is what we call a GV-Frobenius algebra.[Egg2, 2.4]

Definition 5.1.24. [FSSW1, 4.4] A GV-Frobenius algebra is a tuple $(A, \mu, \eta, \Delta, \varepsilon)$ where $A \in \mathcal{C}$ is an object and we have four morphisms:

$$\begin{aligned}
\mu &: A \otimes A \to A & \Delta &: A \to A \parr A \\
\eta &: \mathbb{I} \to A & \varepsilon &: A \to K.
\end{aligned}$$

We need that (A, μ, η) is an algebra in $(\mathcal{C}, \otimes, \mathbb{I})$ and (A, Δ, ε) is a coalgebra in $(\mathcal{C}, \parr, K)$. Finally, these morphisms satisfy the diagram (5.6). Note that this diagram makes use of the distributors.

$$\begin{tikzcd}
& A \otimes A \arrow[dl, "\mathrm{id}\otimes\Delta"'] \arrow[dd, "\mu"] \arrow[dr, "\Delta\otimes\mathrm{id}"] & \\
A \otimes (A \parr A) \arrow[dd, "\delta^\ell"] & & (A \parr A) \otimes A \arrow[dd, "\delta^r"] \\
& A \arrow[dd, "\Delta"] & \\
(A \otimes A) \parr A \arrow[dr, "\mu\parr\mathrm{id}"'] & & A \parr (A \otimes A) \arrow[dl, "\mathrm{id}\parr\mu"] \\
& A \parr A &
\end{tikzcd} \tag{5.6}$$

As in the monoidal case, there are several equivalent definitions of Frobenius algebras in GV-categories, as discussed in [FSSW1]. For this we need some auxiliary notions.

Definition 5.1.25 (GV-pairing and GV-copairing)**.** [FSSW1, 4.7] Let $x \in \mathcal{C}$ be an object in a GV category. Then a GV-pairing on x is a morphism $\kappa_x \in \mathrm{Hom}_{\mathcal{C}}(x \otimes x, K)$, and a GV-copairing on x is a morphism $\bar{\kappa}_x \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, x \parr x)$. We say a GV-pairing κ_x on x is called non-degenerate if and only if there exists some GV-copairing $\bar{\kappa}_x$ on x such that diagram (5.7) holds.

$$\begin{tikzcd}
 & x \otimes \mathbb{I} \arrow[dl, "\mathrm{id} \otimes \bar{\kappa}_x"'] & & \mathbb{I} \otimes x \arrow[dr, "\bar{\kappa}_x \otimes \mathrm{id}"] & \\
x \otimes (x \parr x) \arrow[d, "\delta^{\ell}"] & & x \arrow[ul, "\cong"'] \arrow[ur, "\cong"] \arrow[d, equal] & & (x \parr x) \otimes x \arrow[d, "\delta^{r}"] \\
(x \otimes x) \parr x \arrow[dr, "\kappa_x \parr \mathrm{id}"'] & & x & & x \parr (x \otimes x) \arrow[dl, "\mathrm{id} \parr \kappa_x"] \\
 & K \parr x \arrow[ur, "\cong"] & & x \parr K \arrow[ul, "\cong"'] &
\end{tikzcd} \tag{5.7}$$

Definition 5.1.26 (Invariant Pairing)**.** [FSSW1, 4.8]

Let A be an algebra in a GV-category $\mathcal{C}$. Then an invariant GV-pairing on A is a GV-pairing κ_A on A such that diagram (5.8) commutes.

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\alpha^{\otimes}_{A,A,A}"] \arrow[d, "\mu \otimes \mathrm{id}"'] & & A \otimes (A \otimes A) \arrow[d, "\mathrm{id} \otimes \mu"] \\
A \otimes A \arrow[dr, "\kappa_A"'] & & A \otimes A \arrow[dl, "\kappa_A"] \\
 & K &
\end{tikzcd} \tag{5.8}$$

These notions let us state the following without proof.

Proposition 5.1.27. *[FSSW1, 4.9] For an algebra A in $\mathcal{C}$ there is a bijection between GV-Frobenius algebras and the invariant non-degenerate pairings on A.*

The reader is reminded that in Chapter 3, we considered symmetric Frobenius algebras. As such, we should have some sensible notion of a symmetric GV-Frobenius algebra. Since the notion of a symmetric Frobenius algebra presupposes that the ambient monoidal category is pivotal, we should expect that the ambient GV category should also be pivotal. This is the case.

Proposition 5.1.28. *[FSSW1, 4.18] We say a GV-Frobenius algebra $(A, \mu, \eta, \Delta, \varepsilon)$ in a pivotal GV-category is symmetric if and only if the related invariant GV-pairing κ_A is symmetric. That is, if the* *diagram* (5.9) *commutes.*

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\mathrm{id} \otimes (\mathrm{id} \otimes \mathrm{coeval}^{\ell})"] \arrow[ddd, "\kappa_A"'] & & A \otimes (A \otimes (A \parr GA)) \arrow[d, dashed, "\mathrm{id} \otimes \delta^{\ell}"] \\
& & A \otimes ((A \otimes A) \parr GA) \arrow[d, dashed, "\mathrm{id} \otimes (\kappa_A \parr \mathrm{id})"] \\
& & A \otimes (K \parr GA) \arrow[d, dashed, "\cong"] \\
K & & A \otimes GA \arrow[dl, "\mathrm{id} \otimes \varpi^{-1}"] \\
& A \otimes G^{-1}A \arrow[ul, "\mathrm{eval}_A"] &
\end{tikzcd} \tag{5.9}$$

5.2 Functors between Grothendieck-Verdier Categories

The following section is almost entirely novel results. Similar results are present in several papers in the literature, namely [Egg1; Egg2], but this section uses explicit algebraic constructions.

While the Day convolution is present in the literature, the coDay convolution of $\mathbb{K}$-linear functors is a novel development. A similar notion is mentioned in [Egg1, 3], but is not given explicitly.

Definition 5.2.1. Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear GV-categories with finite-dimensional hom spaces, $\mathcal{C}$ small, $\mathcal{D}$ finitely copowered, and let $\mathcal{D}$ have all limits and colimits of size $\mathcal{C}$. Then

1. The Day Convolution is defined as the coend
$$F_1 \sun F_2 := \int^{XY} \mathrm{Hom}_{\mathcal{C}}(X \otimes_{\mathcal{C}} Y, -) \star F_1X \otimes_{\mathcal{D}} F_2Y \tag{5.10}$$
2. The coDay Convolution is defined as the end
$$F_1 \RIGHTcircle F_2 := \int_{XY} \mathrm{Hom}_{\mathcal{C}}(-, X \parr_{\mathcal{C}} Y)^* \star F_1X \parr_{\mathcal{D}} F_2Y \tag{5.11}$$

While the definition in terms of (co)ends is useful for direct computation, the following lemma provides a deeper insight, where the Day and coDay convolutions are understood as Kan extensions.

Lemma 5.2.2. *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear GV-categories with finite-dimensional hom spaces, $\mathcal{C}$ small, $\mathcal{D}$ finitely copowered, and let $\mathcal{D}$ have all limits and colimits of size $\mathcal{C}$. Then it is the case that*

- $F_1 \sun F_2 = \mathrm{Lan}_{\otimes_{\mathcal{C}}}(\otimes_{\mathcal{D}} \circ (F_1 \times F_2))$
- $F_1 \RIGHTcircle F_2 = \mathrm{Ran}_{\parr_{\mathcal{C}}}(\parr_{\mathcal{D}} \circ (F_1 \times F_2))$.

Proof. This is a corollary of Lemma 4.1.5. □

Corollary 5.2.3. *The operations of the Day and coDay convolutions are universally characterized by the property there are natural isomorphisms:*

$$\mathrm{Funct}(\mathcal{C}, \mathcal{D})(F_1 \sun F_2, F_3) \cong \mathrm{Funct}(\mathcal{C} \times \mathcal{C}, \mathcal{D})(\otimes_{\mathcal{D}} \circ (F_1 \times F_2), F_3 \circ \otimes_{\mathcal{C}})$$
$$\mathrm{Funct}(\mathcal{C}, \mathcal{D})(F_1, F_2 \RIGHTcircle F_3) \cong \mathrm{Funct}(\mathcal{C} \times \mathcal{C}, \mathcal{D})(F_1 \circ \parr_{\mathcal{C}}, \parr_{\mathcal{D}} \circ (F_2 \times F_3))$$

Lemma 5.2.4. *Say that $\mathcal{C}$ is a $\mathbb{K}$-linear GV-category with finite-dimensional hom spaces and finitely copowered, and take $U \in \mathbf{FinVect}_{\mathbb{K}}$ and $X \in \mathcal{C}$. Then*

$$G_{\mathcal{C}}^{-1}(U \star X) \cong U^* \star G_{\mathcal{C}}^{-1}(X)$$

and

$$G_{\mathcal{C}}(U \star X) \cong U^* \star G(X)$$

Proof. Let $Y \in \mathcal{C}$ be an arbitrary object.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(Y, G_{\mathcal{C}}^{-1}(U \star X)) &\cong \mathrm{Hom}_{\mathcal{C}}((U \star X) \otimes Y, K_{\mathcal{C}}) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(U \star (X \otimes Y), K_{\mathcal{C}}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(U, \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, K_{\mathcal{C}})) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(U, \mathrm{Hom}_{\mathcal{C}}(Y, G_{\mathcal{C}}^{-1}X)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(Y, U^* \star G_{\mathcal{C}}^{-1}X)
\end{aligned}$$

By Yoneda, this gives $G_{\mathcal{C}}^{-1}(U \star X) \cong U^* \star G_{\mathcal{C}}^{-1}(X)$.

For the other case:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(Y, G_{\mathcal{C}}(U \star X)) &\cong \mathrm{Hom}_{\mathcal{C}}(Y \otimes (U \star X), K_{\mathcal{C}}) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(U \star (Y \otimes X), K_{\mathcal{C}}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(U, \mathrm{Hom}_{\mathcal{C}}(Y \otimes X, K_{\mathcal{C}})) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(U, \mathrm{Hom}_{\mathcal{C}}(Y, G_{\mathcal{C}}X)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(Y, U^* \star G_{\mathcal{C}}X)
\end{aligned}$$

By Yoneda, this gives $G_{\mathcal{C}}(U \star X) \cong U^* \star G(X)$. $\square$

With these lemmas on board, we get the following result. Note that we take <u>all</u> $\mathbb{K}$-linear functors between the relevant GV-categories.

Theorem 5.2.5 (Functors between GV-categories form a GV-category)**.** *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear GV-categories with finite dimensional hom-spaces, $\mathcal{C}$ small, $\mathcal{D}$ having all limits and colimits of size $\mathcal{C}$ and $\mathcal{D}$ finitely copowered. As well, we assume that for all functors $F, G : \mathcal{C} \to \mathcal{D}$, $\mathrm{Nat}(\mathcal{C}, \mathcal{D})$ is finite-dimensional.*

Then with the Day and coDay convolutions defined above, $(\mathrm{Funct}(\mathcal{C}, \mathcal{D}), \varoast, ◑)$ forms a GV-category with unit object $\tilde{\mathbb{I}} := \mathcal{C}_{-}^{\mathbb{I}_{\mathcal{C}}} \star \mathbb{I}_{\mathcal{D}}$ and dualizing object $\tilde{K} := (\mathcal{C}_{K_{\mathcal{C}}}^{-})^* \star K_{\mathcal{D}}$.

Proof. We just need to find the dualizing functor, then the rest falls into place. We will construct it by working directly with homomorphisms into our dualizing object $\tilde{K} = (\mathcal{C}_{K_{\mathcal{C}}}^{-})^* \star K_{\mathcal{D}}$. Using the coend calculus:

$$\begin{aligned}
\mathrm{Nat}(F_1 \mathbin{\text{☼}} F_2, \tilde{K}) &\cong \mathrm{Nat}(F_1 \mathbin{\text{☼}} F_2, (\mathcal{C}^{-}_{K_{\mathcal{C}}})^* \star K_{\mathcal{D}}) && \text{def. of } \tilde{K}.\\
&\cong \int_A \mathrm{Hom}_{\mathcal{D}}((F_1 \mathbin{\text{☼}} F_2)(A), (\mathcal{C}^{A}_{K_{\mathcal{C}}})^* \star K_{\mathcal{D}}) && \text{def. of nat. trans. as end.}\\
&\cong \int_A \mathrm{Hom}_{\mathcal{D}}\left(\int^{XY} \mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{A} \star F_1X \otimes_{\mathcal{D}} F_2Y, (\mathcal{C}^{A}_{K_{\mathcal{C}}})^* \star K_{\mathcal{D}}\right) && \text{def. of conv.}\\
&\cong \int_{AXY} \mathrm{Hom}_{\mathcal{D}}\left(\mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{A} \star F_1X \otimes_{\mathcal{D}} F_2Y, (\mathcal{C}^{A}_{K_{\mathcal{C}}})^* \star K_{\mathcal{D}}\right) && \text{pull out coend.}\\
&\cong \int_{AXY} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{A}, \mathrm{Hom}_{\mathcal{D}}\left(F_1X \otimes_{\mathcal{D}} F_2Y, (\mathcal{C}^{A}_{K_{\mathcal{C}}})^* \star K_{\mathcal{D}}\right)\right) && \text{nat. iso. of copower.}\\
&\cong \int_{AXY} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{A}, \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^{A}_{K_{\mathcal{C}}}, \mathrm{Hom}_{\mathcal{D}}(F_1X \otimes_{\mathcal{D}} F_2Y, K_{\mathcal{D}})\right)\right) && \text{nat. iso. of power.}\\
&\cong \int_{AXY} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{A} \otimes \mathcal{C}^{A}_{K_{\mathcal{C}}}, \mathrm{Hom}_{\mathcal{D}}(F_1X \otimes_{\mathcal{D}} F_2Y, K_{\mathcal{D}})\right) && \text{tensor-hom adj.}\\
&\cong \int_{XY} \mathrm{Hom}_{\mathbb{K}}\left(\int^{A} \mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{A} \otimes \mathcal{C}^{A}_{K_{\mathcal{C}}}, \mathrm{Hom}_{\mathcal{D}}(F_1X \otimes_{\mathcal{D}} F_2Y, K_{\mathcal{D}})\right) && \text{pull in coend.}\\
&\cong \int_{XY} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^{X\otimes_{\mathcal{C}} Y}_{K_{\mathcal{C}}}, \mathrm{Hom}_{\mathcal{D}}(F_1X \otimes_{\mathcal{D}} F_2Y, K_{\mathcal{D}})\right) && \text{Yoneda lemma}\\
&\cong \int_{XY} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^{X}_{G_{\mathcal{C}}Y}, \mathrm{Hom}_{\mathcal{D}}(F_1X, G_{\mathcal{D}}(F_2Y))\right) && \text{dual. funct.}\\
&\cong \int_{XY} \mathrm{Hom}_{\mathcal{D}}(\mathcal{C}^{X}_{G_{\mathcal{C}}Y} \star F_1X, G_{\mathcal{D}}(F_2Y)) && \text{nat. iso of copower}\\
&\cong \int_{Y} \mathrm{Hom}_{\mathcal{D}}\left(\int^{X} \mathcal{C}^{X}_{G_{\mathcal{C}}Y} \star F_1X, G_{\mathcal{D}}(F_2Y)\right) && \text{pull in coend}\\
&\cong \int_{Y} \mathrm{Hom}_{\mathcal{D}}(F_1(G_{\mathcal{C}}Y), G_{\mathcal{D}}(F_2Y)) && \text{Yoneda lemma}\\
&\cong \mathrm{Nat}(F_1G_{\mathcal{C}}, G_{\mathcal{D}}F_2) && \text{def. of nat. trans. as end.}
\end{aligned}$$

Now, $F_1G_{\mathcal{C}}$ and $G_{\mathcal{D}}F_2$ are contravariant functors. By precomposing everything with $G_{\mathcal{C}}^{-1}$, we can get back to covariant functors. This gives us:

$$\mathrm{Nat}(F_1 \mathbin{\text{☼}} F_2, \tilde{K}) \cong \mathrm{Nat}(F_1, G_{\mathcal{D}}F_2G_{\mathcal{C}}^{-1}).$$

Let $\tilde{G}(F) := G_{\mathcal{D}}FG_{\mathcal{C}}^{-1}$. □

The following result checks that we have chosen the appropriate second monoidal product.

Theorem 5.2.6. *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear GV-categories with finite dimensional hom-spaces, $\mathcal{C}$ small, $\mathcal{D}$ having all limits and colimits of size $\mathcal{C}$ and $\mathcal{D}$ finitely copowered. As well, we assume that for all functors $F, G : \mathcal{C} \to \mathcal{D}$, $\mathrm{Nat}(\mathcal{C}, \mathcal{D})$ is finite-dimensional. Let the Day and coDay convolutions be defined as above. Then $F_1 \mathbin{◑} F_2 \cong \tilde{G}^{-1}(\tilde{G}F_2 \mathbin{\text{☼}} \tilde{G}F_1)$, $\tilde{G}^{-1}\tilde{K} \cong \tilde{\mathbb{I}}$ and $\tilde{G}\tilde{K} \cong \tilde{\mathbb{I}}$.*

Proof. The proof will proceed directly. Note that as $G_{\mathcal{D}}^{-1}$ is an anti-equivalence, it swaps limits and colimits, and it swaps ends and coends. Note as well since we have

$$A \mathbin{⅋}_{\mathcal{D}} B \cong G_{\mathcal{D}}^{-1}(G_{\mathcal{D}}B \otimes_{\mathcal{D}} G_{\mathcal{D}}A) \tag{5.12}$$

that implies

$$G_{\mathcal{D}}(A \mathbin{⅋}_{\mathcal{D}} B) \cong G_{\mathcal{D}}B \otimes_{\mathcal{D}} G_{\mathcal{D}}A. \tag{5.13}$$

First consider the monoidal products:

$$\begin{aligned}
\tilde{G}^{-1}(\tilde{G}F_2 \mathbin{☼} \tilde{G}F_1)(A) &\cong \tilde{G}^{-1}(G_{\mathcal{D}}F_2G_{\mathcal{C}}^{-1} \mathbin{☼} G_{\mathcal{D}}F_1G_{\mathcal{C}}^{-1})(A) && \text{def. of } \tilde{G} \\
&\cong \tilde{G}^{-1}\left(\int^{XY} \mathcal{C}_{A}^{X\otimes_{\mathcal{C}} Y} \star G_{\mathcal{D}}F_2G_{\mathcal{C}}^{-1}X \otimes_{\mathcal{D}} G_{\mathcal{D}}F_1G_{\mathcal{C}}^{-1}Y\right) && \text{def. of conv.} \\
&\cong G_{\mathcal{D}}^{-1}\left(\int^{XY} \mathcal{C}_{G_{\mathcal{C}}A}^{X\otimes_{\mathcal{C}} Y} \star G_{\mathcal{D}}(F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X)\right) && \text{eq. (5.13)} \\
&\cong G_{\mathcal{D}}^{-1}\left(\int^{XY} \mathcal{C}_{G_{\mathcal{C}}A}^{X\otimes_{\mathcal{C}} Y} \star G_{\mathcal{D}}(F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X)\right) && \text{def. of } \tilde{G} \\
&\cong G_{\mathcal{D}}^{-1}\left(\int^{XY} \mathcal{C}_{K_{\mathcal{C}}}^{X\otimes_{\mathcal{C}} Y\otimes_{\mathcal{C}} A} \star G_{\mathcal{D}}(F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X)\right) && G_{\mathcal{C}} \text{ dual. funct.} \\
&\cong G_{\mathcal{D}}^{-1}\left(\int^{XY} \mathcal{C}^{A}_{G_{\mathcal{C}}^{-1}Y\parr_{\mathcal{C}} G_{\mathcal{C}}^{-1}X} \star G_{\mathcal{D}}(F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X)\right) && G_{\mathcal{C}} \text{ dual. funct.} \\
&\cong \int_{XY} G_{\mathcal{D}}^{-1}\left(\mathcal{C}^{A}_{G_{\mathcal{C}}^{-1}Y\parr_{\mathcal{C}} G_{\mathcal{C}}^{-1}X} \star G_{\mathcal{D}}(F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X)\right) && G_{\mathcal{D}}^{-1} \text{ anti-equiv.} \\
&\cong \int_{XY} \left(\mathcal{C}^{A}_{G_{\mathcal{C}}^{-1}Y\parr_{\mathcal{C}} G_{\mathcal{C}}^{-1}X}\right)^{*} \star G_{\mathcal{D}}^{-1}G_{\mathcal{D}}(F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X) && \text{lem. 5.2.4} \\
&\cong \int_{XY} \left(\mathcal{C}^{A}_{G_{\mathcal{C}}^{-1}Y\parr_{\mathcal{C}} G_{\mathcal{C}}^{-1}X}\right)^{*} \star F_1G_{\mathcal{C}}^{-1}Y \parr_{\mathcal{D}} F_2G_{\mathcal{C}}^{-1}X && \text{inverses} \\
&\cong \int_{UV} (\mathcal{C}^{A}_{U\parr_{\mathcal{C}} V})^{*} \star F_1U \parr_{\mathcal{D}} F_2V && \text{relabel} \\
&\cong (F_1 \mathbin{◑} F_2)(A). && \text{def. of conv.}
\end{aligned}$$

Now consider the monoidal units:

$$\begin{aligned}
\tilde{G}^{-1}\tilde{K}(X) &\cong \tilde{G}^{-1}\left((\mathcal{C}^{X}_{K_{\mathcal{C}}})^{*} \star K_{\mathcal{D}}\right) \\
&\cong G_{\mathcal{D}}^{-1}\left((\mathcal{C}^{G_{\mathcal{C}}X}_{K_{\mathcal{C}}})^{*} \star K_{\mathcal{D}}\right) \\
&\cong (\mathcal{C}^{G_{\mathcal{C}}X}_{K_{\mathcal{C}}})^{**} \star G_{\mathcal{D}}^{-1}K_{\mathcal{D}} \\
&\cong (\mathcal{C}^{G_{\mathcal{C}}X}_{K_{\mathcal{C}}}) \star G_{\mathcal{D}}^{-1}K_{\mathcal{D}} \\
&\cong \mathcal{C}^{\mathbb{I}_{\mathcal{C}}}_{X} \star \mathbb{I}_{\mathcal{D}} = \tilde{\mathbb{I}}(X).
\end{aligned}$$

Note that here all the steps are either via the definition or the pivotality of $\mathbf{FinVect}_{\mathbb{K}}$ up until the final step. There, the step is given by applying the isomorphisms $G_{\mathcal{D}}^{-1}K_{\mathcal{D}} \cong \mathbb{I}_{\mathcal{D}}$ and $\mathrm{Hom}_{\mathcal{C}}(G_{\mathcal{C}}X, K_{\mathcal{C}}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}_{\mathcal{C}}, X)$.

And the second monoidal unit condition:

$$\begin{aligned}
\tilde{G}\tilde{K}(X) &\cong \tilde{G}\left((\mathcal{C}^{X}_{K_{\mathcal{C}}})^{*} \star K_{\mathcal{D}}\right) \\
&\cong G_{\mathcal{D}}\left((\mathcal{C}^{G_{\mathcal{C}}X}_{K_{\mathcal{C}}})^{*} \star K_{\mathcal{D}}\right) \\
&\cong (\mathcal{C}^{G_{\mathcal{C}}^{-1}X}_{K_{\mathcal{C}}})^{**} \star G_{\mathcal{D}}K_{\mathcal{D}} \\
&\cong (\mathcal{C}^{G_{\mathcal{C}}^{-1}X}_{K_{\mathcal{C}}}) \star G_{\mathcal{D}}K_{\mathcal{D}} \\
&\cong \mathcal{C}^{\mathbb{I}_{\mathcal{C}}}_{X} \star \mathbb{I}_{\mathcal{D}} = \tilde{\mathbb{I}}(X).
\end{aligned}$$

Similarly to above, note that here all the steps are either via the definition or the pivotality of $\mathbf{FinVect}_{\mathbb{K}}$ up until the final step. There, the step is given by applying the isomorphisms $G_{\mathcal{D}}K_{\mathcal{D}} \cong \mathbb{I}_{\mathcal{D}}$ and $\mathrm{Hom}_{\mathcal{C}}(G_{\mathcal{C}}^{-1}X, K_{\mathcal{C}}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}_{\mathcal{C}}, X)$. □

Here we prove a small lemma in preparation for demonstrating sufficient conditions for the GV-category $(\mathrm{Funct}(\mathcal{C},\mathcal{D}), \star, ◑)$ to be pivotal.

Lemma 5.2.7. *Let $\mathcal{C},\mathcal{D}$ be $\mathbb{K}$-linear monoidal categories with finite-dimensional hom spaces, $\mathcal{C}$ small, $\mathcal{D}$ finitely copowered and $\mathcal{D}$ having all colimits of size $\mathcal{C}$.*

- *If $\Phi : \mathcal{C} \to \mathcal{C}$ is a strong $\otimes$-monoidal functor together with a monoidal natural isomorphism $\phi : \mathrm{id}_{\mathcal{C}} \Rightarrow \Phi$, then $F \mapsto F \circ \Phi$ is a strong $\star$-monoidal functor in* $\mathrm{Funct}(\mathcal{C},\mathcal{D})$ *with a monoidal natural isomorphism $\tilde{\phi} : \mathrm{id} \Rightarrow - \circ \Phi$.*

- *If $\Psi : \mathcal{D} \to \mathcal{D}$ is a $\mathbb{K}$-linear strong $\otimes$-monoidal functor together with a monoidal natural isomorphism $\psi : \mathrm{id}_{\mathcal{D}} \Rightarrow \Psi$, then $F \mapsto \Psi \circ F$ is a strong $\star$-monoidal functor in* $\mathrm{Funct}(\mathcal{C},\mathcal{D})$ *with a monoidal natural isomorphism $\tilde{\psi} : \mathrm{id} \Rightarrow \Psi \circ -$.*

Proof. We will proceed using coend calculus. Note that due to the assumption that Φ and Ψ are naturally isomorphic to the identity, in particular they are equivalences of categories. This allows us to relabel indexes by the universality of (co)ends.

First we prove the statement for precomposition. This entails showing a compatible natural isomorphism $(F_1 \circ \Phi) \star (F_2 \circ \Phi) \to (F_1 \star F_2) \circ \Phi$.

$$\begin{aligned}
((F_1 \circ \Phi) \star (F_2 \circ \Phi))(X) &\cong \int^{AB} \mathcal{C}_X^{A\otimes B} \star F_1\Phi A \otimes F_2 \Phi B && \text{def. of conv.}\\
&\cong \int^{AB} \mathcal{C}_{\Phi X}^{\Phi(A\otimes B)} \star F_1\Phi A \otimes F_2 \Phi B && \Phi \text{ equiv. of cat.}\\
&\cong \int^{AB} \mathcal{C}_{\Phi X}^{\Phi A\otimes \Phi B} \star F_1\Phi A \otimes F_2 \Phi B && \Phi \text{ str. mon.}\\
&\cong \int^{XY} \mathcal{C}_{\Phi X}^{A\otimes B} \star F_1 A \otimes F_2 B && \text{apply. } \phi^{-1}\\
&\cong ((F_1 \star F_2) \circ \Phi)(X) && \text{def. of conv.}
\end{aligned}$$

This also tells us that $\tilde{\phi} = \mathrm{id} \bullet \phi$. Coherence of this with unitors and associators is routine via coend calculus. This is the horizontal composition of monoidal natural isomorphisms and is thus a monoidal natural isomorphism itself.

Now for the post-composition. Similarly, we need to show a compatible natural isomorphism $(\Psi \circ F_1) \star (\Psi \circ F_2) \to \Psi \circ (F_1 \star F_2)$.

$$\begin{aligned}
((\Psi \circ F_1) \star (\Psi \circ F_2))(X) &\cong \int^{AB} \mathcal{C}_X^{A\otimes B} \star \Psi F_1 A \otimes \Psi F_2 B && \text{def. of conv.}\\
&\cong \int^{AB} \mathcal{C}_X^{A\otimes B} \star \Psi(F_1 A \otimes F_2 B) && \Psi\text{. str. mon.}\\
&\cong \int^{AB} \Psi(\mathcal{C}_X^{A\otimes B} \star F_1 A \otimes F_2 B) && \Psi\text{. } \mathbb{K}\text{-lin.\& pres. colim}\\
&\cong \Psi \int^{AB} (\mathcal{C}_X^{A\otimes B} \star F_1 A \otimes F_2 B) && \Psi\text{. equiv. of. cat.}\\
&\cong (\Psi \circ (F_1 \star F_2))(X) && \text{def. of conv.}
\end{aligned}$$

This also tells us that $\tilde{\psi} = \psi \bullet \mathrm{id}$. Coherence of this with unitors and associators is routine via coend calculus. This is the horizontal composition of monoidal natural isomorphisms and is thus a monoidal natural isomorphism itself. $\square$

Lemma 5.2.8. *If $\mathcal{C}, \mathcal{D}$ are $\mathbb{K}$-linear pivotal GV-categories, with finite dimensional hom-spaces, $\mathcal{C}$ small, $\mathcal{D}$ having all limits and colimits of size $\mathcal{C}$ and $\mathcal{D}$ finitely copowered.Then* $(\mathrm{Funct}(\mathcal{C}, \mathcal{D}), \varnothing, ◑)$ *is a pivotal GV-category.*

Proof. By Proposition 5.1.9, we have unique natural isomorphisms $\varpi^{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \Rightarrow G^2_{\mathcal{C}}$ and $\varpi^{\mathcal{D}} : \mathrm{id}_{\mathcal{D}} \Rightarrow G^2_{\mathcal{D}}$. Then for a functor $F \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$, we have

$$\varpi^{\mathcal{D}} \bullet F \bullet \varpi^{\mathcal{C}} : F \Rightarrow G^2_{\mathcal{D}} F G^2_{\mathcal{C}} \cong \tilde{G}^2(F)$$

Define $\tilde{\varpi}(F) := \varpi^{\mathcal{D}} \bullet F \bullet \varpi^{\mathcal{C}}$. This is our pivotal structure on $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$. We need to prove that $\tilde{\varpi}$ is monoidal and that $\tilde{\varpi}(\tilde{K})$ is equal to the composition

$$\tilde{K} \xrightarrow{\cong} \tilde{G}\tilde{\mathbb{I}} \xrightarrow{\cong} \tilde{G}^2 \tilde{G}^{-1} \tilde{\mathbb{I}} \xrightarrow{\cong} \tilde{G}^2 \tilde{K}.$$

Note that $\tilde{\varpi}$ is monoidal by Lemma 5.2.7 above. It just remains to show the equality of the composition. This can be seen by expanding the observations made at the end of the proof of Theorem 5.2.6. □

Proposition 5.2.9 (Module Categories of the GV-category $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$)**.** *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{K}$-linear GV-categories with finite dimensional hom-spaces, $\mathcal{C}$ small, $\mathcal{D}$ having all limits and colimits of size $\mathcal{C}$ and $\mathcal{D}$ finitely copowered, and for all functors* $F, G \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$, $\mathrm{Nat}(F, G)$ *is finite-dimensional. Furthermore, let $\mathcal{M}$ be a right $\mathcal{C}$ GV-module category with action $\triangleleft$, right adjoint to $- \triangleleft c$ denoted $\Gamma^{\mathcal{M}}_c$ and internal hom $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}$, $\mathcal{N}$ be a right $\mathcal{D}$ GV-module category with action $\triangleleft$, right adjoint to $- \triangleleft d$ denoted $\Gamma^{\mathcal{N}}_d$ and internal hom $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}$, with all limits and colimits of size $\mathcal{M}$, and finitely copowered. As well, assume that for all functors* $H, G \in \mathrm{Funct}(\mathcal{M}, \mathcal{N})$, $\mathrm{Nat}(H, G)$ *is finite-dimensional.*

Then $\mathrm{Funct}(\mathcal{M}, \mathcal{N})$ *is a right* $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *GV-module category with right $\varnothing$-action $\triangleleft$, right adjoint to $- \triangleleft F$ denoted $\tilde{\Gamma}_F$ and internal hom $\underset{\sim}{\mathrm{Nat}}^{\ell}$. We call this the left inner nat.*

Proof. We will first define the action, and then we will show the adjoints. The $\varnothing$-action is defined by a coend. For $H \in \mathrm{Funct}(\mathcal{M}, \mathcal{N})$, $F \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$, $x \in \mathcal{M}$ we have:

$$(H \triangleleft F)(x) = \int^{mc} \mathrm{Hom}_{\mathcal{M}}(m \triangleleft c, x) \star (Hm \triangleleft Fc)$$

Following [EGNO, 7.1.1], I will prove the two basic requirements for the action: $H \triangleleft (F \varnothing G) \cong (H \triangleleft F) \triangleleft G$ and $H \triangleleft \tilde{\mathbb{I}} \cong H$. I will leave the pentagon axiom as an instructive proof for the reader.

First, that $H \triangleleft (F \medstar G) \cong (H \triangleleft F) \triangleleft G$. Here I will use coend calculus.

$$\begin{aligned}
H \triangleleft (F \medstar G)(x) &\cong \int^{mc} \mathcal{M}_x^{m \triangleleft c} \star (Hm \triangleleft (F \medstar G)(c)) && \text{def. of act.} \\
&\cong \int^{mc} \mathcal{M}_x^{m \triangleleft c} \star \left(Hm \triangleleft \int^{ab} \mathcal{C}_c^{a \otimes b} \star (Fa \otimes Gb) \right) && \text{def. of. conv.} \\
&\cong \int^{mab} \int^{c} (\mathcal{M}_x^{m \triangleleft c} \otimes \mathcal{C}_c^{a \otimes b}) \star (Hm \triangleleft (Fa \otimes Gb)) && \text{rearrange} \\
&\cong \int^{mab} \mathcal{M}_x^{m \triangleleft (a \otimes b)} \star (Hm \triangleleft (Fa \otimes Gb)) && \text{Yoneda lemma} \\
&\cong \int^{mab} \mathcal{M}_x^{(m \triangleleft a) \triangleleft b} \star ((Hm \triangleleft Fa) \triangleleft Gb) && \text{nat. iso. of act.} \\
&\cong \int^{yb} \int^{ma} (\mathcal{M}_y^{m \triangleleft a} \otimes \mathcal{M}_x^{y \triangleleft b}) \star ((Hm \triangleleft Fa) \triangleleft Gb) && \text{Yoneda lemma} \\
&\cong \int^{yb} \mathcal{M}_x^{y \triangleleft b} \star \left(\int^{ma} \mathcal{M}_y^{m \triangleleft a} \star (Hm \triangleleft Fa) \triangleleft Gb \right) && \text{rearrange} \\
&\cong \int^{yb} \mathcal{M}_x^{y \triangleleft b} \star ((H \triangleleft F)(y) \triangleleft Gb) && \text{def. of act.} \\
&\cong ((H \triangleleft F) \triangleleft G)(x) && \text{def. of act.}
\end{aligned}$$

Second, that $H \triangleleft \tilde{\mathbb{I}} \cong H$:

$$\begin{aligned}
(H \triangleleft \mathbb{I})(x) &\cong \int^{mc} \mathcal{M}_x^{m \triangleleft c} \star Hm \triangleleft (\mathcal{C}_c^{\mathbb{I}_{\mathcal{C}}} \star \mathbb{I}_{\mathcal{D}}) && \text{def. of act. \& unit} \\
&\cong \int^{mc} (\mathcal{C}_c^{\mathbb{I}_{\mathcal{C}}} \otimes \mathcal{M}_x^{m \triangleleft c}) \star (Hm \triangleleft \mathbb{I}_{\mathcal{D}}) && \text{rearrange} \\
&\cong \int^{m} \mathcal{M}_x^{m \triangleleft \mathbb{I}_{\mathcal{C}}} \star Hm \triangleleft \mathbb{I}_{\mathcal{D}} && \text{Yoneda lemma} \\
&\cong \int^{m} \mathcal{M}_x^{m} \star Hm && \text{unit const.} \\
&\cong Hx && \text{Yoneda lemma}
\end{aligned}$$

Now I construct the adjoints. First, I construct the adjoint to $-\triangleleft F$:

$$
\begin{aligned}
\mathrm{Nat}(H_0 \triangleleft F, H_1) &\cong \int_t \mathrm{Hom}_{\mathcal{N}}((H_0 \triangleleft F)(t), H_1(t)) && \text{def. of nat. trans. as end.}\\
&\cong \int_t \mathrm{Hom}_{\mathcal{N}}\left(\int^{mc} \mathcal{M}_t^{m\triangleleft c} \star H_0(m) \triangleleft F(c), H_1(t)\right) && \text{def. of act.}\\
&\cong \int_{mct} \mathrm{Hom}_{\mathcal{N}}(\mathcal{M}_t^{m\triangleleft c} \star H_0(m) \triangleleft F(c), H_1(t)) && \text{pull out coend}\\
&\cong \int_{mct} \mathrm{Hom}_{\mathbb{K}}(\mathcal{M}_t^{m\triangleleft c}, \mathrm{Hom}_{\mathcal{N}}(H_0(m) \triangleleft F(c), H_1(t))) && \text{nat iso. of copower.}\\
&\cong \int_{mct} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{M}^m_{\Gamma_c^{\mathcal{M}}(t)}, \mathrm{Hom}_{\mathcal{N}}(H_0 m, \Gamma_{Fc}^{\mathcal{N}}(H_1 t))\right) && \text{apply adj.}\\
&\cong \int_{mct} \mathrm{Hom}_{\mathcal{N}}\left(H_0 m, \left(\mathcal{M}^m_{\Gamma_c^{\mathcal{M}}(t)}\right)^* \star \Gamma_{Fc}^{\mathcal{N}}(H_1 t)\right) && \text{nat iso. of power.}\\
&\cong \int_{m} \mathrm{Hom}_{\mathcal{N}}\left(H_0 m, \int_{ct} \left(\mathcal{M}^m_{\Gamma_c^{\mathcal{M}}(t)}\right)^* \star \Gamma_{Fc}^{\mathcal{N}}(H_1 t)\right) && \text{pull in end.}\\
&\cong \int_{m} \mathrm{Hom}_{\mathcal{N}}(H_0(m), \tilde{\Gamma}_F(H_1)(m)) && \text{def. of } \tilde{\Gamma}\\
&\cong \mathrm{Nat}(H_0, \tilde{\Gamma}_F(H_1)) && \text{def. of nat. trans. as end.}
\end{aligned}
$$

I define the right adjoint in the second to last step. That is, I define $\tilde{\Gamma}_F(H) := \int_{ct} \left(\mathcal{M}^{-}_{\Gamma_c^{\mathcal{M}}(t)}\right)^* \star \Gamma_{Fc}^{\mathcal{N}}(H_1 t)$. By construction, it has the appropriate structure of a right adjoint.

With that adjoint defined, I next find the appropriate inner hom. It suffices to find the right adjoint to $H \triangleleft -$. I will do this with coend calculus.

$$
\begin{aligned}
\mathrm{Nat}(H \triangleleft F, M) &\cong \int_t \mathrm{Hom}_{\mathcal{N}}((H \triangleleft F)(t), M(t)) && \text{def. of nat trans. as end}\\
&\cong \int_t \mathrm{Hom}_{\mathcal{N}}\left(\int^{mc} \mathcal{M}_t^{m\triangleleft c} \star (Hm \triangleleft Fc), Mt\right) && \text{def. of act.}\\
&\cong \int_{tmc} \mathrm{Hom}_{\mathcal{N}}(\mathcal{M}_t^{m\triangleleft c} \star (Hm \triangleleft Fc), Mt) && \text{pull out coend.}\\
&\cong \int_{tmc} \mathrm{Hom}_{\mathbb{K}}(\mathcal{M}_t^{m\triangleleft c}, \mathrm{Hom}_{\mathcal{N}}(Hm \triangleleft Fc, Mt)) && \text{nat iso. of copower.}\\
&\cong \int_{tmc} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{M}_t^{m\triangleleft c}, \mathrm{Hom}_{\mathcal{D}}\left(Fc, \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hm, Mt)\right)\right) && \text{act.-hom adj.}\\
&\cong \int_{tmc} \mathrm{Hom}_{\mathbb{K}}\left(\mathcal{C}^c_{\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m,t)}, \mathrm{Hom}_{\mathcal{D}}\left(Fc, \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hm, Mt)\right)\right) && \text{act.-hom adj.}\\
&\cong \int_{tmc} \mathrm{Hom}_{\mathcal{D}}\left(Fc, \left(\mathcal{C}^c_{\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m,t)}\right)^* \star \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hm, Mt)\right) && \text{nat. iso of power.}\\
&\cong \int_{c} \mathrm{Hom}_{\mathcal{D}}\left(Fc, \int_{mt} \left(\mathcal{C}^c_{\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m,t)}\right)^* \star \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hm, Mt)\right) && \text{pull end in.}\\
&\cong \mathrm{Nat}\left(F, \int_{mt} \left(\mathcal{C}^{-}_{\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m,t)}\right)^* \star \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hm, Mt)\right) && \text{def. of nat trans. as end}
\end{aligned}
$$

I define the right adjoint as the result of that final step. That is, I define

$$
\underset{\sim}{\underline{\mathrm{Nat}}}^{\ell}(H, M)(x) := \int_{m_0, m_1} \left(\mathcal{C}^x_{\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m_0,m_1)}\right)^* \star \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hm_0, Mm_1).
$$

Note that by Lemma 4.1.9 this left inner nat is a right Kan extension. $\square$

The following two results are important for Theorem 5.3.7. However, they are of some independent interest. Lemma 5.2.10 is in essence a generalization of the property of a module category. We have by the definition of a module category that for $H \in \mathrm{Funct}(\mathcal{M}, \mathcal{N})$ and $F_1, F_2 \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ we have the isomorphism of functors $H \triangleleft (F_1 \varoast F_2) \cong (H \triangleleft F_1) \triangleleft F_2$. But that does not mean that this isomorphism is compatible with the structure morphisms $\otimes \circ (F_0 \times F_1) \to (F_0 \varoast F_1) \circ \otimes$ and $\triangleleft \circ (H \times F) \to (H \triangleleft F) \circ \triangleleft$. For that, we need the following lemma.

Lemma 5.2.10. *Let* $\mathcal{C}, \mathcal{D}, \mathcal{M}, \mathcal{N}$ *be as in Proposition 5.2.9. Consider functors* $F_0 : \mathcal{M} \to \mathcal{N}$ *and* $F_1, F_2 : \mathcal{C} \to \mathcal{D}$. *Then the two compositions in diagram* (5.14) *are equal.*

$$
\begin{array}{ccc}
\triangleleft \circ (\mathrm{id} \times \otimes) \circ (F_0 \times F_1 \times F_2) & & \triangleleft \circ (\mathrm{id} \times \otimes) \circ (F_0 \times F_1 \times F_2) \\
\downarrow & & \downarrow \\
\triangleleft \circ (\triangleleft \times \mathrm{id}) \circ (F_0 \times F_1 \times F_2) & & \triangleleft \circ (F_0 \times (F_1 \varoast F_2)) \circ (\mathrm{id} \times \otimes) \\
\downarrow & & \downarrow \\
\triangleleft \circ ((F_0 \triangleleft F_1) \times F_2) \circ (\triangleleft \times \mathrm{id}) & & (F_0 \triangleleft (F_1 \varoast F_2)) \circ \triangleleft \circ (\mathrm{id} \times \otimes) \\
\downarrow & & \downarrow \\
((F_0 \triangleleft F_1) \triangleleft F_2) \circ \triangleleft \circ (\triangleleft \times \mathrm{id}) & & ((F_0 \triangleleft F_1) \triangleleft F_2) \circ \triangleleft \circ (\triangleleft \times \mathrm{id})
\end{array}
\tag{5.14}
$$

Proof. We prove this using the properties of Kan extensions. We express the first composition as a composition of Kan extensions in diagram (5.15). This is a pasting diagram consisting of a natural isomorphism and two left Kan extensions. That is, this diagram has the universal property of a left Kan extension.

$$
\begin{tikzcd}
 & & \mathcal{N}\times\mathcal{D} \arrow[d, Rightarrow, "\cong"] \arrow[dr, "\triangleleft"] & \\
\mathcal{M}\times\mathcal{C}\times\mathcal{C} \arrow[r, "F_0\times F_1\times F_2"] \arrow[d, "\triangleleft\times\mathrm{id}"'] & \mathcal{N}\times\mathcal{D}\times\mathcal{D} \arrow[ur, "\mathrm{id}\times\otimes"] \arrow[r, "\triangleleft\times\mathrm{id}"] \arrow[d, phantom, "\Downarrow"] & \mathcal{N}\times\mathcal{D} \arrow[r, "\triangleleft"] & \mathcal{N} \\
\mathcal{M}\times\mathcal{C} \arrow[urr, dashed, "(F_0\triangleleft F_1)\times F_2"'] \arrow[d, "\triangleleft"'] & \arrow[r, phantom, "\Downarrow"] & & \\
\mathcal{M} \arrow[uurrr, dashed, "(F_0\triangleleft F_1)\triangleleft F_2"'] & & &
\end{tikzcd}
\tag{5.15}
$$

Now, express the second composition as a composition of Kan extensions, as in diagram (5.16). This is also a pasting diagram consisting of left Kan extensions and natural isomorphisms. It will also have the universal property of a left Kan extension. Since the outer legs of the diagrams are the same, this implies that the filling must be the same by the universal property of Kan extensions.

$$
\begin{tikzcd}
 & \mathcal{M}\times\mathcal{C}\times\mathcal{C} \arrow[r, "F_0\times F_1\times F_2"] \arrow[dl, "\triangleleft\times\mathrm{id}"'] \arrow[dd, "\mathrm{id}\times\otimes"] & \mathcal{N}\times\mathcal{D}\times\mathcal{D} \arrow[r, "\mathrm{id}\times\otimes"] \arrow[d, Rightarrow] & \mathcal{N}\times\mathcal{D} \arrow[r, "\triangleleft"] & \mathcal{N} \\
 & & F_0\times(F_1 \mathbin{\text{☼}} F_2) \arrow[ur, dashed] \arrow[d, Rightarrow] & & \\
\mathcal{M}\times\mathcal{C} \arrow[ddr, "\triangleleft"'] & \mathcal{M}\times\mathcal{C} \arrow[l, Rightarrow, "\cong"'] \arrow[dd, "\triangleleft"] \arrow[ur, dashed, no head] & F_0\triangleleft(F_1 \mathbin{\text{☼}} F_2) \arrow[uurr, dashed] \arrow[d, Rightarrow, "\cong"] & & \\
 & & (F_0\triangleleft F_1)\triangleleft F_2 \arrow[uuurr, dashed, bend right] & & \\
 & \mathcal{M} \arrow[uur, dashed, no head] \arrow[ur, dashed, no head, bend right] & & &
\end{tikzcd}
\tag{5.16}
$$

□

Since the left inner nat, $\underset{\sim}{\mathrm{Nat}}^{\ell}$, defined in Proposition 5.2.9 is a right Kan extension, it has a structure morphism. In order to use this structure morphism in the final part of this thesis, we need an explicit description of this structure morphism in terms of adjoints and the structure morphism of the action. The following lemma gives this explicit construction.

Lemma 5.2.11. *Fix* $H \in \mathrm{Funct}(\mathcal{M},\mathcal{N})$ *and* $z \in \mathcal{M}$. *Then the natural transformation*

$$\rho_{G,y} : \underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z,y) \to \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hz,Gy)$$

is equal to the composition

$$
\begin{array}{c}
\underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z,y) \\
\big\downarrow \\
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hz, Hz \triangleleft (\underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z,y))) \\
\big\downarrow \\
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hz, (H \triangleleft \underset{\sim}{\mathrm{Nat}}^{\ell}(H,G)) \circ (z \triangleleft \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z,y))) \\
\big\downarrow \\
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hz, Gy)
\end{array}
$$

Proof. This follows immediately by part 4 of Lemma 4.1.11. □

5.3 Frobenius Algebras in the Functor Category

The ultimate goal of this section is Theorem 5.3.7. This theorem proves that a GV-Frobenius algebra in $(\mathrm{Funct}(\mathcal{C},\mathcal{D}), \text{☼}, ◑)$ is an Frobenius LD functor in $\mathrm{Funct}(\mathcal{C},\mathcal{D})$ and vice versa.

For ease of certain calculations, we use the graphical calculus developed by Max Demirdilek in [Dem1; DS]. We refer the reader to those sources for a deeper discussion of surface diagrams. Here we will simply explain the necessary details. The following discussion and diagrams are drawn primarily from [Dem1, 2.1.6].

These surface diagrams depict natural transformations. Our canvas is the framed unit cube in $\mathbb{R}^3$, that is, the set $[0,1]\times[0,1]\times[0,1]$. See figure 5.1 for the an illustration of the framing. The red face is

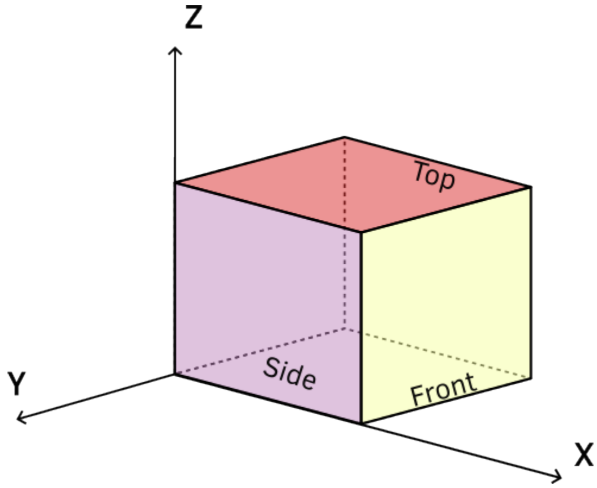


Figure 5.1: The Canvas

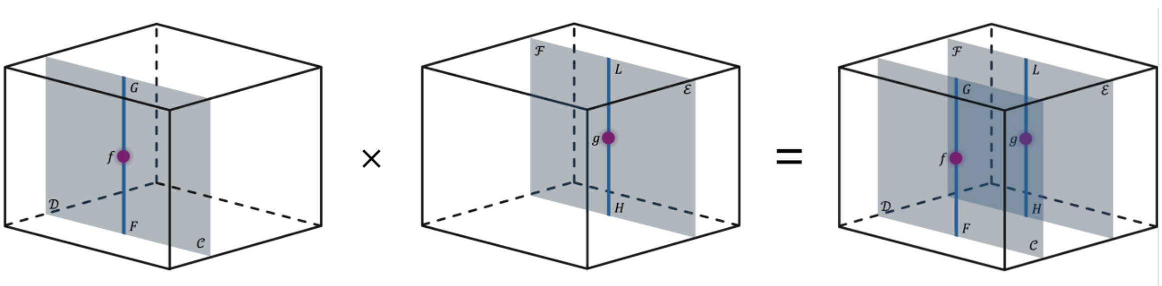


Figure 5.2: Cartesian Product

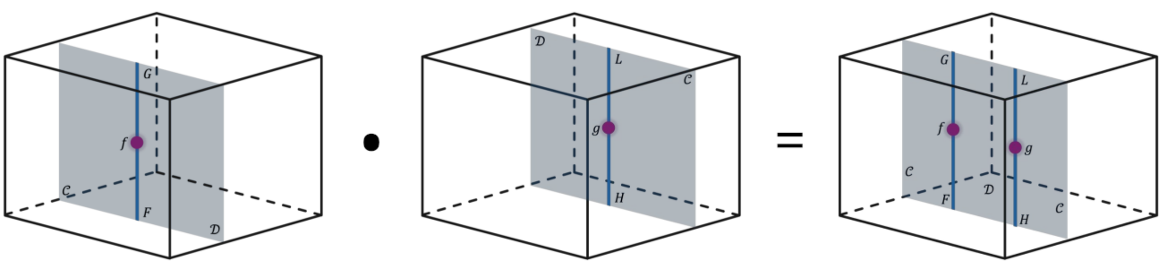


Figure 5.3: Horizontal Composition

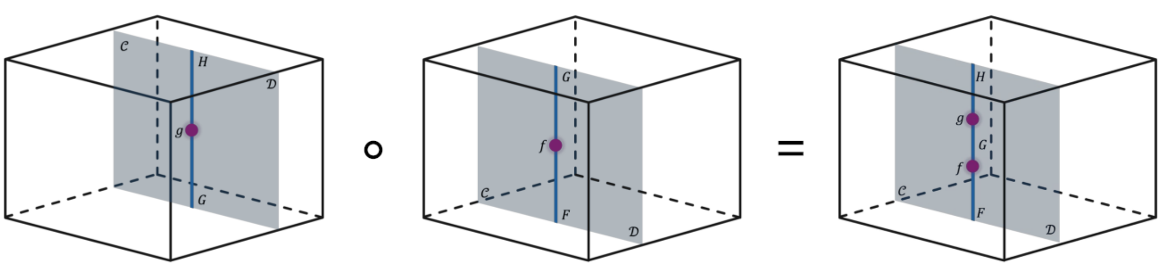


Figure 5.4: Vertical Composition

the top face, the violet face is one of the side faces, and the yellow face is the front face. In this canvas, categories are closed 2-cells, functors are closed 1-cells, and natural transformations are closed 0-cells. We have three forms of composition involved here, one for each direction. These are the Cartesian product in the Y direction, horizontal composition in the X direction, and vertical composition in the Z direction. These are depicted in figures 5.2, 5.3 and 5.4 respectively.

For the Cartesian product in figure 5.2, the figure depicts the Cartesian product of two natural transformations f and g. This is the juxtaposition of two sheets, or gluing the boxes together along the side faces where there are no object sheets nor functor lines.

For horizontal composition in figure 5.3, the figure depicts the horizontal composition of two natural transformations f and g. This is doing one natural transformation after the other along the object sheet. This can be conceptualised as gluing the boxes together along the faces where the object sheets connect to the front and back faces.

For vertical composition in figure 5.4, the figure depicts the vertical composition of two natural transformations f and g. This is putting one natural transformation after the other along the functor line. This can be conceptualised as gluing the boxes together along the top and bottom faces where the object sheets and the functor lines line up.

In the surface diagram calculations in the remainder of the thesis, we suppress the framing box for greater legibility.

We will now move through several earlier results, expressing them in terms of the graphical calculus. This section is difficult, and the reader is encouraged to take their time, and possibly check [Dem1; DS] for additional information.

We denote:

- Our general functors by the magenta lines.
- The $\otimes$-monoidal product by the violet lines.
- The inner hom by the cyan lines.
- The monoidal action by the yellow lines.
- The edge of our object sheet by the wavy red line.
- Natural transformations by black dots.

First, Lemma 5.2.10. This gives us the equality of the two surface diagrams in 5.5. The three input sheets, from left to right, are $\mathcal{M}, \mathcal{C}, \mathcal{C}$. The magenta functor lines are H, F_0, F_1. The output sheet is $\mathcal{N}$.

Consider the left surface diagram. At the very bottom, this depicts $\triangleleft \circ (\mathrm{id} \times \otimes) \circ (H \times F_0 \times F_1)$. After the first natural transformation dot, we have $\triangleleft \circ (\triangleleft \times \mathrm{id}) \circ (H \times F_0 \times F_1)$. Then after the next natural transformation dot, which connects the magenta lines from the first two input sheets, we have $\triangleleft \circ ((H \triangleleft F_0) \times F_1) \circ (\triangleleft \times \mathrm{id})$. Finally, after the last natural transformation dot, we have $((H \triangleleft F_0) \triangleleft F_1) \circ \triangleleft \circ (\triangleleft \times \mathrm{id})$.

Consider the right surface diagram. At the very bottom, this again depicts $\triangleleft \circ (\mathrm{id} \times \otimes) \circ (H \times F_0 \times F_1)$. Once we pass the first natural transformation dot as we move upwards, we instead have $\triangleleft \circ (H \times (F_0 \mathbin{☼} F_1)) \circ (\mathrm{id} \times \otimes)$, as we combine the two magenta functor lines using the yellow monoidal product line via the structure morphism of the Day convolution. Then after the next natural transformation dot up, we have $(H \triangleleft (F_0 \mathbin{☼} F_1)) \circ \triangleleft \circ (\mathrm{id} \times \otimes)$. The next natural transformation dot, resting solely on the magenta functor line, takes us to $((H \triangleleft F_0) \triangleleft F_1) \circ \triangleleft \circ (\mathrm{id} \times \otimes)$. Then, after the final natural transformation dot we are again at $((H \triangleleft F_0) \triangleleft F_1) \circ \triangleleft \circ (\triangleleft \times \mathrm{id})$.

This looks like taking the functor lines from one side of the structure morphism of the module category to the other side, passing two natural transformation dots across one other one. Since this diagram expresses the equality of two compositions of natural transformations, everything is happening within the surfaces, and not on the front face.

Next, we need to express Corollary 5.1.18 explicitly.

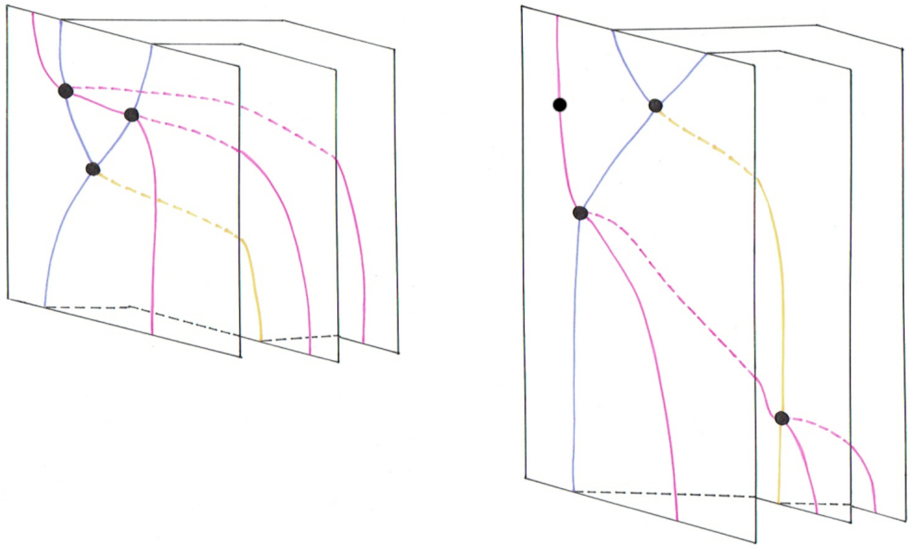


Figure 5.5: Surface Diagram depiction of Lemma 5.2.10

Lemma 5.3.1. *[Dem2] Let $(\mathcal{C}, \otimes)$ be a monoidal category. Let $(\mathcal{M}, \triangleleft)$ be a right module category, with structure morphisms $s_{M,X,Y}$, and possessing all left internal homs. Let $z \in \mathcal{M}$ be some fixed element, and $M \in \mathcal{M}$ and $C \in \mathcal{C}$ be arbitrary. Then the functor $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, -)$ can be given the structure of a lax right $\mathcal{C}$-module functor by the composition (5.17). This is represented in the graphical calculus by the surface diagram 5.6.*

$$\begin{array}{c}
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M) \otimes C \\
\Big\downarrow \scriptstyle \mathrm{coeval}_{\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z,M)\otimes C} \\
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, z \triangleleft (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M) \otimes C)) \\
\Big\downarrow \scriptstyle \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, s_{z, \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z,M), C}) \\
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, (z \triangleleft \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M)) \triangleleft C) \\
\Big\downarrow \scriptstyle \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, \mathrm{eval}_z \otimes \mathrm{id}_C) \\
\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M \triangleleft C)
\end{array} \tag{5.17}$$

Proof. This is a direct consequence of Lemma 5.1.17. □

This gives us the surface diagram 5.6. Start from the bottom again. Here we have three sheets where the fixed object z is represented by the jagged red edge of the first sheet. The second sheet is the category $\mathcal{M}$ and the third sheet is the category $\mathcal{C}$. For ease of notation we denote these sheets by $M \in \mathcal{M}$ and $C \in \mathcal{C}$. Below the first natural transformation dot we have $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M) \otimes C$. The first natural transformation dot is a coevaluation at the object z, giving us $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, z \triangleleft (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M) \otimes C))$. Notice how we introduce a cyan, a violet and a red line in the diagram, corresponding to the introduction of the inner hom $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}$, action $\triangleleft$ and object z. Then after the next natural transformation dot, we have $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, (z \triangleleft \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M)) \triangleleft C)$. Finally, we apply the evaluation natural transformation dot, eliminating a cyan, a violet and a red line, which corresponds to $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M \triangleleft C)$.

Here, we can see some non-trivial behaviour from the front face, where it resembles the zig-zag

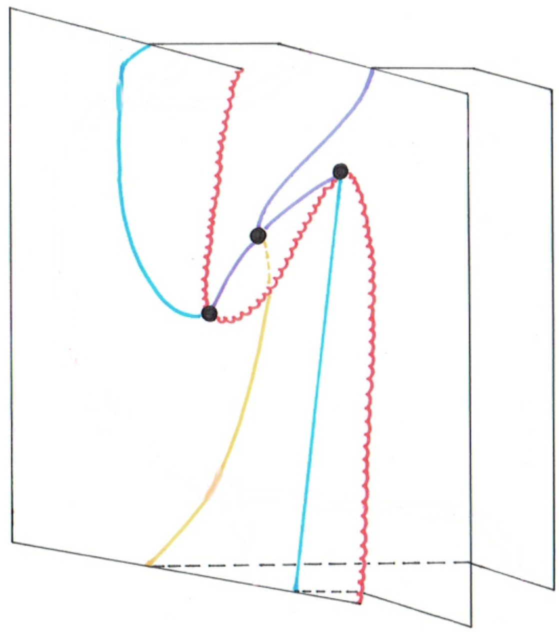


Figure 5.6: Lax Module Functor Structure of the Inner Hom

relation from the graphical calculus of rigid monoidal categories. We have essentially bent the leftmost sheet into a sideways S shape. The point where the lines where we have glued the sheets together is in the middle of the S; this should remind the reader of taking the dual of a morphism.

Finally, we need to express Lemma 5.2.11 as a diagram, namely that the two surface diagrams of 5.7 are equal. Again let $z \in \mathcal{M}$ be our fixed object and let $M \in \mathcal{M}$ act as a dummy variable. The left sheet is simple, consisting of a single natural transformation dot that depicts the natural transformation $\utilde{\mathrm{Nat}}^{\ell}(H_0, H_1) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M) \to \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}} \circ (H_0 z \times H_1 M)$.

The right hand surface diagram in 5.7 is basically the same as the surface diagram 5.6, but instead of a module structure line connecting three object sheets, there is a functor line connecting two object sheets. Moving from the bottom to the top, we start, as expected, at $\utilde{\mathrm{Nat}}^{\ell}(H_0, H_1) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M)$. Above the first natural transformation dot, we have $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(H_0 z, H_0 z \triangleleft (\utilde{\mathrm{Nat}}^{\ell}(H_0, H_1) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M)))$. Notice how we have introduced a cyan, a violet, and two magenta lines. Then after the next natural transformation dot we have $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(H_0 z, (H_0 \triangleleft \utilde{\mathrm{Nat}}^{\ell}(H_0, H_1)) \circ (z \triangleleft \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, M)))$. Finally, after the third natural transformation dot we arrive at $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(H_0 z, H_1 M)$.

All of this graphical calculus is used in the following lemma. We want to prove that the lax module structure of $\utilde{\mathrm{Nat}}^{\ell}(H, -)$ is related to the lax module structure of $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(z, -)$ and $\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(Hz, -)$ in a clean way. Via Proposition 5.1.19, this means that the distributors of the three GV-categories $\mathcal{C}$, $\mathcal{D}$ and $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ are also related in this clean way. That fills in a crucial part of the commutative diagram 5.11.

Remark 5.3.2. In the following section, for ease of reading, I will only denote the essential part of each

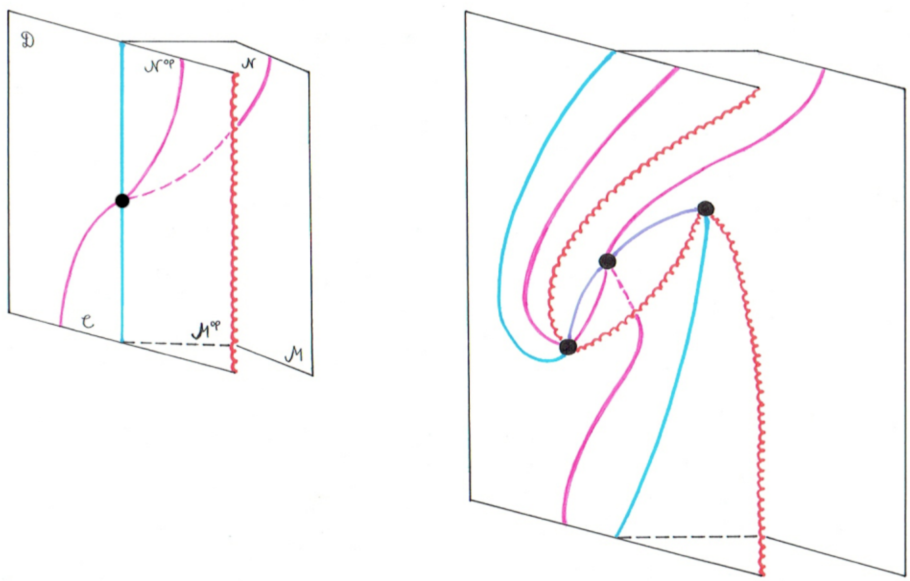


Figure 5.7: Expression of Lemma 5.2.11

arrow in the commutative diagrams. The morphisms will be as follows:

$$
\begin{aligned}
\lambda &: \otimes \circ (F_1 \times F_2) \Rightarrow (F_1 \mathbin{☼} F_2) \circ \otimes \\
\zeta &: \triangleleft \circ (H \times F) \Rightarrow (H \triangleleft F) \circ \triangleleft \\
\nu &: (F_1 \mathbin{◑} F_2) \circ \parr \Rightarrow \parr \circ (F_1 \times F_2) \\
\rho &: \underline{\mathrm{Nat}}^{\ell}(H, G) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \Rightarrow \underline{\mathrm{Nat}}^{\ell}_{\mathcal{N}} \circ (H \times G) \\
\sigma_{\mathcal{M}} &: \otimes \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m, -) \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}}(m, -) \circ \triangleleft \\
\sigma_{\mathcal{N}} &: \otimes \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(n, -) \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}}(n, -) \circ \triangleleft \\
\tilde{\sigma} &: ☼ \circ (\underline{\mathrm{Nat}}^{\ell}(H, -) \times \mathrm{id}) \Rightarrow \underline{\mathrm{Nat}}^{\ell}(H, -) \circ \triangleleft
\end{aligned}
$$

For the structure morphisms of a functor, we will have:

$$
\begin{aligned}
\mu &: F \mathbin{☼} F \Rightarrow F \\
\tilde{\mu} &: \otimes \circ (F \times F) \Rightarrow F \circ \otimes \\
\Delta &: F \Rightarrow F \mathbin{◑} F \\
\tilde{\Delta} &: F \circ \parr \Rightarrow \parr \circ (F \times F)
\end{aligned}
$$

Remark 5.3.3. The following results of this section are closely related to the results of [Egg1]. The proof techniques and Theorem 5.3.4 are novel. The other results are included here for completeness, and to bring the results into a unified framework with the remainder of the thesis.

Theorem 5.3.4. *With $\mathcal{C}, \mathcal{D}, \mathcal{M}, \mathcal{N}$ as in Proposition 5.2.9, then the lax module functor structure of the* *internal homs, are related by the commutative diagram* (5.18).

$$
\begin{tikzcd}[column sep=small]
 & & (\underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \mathbin{☼} F) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \circ (\mathrm{id} \times \triangleleft) \arrow[drr, "\tilde{\sigma}"] & & \\
(\underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \mathbin{☼} F) \circ \otimes \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \times \mathrm{id}) \arrow[urr, "\sigma_{\mathcal{M}}"] \arrow[dr, "\tilde{\sigma}"] & & & & \underset{\sim}{\mathrm{Nat}}^{\ell}(H, G \triangleleft F) \circ \underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \circ (\mathrm{id} \times \triangleleft) \arrow[d, "\rho"] \\
\otimes \circ (\underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \times F) \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \times \mathrm{id}) \arrow[u, "\lambda"] \arrow[d, "\rho"] & \underset{\sim}{\mathrm{Nat}}^{\ell}(H, G \triangleleft F) \circ \otimes \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \times \mathrm{id}) \arrow[urrr, "\sigma_{\mathcal{M}}"] & & & \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}} \circ (H \times (G \triangleleft F)) \circ (\mathrm{id} \times \triangleleft) \\
\otimes \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}} \times \mathrm{id}) \circ (H \times G \times F) \arrow[rrrr, "\sigma_{\mathcal{N}}"] & & & & \underline{\mathrm{Hom}}^{\ell}_{\mathcal{N}} \circ (\mathrm{id} \times \triangleleft) \circ (H \times G \times F) \arrow[u, "\zeta"]
\end{tikzcd}
\tag{5.18}
$$

Proof. We proceed with the graphical calculus. The first surface diagram 5.8a depicts the top line of the commutative diagram 5.18. Reading from bottom to top, we start at $\otimes \circ (\underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \times F) \circ (\underline{\mathrm{Hom}}^{\ell}_{\mathcal{M}} \times \mathrm{id})$. Then we pass the structure morphism λ. Then in the front of the diagram we see the structure of diagram 5.6. Then we see a black dot that represents the lax module functor structure $\tilde{\sigma} : \underset{\sim}{\mathrm{Nat}}^{\ell}(H,G) \mathbin{☼} F \to \underset{\sim}{\mathrm{Nat}}^{\ell}(H, G \triangleleft F)$. Finally we see structure morphism of $\underset{\sim}{\mathrm{Nat}}^{\ell}$, identical to the left hand side of the surface diagram 5.7.

We then apply that the two surface diagrams 5.7 are equal to rewrite the structure morphism of the internal hom in terms of evaluation, coevaluation and the other Kan extension to get the surface diagram 5.8b.

We can then move the dot that denotes $\tilde{\sigma}$ up above the natural transformations via the lax module structure. This gives us the surface diagram 5.8c.

Then, we cancel out an evaluation and a coevaluation. That gives us the surface diagram 5.9a.

By applying the equality of surface diagrams 5.5 we get the surface diagram 5.9b.

We then introduce an evaluation-coevaluation pair, giving us the surface diagram 5.9c.

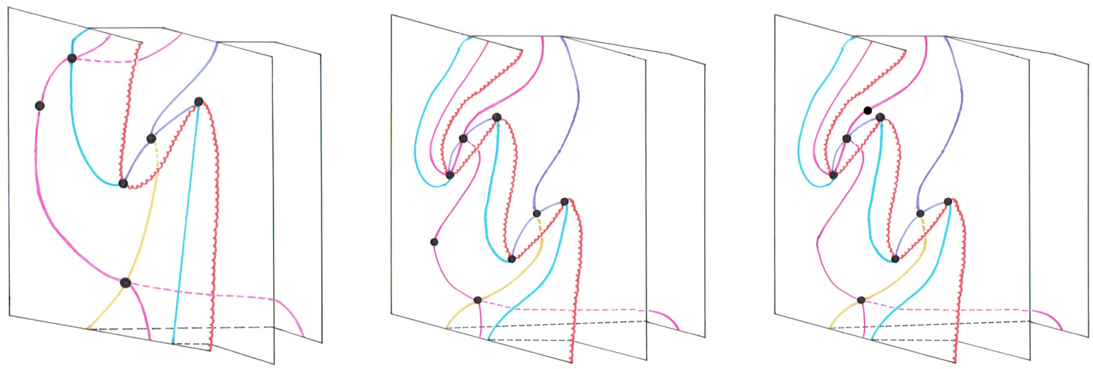
Finally, we apply the equality of surface diagrams 5.7 to get the final surface diagram 5.10. This final surface diagram is the lower line of the commutative diagram. ☐

Since the distributors of the GV-category are related to these inner homs as described in Proposition 5.1.19, we can specialize this to the following corollary.

Corollary 5.3.5. *In particular, if $\mathcal{M}$ is $\mathcal{C}$ and $\mathcal{N}$ is $\mathcal{D}$, then we see that the distributors of the GV-categories are related by the diagram* (5.19).

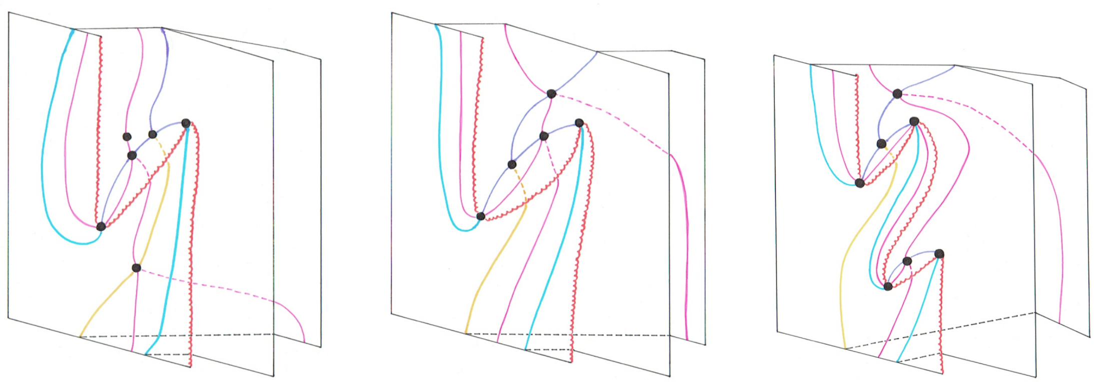
$$
\begin{tikzcd}[column sep=small]
 & & ((F_1 \mathbin{◑} F_2) \mathbin{☼} F_3) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[drr, "\delta^r_{\mathrm{Funct}(\mathcal{C},\mathcal{D})}"] & & \\
((F_1 \mathbin{◑} F_2) \mathbin{☼} F_3) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[urr, "\delta^r_{\mathcal{C}}"] \arrow[dr, "\delta^r_{\mathrm{Funct}(\mathcal{C},\mathcal{D})}"] & & & & (F_1 \mathbin{◑} (F_2 \mathbin{☼} F_3)) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[d, "\nu"] \\
\otimes \circ ((F_1 \mathbin{◑} F_2) \times F_3) \circ (\parr \times \mathrm{id}) \arrow[u, "\lambda"] \arrow[d, "\nu"] & (F_1 \mathbin{◑} (F_2 \mathbin{☼} F_3)) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[urrr, "\delta^r_{\mathcal{C}}"] & & & \parr \circ (F_1 \times (F_2 \mathbin{☼} F_3)) \circ (\mathrm{id} \times \otimes) \\
\otimes \circ (\parr \times \mathrm{id}) \circ (F_1 \times F_2 \times F_3) \arrow[rrrr, "\delta^r_{\mathcal{D}}"] & & & & \parr \circ (\mathrm{id} \times \otimes) \circ (F_1 \times F_2 \times F_3) \arrow[u, "\lambda"]
\end{tikzcd}
\tag{5.19}
$$

We will begin with the assumption that F is a Frobenius LD Functor. We will eventually start with the other assumption, that F is a GV Frobenius algebra, but this section of the proof is easier to grasp with our initial assumption.



(a) Step 1 (b) Step 2 (c) Step 3

Figure 5.8: Steps 1-3



(a) Step 4 (b) Step 5 (c) Step 6

Figure 5.9: Steps 4-6

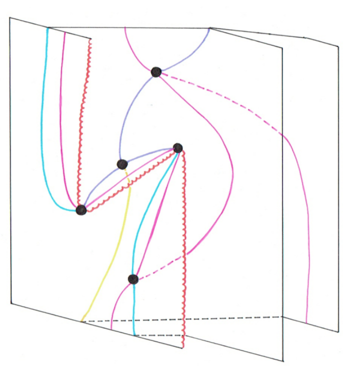


Figure 5.10: Step 7

Lemma 5.3.6. *Let* $F \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *be a Frobenius LD Functor. Then the commutative diagrams 5.11 and* (5.20) *commute.*

$$\begin{tikzcd}
 & (F \mathbin{☼} F) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[dl, "\mu"'] \arrow[dr, "\Delta ☼ \mathrm{id}"] & \\
F \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[dd, "\delta^r"] & & ((F \mathbin{◑} F) \mathbin{☼} F) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[dl, "\delta^r"'] \arrow[d, dashed, "\delta^r"] \\
 & ((F \mathbin{◑} F) \mathbin{☼} F) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[dr, "\delta^r"] & (F \mathbin{◑} (F \mathbin{☼} F)) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[d, dashed, "\delta^r"] \\
F \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[dr, "\Delta"] & & (F \mathbin{◑} (F \mathbin{☼} F)) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[dl, "\mathrm{id} ◑ \mu"'] \\
 & (F \mathbin{◑} F) \circ \parr \circ (\mathrm{id} \times \otimes) &
\end{tikzcd} \tag{5.20}$$

Proof. First, note that the commutative diagram (5.20) is the outermost part of the commutative diagram 5.11. We will thus start by proving that commutative diagram 5.11 commutes. Note that each part of the diagram commutes in turn:

(i) These commute by naturality.

(ii) These commute by definition of the Day and coDay convolutions as Kan extensions.

(iii) This commutes by F being a Frobenius LD-functor.

(iv) This commutes by Corollary 5.3.5.

We now want to isolate the outer section of the diagram.

Note that the arrow $\otimes \circ (F \times F) \circ (\mathrm{id} \times \parr) \to (F \mathbin{☼} F) \circ \otimes \circ (\mathrm{id} \times \parr)$ is the structure morphism of a left Kan extension. Similarly, the arrow $(F \mathbin{◑} F) \circ \parr \circ (\otimes \times \mathrm{id}) \to \parr \circ (F \times F) \circ (\otimes \times \mathrm{id})$ is the structure morphism of a right Kan extension. By Lemma 4.1.12, this means we can cancel these arrows out on the left and the right respectively. That will give us that the outer commutative diagram (5.20) commutes. □

All of this comes together into this final proof, that Frobenius LD-functors are in bijection with GV-Frobenius algebras in the appropriate functor category.

Theorem 5.3.7 (Frobenius LD-Functors are in bijection with GV-Frobenius Algebras)**.** *Let* $\mathcal{C}, \mathcal{D}$ *be* $\mathbb{K}$-*linear GV-categories with finite dimensional hom-spaces,* $\mathcal{C}$ *small,* $\mathcal{D}$ *having all limits and colimits of size* $\mathcal{C}$ *and* $\mathcal{D}$ *finitely copowered. As well, we assume that for all functors* $F, G : \mathcal{C} \to \mathcal{D}$, $\mathrm{Nat}(\mathcal{C}, \mathcal{D})$ *is finite-dimensional.*

A functor $F \in \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *is a Frobenius LD functor if and only if* $F \in (\mathrm{Funct}(\mathcal{C}, \mathcal{D}), ☼, ◑)$ *is an GV-Frobenius algebra.*

Proof. We will proceed in two parts: first, we will assume that F is a Frobenius LD functor and prove that F has the structure of a GV Frobenius algebra. Then we will assume that F has the structure of a GV Frobenius algebra, and prove it has the structure of a Frobenius LD functor.

Assume F is a Frobenius LD functor. Then by Lemma 5.3.6, we know that commutative diagram (5.20) commutes. Precompose every entry in the commutative diagram (5.20) with the functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ defined by $X \mapsto K \times X \times \mathbb{I}$. This will give us a new valid commutative diagram. It will suffice to show two small facts:

$$[\parr \circ (\mathrm{id} \times \otimes)](K \times X \times \mathbb{I}) \cong (K \parr X) \otimes \mathbb{I} \cong X \otimes \mathbb{I} \cong X$$
$$[\otimes \circ (\parr \times \mathrm{id})(K \times X \times \mathbb{I})] \cong K \parr (X \otimes \mathbb{I}) \cong K \parr X \cong X$$

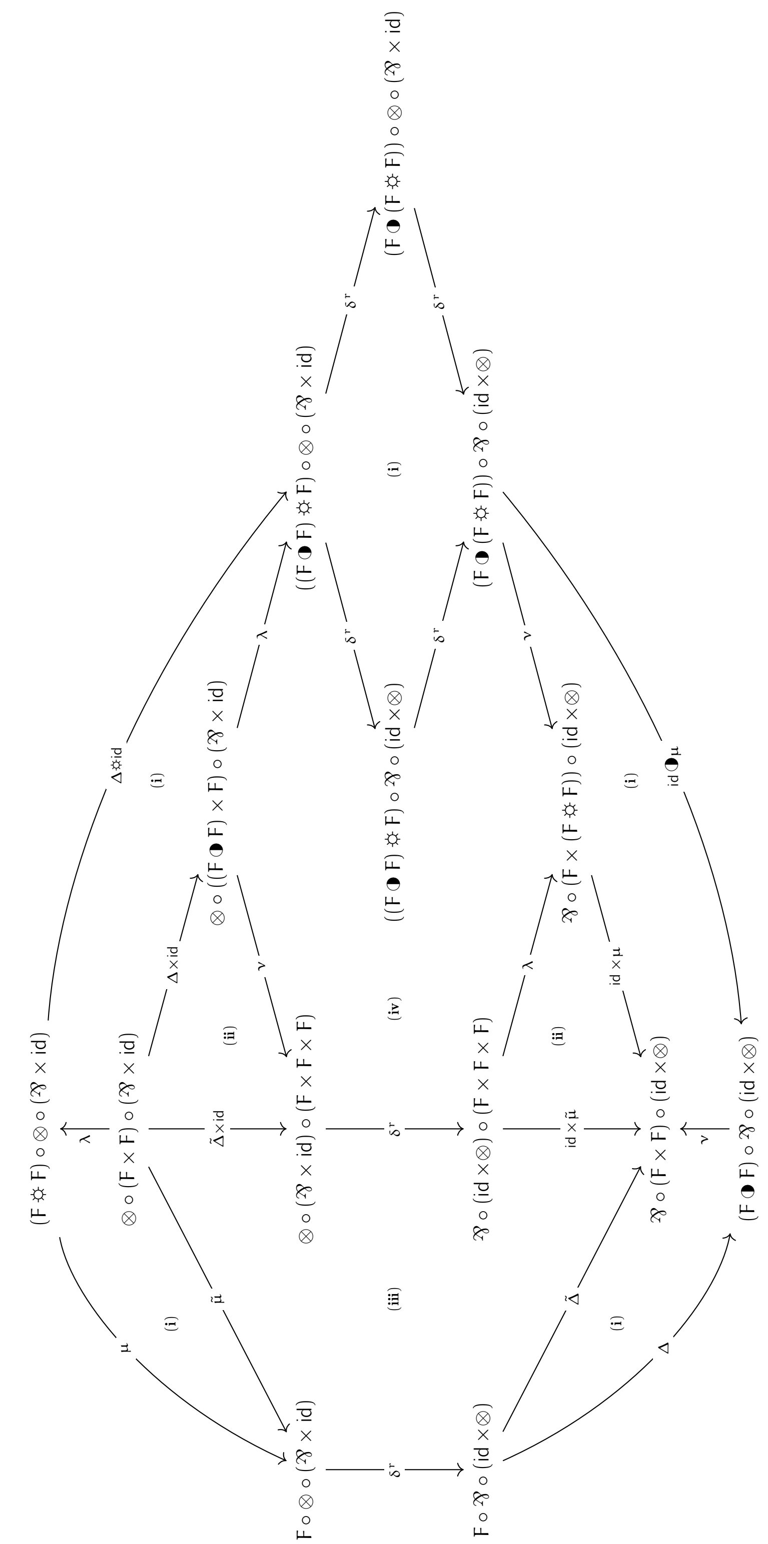


Figure 5.11: Horseshoe Crab Diagram

Applying this to the commutative diagram (5.20) gives us the the commutative diagram (5.21).

$$\begin{tikzcd}
 & F \mathbin{☼} F \arrow[ddl, "\mu"'] \arrow[dr, "\Delta ☼ \mathrm{id}"] & \\
 & & (F \mathbin{◑} F) \mathbin{☼} F \arrow[dd, "\delta^r"] \\
F \arrow[ddr, "\Delta"'] & & \\
 & & F \mathbin{◑} (F \mathbin{☼} F) \arrow[dl, "\mathrm{id} \mathbin{◑} \mu"] \\
 & F \mathbin{◑} F &
\end{tikzcd} \tag{5.21}$$

This commutative diagram is one side of the commutative diagram (5.6); the other side of the commutative diagram comes from regarding $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ as the left regular module category and repeating the above proofs.

Now, assume that F has the structure of a GV Frobenius algebra. This lets us begin with commutative diagram (5.21). Recall this is a diagram of natural transformations. Consider the natural transformation

$$\delta^r : \otimes \circ (\parr \times \mathrm{id}) \Rightarrow \parr \circ (\mathrm{id} \times \otimes).$$

Combining these two, we get the commutative diagram (5.22). From this commutative diagram, we can extract the smaller commutative diagram (5.20).

$$\begin{tikzcd}[column sep=small]
 & & (F \mathbin{☼} F) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[dddll, bend right, "\mu"'] \arrow[ddrr, "\Delta ☼ \mathrm{id}"] & & \\
 & & (F \mathbin{☼} F) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[u, "\delta^r"] \arrow[dr, "\Delta ☼ \mathrm{id}"] \arrow[ddl, "\mu"'] & & \\
 & & & ((F \mathbin{◑} F) \mathbin{☼} F) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[r, "\delta^r"] \arrow[dd, "\delta^r"] & ((F \mathbin{◑} F) \mathbin{☼} F) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[dd, "\delta^r"] \\
F \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[dddrr, bend right, "\Delta"'] & F \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[l, "\delta^r"'] \arrow[ddr, "\Delta"] & & & \\
 & & & (F \mathbin{◑} (F \mathbin{☼} F)) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[r, "\delta^r"] \arrow[dl, "\mathrm{id} \mathbin{◑} \mu"'] & (F \mathbin{◑} (F \mathbin{☼} F)) \circ \parr \circ (\mathrm{id} \times \otimes) \arrow[ddll, "\mathrm{id} \mathbin{◑} \mu"] \\
 & & (F \mathbin{◑} F) \circ \otimes \circ (\parr \times \mathrm{id}) \arrow[d, "\delta^r"] & & \\
 & & (F \mathbin{◑} F) \circ \parr \circ (\mathrm{id} \times \otimes) & &
\end{tikzcd} \tag{5.22}$$

This implies that if F is a GV-Frobenius algebra, we know that everything in comuutative diagram 5.11 commutes except the cell labelled (**iii**). However, that cell commutes because the rest of the diagram commutes. That is one side of the condition that F is a Frobenius LD functor. The other side again comes from regarding $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ as the left regular module category and repeating the above proofs. That implies that F is a Frobenius LD Functor. □

Appendix A

Spectral Sequence Argument

The following account of spectral sequences is mostly drawn from [Wei]. We have dualized some sections.

Definition A.0.1. We define a cohomology spectral sequence in $\mathcal{C}$ via the following collection:

- A family of objects $E_r^{p,q}$ defined for all integers p, q and $r \geqslant 0$. We call $p + q$ the total degree of $E_r^{p,q}$.
- A family of maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that behave as "differentials" in that $d_r \circ d_r = 0$. Note that these give us chain complexes along diagonal lines.
- A family of isomorphisms

$$E_{r+1}^{p,q} \cong \ker(E_r^{p,q} \to E_r^{p+r,q-r+1}) \big/ \mathrm{im}(E_r^{p-r,q+r-1} \to E_r^{p,q})$$

Remark A.0.2. Due to convention, we will generally consider spectral sequences as having reversed coordinates. This arises out of how indexes are handled with cohomological spectral sequences.

Definition A.0.3 (Bounded Convergence)**.** We say a spectral sequence is bounded if for each r, and any n there are only finitely many nonzero objects of total degree n in $E_r^{\bullet\bullet}$. If so, for each (p, q) there exists an r_0 such that $E_{r+1}^{p,q} \cong E_r^{p,q}$ for all $r \geqslant r_0$. We write $E_\infty^{p,q}$ for this stable value.

We say a bounded cohomology spectral sequence $E_r^{p,q}$ converges to $H^\bullet$ if, for a family of objects H^n in $\mathcal{C}$, each have a finite filtration:

$$H^n = F^s H^n \supset \cdots \supset F^p H^n \supset F^{p+1} H^n \supset \cdots \supset F^t H^n = 0.$$

And we have isomorphisms

$$E_\infty^{p,q} \cong F^p H^{p+q} \big/ F^{p+1} H^{p+q}$$

With these definitions we are catching a glance of an immense and complicated world of spectral sequences. We are only interested in one of the most standard uses of spectral sequences; approximating the cohomology of a double complex. We will take a specific path through the topic. First, given a filtered cochain complex we will construct a spectral sequence. Using this construction, if given a double complex we can construct a pair of spectral sequences. Then, we will use some general theorems about the convergence of spectral sequences constructed from filtered cochain complexes to show that the appropriate spectral sequence will always converge. Finally, we will apply that construction to Tsygan's double complex, which will prove the theorem we need.

Lemma A.0.4. *[Wei, 5.4.6.] Let $X^\bullet$ be a cochain complex in an abelian category $\mathcal{C}$. We will assume, for ease of computation and without loss of generality, that $\mathcal{C}$ is a concrete category.*

Furthermore, let $X^\bullet$ have a filtration, that is, we have a family of subcomplexes such that

$$\cdots \subset F^{p+1}X^\bullet \subset F^pX^\bullet \subset F^{p-1}X^\bullet \subset \cdots.$$

Furthermore, let the filtration be bounded, that is, for each n there exists s, t such that

$$0 = F^tX^n \subset \cdots \subset F^{p+1}X^n \subset F^pX^n \subset F^{p-1}X^n \subset \cdots \subset F^sX^n = X^n.$$

Then, there exists a spectral sequence defined by $E_0^{p,q} = {F^pX^{p+q}}/{F^{p+1}X^{p+q}}$ and $E_1^{p,q} = H^{p+q}(E_0^{p\bullet})$.

Proof. To make the following construction easier to read, we will suppress the superscript q. Now define $\eta_p : F^pX \to {F^pX}/{F^{p+1}X} = E_0^p$ to be the standard surjection. Then consider

$$A_r^p = \{x \in F^pX : d(x) \in F^{p+r}X\}$$

which are elements of F^pX that are cocycles up to $F^{p+r}X$. If these are morally "almost cocycles", then we can try to define cohomology with them. So, set $Z_r^p = \eta_p(A_r^p) \subset E_0^p$ and $B_{r+1}^{p+r} = \eta_{p+r}(d(A_r^p)) \subset E_0^{p+r}$. Notice that by our choice of indexes we get that Z_r^p and $B_r^p = \eta_p(d(A_{r-1}^{p-r+1}))$ are subobjects of E_0^p. So, if we hold p fixed, we get two more objects, $Z_\infty^p = \bigcap_{r=1}^\infty Z_r^p$ and $B_\infty^p = \bigcup_{r=1}^\infty B_r^p$. This gives us a tower of subobjects for each E_0^p.

$$0 = B_0^p \subset B_1^p \subset \cdots \subset B_r^p \subset \cdots \subset B_\infty^p \subset Z_\infty^p \subset \cdots \subset Z_r^p \subset \cdots \subset Z_1^p \subset Z_0^p = E_0^p$$

Now, consider that $A_r^p \cap F^{p+1}X = A_{r-1}^{p+1}$, which gives $Z_r^p \cong {A_r^p}/{A_{r-1}^{p+1}}$. By the dual to [Wei, 5.2.8], we have that

$$E_r^p = {Z_r^p}/{B_r^p} \cong \frac{A_r^p + F^{p+1}X}{d(A_{r-1}^{p-r+1}) + F^{p+1}X} \cong \frac{A_r^p}{d(A_{r-1}^{p-r+1}) + A_{r-1}^{p+1}}$$

So let our $d_r^p : E_r^p \to E_r^{p+r}$ be the map induced by the differential of X. Then all we need to define our spectral sequence is to give the isomorphism between E_{r+1} and $H^\bullet(E_r)$.

Lemma A.0.5. *The map d determines isomorphisms*

$$ {Z_r^p}/{Z_{r+1}^p} \xrightarrow{\cong} {B_{r+1}^{p+r}}/{B_r^{p+r}}$$

Proof. We begin by unwrapping our notation. First, we have that $d(A_r^p) \cap F^{p+r+1}X = d(A_{r+1}^p)$ by definition of A_r^p. That implies that $B_{r+1}^{p+r} \cong {d(A_r^p)}/{d(A_{r+1}^p)}$. That gives us

$$ {B_{r+1}^{p+r}}/{B_r^{p+r}} \cong \frac{d(A_r^p)}{d(A_{r+1}^p + A_{r-1}^{p+1})}.$$

We now need the other term. By above arguments, we know that

$$ {Z_r^p}/{Z_{r+1}^p} \cong \frac{A_r^p}{A_{r+1}^p + A_{r-1}^{p+1}}.$$

It remains to say that because the kernel of $d : A_r^p \to F^{p+r}X$ is contained in A_{r+1}^p, the two sides are isomorphic. $\square$

With that lemma proven, we can consider the kernel of d_r^p. That is,

$$\frac{\left\{x \in A_p^r : d(x) \in d(A_{r-1}^{p+1}) + A_{r-1}^{p-r+1}\right\}}{d(A_{r-1}^{p-r+1}) + A_{r-1}^{p+1}} \cong \frac{A_{r-1}^{p+1} + A_{r+1}^{p}}{d(A_{r-1}^{p-r+1}) + A_{r-1}^{p+1}} \cong \frac{Z_{r+1}^p}{B_r^p}$$

By Lemma A.0.5 we have the map d_r^p factors

$$E_r^p = Z_r^p / B_r^p \to Z_r^p / Z_{r+1}^p \xrightarrow{\cong} B_{r+1}^{p+r} / B_r^{p+r} \hookrightarrow Z_r^{p+r} / B_r^{p+r} = E_r^{p+r}.$$

So the image of d_r^p is $B_{r+1}^{p+r} / B_r^{p+r}$. Similarly, the image of d_r^{p-r} is B_{r+1}^p / B_r^p. This gives us the isomorphism we need, that being

$$E_{r+1}^p = Z_{r+1}^p / B_{r+1}^p \cong \ker(d_r^p) / \operatorname{im}(d_r^{p-r}).$$

□

We cite the following theorem without proof.

Theorem A.0.6 (Classical Convergence Theorem)**.** *[Wei, 5.5.1] Suppose we have a cochain complex* $X^\bullet$ *that has a bounded filtration. Then the spectral sequence constructed above converges to* $H^\bullet(X)$.

We will be filtering Tsygan's double complex by rows. The following lemma will make our computations much easier.

Lemma A.0.7. *Let* $G = \mathbb{Z} / n\mathbb{Z}$ *be the cyclic group of order* n *with generator* g. *Let* $\mathcal{C}$ *be an abelian category and* $A \in \mathcal{C}$ *be an object with some nontrivial* G*-action on* A. *We know that* $\operatorname{Hom}_{\mathcal{C}}(A, A)$ *is naturally an abelian group. Let* $N = \sum_{i=0}^{n-1} g^i$ *be the norm. If the action of* n *on* $\operatorname{Hom}_{\mathcal{C}}(A, A)$ *is invertible, then the cochain complex*

$$C : 0 \to A \xrightarrow{1-g} A \xrightarrow{N} A \xrightarrow{1-g} A \xrightarrow{N} \cdots$$

is contractible, giving us the cohomology groups, $H^0(C) = A^G$ *is the invariants of* A *with respect to the* G*-action, and* $H^i(C) = 0$ *for all* $i > 0$.

Proof. First we will show that $H^0(C)$ is the invariants of A with respect to the G-action. The image of any map out of 0 is 0, so $H^0(C)$ is exactly the kernel of $1 - g$, or the invariants of the G-action on A.

To see that C is contractible, we take $h' = \frac{1}{n}$ and $h = -\frac{1}{n}\sum_{i=1}^{n-1} i g^i$. We then need that $h'N + (1-g)h = 1$ and $Nh' + h(1-g) = 1$. We will only work through the first one, as the second one is essentially the same.

$$\begin{aligned}
h'N + (1-g)h &= \frac{1}{n}\sum_{i=0}^{n-1} g^i - \frac{1}{n}(1-g)\sum_{i=1}^{n-1} i g^i \\
&= \frac{1}{n}\left[\sum_{i=0}^{n-1} g^i - \sum_{i=1}^{n-1} i g^i + \sum_{i=1}^{n-1} i g^{i+1}\right] \\
&= \frac{1}{n}\left[1 + \sum_{i=1}^{n-1} (1-i) g^i + \sum_{i=2}^{n} (i-1) g^i\right] \\
&= \frac{1}{n}\left[1 + (n-1) + \sum_{i=2}^{n-1} (1-i) g^i + \sum_{i=2}^{n-1} (i-1) g^i\right] \\
&= 1
\end{aligned}$$

□

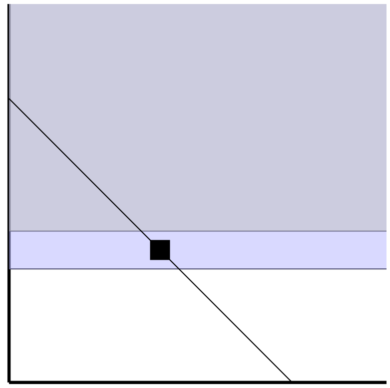


Figure A.1: Filtration by rows

Note that the proof above does not need that the action of n on $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is invertible to show that $H^0(C) = A^G$.

Finally, we are ready to prove Theorem 3.3.10.

Theorem A.0.8. *[Lod, 2.4.2] Let X be a cocyclic object in an abelian category. Let k be the maximal integer such that for all $m < k$, the $\mathbb{Z}$-action of m on $\mathrm{Hom}_{\mathcal{C}}(X^{m-1}, X^{m-1})$ is invertible. Then the injection from Lemma 3.3.9 induces a collection of isomorphisms for all $n \leqslant k$:*

$$H^n_\lambda(X) \xrightarrow{\cong} HC^n(X), \quad n \geqslant 0. \tag{A.1}$$

If there is no such maximal k, then this isomorphism holds for all n.

Proof. Define a filtration on Tsygan's double complex by $F^p\,\mathrm{Tsy}(X) = \sum_{i \geqslant p} \mathrm{Tsy}^{q,i}(X)$. That is, our filtration is given by taking all rows above the p-th row. This filtration passes down to the total complex. Note that the filtration on $\mathrm{Tot}(\mathrm{Tsy}(X))$ is bounded. This gives us a spectral sequence by the theorem above.

We want to construct part of the E_∞ page of the spectral sequence, in particular the part at and below the $(k+1)$-th row.

By the filtration, we get that

$$E_0^{p,q} = \frac{F^p(\mathrm{Tot}(\mathrm{Tsy}(X))^{p+q})}{F^{p+1}(\mathrm{Tot}(\mathrm{Tsy}(X))^{p+q})} = \frac{\mathrm{Tsy}^{q,p}(X) \oplus \mathrm{Tsy}^{q-1,p+1}(X) \oplus \cdots}{\mathrm{Tsy}^{q-1,p+1}(X) \oplus \mathrm{Tsy}^{q-2,p+2}(X) \oplus \cdots} \cong \mathrm{Tsy}^{q,p}(X) \cong X^p.$$

You can easily see this from the diagram A.1. The grey region is F^{p+1}, the blue region is F^p. The diagonal line lies along $\mathrm{Tsy}^{i,j}$ with $i+j = n$. The black square is the only term which F^p doesn't exclude and F^{p+1} doesn't quotient out.

This gives us that $E_1^{p,q} = H^q(X^p, d_h)$. If $p < k$, then $H^0(X^p, d_h) = C^p_\lambda(X)$ and $H^q(X^p, d_h) = 0$ for $q \neq 0$ by Lemma A.0.7.

The next page of our spectral sequence is $E_2^{p,q} = \ker(E_1^{p,q} \to E_1^{p+1,q}) / \mathrm{im}(E_1^{p-1,q} \to E_1^{p,q})$. Along $q = 0$ this gives us $E_2^{p,0} = H^p_\lambda(X)$ for all p.

We now want that $E_\infty^{p,0} = E_2^{p,0}$, that is, for all $p \leqslant k$ and $r > 2$ we have that $E_r^{p,0} = E_2^{p,0}$. By definition of $E_{r+1}^{p,q}$, we just need that $E_r^{p+r,-r+1} = 0$ and $E_r^{p-r,r-1} = 0$. Since $E_s^{p,q} = 0$ implies that $E_r^{p,q} = 0$ for all $r > s$, it suffices to show that for any $r > 2$, $E_2^{p+r,-r+1} = 0$ and $E_2^{p-r,r-1} = 0$.

Well, $r > 2$ implies $-r+1 < 0$, and since our spectral sequence is a first quadrant spectral sequence, this implies $E_2^{p+r,-r+1} = 0$. Next, $p-r < k$ and $r-1 \neq 0$, so by the facts we proved above, $E_2^{p-r,r-1} = 0$.

$$
\begin{array}{rlllll}
(k+1)\text{th row:} & * & * & * & * & * \\
k\text{th row:} & H^k_\lambda & * & * & * & * \\
(k-1)\text{th row:} & H^{k-1}_\lambda & 0 & 0 & 0 & 0 \\
(k-2)\text{th row:} & H^{k-2}_\lambda & 0 & 0 & 0 & 0 \\
 & \vdots & & \vdots & & \\
2\text{nd row:} & H^2_\lambda & 0 & 0 & 0 & 0 \\
1\text{st row:} & H^1_\lambda & 0 & 0 & 0 & 0 \\
0\text{th row:} & H^0_\lambda & 0 & 0 & 0 & 0
\end{array}
$$

Figure A.2: E_∞-page of the spectral sequence

We now know the terms $E_\infty^{p,0}$ for $p \leqslant k$ and $E_\infty^{p,q}$ for $p < k$. This gives us the fragment of the $E_\infty^{p,q}$ page in A.2

Now, $E_\infty^{p,q} = {F^pHC^{p+q}}/{F^{p+1}HC^{p+q}}$. Fix $n \leqslant k$. Then there is only one nontrivial filtered component, $E_\infty^{n,0} = {F^nHC^n}/{F^{n+1}HC^n}$. So, $F^{n+1}HC^n = 0$, and $F^nHC^n = HC^n$ Putting this all together we get that

$$HC^n \cong {HC^n}/{0} \cong {F^nHC^n}/{F^{n+1}HC^n} \cong E_\infty^{n,0} \cong H^n_\lambda.$$

It just remains to prove that the injection ι_X induces this isomorphism. But this is clear from the construction of $E_2^{n,0}$.

And that proves what needs to be proven. $\square$

Remark A.0.9. In the particular cases we are working with, that is algebras over fields, $k = \text{char}(\mathbb{K})$ if $\text{char}(\mathbb{K}) \neq 0$, or k does not exist.

Appendix B

Computation of the cases $\mathbb{K}_2[\mathbb{Z}_3]$ and $\mathbb{K}_3[\mathbb{Z}_2]$.

Theorem 3.4.11 has two notable exceptions, namely $\mathrm{char}(\mathbb{K}) = 2$ and $\mathrm{char}(\mathbb{K}) = 3$.

B.1 Connes' cohomology of $\mathbb{K}_2[\mathbb{Z}_3]$.

Let $A = \mathbb{K}_2[\mathbb{Z}_3]$ be the group algebra of $\mathbb{Z}_3$ over the field of two elements. Note that this is semisimple. Furthermore, as $0! = 1! = 1$ is invertible in $\mathbb{K}_2$, we just need to compute $H^2_\lambda(A)$. Following [PS] we get that we need to compute

$$H^2_\lambda(A) = \frac{\ker(d_\lambda : C^2_\lambda \to C^3_\lambda)}{\mathrm{im}(d_\lambda : C^1_\lambda \to C^2_\lambda)}$$

Starting with computing $\mathrm{im}(d_\lambda)$. By [PS] we have that

$$(d_\lambda f)(a_1, a_2, a_3) := f(a_3a_1, a_2) + f(a_1a_2, a_3) + f(a_2a_3, a_1)$$

Working out the kernel, we get the system of equations:

$$\begin{aligned} f(1,1) &= 0 & f(1,x) &= 0 & f(1,x^2) &= 0 \\ f(x,x^2) &= 0 & f(x,x) &= 0 & f(x^2,x^2) &= 0 \end{aligned}$$

That means this kernel is zero. Note that $\dim(C^1_\lambda) = 6$ so $\dim(\mathrm{im}(d_\lambda)) = 6$.

Now we need to work out $\ker(d_\lambda)$.

$$(d_\lambda f)(a_1, a_2, a_3, a_4) := f(a_4a_1, a_2, a_3) + f(a_1a_2, a_3, a_4) + f(a_2a_3, a_4, a_1) + f(a_3a_4, a_1, a_2)$$

Working out the kernel, we find it's generated by:

$$\begin{aligned} f(1,x,x) + f(1,1,x^2) &= 0 & f(1,x^2,x^2) + f(1,1,x) &= 0 \\ f(1,x,x^2) + f(1,1,1) &= 0 & f(1,x^2,x) + f(1,1,1) &= 0 \\ f(1,x,x^2) + f(1,x,x) &= 0 & f(1,x^2,x) + f(1,x^2,x^2) &= 0 \\ f(x^2,x^2,x^2) + f(x,x,x) &= 0 \end{aligned}$$

This gives us that $H^2_\lambda(A) = \mathbb{K}_2$. Now, $HC^2(A) \cong HC^0(A) \cong HH^0(A) \cong \mathbb{K}_2^3$, and $\mathbb{K}_2 \ncong \mathbb{K}_2^3$.

B.2 Connes' cohomology of $\mathbb{K}_3[\mathbb{Z}_2]$.

Let $A = \mathbb{K}_3[\mathbb{Z}_2]$. Now, we do know that $H^i_\lambda(A) \cong HC^i(A)$ for $i = 0, 1, 2$. We just need to work out $H^3_\lambda(A)$. We can work out the image of $d_\lambda : C^2_\lambda \to C^3_\lambda$ without explicit computation. Recalling the definition of cohomology over a field and rank-nullity, we get the two identities:

$$\dim(H^k_\lambda(A)) = \dim(\ker(d_\lambda : C^k_\lambda \to C^{k+1}_\lambda)) - \dim(\mathrm{im}(d_\lambda : C^{k-1}_\lambda \to C^k_\lambda))$$
$$\dim(C^k_\lambda) = \dim(\ker(d_\lambda : C^k_\lambda \to C^{k+1}_\lambda)) - \dim(\mathrm{im}(d_\lambda : C^k_\lambda \to C^{k+1}_\lambda))$$

This lets us fill out the table:

$\dim(H^0_\lambda(A)) = 2$	$\dim(C^0_\lambda) = 2$	$\dim(\ker(d_\lambda)) = 2$	$\dim(\mathrm{im}(d_\lambda)) = 0$
$\dim(H^1_\lambda(A)) = 0$	$\dim(C^1_\lambda) = 3$	$\dim(\ker(d_\lambda)) = 2$	$\dim(\mathrm{im}(d_\lambda)) = 2$
$\dim(H^2_\lambda(A)) = 2$	$\dim(C^2_\lambda) = 4$	$\dim(\ker(d_\lambda)) = 3$	$\dim(\mathrm{im}(d_\lambda)) = 1$

That means that $\dim(\mathrm{im}(d_\lambda : C^2_\lambda \to C^3_\lambda)) = 1$. So, now we just need to work out the kernel $\ker(d_\lambda : C^3_\lambda \to C^4_\lambda)$. Well,

$$\begin{aligned} d_\lambda f(a_1, a_2, a_3, a_4, a_5) = {} & f(a_2, a_3, a_4, a_5 a_1) - f(a_1 a_2, a_3, a_4, a_5) \\ & + f(a_1, a_2 a_3, a_4, a_5) - f(a_1, a_2, a_3 a_4, a_5) + f(a_1, a_2, a_3, a_4 a_5) \end{aligned}$$

Working out the kernel we get:

$$\begin{aligned} 0 &= f(1,1,1,1) & 0 &= f(1,1,1,x) \\ 0 &= f(1,x,1,x) & 0 &= f(1,x,x,x) \end{aligned}$$

This gives us that $H^3_\lambda(A) \cong \mathbb{K}_3^3 \not\cong 0$.

And so we have our two examples of why the theorem doesn't work for fields of these characteristics.

Appendix C

Coherence Conditions for the Distributors

The following coherence conditions are discussed in depth in [CS1][Dem1, Appendix A]. We reproduce them here, with some discussion, for completeness.

Let $\mathcal{C}$ be a category, and both $(\mathcal{C}, \otimes, \mathbb{I})$ and $(\mathcal{C}, \parr, K)$ are both monoidal categories on their own. We have two distributors.

$$
\begin{aligned}
\delta^{\ell} &: \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \parr) \Rightarrow \parr \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \\
\delta^{r} &: \otimes \circ (\parr \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \parr \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)
\end{aligned}
$$

The intuition behind the distributors is that they behave like "one-directional associators". So, they should satisfy similar coherency conditions as the associators do.

First, the distributors need to behave appropriately with the unitors. That gives us four triangle diagrams.

$$
\begin{tikzcd}
\mathbb{I} \otimes (A \parr B) \arrow[r, "u"] \arrow[d, "\delta^{\ell}"'] & A \parr B \\
(\mathbb{I} \otimes A) \parr B \arrow[ur, "u \parr \mathrm{id}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \parr B) \otimes \mathbb{I} \arrow[r, "u"] \arrow[d, "\delta^{r}"'] & A \parr B \\
A \parr (B \otimes \mathbb{I}) \arrow[ur, "\mathrm{id} \parr u"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes (B \parr K) \arrow[d, "\delta^{\ell}"'] \arrow[dr, "\mathrm{id} \otimes u"] & \\
(A \otimes B) \parr K \arrow[r, "u"] & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
(K \parr A) \otimes B \arrow[d, "\delta^{r}"'] \arrow[dr, "u \otimes \mathrm{id}"] & \\
K \parr (A \otimes B) \arrow[r, "u"] & A \otimes B
\end{tikzcd}
\tag{C.1}
$$

Next, the distributors should behave nicely with the associators. That gives us four more diagrams, as below. These diagrams are similar to the pentagon conditions for the associator.

$$
\begin{tikzcd}
(A \otimes B) \otimes (C \parr D) \arrow[r, "\alpha"] \arrow[dd, "\delta^{\ell}"] & A \otimes (B \otimes (C \parr D)) \arrow[d, dashed, "\mathrm{id} \otimes \delta^{\ell}"] \\
& A \otimes ((B \otimes C) \parr D) \arrow[d, dashed, "\delta^{\ell}"] \\
((A \otimes B) \otimes C) \parr D \arrow[r, "\alpha \parr \mathrm{id}"] & (A \otimes (B \otimes C)) \parr D
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes (B \parr (C \parr D)) \arrow[r, "\mathrm{id} \otimes \alpha"] \arrow[dd, "\delta^{\ell}"] & A \otimes ((B \parr C) \parr D) \arrow[d, dashed, "\delta^{\ell}"] \\
& (A \otimes (B \parr C)) \parr D \arrow[d, dashed, "\delta^{\ell} \parr \mathrm{id}"] \\
(A \otimes B) \parr (C \parr D) \arrow[r, "\alpha"] & ((A \otimes B) \parr C) \parr D
\end{tikzcd}
$$

(C.2)

$$
\begin{tikzcd}
((A \parr B) \parr C) \otimes D \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[dd, "\delta^{r}"] & (A \parr (B \parr C)) \otimes D \arrow[d, dashed, "\delta^{r}"] \\
& A \parr ((B \parr C) \otimes D) \arrow[d, dashed, "\mathrm{id} \parr \delta^{r}"] \\
(A \parr B) \parr (C \otimes D) \arrow[r, "\alpha"] & A \parr (B \parr (C \otimes D))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \parr B) \otimes (C \otimes D) \arrow[r, "\alpha"] \arrow[dd, "\delta^{r}"] & ((A \parr B) \otimes C) \otimes D \arrow[d, dashed, "\delta^{r} \otimes \mathrm{id}"] \\
& (A \parr (B \otimes C)) \otimes D \arrow[d, dashed, "\delta^{r}"] \\
A \parr (B \otimes (C \otimes D)) \arrow[r, "\mathrm{id} \parr \alpha"] & A \parr ((B \otimes C) \otimes D)
\end{tikzcd}
$$

Finally, the left and right distributors should have some coherency conditions with each other. These are also pentagon diagrams, as below.

$$
\begin{tikzcd}
 & (A \parr B) \otimes (C \parr D) \arrow[dl, "\delta^\ell"'] \arrow[dr, "\delta^r"] & \\
((A \parr B) \otimes C) \parr D \arrow[d, "\delta^r \parr \mathrm{id}"'] & & A \parr (B \otimes (C \parr D)) \arrow[d, "\mathrm{id} \parr \delta^\ell"] \\
(A \parr (B \otimes C)) \parr D \arrow[rr, "\alpha"] & & A \parr ((B \otimes C) \parr D)
\end{tikzcd}
\tag{C.3}
$$

$$
\begin{tikzcd}
(A \otimes (B \parr C)) \otimes D \arrow[rr, "\alpha"] \arrow[d, "\delta^\ell \otimes \mathrm{id}"'] & & A \otimes ((B \parr C) \otimes D) \arrow[d, "\mathrm{id} \otimes \delta^r"] \\
((A \otimes B) \parr C) \otimes D \arrow[dr, "\delta^r"'] & & A \otimes (B \parr (C \otimes D)) \arrow[dl, "\delta^\ell"] \\
 & (A \otimes B) \parr (C \otimes D) &
\end{tikzcd}
$$

Bibliography

[Bar1] M. Barr. *-Autonomous categories. Vol. 752. Springer, 1979.

[Bar2] M. Barr. "Nonsymmetric *-autonomous categories". In: Theoretical Computer Science 139.1 (1995), pp. 115–130. ISSN: 0304-3975. DOI: https://doi.org/10.1016/0304-3975(94)00089-2.

[BD] M. Boyarchenko and V. Drinfeld. "A duality formalism in the spirit of Grothendieck and Verdier". In: Quantum Topology 4.4 (2013), pp. 447–489. arXiv: 1108.6020 [math.QA]. URL: https://arxiv.org/abs/1108.6020.

[Bor1] F. Borceux. Handbook of categorical algebra: volume 1, Basic category theory. Vol. 1. Cambridge University Press, 1994.

[Bor2] F. Borceux. Handbook of Categorical Algebra: Volume 2, Categories and Structures. Vol. 2. Cambridge University Press, 1994.

[CCEKS] J.S. Carter, A.S. Crans, M. Elhamdadi, E. Karadayi, and M. Saito. "Cohomology of Frobenius Algebras and the Yang-Baxter Equation". In: Communications in Contemporary Mathematics 10.supp01 (2008), pp. 791–814. arXiv: 0801.2567 [math.QA].

[CS1] J. R. B. Cockett and R. A. G. Seely. "Weakly distributive categories". In: Journal of Pure and Applied Algebra 114.2 (1997), pp. 133–173. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(95)00160-3. URL: https://www.math.mcgill.ca/rags/linear/wdc-fix.pdf.

[CS2] J. R. B. Cockett and R. A. G. Seely. "Linearly distributive functors". In: Journal of Pure and Applied Algebra 143.1 (1999), pp. 155–203. ISSN: 0022-4049. DOI: https://doi.org/10.1016/S0022-4049(98)00110-8. URL: https://www.sciencedirect.com/science/article/pii/S0022404998001108.

[Day1] B. Day. "Construction of biclosed categories". PhD thesis. University of New South Wales, 1970.

[Day2] B. Day. "On closed categories of functors". In: Reports of the Midwest Category Seminar IV. Springer. 2006, pp. 1–38.

[DE] A. Davydov and M. Elbehiry. "Deformation cohomology of Schur–Weyl categories: Free symmetric categories". In: Selecta Mathematica 29.1 (2023), p. 1.

[Dem1] M. Demirdilek. "Surface Diagrams for Grothendieck-Verdier Categories". Oct. 2024. URL: https://maxdemirdilek.github.io/Research/ (visited on 11/11/2024).

[Dem2] M. Demirdilek. Personal Communication. 2025.

[DP] B. Day and C. Pastro. Note on Frobenius monoidal functors. 2008. arXiv: 0801.4107 [math.CT]. URL: https://arxiv.org/abs/0801.4107.

[DS] M. Demirdilek and C. Schweigert. "Surface Diagrams for Frobenius Algebras and Frobenius-Schur Indicators in Grothendieck-Verdier Categories". In: arXiv preprint arXiv:2503.13325 (2025).

[Egg1] J. M. Egger. "Star-autonomous functor categories". In: Theory and Applications of Categories 20.11 (2008), pp. 307–333. URL: http://emis.icm.edu.pl/journals/TAC/volumes/20/11/20-11.pdf.

[Egg2] J. M. Egger. "The Frobenius relations meet linear distributivity." In: Theory and Applications of Categories 24 (2010), pp. 25–38.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor categories. Vol. 205. American Mathematical Soc., 2015.

[FS] J. Fuchs and C. Stigner. "On Frobenius algebras in rigid monoidal categories". In: (2009). arXiv: 0901.4886 [math.CT]. URL: https://arxiv.org/abs/0901.4886.

[FSSW1] J. Fuchs, G. Schaumann, C. Schweigert, and S. Wood. "Grothendieck-Verdier module categories, Frobenius algebras and relative Serre functors". In: Advances in Mathematics 475 (2025), p. 110325. arXiv: 2405.20811 [math.CT]. URL: https://arxiv.org/abs/2405.20811.

[FSSW2] J. Fuchs, G. Schaumann, C. Schweigert, and S. Wood. "Grothendieck-Verdier duality in categories of bimodules and weak module functors". In: (2024). arXiv: 2306.17668 [math.CT]. URL: https://arxiv.org/abs/2306.17668.

[FSY] J. Fuchs, C. Schweigert, and Y. Yang. String-net models for pivotal bicategories. 2025. arXiv: 2302.01468 [math.QA]. URL: https://arxiv.org/abs/2302.01468.

[Ger] M. Gerstenhaber. "On the deformation of rings and algebras". In: Annals of Mathematics 79.1 (1964), pp. 59–103.

[GHS] A. M. Gainutdinov, J. Haferkamp, and C. Schweigert. "Davydov-Yetter cohomology, comonads and Ocneanu rigidity". In: Advances in Mathematics 414 (Feb. 2023), p. 45. ISSN: 0001-8708. DOI: 10.1016/j.aim.2022.108853. URL: http://dx.doi.org/10.1016/j.aim.2022.108853.

[GS] M. Gerstenhaber and S.D. Schack. "Algebraic cohomology and deformation theory". In: Deformation theory of algebras and structures and applications. Springer, 1988, pp. 11–264.

[Lei] T. Leinster. Higher Operads, Higher Categories. 2003. arXiv: math/0305049 [math.CT]. URL: https://arxiv.org/abs/math/0305049.

[Lod] J-L. Loday. Cyclic homology. 2nd ed. Springer, 1998.

[Lor] F. Loregian. (Co)end Calculus. Cambridge University Press, June 2021. ISBN: 9781108746120. DOI: 10.1017/9781108778657. URL: http://dx.doi.org/10.1017/9781108778657.

[LvdB] W. T. Lowen and M. van den Bergh. "Deformation theory of abelian categories". In: Trans. Am. Math. Soc. 358.12 (2006), pp. 5441–5483. arXiv: math/0405226 [math.CT]. URL: https://arxiv.org/abs/math/0405226.

[Mac] S. Mac Lane. Categories for the working mathematician. Vol. 5. Springer Science & Business Media, 1998.

[MS] M.B. McCurdy and R. Street. What Separable Frobenius Monoidal Functors Preserve. 2016. arXiv: 0904.3449 [math.CT]. URL: https://arxiv.org/abs/0904.3449.

[PS] M. Penkava and A. Schwarz. "A_∞ Algebras and the Cohomology of Moduli Spaces". In: (1995). arXiv: hep-th/9408064 [hep-th].

[Sel] P. Selinger. "A Survey of Graphical Languages for Monoidal Categories". In: New Structures for Physics. Ed. by B. Coecke. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 289–355. ISBN: 978-3-642-12821-9. DOI: 10.1007/978-3-642-12821-9_4. URL: https://doi.org/10.1007/978-3-642-12821-9_4.

[Shi] K. Shimizu. “Relative Serre functor for comodule algebras”. In: (2023). arXiv: 1904.00376 [math.CT]. URL: https://arxiv.org/abs/1904.00376.

[Swe] M. E. Sweedler. Hopf algebras. W.A. Benjamin, 1969.

[Wei] C.A. Weibel. An introduction to homological algebra. Cambridge University Press, 1997.

[Yet1] D. N. Yetter. “Braided Deformations of Monoidal Categories and Vassiliev Invariants”. In: (1997). arXiv: q-alg/9710010 [q-alg]. URL: https://arxiv.org/abs/q-alg/9710010.

[Yet2] D. N. Yetter. “Abelian Categories of Modules over a (Lax) Monoidal Functor”. In: (2001). arXiv: math/0112307 [math.CT]. URL: https://arxiv.org/abs/math/0112307.

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