1. The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+8y(t)=2x(t)$$

(*a*) Find the impulse response of this system.

(*b*) What is the response of this system if $x(t)=te^{-2t}u(t)$?

(*c*) Repeat part (*a*) for the stable and causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2}+\sqrt{2}\frac{dy(t)}{dt}+y(t)=2\frac{d^2x(t)}{dt^2}-2x(t)$$

作答

Solution:

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+8y(t)=2x(t)$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\xleftrightarrow{F}\frac{1}{(a+j\omega)^n}$$

$$(a)\quad (j\omega)^2Y(j\omega)+6j\omega Y(j\omega)+8Y(j\omega)=2X(j\omega)$$

$$\Rightarrow H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{2}{(j\omega)^2+6j\omega+8}=\frac{2}{(j\omega+2)(j\omega+4)}=\frac{1}{j\omega+2}-\frac{1}{j\omega+4}$$

$$\Rightarrow h(t)=e^{-2t}u(t)-e^{-4t}u(t)$$

$$(b)\ x(t)=te^{-2t}u(t)\xleftrightarrow{F}\frac{1}{(j\omega+2)^2}=X(j\omega)$$

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{2}{(j\omega+2)(j\omega+4)}\frac{1}{(j\omega+2)^2}$$

$$=\frac{A}{(j\omega+2)^3}+\frac{B}{(j\omega+2)^2}+\frac{C}{j\omega+2}+\frac{D}{j\omega+4}$$

$$=\frac{1}{(j\omega+2)^3}+\frac{-1/2}{(j\omega+2)^2}+\frac{1/4}{j\omega+2}+\frac{-1/4}{j\omega+4}$$

$$\Rightarrow y(t)=\frac{t^2}{2}e^{-2t}u(t)-\frac{1}{2}te^{-2t}u(t)+\frac{1}{4}e^{-2t}u(t)-\frac{1}{4}e^{-4t}u(t)$$

$$A=Y(j\omega)(j\omega+2)^3\Big|_{j\omega=-2}=\frac{2}{j\omega+4}\Big|_{j\omega=-2}=1$$

$$B=\frac{d}{d(j\omega)}\left[Y(j\omega)(j\omega+2)^3\right]\Big|_{j\omega=-2}=-\frac{2}{(j\omega+4)^2}\Big|_{j\omega=-2}=-\frac{1}{2}$$

$$C=\frac{1}{2}\frac{d^2}{d(j\omega)^2}\left[Y(j\omega)(j\omega+2)^3\right]\Big|_{j\omega=-2}=\frac{1}{2}(-2)\frac{-2}{(j\omega+4)^3}\Big|_{j\omega=-2}=\frac{1}{4}$$

$$D=Y(j\omega)(j\omega+4)\Big|_{j\omega=-4}=\frac{2}{(j\omega+2)^3}\Big|_{j\omega=-4}=-\frac{1}{4}$$

$$te^{-2t}u(t)\xleftrightarrow{F}\frac{1}{(j\omega+2)^2},\qquad \frac{t^2}{2}e^{-2t}u(t)\xleftrightarrow{F}\frac{1}{(j\omega+2)^3}$$

补充：

$$e^{-at}\cos\omega_0 t\ u(t) \xleftrightarrow{F} \frac{a+j\omega}{(a+j\omega)^2+\omega_0^{\,2}},\quad e^{-at}\sin\omega_0 t\ u(t) \xleftrightarrow{F} \frac{\omega_0}{(a+j\omega)^2+\omega_0^{\,2}}$$

$$\because e^{-at}\cos\omega_0 t\ u(t) = e^{-at}\frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})u(t) = \frac{1}{2}e^{-(a-j\omega_0)t}u(t)+\frac{1}{2}e^{-(a+j\omega_0)t}u(t)$$

$$\Rightarrow F\{e^{-at}\cos\omega_0 t\ u(t)\} = \frac{1}{2}F\{e^{-(a-j\omega_0)t}u(t)\}+\frac{1}{2}F\{e^{-(a+j\omega_0)t}u(t)\}$$

$$= \frac{1}{2}\left[\frac{1}{(a-j\omega_0)+j\omega}+\frac{1}{(a+j\omega_0)+j\omega}\right] = \frac{1}{2}\left[\frac{1}{(a+j\omega)-j\omega_0}+\frac{1}{(a+j\omega)+j\omega_0}\right]$$

$$= \frac{a+j\omega}{(a+j\omega)^2+\omega_0^{\,2}}$$

$$F\{e^{-at}\sin\omega_0 t\ u(t)\} = \frac{1}{2j}F\{e^{-(a-j\omega_0)t}u(t)\}-\frac{1}{2j}F\{e^{-(a+j\omega_0)t}u(t)\}$$

$$= \frac{1}{2j}\left[\frac{1}{(a-j\omega_0)+j\omega}-\frac{1}{(a+j\omega_0)+j\omega}\right] = \frac{1}{2j}\left[\frac{1}{(a+j\omega)-j\omega_0}-\frac{1}{(a+j\omega)+j\omega_0}\right]$$

$$= \frac{\omega_0}{(a+j\omega)^2+\omega_0^{\,2}}$$

$$e^{-at}\cos\omega_0 t\ u(t) \xleftrightarrow{F} \frac{a+j\omega}{(a+j\omega)^2+\omega_0{}^2},\quad e^{-at}\sin\omega_0 t\ u(t) \xleftrightarrow{F} \frac{\omega_0}{(a+j\omega)^2+\omega_0{}^2}$$

$$(c)\quad (j\omega)^2 Y(j\omega)+\sqrt{2}\,j\omega Y(j\omega)+Y(j\omega)=2(j\omega)^2 X(j\omega)-2X(j\omega)$$

$$\Rightarrow H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{2(j\omega)^2-2}{(j\omega)^2+\sqrt{2}\,j\omega+1}=2+\frac{-2\sqrt{2}\,j\omega-4}{(j\omega)^2+\sqrt{2}\,j\omega+1}=2+\frac{-2\sqrt{2}\,j\omega-4}{(j\omega)^2+\sqrt{2}\,j\omega+(\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}$$

$$=2+\frac{-2\sqrt{2}(j\omega+\frac{\sqrt{2}}{2})-2}{(j\omega+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}=2-2\sqrt{2}\frac{j\omega+\frac{\sqrt{2}}{2}}{(j\omega+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}-2\sqrt{2}\frac{\frac{\sqrt{2}}{2}}{(j\omega+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}$$

$$\Rightarrow h(t)=2\delta(t)-2\sqrt{2}e^{-\frac{\sqrt{2}}{2}t}\cos\frac{\sqrt{2}}{2}t\ u(t)-2\sqrt{2}e^{-\frac{\sqrt{2}}{2}t}\sin\frac{\sqrt{2}}{2}t\ u(t)$$

$$=2\delta(t)-2\sqrt{2}e^{-\frac{\sqrt{2}}{2}t}\left(\cos\frac{\sqrt{2}}{2}t+\sin\frac{\sqrt{2}}{2}t\right)u(t)$$

$$(c)' \quad (j\omega)^2 Y(j\omega) + \sqrt{2}\, j\omega Y(j\omega) + Y(j\omega) = 2(j\omega)^2 X(j\omega) - 2X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2(j\omega)^2 - 2}{(j\omega)^2 + \sqrt{2}\, j\omega + 1} = 2 + \frac{-2\sqrt{2}\, j\omega - 4}{(j\omega)^2 + \sqrt{2}\, j\omega + 1} = 2 + \frac{-2\sqrt{2}\, j\omega - 4}{(j\omega)^2 + \sqrt{2}\, j\omega + (\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2}$$

$$= 2 + \frac{-2\sqrt{2}\, j\omega - 4}{(j\omega + \frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2} = 2 + \frac{-2\sqrt{2}\, j\omega - 4}{(j\omega + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j)(j\omega + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j)}$$

$$= 2 + \frac{A}{j\omega + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j} + \frac{B}{j\omega + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j}$$

$$A = \left.\frac{-2\sqrt{2}\, j\omega - 4}{j\omega + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j}\right|_{j\omega = -(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j)} = -\sqrt{2} - \sqrt{2} j, \quad B = \left.\frac{-2\sqrt{2}\, j\omega - 4}{j\omega + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j}\right|_{j\omega = -(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j)} = -\sqrt{2} + \sqrt{2} j$$

$$\Rightarrow h(t) = 2\delta(t) - \sqrt{2}(1 + j) e^{-(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} j)t} u(t) - \sqrt{2}(1 - j) e^{-(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j)t} u(t)$$

$$= 2\delta(t) - \sqrt{2}(1 + j) e^{\frac{-(1+j)}{\sqrt{2}} t} u(t) - \sqrt{2}(1 - j) e^{\frac{-(1-j)}{\sqrt{2}} t} u(t)$$

2. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{dy(t)}{dt}+10y(t)=\int_{-\infty}^{+\infty} x(\tau)z(t-\tau)d\tau - x(t)$$

where $z(t)=e^{-t}u(t)+3\delta(t)$.

(a) Find the frequency response $H(j\omega)=Y(j\omega)/X(j\omega)$ of this system.

(b) Determine the impulse response of the system.

Solution:

$$\frac{dy(t)}{dt}+10y(t)=\int_{-\infty}^{+\infty}x(\tau)z(t-\tau)d\tau-x(t)\quad \text{where}\quad z(t)=e^{-t}u(t)+3\delta(t).$$

$$Z(j\omega)=\frac{1}{1+j\omega}+3,\qquad \frac{dy(t)}{dt}+10y(t)=x(t)*z(t)-x(t)$$

$$\Rightarrow j\omega Y(j\omega)+10Y(j\omega)=X(j\omega)Z(j\omega)-X(j\omega)$$

$$\Rightarrow H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{Z(j\omega)-1}{j\omega+10}=\frac{\frac{1}{1+j\omega}+2}{j\omega+10}=\frac{2j\omega+3}{(1+j\omega)(j\omega+10)}=\frac{A}{j\omega+1}+\frac{B}{j\omega+10}$$

$$A=\left.\frac{2j\omega+3}{j\omega+10}\right|_{j\omega=-1}=\frac{1}{9},\qquad B=\left.\frac{2j\omega+3}{j\omega+1}\right|_{j\omega=-10}=\frac{17}{9},\qquad \cdots\cdots\cdots\cdots\cdots\cdots(a)$$

$$\Rightarrow H(j\omega)=\frac{1/9}{j\omega+1}+\frac{17/9}{j\omega+10}$$

$$\Rightarrow h(t)=\frac{1}{9}e^{-t}u(t)+\frac{17}{9}e^{-10t}u(t)\qquad \cdots\cdots\cdots\cdots\cdots\cdots(b)$$

3. Compute the Fourier transform of the signal $x[n] = 2^n \sin(\frac{\pi}{4} n) u[-n]$.

4. $X(e^{j\omega})$ is the Fourier transform of discrete-time signal $x[n]$, determine the signal $x[n]$ corresponding to the following transform

$$X(e^{j\omega}) = \frac{1 - \frac{1}{3} e^{-j\omega}}{1 - \frac{1}{4} e^{-j\omega} - \frac{1}{8} e^{-2j\omega}}$$

3. Compute the Fourier transform of the signal $x[n]=2^n\sin(\frac{\pi}{4}n)u[-n]$.

Solution:

$$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}=\sum_{n=-\infty}^{0}2^n\sin(\frac{\pi}{4}n)e^{-j\omega n}=\sum_{n=0}^{+\infty}(\frac{1}{2})^n\sin(-\frac{\pi}{4}n)e^{j\omega n}$$

$$=\sum_{n=0}^{+\infty}(\frac{1}{2})^n\frac{1}{2j}(e^{-j\frac{\pi}{4}n}-e^{j\frac{\pi}{4}n})e^{j\omega n}=\frac{-1}{2j}\left[\sum_{n=0}^{+\infty}(\frac{1}{2}e^{j\frac{\pi}{4}})^n e^{j\omega n}-\sum_{n=0}^{+\infty}(\frac{1}{2}e^{-j\frac{\pi}{4}})^n e^{j\omega n}\right]$$

$$=\frac{-1}{2j}\left[\frac{1}{1-\frac{1}{2}e^{j\frac{\pi}{4}}e^{j\omega}}-\frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{4}}e^{j\omega}}\right]=-\frac{1}{2}\frac{(\sin\frac{\pi}{4})e^{j\omega}}{1-(\cos\frac{\pi}{4})e^{j\omega}+\frac{1}{4}e^{j2\omega}}$$

$$=-\frac{1}{2}\frac{\frac{\sqrt{2}}{2}e^{j\omega}}{1-\frac{\sqrt{2}}{2}e^{j\omega}+\frac{1}{4}e^{j2\omega}}=\frac{-\sqrt{2}e^{j\omega}}{4-2\sqrt{2}e^{j\omega}+e^{j2\omega}}$$

4. Solution:

$$X(e^{j\omega}) = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}} = \frac{1-\frac{1}{3}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{4}e^{-j\omega}\right)} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1+\frac{1}{4}e^{-j\omega}}$$

$$A = X(e^{j\omega})\left(1-\frac{1}{2}e^{-j\omega}\right)\bigg|_{e^{-j\omega}=2} = \frac{2}{9},$$

$$B = X(e^{j\omega})\left(1+\frac{1}{4}e^{-j\omega}\right)\bigg|_{e^{-j\omega}=-4} = \frac{7}{9},$$

$$\Rightarrow \quad x[n] = \frac{2}{9}(\frac{1}{2})^2 u[n] + \frac{7}{9}(-\frac{1}{4})^2 u[n]$$

5. Use the Fourier transform synthesis equation to determine the inverse Fourier transform of $X(e^{j\omega})=|X(e^{j\omega})|e^{j\measuredangle X(e^{j\omega})}$, where

$$|X(e^{j\omega})|=\begin{cases}1, & 0\le|\omega|<\dfrac{\pi}{4}\\[2ex] 0, & \dfrac{\pi}{4}\le|\omega|<\pi\end{cases}\quad \text{and}\quad \measuredangle X(e^{j\omega})=-\frac{3\omega}{2}.$$

Use your answer to determine the values of n for which $x[n]=0$.

作答

5. Solution:

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|e^{j\measuredangle X(e^{j\omega})}e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}e^{-j\frac{3}{2}\omega}e^{j\omega n}d\omega=\frac{1}{2\pi}\frac{1}{(-\frac{3}{2}+n)j}e^{(-\frac{3}{2}+n)j\omega}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$=\frac{\sin[(n-\frac{3}{2})\frac{\pi}{4}]}{\pi(n-\frac{3}{2})}$$

(contradictory)

$$\sin k\pi=0 \;\Rightarrow\; (n-\frac{3}{2})\frac{1}{4}=k\ \text{(integer)} \;\Rightarrow\; \text{(integer)}\ \boxed{n=4k+\frac{3}{2}}\ \text{(Non integer)}$$

$$\Rightarrow\ \text{Therefore, } x[n]=0 \text{ only for } n=\pm\infty.$$

6. The following four facts are given about a real signal $x[n]$ with Fourier transform $X(e^{j\omega})$:

(1) $x[n]=0$ for $n>0$;

(2) $x[0]>0$;

(3) $\operatorname{Im}\left\{X(e^{j\omega})\right\}=\sin\omega-\sin 2\omega$;

(4) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega=3.$

Determine $x[n]$.

作答

Solution 1: $x[n]$ is real, $x[n]=x_e[n]+x_o[n]$,

$$\Rightarrow\ x_e[n]\xleftrightarrow{F}\mathrm{Re}\{X(j\omega)\},\quad x_o[n]\xleftrightarrow{F} j\,\mathrm{Im}\{X(j\omega)\},$$

$$j\,\mathrm{Im}\{X(e^{j\omega})\}=j(\sin\omega-\sin 2\omega)=\frac{1}{2}(e^{j\omega}-e^{-j\omega}-e^{j2\omega}+e^{-j2\omega})$$

$$\because\ \delta[n]\xleftrightarrow{F}1,\quad x[n-n_0]\xleftrightarrow{F}X(e^{j\omega})e^{-j\omega n_0}$$

$$\therefore\ \delta[n-1]\xleftrightarrow{F}e^{-j\omega},\ \delta[n+1]\xleftrightarrow{F}e^{j\omega},\ \delta[n-2]\xleftrightarrow{F}e^{-j2\omega},\ \delta[n+2]\xleftrightarrow{F}e^{j2\omega}$$

$$\Rightarrow\ x_o[n]=\frac{1}{2}\left(\delta[n+1]-\delta[n-1]-\delta[n+2]+\delta[n-2]\right)$$

$$x_o[n]=\frac{x[n]-x[-n]}{2}\ \Rightarrow\ x[n]=2x_o[n]+x[-n]\ \overset{\because x[n]=0\text{ for }n>0}{\Rightarrow}\ x[n]=2x_o[n]\text{ for }n<0$$

$$\Rightarrow\ x[n]=k\delta[n]+\delta[n+1]-\delta[n+2]$$

$$\because\ \sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2d\omega=3$$

$$\therefore k^2+1+1=3\ \Rightarrow\ k^2=1\ \because x[0]=k>0\ \Rightarrow\ k=1.$$

Solution 2: $x[n]$ is real, $x[n]=x_e[n]+x_o[n]$,

$$\Rightarrow x_e[n] \xleftrightarrow{F} \mathrm{Re}\{X(j\omega)\},\quad x_o[n] \xleftrightarrow{F} j\,\mathrm{Im}\{X(j\omega)\},$$

$$j\,\mathrm{Im}\{X(e^{j\omega})\} = j(\sin\omega - \sin 2\omega) = \frac{1}{2}(e^{j\omega} - e^{-j\omega} - e^{j2\omega} + e^{-j2\omega})$$

$$\because\ \delta[n] \xleftrightarrow{F} 1,\quad x[n-n_0] \xleftrightarrow{F} X(e^{j\omega})e^{-j\omega n_0}$$

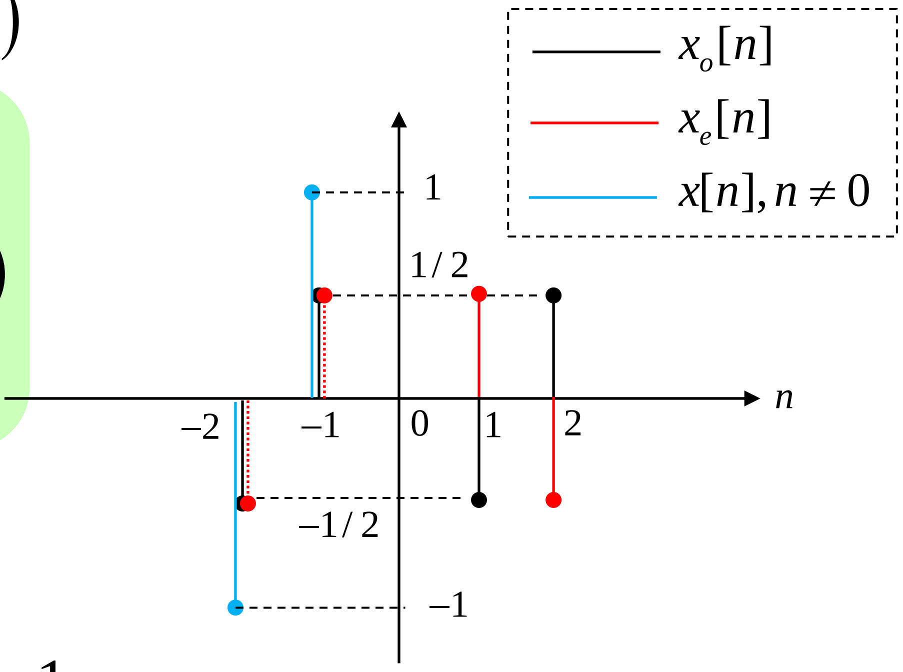
$$\therefore\ \delta[n-1] \xleftrightarrow{F} e^{-j\omega},\ \delta[n+1] \xleftrightarrow{F} e^{j\omega},\ \delta[n-2] \xleftrightarrow{F} e^{-j2\omega},\ \delta[n+2] \xleftrightarrow{F} e^{j2\omega}$$

$$\Rightarrow x_o[n] = \frac{1}{2}\left(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2]\right)$$

$$\because\ x[n]=0 \ \text{ for } \ n>0$$

$$\therefore\ x_e[n] = \frac{1}{2}\left(\delta[n+1] + \delta[n-1] - \delta[n+2] - \delta[n-2]\right)$$

$$\Rightarrow x[n] = x_o[n] + x_e[n] = k\delta[n] + \delta[n+1] - \delta[n+2]$$



$$\because\ \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 3$$

$$\therefore\ k^2 + 1 + 1 = 3 \ \Rightarrow\ k^2 = 1 \ \because\ x[0] = k > 0 \ \Rightarrow\ k = 1.$$

主观题 10分

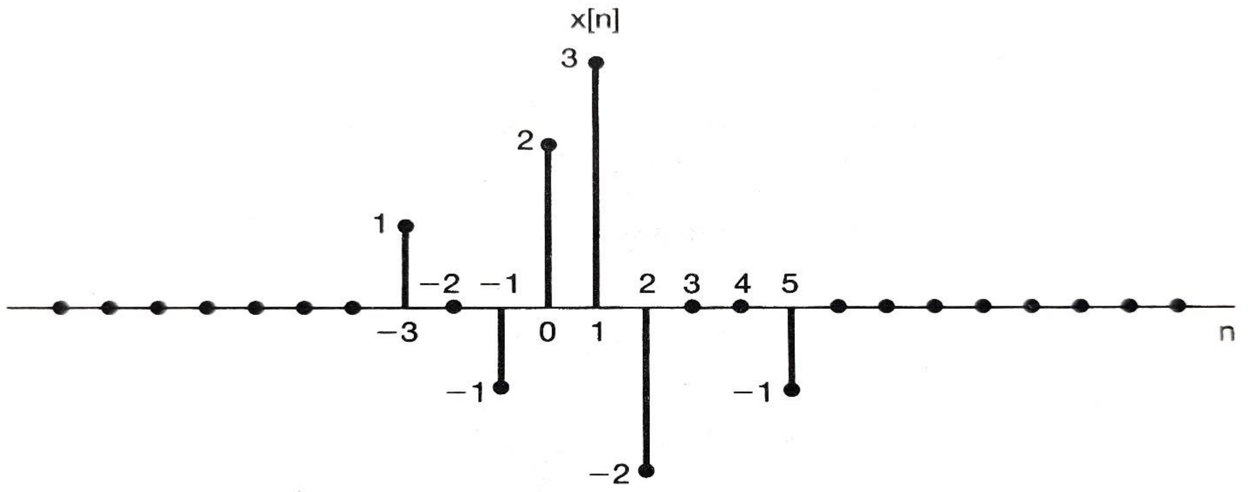
设置

7. Consider the signal depicted in the following figure. Let the Fourier transform of this signal be written in rectangular form as

$$X(e^{j\omega}) = A(\omega) + jB(\omega).$$

Sketch the function of time corresponding to the transform

$$Y(e^{j\omega}) = B(\omega) + A(\omega)e^{j\omega}.$$

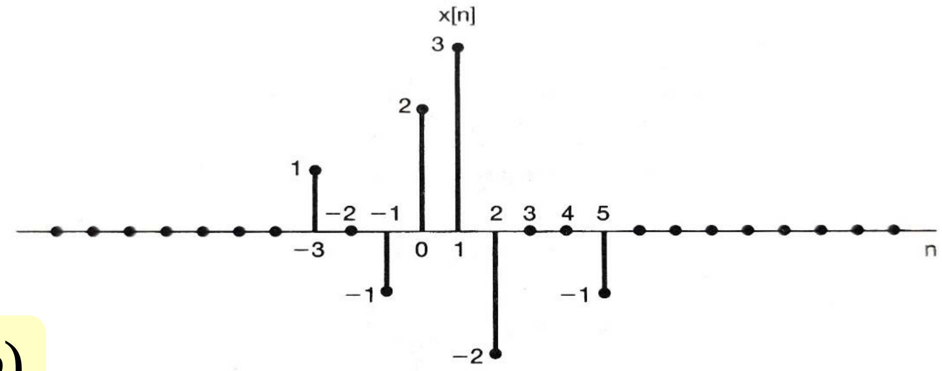


作答

Solution:

$$x_e[n] \overset{F}{\longleftrightarrow} A(\omega) \qquad x_o[n] \overset{F}{\longleftrightarrow} jB(\omega)$$

$$x_e[n+1] \overset{F}{\longleftrightarrow} A(\omega)e^{j\omega} \qquad -jx_o[n] \overset{F}{\longleftrightarrow} B(\omega)$$



$$\Rightarrow y[n] = x_e[n+1] - jx_0[n] \overset{F}{\longleftrightarrow} A(\omega)e^{j\omega} + B(\omega)$$

$$x[n] = [1,\ 0, -1, \overset{\downarrow}{2},\ 3, -2, 0, 0, -1]$$

$$x[-n] = [-1, 0, 0, -2,\ 3, \overset{\downarrow}{2}, -1,\ 0, 1]$$

$$\Rightarrow x_e[n] = \frac{1}{2}[-1, 0, 1, -2, 2, \overset{\downarrow}{4}, 2, -2, 1, 0, -1]$$

$$\Rightarrow x_o[n] = \frac{1}{2}[1, 0, 1, 2, -4, \overset{\downarrow}{0}, 4, -2 -1, 0, -1]$$

$$x_e[n+1] = [-\frac{1}{2}, 0, \frac{1}{2}, -1, 1, 2, \overset{\downarrow}{1}, -1, \frac{1}{2}, 0, -\frac{1}{2}] = \mathrm{Re}\{y[n]\}$$

$$-x_o[n] = [-\frac{1}{2}, 0, -\frac{1}{2}, -1, 2, \overset{\downarrow}{0}, -2, 1, \frac{1}{2}, 0, \frac{1}{2}] = \mathrm{Im}\{y[n]\}$$