Seat No. ______ Major/Class ______ School ______ Student ID ______ Name ______

(DO NOT WRITE YOUR ANSWER IN THIS AREA)

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WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

Probability and Statistics Exam paper B (2021–2022-2)

Notice:
1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam answer sheet .**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

Question No.	I	II	III	IV	V	VI	VII	VIII	IX	Sum
Score										

I. (10 points) We roll a die n times. Let A_{ij} be the event that the i th and j th rolls produce the same number. Show that the events $\{A_{ij} : 1 \le i < j \le n\}$ are pairwise independent but not independent.

Score

Solution.
Suppose $i < j$ and $m < n$. If $j < m$, then A_{ij} and A_{mn} are determined by distinct independent rolls, and are therefore independent. For the case $j = m$ we have that

$$P(A_{ij} \cap A_{jn}) = P(i \text{ th}, j \text{ th, and } n \text{ th rolls show same number}).$$

$$= \sum_{r=1}^{6} \frac{1}{6} P(j \text{ th and } n \text{ th rolls both show } r \mid i \text{ th shows } r) = \frac{1}{36} = P(A_{ij}) P(A_{jn})$$

as required. However, if $i \ne j \ne k$,

$$P(A_{ij} \cap A_{jk} \cap A_{ik}) = \frac{1}{36} \ne \frac{1}{216} = P(A_{ij}) P(A_{jk}) P(A_{ik}).$$

II. (10 points) Two fair dice are rolled.

(a) Compute the probability of the two dice have different scores.

(b) Show that the event that their sum is 7 is independent of the score shown by the first die.

Score

Solution.
(a) The probability of the two dice have different scores is $\frac{6 \cdot 5}{6^2} = \frac{5}{6}$.

(b) It is implied by

$$P(1 \text{ st shows } r \text{ and sum is } 7) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(1 \text{ st shows } r) P(\text{ sum is } 7).$$

III. (10 points) Individuals **A** and **B** begin to play a sequence of chess games. Let $S = \{\text{A wins a game}\}$, and suppose that outcomes of successive games are independent with $P(S) = p$ and $P(F) = 1 - p$ (they never draw). They will play until one of them wins ten games. Let $X =$ the numher of games played (with possible values $10, 11, \cdots, 19$).

Score

(a) For $x = 10, 11, \ldots, 19$, obtain an expression for $p(x) = P(X = x)$

(b) If a draw is possible, with $p = P(S), q = P(F), 1 - p - q = P(\text{draw})$, what is $P(20 \leq X)$?

Solution.

(a) $P(X = x) = P(\text{ A wins in } x \text{ games }) + P(\text{ B wins in } x \text{ games })$

$=P\left(9S'\text{s in } 1^{\text{st}}\, x-1 \cap S \text{ on the } x^{\text{th}}\right) + P\left(9F' \text{ s in } 1^{\text{st}}\, x-1 \cap F \text{ on the } x^{\text{th}}\right)$

$$= \binom{x-1}{9} p^9 (1-p)^{(x-1)-9} \cdot p + \binom{x-1}{9} (1-p)^9 p^{(x-1)-9} \cdot (1-p)$$

$$= \binom{x-1}{9} \left[p^{10}(1-p)^{x-10} + (1-p)^{10} p^{x-10}\right].$$

(b) Possible values of X are now all positive integers $\geq 10 : 10, 11, 12, \cdots$.

Similar to case (a), we have

$P(X = x) = P(\text{ A wins in } x \text{ games }) + P(\text{ B wins in } x \text{ games })$

$= P\left(9S's \text{ in } 1^{\text{st}}\, x-1 \cap S \text{ on the } x^{\text{th}}\right) + P\left(9F' \text{ s in } 1^{\text{st}}\, x-1 \cap F \text{ on the } x^{\text{th}}\right)$

$$= \binom{x-1}{9} p^9 (1-p)^{(x-1)-9} \cdot p + \binom{x-1}{9} q^9 (1-q)^{(x-1)-9} \cdot q$$

$$= \binom{x-1}{9} \left[p^{10}(1-p)^{x-10} + q^{10}(1-q)^{x-10}\right]. \text{ Finally,}$$

$$P(X \geq 20) = 1 - P(X < 20) = 1 - \sum_{x=10}^{19} \binom{x-1}{9} \left[p^{10}(1-p)^{x-10} + q^{10}(1-q)^{x-10}\right].$$

IV. (10 points) A 12-in. bar that is clamped at both ends is to be subjected to an increasing amount of stress until it snaps. Let $Y =$ the distance from the left end at which the break occurs. Suppose Y has pdf

Score

$$f(y) = \begin{cases} \left(\frac{1}{24}\right) y \left(1 - \frac{y}{12}\right) & 0 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

Compute the following:

(a) The cdf of Y.

(b) The expected length of the shorter segment $\min(Y, 12 - Y)$ when the break occurs.

Solution.

(a) For $0 \leq y \leq 12$, $F(y) = \frac{1}{24} \int_0^y \left(u - \frac{u^2}{12}\right) du = \frac{1}{24} \left(\frac{u^2}{2} - \frac{u^3}{36}\right)\Big|_0^y = \frac{y^2}{48} - \frac{y^3}{864}$.

(b) The shorter segment has length equal to $\min(Y, 12 - Y)$, and

$$E[\min(Y, 12 - Y)] = \int_0^{12} \min(y, 12-y) \cdot f(y) dy = \int_0^6 \min(y, 12-y) \cdot f(y) dy$$

$$+ \int_6^{12} \min(y, 12-y) \cdot f(y) dy = \int_0^6 y \cdot f(y) dy + \int_6^{12} (12-y) \cdot f(y) dy = \frac{90}{24} = 3.75 \text{ inches.}$$

V. (20 points) Suppose that X and Y are two independent rv' s, both of which has uniform distribution in $(0, 2)$.

Score

(a) Determine the joint pdf of X and Y.
(b) Compute the probability $P(X + Y \leq 1)$.
(c) Compute the probability $P(X \leq Y)$.
(d) Compute $V(X - Y)$.

Solution.
(a) Since X and Y are independent, their joint pdf is

$$f(x, y) = \begin{cases} f_X(x)f_Y(y) = \frac{1}{4}, & 0 \leq x \leq 2,\ 0 \leq y \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

(b) $P(X + Y \leq 1) = \int_0^1 \left[\int_0^{1-x} \frac{1}{4} dy\right] dx = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.
(c) $P(X \leq Y) = \int_0^2 \left[\int_x^2 \frac{1}{4} dy\right] dx = \frac{1}{4} \times 2 = \frac{1}{2}$.
(d) $V(X - Y) = V(X) + V(Y) = 2V(X) = 2 \times \frac{2^2}{12} = \frac{2}{3}$.

VI. (10 points) Suppose the expected tensile strength of type-A steel is 100ksi and the standard deviation of tensile strength is 8ksi. For type-B steel, suppose the expected tensile strength is 95ksi and the standard deviation of tensile strength is 7ksi, respectively. Let $\overline{X}$ = the sample average tensile strength of a random sample of 40 type-A specimens, and let $\overline{Y}$ = the sample average tensile strength of a random sample of 35 type-B specimen. Use the Central Limit Theorem to answer the following questions.

Score

(a) What are the approximate distributions of $\overline{X}$ and $\overline{Y}$ respectively?
(b) Calculate $P(\overline{X} - \overline{Y} \geq 10)$. ($\Phi(1.65) = 0.95, \Phi(2.89) = 0.998, \Phi(1.96) = 0.975$.)

Solution.
(a) According to the CLT, $\overline{X}$ has approximately a normal distribution $N\left(100, \frac{8^2}{40}\right)$, i.e. $N(100, 1.6)$, $\overline{Y}$ has approximately a normal distribution $N\left(95, \frac{7^2}{35}\right)$, i.e. $N(95, 1.4)$.
(b) According to the CLT, $\overline{X} - \overline{Y}$ has approximately $N(5, 3)$. $P(\overline{X} - \overline{Y} \geq 10) = 1 - P(\overline{X} - \overline{Y} < 10)$
$\approx 1 - \Phi\left(\frac{10-5}{\sqrt{3}}\right) = 1 - \Phi\left(\frac{5}{\sqrt{3}}\right) = 1 - \Phi(2.89) = 1 - 0.998 = 0.002$.

VII. (10 points) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the pdf

Score

$$f(x; \theta) = \begin{cases} \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Use the method of moments to find an estimator for θ.
(b) Find the maximum likelihood estimator for θ.

Solution.

(a)

$$E(X)=\int_{-\infty}^{+\infty}xf(x,\theta)dx=\int_{0}^{+\infty}\frac{x^2}{\theta}e^{-\frac{x^2}{2\theta}}dx$$
$$=\int_{0}^{+\infty}xe^{-\frac{x^2}{2\theta}}d\frac{x^2}{2\theta}=x\left(-e^{-\frac{x^2}{2\theta}}\right)\Big|_0^{+\infty}+\int_{0}^{+\infty}e^{-\frac{x^2}{2\theta}}dx$$
$$=0+\frac{1}{2}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2\theta}}dx=\sqrt{\frac{\theta\cdot\pi}{2}}$$
$$\therefore\theta=\frac{2E^2(x)}{\pi}.$$

(or)

$$E(X)=\int_{-\infty}^{+\infty}xf(x,\theta)dx=\int_{0}^{+\infty}\frac{x^2}{\theta}e^{-\frac{x^2}{2\theta}}dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2}{\theta}e^{-\frac{x^2}{2\theta}}dx$$
$$=\frac{\sqrt{2\pi}}{2\sqrt{\theta}}\int_{-\infty}^{+\infty}\frac{x^2}{\sqrt{2\pi}\sqrt{\theta}}e^{-\frac{x^2}{2\theta}}dx=\sqrt{\frac{\pi}{2\theta}}\cdot V(Y),\ (Y\sim N(0,\theta))$$
$$=\sqrt{\frac{\pi}{2\theta}}\cdot(\sqrt{\theta})^2=\sqrt{\frac{\theta\cdot\pi}{2}}$$
$$\therefore\theta=\frac{2E^2(x)}{\pi}.$$

Thus, the moment estimator is $\hat{\theta}_M=\frac{2}{\pi}\overline{X}^2$.

(or using 2rd moment)

$$E(X^2)=\int_{-\infty}^{+\infty}x^2f(x,\theta)dx=\int_{0}^{+\infty}\frac{x^3}{\theta}e^{-\frac{x^2}{2\theta}}dx$$
$$=\int_{0}^{+\infty}x^2e^{-\frac{x^2}{2\theta}}d\frac{x^2}{2\theta}=2\theta\cdot\int_{0}^{+\infty}\frac{x^2}{2\theta}e^{-\frac{x^2}{2\theta}}d(\frac{x^2}{2\theta})$$
$$=2\theta\cdot\int_{0}^{+\infty}ue^{-u}du=2\theta$$
$$\therefore\theta=\frac{E(X^2)}{2}$$

The moment estimator is $\hat{\theta}_M=\frac{1}{2n}\sum_{i=1}^{n}X_i^2$.

(b) The likelihood function is

$$L(\theta)=L(x_1,\cdots,x_n,\theta)=\prod_{i=1}^{n}\frac{x_i}{\theta}\cdot e^{-\sum_{i=1}^{n}\frac{x_i^2}{2\theta}}\quad(x_i>0,i=1,\cdots,n).$$

The ln(likelihood) is

$$\ln L(\theta)=\sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right)-\sum_{i=1}^{n}\frac{x_i^2}{2\theta}=\sum_{i=1}^{n}\ln x_i-n\ln\theta-\sum_{i=1}^{n}\frac{x_i^2}{2\theta}\quad(x_i>0,i=1,\cdots,n).$$

Let

$$\frac{d\ln L(x_1,\cdots,x_n,\theta)}{d\theta}=\frac{-n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}\frac{x_i^2}{2}=0.$$

We get $\theta=\frac{1}{2n}\sum_{i=1}^{n}x_i^2$. So the maximum likelihood estimator of θ is

$$\hat{\theta}_L=\frac{1}{2n}\sum_{i=1}^{n}X_i^2.$$

Score

VIII. (10 points) It is reported that for a sample of 49 kitchens with gas cooking appliances monitored during a one-week period, the sample mean CO_2 level (ppm) was 654. Suppose that the population of CO_2 level of all homes is normal.

(a) Calculate a 95% confidence interval for true average CO_2 level with the sample standard deviation $s = 168$. ($t_{0.05,48} = 1.68, t_{0.025,48} = 2.0$.)

(b) Suppose that $\sigma = 175$. What sample size would be necessary to obtain an interval width of at most 50ppm for a confidence level of 95% ? ($z_{0.05} = 1.65, z_{0.025} = 1.96$.)

Solution.

(a) With $n = 49, \overline{x} = 654$ and $t_{\alpha/2,n-1} = t_{0.025,48} = 2.0$, the 95% confidence interval for μ is

$$\overline{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} = 654 \pm 2.0\frac{168}{7} = 654 \pm 48 = (606, 702).$$

(b) With $\sigma = 175$ and $\mu \in \overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$, the width of CI is $w = 2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. So

$$n = \left(\frac{2z_{\alpha/2}\sigma}{w}\right)^2 = \left(\frac{2(1.96)(175)}{50}\right)^2 = (13.72)^2 = 188.24,$$

which rounds up to 189.

Score

IX. (10 points) The desired percentage of SiO_2 in a certain type of aluminous cement is 5.5. In a test 16 independently obtained samples are analyzed. Suppose that the percentage of SiO_2 is normally distributed with $\sigma = 0.3$ and that $\bar{x} = 5.25$.

(a) Does this indicate conclusively that the true average percentage less than 5.5? Consider a significance level of $\alpha = 0.01$. ($z_{0.01} = 2.33, z_{0.005} = 2.58$.)

(b) If the true average percentage is $\mu = 5.3$ and a level $\alpha = 0.01$ test based on $n = 16$ is used, what is the probability of rejecting H_0? ($\Phi(0.34) = 0.63, \Phi(0.64) = 0.74$.)

Solution.

The hypotheses are $H_0 : \mu = 5.5$ vs $H_a : \mu < 5.5$. The sample mean is $\bar{x} = 5.25$.

(a) $z = \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{5.25-5.5}{0.3/\sqrt{16}} = -3.33 \le -z_\alpha = -z_{0.01} = -2.33$. Reject H_0.

(b) The probability of making a type II error when $\mu = 5.3$ is

$$\beta(5.3) = 1 - \Phi\left(-z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(-2.33 + \frac{5.5 - 5.3}{0.3/\sqrt{16}}\right) = 1 - \Phi(0.34).$$

So the probability of rejecting H_0 is $1 - \beta(5.3) = \Phi(0.34) = 0.63$.