Seat NO.________ Major________ Student ID □□□□□□□□□□□□ Chinese Name________

（DONOT WRITE YOUR ANSWER IN THIS AREA）

Seal Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

2022-2023-2《Calculus II》 Exam Paper A

Notice:
1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

Question No.	1	2	3	4	5	6	7	8	Sum
Score									

Score

1. Answer the following questions (30 points):

(1) Classify the following series as absolutely convergent, conditionally convergent or divergent: $\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n)}{n\sqrt{n}}$.

Solution:

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{sin(n)}{n\sqrt{n}}\right|=\sum_{n=1}^{\infty}\frac{|sin(n)|}{n\sqrt{n}}.$$ (3 points)

$$\frac{|sin(n)|}{n\sqrt{n}}\le\frac{1}{n\sqrt{n}}=\frac{1}{n^{\frac{3}{2}}}.$$ (6 points)

Thus, the series is absolutely convergent.

(2) Find the distance between the parallel planes $2x-3y+\sqrt{3}z=4$ and $2x-3y+\sqrt{3}z=9$.

Solution: Distance between the parallel planes:

$$d=\frac{|4-9|}{\sqrt{2^2+(-3)^2+\sqrt{3}^2}}=\frac{5}{4}.$$ (6 points)

(3) Let G be the spherical surface $x^2+y^2+z^2=a^2$. Evaluate the following surface integral $\iint_G\frac{x+y^3+\sin z}{1+z^4}dS$.

Solution:

$$\iint_G \frac{x+y^3+\sin z}{1+z^4}\mathrm{d}S$$

(1 points)

$$=\iint_G \frac{x}{1+z^4}\mathrm{d}S+\iint_G \frac{y^3}{1+z^4}\mathrm{d}S+\iint_G \frac{\sin z}{1+z^4}\mathrm{d}S=0.$$

For G is symmetric about the three coordinate planes, and set:

$$f(x,z)=\frac{x}{1+z^4},\ g(y,z)=\frac{y^3}{1+z^4},\ h(z)=\frac{sinz}{1+z^4}.$$

(4 points)

$f(x,z)$ is odd w.r.t. x, $g(y,z)$ is odd w.r.t. y, $h(z)$ is odd w.r.t. z.

Thus $\iint_G \frac{x+y^3+\sin z}{1+z^4}\mathrm{d}S=0.$ (6 points)

(4) Change the order of integration of $\int_{\frac{1}{2}}^{1}[\int_{x^3}^{x} f(x,y)\mathrm{d}y]\mathrm{d}x.$

Solution:

The integration region:

$$R=\left\{(x,y):x^3\le y\le x,\quad \frac{1}{2}\le x\le 1\right\}.$$

(2 points)

Changing the order, the integral is:

$$\int_{\frac{1}{2}}^{1}\left[\int_{x^3}^{x} f(x,y)dy\right]dx=\int_{\frac{1}{8}}^{\frac{1}{2}}\left[\int_{\frac{1}{2}}^{y^{\frac{1}{3}}} f(x,y)dx\right]dy+\int_{\frac{1}{2}}^{1}\left[\int_{y}^{y^{\frac{1}{3}}} f(x,y)dx\right]dy.$$

(6 points)

(5) Find $\frac{\partial z}{\partial x}$, if equation $3x^2z+y^3-xyz^3=0$ defines an implicit function $z=f(x,y)$.

Solution:

Method 1

Differentiate both sides of the above equation with respect to variable x:

$$6xz+3x^2\frac{\partial z}{\partial x}-yz^3-3xyz^2\frac{\partial z}{\partial x}=0.$$

(3 points)

Thus,

$$\frac{\partial z}{\partial x}=\frac{6xz-yz^3}{3xyz^2-3x^2}.$$

(6 points)

Method 2

Set:

$$F(x,y,z)=3x^2z+y^3-xyz^3,$$

$$\frac{\partial F(x,y,z)}{\partial x}=6xz-yz^3,\frac{\partial F(x,y,z)}{\partial z}=3x^2-3xyz^2$$ (4 points)

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=\frac{6xz-yz^3}{3xyz^2-3x^2}$$ (6 points)

2. Evaluate the following problems (30 points):

Score

(1) Evaluate $\int_1^2\int_0^{\sqrt{2x-x^2}}\frac{1}{\sqrt{x^2+y^2}}dydx.$

Solution:

Integral region:

$$R:\left\{(x,y):1\le x\le 2,\quad 0\le y\le\sqrt{2x-x^2}\right\}.$$

Taking polar coordinates:

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases},\quad \theta\in[0,\frac{\pi}{4}].$$ (2 points)

The boundary of r is decided by:

$$r\cos\theta=1,\quad \therefore r=\frac{1}{\cos\theta};\quad r\sin\theta=\sqrt{2r\cos\theta-r^2\cos\theta}\quad \therefore r=2\cos\theta.$$

Thus:

$$\theta\in[0,\frac{\pi}{4}],\quad r\in[\frac{1}{\cos\theta},2\cos\theta].$$ (4 points)

$$\int_1^2\int_0^{\sqrt{2x-x^2}}\frac{1}{\sqrt{x^2+y^2}}dydx=\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\theta}}^{2\cos\theta}\frac{1}{r}\cdot rdrd\theta=\int_0^{\frac{\pi}{4}}\left(2\cos\theta-\frac{1}{\cos\theta}\right)d\theta.$$

$$=2\int_0^{\frac{\pi}{4}}\cos\theta d\theta-\int_0^{\frac{\pi}{4}}\frac{1}{\cos\theta}d\theta=I_1-I_2.$$

$$I_1=2\int_0^{\frac{\pi}{4}}\cos\theta d\theta=2sin\theta|_0^{\frac{\pi}{4}}=\sqrt{2}.$$ (5 points)

$$I_2=\int_0^{\frac{\pi}{4}}\frac{\cos\theta}{1-\sin^2\theta}d\theta=\int_0^{\frac{\pi}{4}}\frac{d\sin\theta}{1-\sin^2\theta}=\int_0^{\frac{\pi}{4}}\frac{1}{(1+\sin\theta)(1-\sin\theta)}d\sin\theta=\int_0^{\frac{\sqrt{2}}{2}}\frac{dt}{(1+t)(1-t)}.$$

$$\therefore I_2=\frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt=\frac{\ln(3+2\sqrt{2})}{2}=\ln(1+\sqrt{2}).$$

Therefore, the result is:

$$\int_1^2\int_0^{\sqrt{2x-x^2}}\frac{1}{\sqrt{x^2+y^2}}dydx=\sqrt{2}-\ln(1+\sqrt{2}).$$ (6 points)

(2) Find the convergence set for the power series $\sum_{n=1}^{\infty}\frac{(3x+1)^n}{n2^n}$.

Solution:

Set: $a_n=\frac{(3x+1)^n}{n2^n}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(3x+1)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(3x+1)^n}\right|=\lim_{n\to\infty}\left|\frac{(3x+1)}{2}\cdot\frac{n}{n+1}\right|=\left|\frac{3x+1}{2}\right|.$$ (1 points)

$$\left|\frac{3x+1}{2}\right|<1 \quad\Rightarrow -1<x<\frac{1}{3}.$$ (3 points)

When $x=-1$:

$$\sum_{n=1}^{\infty}\frac{(3x+1)^n}{n2^n}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n},\quad \text{convergent.}$$ (4 points)

When $x=\frac{1}{3}$:

$$\sum_{n=1}^{\infty}\frac{(3x+1)^n}{n2^n}=\sum_{n=1}^{\infty}\frac{1}{n},\quad \text{divergent.}$$ (5 points)

Thus, the convergence set is $[-1,\frac{1}{3})$. (6 points)

(3) Find the symmetric equation of the tangent line to the curve with equation $\vec{r}=2\cos t\vec{i}+6\sin t\vec{j}+t\vec{k}$, at $t=\frac{\pi}{3}$.

Solution:

$$\frac{d\vec{r}}{dt}=\frac{d}{dt}\left(2\cos t\vec{i}+6\sin t\vec{j}+t\vec{k}\right)=-2sint\vec{i}+6cost\vec{j}+\vec{k}.$$ (2 points)

The derivative at $t=\frac{\pi}{3}$:

$$\frac{d\vec{r}}{dt}\Big|_{t=\frac{\pi}{3}}=-\sqrt{3}\vec{i}+3\vec{j}+\vec{k}=<-\sqrt{3},3,1>.$$

The value on the curve when $t=\frac{\pi}{3}$:

$$\vec{r}\,|_{t=\frac{\pi}{3}}=\vec{i}+3\sqrt{3}\vec{j}+\frac{\pi}{3}\vec{k}=<1,3\sqrt{3},\frac{\pi}{3}>.$$ (4 points)

Thus, the symmetric equation of the tangent line is:

$$\frac{x-1}{-\sqrt{3}}=\frac{x-3\sqrt{3}}{3}=\frac{z-\frac{\pi}{3}}{1}.$$ (6 points)

(4) Find the maximum and minimum values of $f(x,y,z)=-x+2y+2z$ on the ellipse

$$\begin{cases} x^2+y^2=2 \\ y+2z=1 \end{cases}.$$

Solution:

Define the function:

$$L(x,y,z,\lambda_1,\lambda_2)=-x+2y+2z-\lambda_1(x^2+y^2-2)-\lambda_2(y+2z-1).$$ (1 points)

Then:

$$\frac{\partial L}{\partial x}=-1-2\lambda_1 x=0,$$
$$\frac{\partial L}{\partial y}=2-2\lambda_1 y-\lambda_2=0,$$
$$\frac{\partial L}{\partial z}=2-2\lambda_2=0.$$

With conditions $x^2+y^2=2$ and $y+2z=1$, get $\lambda_2=1$.

Case 1: $\lambda_1=1/2 \quad x=-1 \quad y=1 \quad z=0;$

Case 2: $\lambda_1=-1/2 \quad x=1 \quad y=-1 \quad z=1.$

The Critical points are: (-1, 1, 0), (1, -1, 1). (4 points)

$$f_{\max}(-1,1,0)=3, \quad f_{\min}(1,-1,1)=-1.$$ (6 points)

(5) Evaluate the line integrals $\int_{(0,0,0)}^{(1,1,1)}\left(6xy^3+2z^2\right)\mathrm{d}x+9x^2y^2\mathrm{d}y+\left(4xz+1\right)\mathrm{d}z.$

Solution: The line integral is independent to path. (3 points)

$$\int_{(0,0,0)}^{(1,1,1)} \left(6xy^3+2z^2\right)\mathrm{d}x+9x^2y^2\mathrm{d}y+\left(4xz+1\right)\mathrm{d}z$$

$$=\int_{(0,0,0)}^{(1,1,1)} d\left(2xz^2+z+3x^2y^3\right)$$ (6 points)

$$=\left(2xz^2+z+3x^2y^3\right)\Big|_{(0,0,0)}^{(1,1,1)}$$

$$=6$$

3.(6 points) Show the limit $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^3+y^3+z^3}$ does not exist.

Score

Solution:

(1) For Path 1: $x=y=z=t, t\to 0$

$$\lim_{t\to 0}\frac{xyz}{x^3+y^3+z^3}=\lim_{t\to 0}\frac{t\cdot t\cdot t}{t^3+t^3+t^3}=\lim_{t\to 0}\frac{t^3}{3t^3}=\frac{1}{3}.$$ (2 points)

(2) For Path 2: $x=0, y=0, z=t, t\to 0$

$$\lim_{t\to 0}\frac{xyz}{x^3+y^3+z^3}=\lim_{t\to 0}\frac{0\cdot 0\cdot t}{0^3+0^3+t^3}=\lim_{t\to 0}\frac{0}{t^3}=0.$$ (4 points)

Two paths give different limits.

$\lim\limits_{(x,y,z)\to(0,0,0)}\dfrac{xyz}{x^3+y^3+z^3}$ does not exit. (6 points)

4.(8 points) Let S be the solid cylinder bounded by $x^2+y^2=4, z=0, z=3$, and let n be the outer unit normal to the boundary ∂S. If $\vec{F}=\left(x^3+\tan yz\right)\vec{i}+\left(y^3-e^{xz}\right)\vec{j}+\left(3z+x^3\right)\vec{k}$, find the surface integral:

$$\iint_{\partial S}\vec{F}\cdot\vec{n}\,\mathrm{d}S=\iint_{\partial S}\left(x^3+\tan yz\right)\mathrm{d}y\mathrm{d}z+\left(y^3-e^{xz}\right)\mathrm{d}z\mathrm{d}x+\left(3z+x^3\right)\mathrm{d}x\mathrm{d}y.$$

Score

Solution:

$$M=x^3+\tan yz,\quad N=y^3-e^{xz},\quad P=3z+x^3.$$ (4 points)

since S is a closed surface. Using Gauss's theorem:

$$\iint_{\partial S}\left(x^3+\tan yz\right)dydz+\left(y^3-e^x z\right)dzdx+\left(3z+x^3\right)dxdy$$
$$=\iiint_{\Omega}\left(3x^2+3y^2+3\right)dxdydz$$
$$=\int_0^{2\pi}\int_0^2\int_0^3\left(3r^2+3\right)rdzdrd\theta$$
$$=108\pi.$$

(8 points)

5.(6 points) Calculate $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^2(x^2+y^2)^{1/2}\mathrm{d}z\mathrm{d}y\mathrm{d}x.$

Score

Solution:

In polar coordinates, we have

$$x=r\cos\theta,$$
$$y=r\sin\theta.$$

(3 points)

Thus, the integral becomes:

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^2\left(x^2+y^2\right)^{1/2}dzdydx=\int_0^{\frac{\pi}{2}}\int_0^3\int_0^2(r^2)^{\frac{1}{2}}rdzdrd\theta=9\pi.$$

(6 points)

6. (6 points) Find the sum of constant series $\sum_{n=0}^{\infty}\frac{(n+1)^2}{n!}$.

Score

Solution:

$$\sum_{n=0}^{\infty}\frac{(n+1)^2}{n!}=\sum_{n=0}^{\infty}\left(\frac{n^2}{n!}+\frac{2n}{n!}+\frac{1}{n!}\right)$$
$$=\sum_{n=1}^{\infty}\frac{n}{(n-1)!}+\sum_{n=1}^{\infty}\frac{2}{(n-1)!}+\sum_{n=0}^{\infty}\frac{1}{n!}$$
$$=\sum_{n=1}^{\infty}\frac{n-1+1}{(n-1)!}+2\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=0}^{\infty}\frac{1}{n!}$$
$$=\sum_{n=2}^{\infty}\frac{1}{(n-2)!}+3\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=0}^{\infty}\frac{1}{n!}$$
$$=e+3e+e$$
$$=5e$$

(3 points)

(6 points)

7. (8 points) Solve differential equation $y''-3y'+2y=e^x+1.$

Score

Solution:

For $y'' - 3y' + 2y = 0$:

$$\lambda^2 - 3\lambda + 2 = 0 \quad \therefore \lambda = 1, \lambda = 2 \quad \therefore y_h = c_1 e^x + c_2 e^{2x}.$$ (3 points)

Suppose that $y_p = axe^x + b$: $y_p'' - 3y_p' + 2y_p = e^x + 1$

$$-ae^x + 2b = e^x + 1 \quad , \quad a = -1 \quad b = \frac{1}{2}.$$ (6 points)

Therefore:

$$y_p = -xe^x + \frac{1}{2}$$

$$y = y_h + y_p = c_1 e^x + c_2 e^{2x} - xe^x + \frac{1}{2}.$$ (8 points)

8.(6 points) Let C be the positive closed curve formed by $(x-a)^2 + (y-a)^2 = 1$ and f(x) is a positive continuous function.

Score

Prove that the line integral $\oint_C \frac{x}{f(y)} \mathrm{d}y - yf(x)\mathrm{d}x \geq 2\pi$.

Solution:

Set $M(x,y) = -yf(x) \qquad N(x,y) = \frac{x}{f(y)}$ (3 points)

Green's theorem:

$$\oint_c \frac{x}{f(y)} dy - yf(x)dx$$
$$= \iint_S \left[\frac{1}{f(y)} + f(x) \right] dxdy \quad S: (x-a)^2 + (y-a)^2 \leq 1$$
$$= \frac{1}{2} \iint_S \left[\frac{1}{f(y)} + f(x) + \frac{1}{f(x)} + f(y) \right] dxdy$$ (6 points)
$$\geq \frac{1}{2} \iint_S 4dxdy$$
$$= 2\pi$$