$$\begin{array}{ll} \text{I} & \overline{n}_1 = (2x_1+1)(2u_1+1) - 2\Delta x_{1/2}\Delta_1 \\ \text{II} & \overline{n}_2 = (2x_2+1)(2u_2+1) - 2\Delta x_{1/2}\Delta_2 \end{array} \qquad \text{(eq.1)}$$

Be $x_2 > x_1$ $\quad (x_2 = x_1 + \Delta x_{1/2})$

$$x_1 = x_2 - \Delta x_{1/2}$$

$$\Delta x_{1/2} = x_2 - x_1$$

1. Substitute: $x_2 = x_1 + \Delta x_{1/2}$

$$\text{(eq2)} \quad 0 = (2x_2+1)(2u_2+1) - (2x_1+1)(2u_1+1) + 2\Delta x_{1/2}(-\Delta_2+\Delta_1)$$

$$0 = (2(x_1+\Delta x_{1/2})+1)(2u_2+1) - (2x_1+1)(2u_1+1) + 2\Delta x_{1/2}(-\Delta_2+\Delta_1)$$

$$= (2x_1+1)(2u_2+1) + 2\Delta x_{1/2}(2u_2+1) - (2x_1+1)(2u_1+1) + 2\Delta x_{1/2}(-\Delta_2+\Delta_1)$$

$$(2x_1+1)(2u_1+1) = (2x_1+1)(2u_2+1) + 2\Delta x_{1/2}(2u_2+1) + 2\Delta x_{1/2}(-\Delta_2+\Delta_1)$$

$$\text{(eq3.)} \quad (2u_1+1) = (2u_2+1) + 2\Delta x_{1/2} \frac{\cancel{(2u_2+1)+2\Delta x_{1/2}} \; (2u_2+1)+(-\Delta_2+\Delta_1)}{(2x_1+1)}$$

Case: $\Delta x_{1/2} = n\cdot(2x_1+1)$ $\quad (n\in\mathbb{N})$ $\quad$ (That is equal to $(2x_2+1) = (2x_1+1)(2n+1)$)

$$(2u_1+1) = (2u_2+1) + 2\cdot n\cdot\cancel{(2x_1+1)}\frac{(2u_2+1)+(-\Delta_2+\Delta_1)}{\cancel{2x_1+1}}$$

$$(2u_1+1) = (2u_2+1) + 2\cdot n\cdot(2u_2+1) + (-\Delta_2+\Delta_1)$$

Solve this for u_2: $(2u_2+1)$

$$(2u_1+1) = (2u_2+1)(2n+1) + (-\Delta_2+\Delta_1)$$

$$(2u_2+1) = \frac{(2u_1+1)-(-\Delta_2+\Delta_1)}{\cancel{2u_1+\Delta_1} \; 2n+1}$$

We need integer solutions for this!

$\Rightarrow$ Be $u_1 = (-\Delta_2+\Delta_1\mp 1)(2ny_1+n+y_1)$

$$(2u_2+1) = \frac{2\cdot(-\Delta_2+\Delta_1\mp 1)(2ny_1+n+y_1)+1-(-\Delta_2+\Delta_1)}{2n+1}$$

$$= \frac{2\cdot(-\Delta_2+\Delta_1\mp 1)((2n+1)(2y_1+1)-1)\frac{1}{2}+1-(-\Delta_2+\Delta_1)}{2n+1}$$

$$(2u_2+1) = \frac{\cancel{2}\cdot(-\Delta_2+\Delta_1-1)(2n+1)(2y_1+1)\cdot\frac{1}{\cancel{2}} - \cancel{2}\cdot(-\Delta_2+\Delta_1-1)\cdot\frac{1}{\cancel{2}} + 1 - (-\Delta_2+\Delta_1)}{2n+1}$$

$$= -(-\Delta_2+\Delta_1-1) - (-\Delta_2+\Delta_1-1)$$

$$= (-\Delta_2+\Delta_1-1)(2y_1+1) + \frac{-(-\Delta_2+\Delta_1-1)+1-(-\Delta_2+\Delta_1)}{2n+1}$$

$$= (-\Delta_2+\Delta_1-1)(2y_1+1) - 2\,\frac{(-\Delta_2+\Delta_1-1)}{2n+1} \quad ↯$$

<u>or Be:</u> $u_1 = -(-\Delta_2+\Delta_1-1)(2ny_1+n+y_1)$

$$(2u_2+1) = \frac{-\cancel{2}(-\Delta_2+\Delta_1-1)(2n+1)(2y_1+1)\frac{1}{\cancel{2}} + 2(-\Delta_2+\Delta_1-1)\frac{1}{2}+1-(\Delta_2+\Delta_1)}{2n+1}$$

$$= -(-\Delta_2+\Delta_1-1)(2y_1+1) + \frac{\overbrace{(-\Delta_2+\Delta_1-1) - (-\Delta_2+\Delta_1-1)}^{=0}}{2n+1}$$

$$(2u_2+1) = -(-\Delta_2+\Delta_1-1)(2y_1+1)$$

$$u_2 = \left(-(-\Delta_2+\Delta_1-1)(2y_1+1) - 1\right)\frac{1}{2}$$

$$= -(-\Delta_2+\Delta_1-1)\left((2y_1+1)-1\right)\frac{1}{2}$$

$$\underline{u_2 = -(-\Delta_2+\Delta_1-1)\,y_1}$$ arbitrary!! (can e.g. be ~~$y_1 = 2x_1y_1 + x_1+y_1$~~ $y_1 = 2x_1\alpha_1 + x_1 + \alpha_1$)

<u>Case: $\Delta x_{1/2} \neq n\cdot(2x_1+1)$:</u>

(eq.3) $$(2u_1+1) = (2u_2+1) + 2\Delta x_{1/2}\,\frac{(2u_2+1)+(-\Delta_2+\Delta_1)}{(2x_1+1)}$$

<u>Be $u_2 = (-\Delta_2+\Delta_1+1)(2x_1y_1+x_1+y_1)$</u>

$$= (-\Delta_2+\Delta_1+1)\left((2x_1+1)(2y_1+1)-1\right)\frac{1}{2}$$

$$\Rightarrow (2u_1+1) = (2u_2+1) + 2\Delta x_{1/2}\,\frac{\cancel{2}\cdot(-\Delta_2+\Delta_1+1)\overbrace{\cancel{((2x_1+1)(2y_1+1)-1)}}^{2x_1y_1+x_1+y_1}\frac{1}{\cancel{2}}+1+(-\Delta_2+\Delta_1)}{(2x_1+1)}$$

~~$= (2u_2+1) + 2\Delta x_{1/2}\,\frac{(-\Delta_2+\Delta_1+1)(2x_1+1)(2y_1+1) \overbrace{+1}^{-(-\Delta_2+\Delta_1+1)} + (-\Delta_2+\Delta_1)}{2x_1+1}$~~

~~$= (2u_2+1) + 2\Delta x_{1/2}\left[(-\Delta_2+\Delta_1+1)(2y_1+1) + \frac{-(-\Delta_2+\Delta_1+1)}{}\right.$~~

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$$\Rightarrow (2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}\, \frac{2\cdot(-\Delta_2+\Delta_1+1)(2x_1y_1+x_1+y_1)+(1+(-\Delta_2+\Delta_1))}{2x_1+1}$$

$$(2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}\cdot \frac{2\cdot(-\Delta_2+\Delta_1)(2x_1y_1+x_1+y_1)+2(2x_1y_1+x_1+y_1)}{2x_1+1}$$

$$+ 2\Delta x_{1,2}\cdot \frac{(1+1-\Delta_2+\Delta_1)}{2x_1+1}$$

$$= (2u_2+1) + 2\Delta x_{1,2}\left[\cancel{(-\Delta_2+1)}(-\Delta_2+\Delta_1+1)\frac{2\cdot(2x_1y_1+x_1+y_1)+1}{2x_1+1}\right]$$

$$= (2u_2+1) + 2\Delta x_{1,2}(-\Delta_2+\Delta_1+1)\frac{2\cdot[(2x_1+1)(2y_1+1)-1]\cdot\frac{1}{2}+1}{2x_1+1}$$

$$= (2u_2+1) + 2\Delta x_{1,2}(-\Delta_2+\Delta_1+1)\frac{(2x_1+1)(2y_1+1)\cancel{-1+1}}{2x_1+1}$$

$$\boxed{(2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}(-\Delta_2+\Delta_1+1)(2y_1+1)}$$

$$(2u_1+1) = 2\left[(2x_1+1)(2y_1+1)-1\right]\frac{1}{2}+1 + 2(x_2-x_1)(-\Delta_2+\Delta_1+1)(2y_1+1)$$

$$2u_1+1 = (2x_1+1)(2y_1+1)\cancel{-1+1} + 2(x_2-x_1)(-\Delta_2+\Delta_1+1)(2y_1+1)$$

$$(2u_1+1) = \cancel{(2x_1+1)(2y_1+1) + 2x_2(-\Delta_2+\Delta_1+1)(2y_1+1)}$$

$$= (2y_1+1)\left[(2x_1+1) + 2(x_2-x_1)(-\Delta_2+\Delta_1+1)\right]$$

$$\left(u_1 = \left\{(2y_1+1)\left[(2x_1+1) + 2(x_2-x_1)(-\Delta_2+\Delta_1+1)\right]-1\right\}\tfrac{1}{2}\right)$$

$$(2u_1+1) = (2y_1+1)\,2x_1 + (2y_1+1) + (2x_2-2x_1)(-\Delta_2+\Delta_1+1)$$

$$= (2y_1+1)\left[\underbrace{(2x_1+1)+2(x_2-x_1)(-\Delta_2+\Delta_1+1)}\right]$$

$$= 2x_1+1 + 2x_2(-\Delta_2+\Delta_1+1) - 2x_1(-\Delta_2+\Delta_1+1)$$

$$=$$

$$(2u_1+1) = 2\cdot(-\Delta_2+\Delta_1+1)(2x_1y_1+x_1+y_1)+1+ \cancel{2\Delta x_{1,2}}\ 2(x_2-x_1)(-\Delta_2+\Delta_1+1)(2y_1+1)$$

$$= (-\Delta_2+\Delta_1+1)\left[2\underbrace{(2x_1y_1+x_1+y_1)}_{=((2x_1+1)(2y_1+1)-1)\frac{1}{2}} + 2(x_2-x_1)(2y_1+1)\right]+1$$

$$= (-\Delta_2+\Delta_1+1)\left[(2x_1+1)(2y_1+1)-1 + 2(x_2-x_1)(2y\right.$$

$$(2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}(-\Delta_2+\Delta_1+1)(2y_1+1)$$

$$= \left\{2\left((-\Delta_2+\Delta_1+1)\left((2x_1+1)(2y_1+1)-1\right)\tfrac{1}{2}\right)+1\right\}$$

$$+ 2(x_2-x_1)(-\Delta_2+\Delta_1+1)(2y_1+1)$$

$$= (-\Delta_2+\Delta_1+1)(2x_1+1)(2y_1+1) - (-\Delta_2+\Delta_1+1) + 1$$

$$+ 2x_2(-\Delta_2+\Delta_1+1)(2y_1+1) - 2x_1(-\Delta_2+\Delta_1+1)$$

$$= (-\Delta_2+\Delta_1+1)\left[(2x_1+1)(2y_1+1) - 1 + 2x_2 - 2x_1\right] + 1$$

$$(2u_1+1) = (-\Delta_2+\Delta_1+1)\left[2(2x_1y_1+x_1+y_1) \cancel{+1} \cancel{-1} + 2x_2 - 2x_1\right] + 1$$

$$= (-\Delta_2+\Delta_1+1)\left[2^2x_1y_1 + \cancel{2x_1} + 2y_1 + 2x_2 - \cancel{2x_1}\right] + 1$$

$$= (-\Delta_2+\Delta_1+1)\left[2^2x_1y_1 + 2y_1 + 2x_2\right] + 1$$

$$= (-\Delta_2+\Delta_1+1)\left[2\cdot(2x_1y_1+y_1+x_2)\right] + 1 \quad ?$$

$$\boxed{\text{Be } (2u_2+1) = (-\Delta_2+\Delta_1)\cdot 2\cdot(2x_1y_1+x_1+y_1)} + Z\cdot(2x_1+1)$$

$$\Rightarrow \text{(eq.3)} \quad (2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}\,\frac{(2u_2+1)+(-\Delta_2+\Delta_1)+Z\cdot(2x_1+1)}{2x_1+1}$$

$$= (2u_2+1) + 2\Delta x_{1,2}\,\frac{(-\Delta_2+\Delta_1)\cdot 2\cdot(2x_1y_1+x_1+y_1)+(-\Delta_2+\Delta_1)+Z(2x_1+1)}{2x_1+1}$$

$$= (2u_2+1) + 2\Delta x_{1,2}\cdot(-\Delta_2+\Delta_1)\,\frac{2\cdot\left((2x_1+1)(2y_1+1)-1\right)\frac{1}{2}+1+Z(2x_1+1)}{2x_1+1}$$

$$= (2u_2+1) + 2\Delta x_{1,2}(-\Delta_2+\Delta_1)\,\frac{(2x_1+1)(2y_1+1)+Z(2x_1+1)}{(2x_1+1)}$$

$$\boxed{(2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}(-\Delta_2+\Delta_1)\left((2y_1+1)+Z\right)} + Z2\Delta x_{12}$$

$$\cancel{(2u_1+1) = 2\cdot(-\Delta_2+\Delta_1)\cdot 2(2x_1y_1+x_1+y_1)+1+2(x_2-x_1)(-\Delta_1+\Delta_1)\cdot(2y_1+1)}$$

③ x^5

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$$(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2\left[2\left((2x_1+1)(2y_1+1)-1\right)\tfrac{1}{2} + (x_2-x_1)(2y_1+1)\right] + 1$$

$$= (-\Delta_2+\Delta_1)\cdot 2\left[(2x_1+1)(2y_1+1) - 1 + (x_2-x_1)(2y_1+1)\right]+1$$

$$= (-\Delta_2+\Delta_1)\cdot 2\left[(2y_1+1)(2x_1+1+x_2-x_1)-1\right]+1$$

$$= (-\Delta_2+\Delta_1)\cdot 2\cdot\left[(2y_1+1)(x_1+x_2+1)-1\right]+1$$

$$= x_2 - \Delta x_{1,2}$$

$$(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2\cdot(2x_2y_2+x_2+y_2)$$

$$x_2 = x_1+\Delta x_{1,2}$$

$$(2u_1+1) = (2u_2+1) + 2\underbrace{\Delta x_{1,2}}_{(x_2-x_1)}(-\Delta_2+\Delta_1)(2y_1+1)$$

$$= (-\Delta_2+\Delta_1)\cdot 2(2x_1y_1+x_1+y_1)$$

$$+ 2\cdot(x_2-x_1)(-\Delta_2+\Delta_1)(2y_1+1)$$

$$= (-\Delta_2+\Delta_1)\left[2\cdot\underbrace{(2x_1y_1+x_1+y_1)}_{=((2x_1+1)(2y_1+1)-1)\frac{1}{2}} + 2(x_2-x_1)(2y_1+1)\right]$$

be $(2x_1+1)\cdot\beta$

$$= (-\Delta_2+\Delta_1)\left[(2x_1+1)(2y_1+1)-1+2\overbrace{(x_2-x_1)}(2y_1+1)\right]$$

$2\cdot 2x_1y_1 + 2x_1 + 2y_1 + 1 - 1 + 2\cdot(2x_1+1)\beta(2y_1+1)$

$$= (-\Delta_2+\Delta_1)\left[\underline{2\cdot 2x_1y_1} + \cancel{2x_1} + \underline{2y_1} + \cancel{1} - \cancel{1} + \underline{2x_2 2y_1} + \underline{2x_2} - 2^2x_1y_1 - \cancel{2x_1}\right]$$

$$= (-\Delta_2+\Delta_1)\left[2^2\cancel{x_1y_1} + 2y_1 + 2^2x_2y_1 + 2x_2 - 2^2\cancel{x_1y_1}\right]$$

$$= (-\Delta_2+\Delta_1)\left[2y_1 + 2^2x_2y_1 + 2x_2\right]$$

$$\boxed{(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2(2x_2y_1+x_2+y_1)}$$ ✓✓✓

$+\cancel{4}\cdot(2x_2+1)$

$= (-\Delta_2+\Delta_1)\,[\,2\cdot 2x_1y_1 + 2x_1 + 2y_1 + \cancel{1} - \cancel{1} + 2^3 x_1\beta y_1 + 2^2 x_1\beta + 2^2\beta y_1 + 2\beta\,]$

$= (-\Delta_2+\Delta_1)\,[\,2\cdot 2x_1y_1 + x_1(2+2^2\beta) + y_1(2+2^2\beta) + 2^3x_1\beta y_1 + 2\beta\,]$

$= (-\Delta_2+\Delta_1)\cdot 2\,[\,2x_1y_1 + \underbrace{x_1(1+2\beta)}_{=x_1+2x_1\beta} + \underbrace{y_1(1+2\beta)}_{=y_1+2y_1\beta} + 2^2x_1\beta y_1 + \beta\,]$

NR:

$2\cdot x_2y_1 + x_2 + y_1$

$= 2\cdot(2x_1+1)\beta\, y_1 + (2x_1+1)\beta + y_1$

$= 2^2x_1\beta y_1 + 2\beta y_1 + 2x_1\beta + \beta + y_1$

$(2x_1+1)y_1 = 2x_1y_1 + y_1$

$= (-\Delta_2+\Delta_1)\cdot 2\,[\,(2x_2y_1 + x_2 + y_1) + 2x_1y_1 + x_1\,]$

$= (-\Delta_2+\Delta_1)\cdot 2\,[\,(2x_2y_1 + x_2 + y_1) + (2y_1+1)\,x_1\,]$

For $\Delta x_{1/2} = (2x_1+1)\beta$ follows

$(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2\,\underbrace{(2x_2y_1 + x_2 + y_1)}_{((2x_2+1)(2y_1+1)-1)\frac{1}{2}} + \underbrace{2\cdot(2y_1+1)x_1}_{2\cdot(2y_1x_1+x_1)}$

$= 2\cdot((2x_1+1)y_1 - y_1)$

$= 2\cdot(2x_1+1)y_1 - 2y_1$

$= (-\Delta_2+\Delta_1)\,[\,(2x_2+1)(2y_1+1) - 1 + 2\cdot(2y_1+1)x_1\,]$

$= (-\Delta_2+\Delta_1)\,[\,(2y_1+1)[\underbrace{(2x_2+1) + 2x_1}_{2\cdot(x_2+x_1)+1}] - 1\,]$

~~$= (-\Delta_2+\Delta_1)\,[\,(2y_1+1)(2(x_1+1)+1) - 1\,]$~~

~~$= (-\Delta_2+\Delta_1)\,[\,2^2y_1(x_1+1) + 2y_1 + 2(x_1+1) + 1 - 1\,]$~~

~~$= (-\Delta_2+\Delta_1)\cdot 2\cdot(2y_1(x_1+1) + y_1 + (x_1+1))$~~

=

$(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2\cdot(2x_2y_1 + \overset{x_1+\Delta x_{1,2}}{x_2} + y_1)$

$x_2 = (x_1 + \Delta x_{1,2})$, $\Delta x_{1,2} = (2x_1+1)\beta$

$(2x_2+1)y_1 + (2x_1+1)y_1$

$= (2x_2+1+2x_1+1)\,y_1$

$= (2(x_2+x_1)+2)\,y_1$

$= 2((x_2+x_1)+1)\,y_1$

$(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2\,(\underbrace{2x_2y_1 + x_2 + y_1 - y_1}_{(2x_2+1)y_1}) + 2\cdot(2x_1+1)\,y_1$

$$(2u_1+1) = (2u_2+1) + 2\cdot \underbrace{\Delta x_{1,2}}_{u} \frac{(2u_2+1)(-\Delta_2+\Delta_1)}{2\cancel{x_1}+1}$$

$$\Delta x_{1,2} = n(2\cancel{x_1}+1)$$

$$= (2u_2+1) + 2\cdot n\,(2u_2+1)(-\Delta_2+\Delta_1)$$

$$= (2u_2+1) + 2\cdot n\cdot(-\Delta_2+\Delta_1)\cdot 2\,(2x_1y_1+x_1+y_1)(-\Delta_2+\Delta_1)$$

$$= (2u_2+1) + 2\cdot(-\Delta_2+\Delta_1)\,\big(2n\,(2x_1y_1+x_1+y_1)+1\big)$$

$$= (2u_2+1) + 2\cdot(-\Delta_2+\Delta_1)\,\big(\cancel{2}n\,((2x_1+1)(2y_1+1)-1)\tfrac{1}{\cancel{2}}+1\big)$$

$$= (2u_2+1) + 2\cdot(-\Delta_2+\Delta_1)\cdot\big(n\cdot(2x_1+1)(2y_1+1)\cancel{-1+1}\big)$$

$$= (2u_2+1) + 2\cdot(-\Delta_2+\Delta_1)\cdot n\cdot(2x_1+1)(2y_1+1)$$

$$(2u_1+1) = (2u_2+1) + 2\cdot\big(n\cdot(2\underbrace{\cancel{x_1}}_{x_2-\Delta x_{1,2}}+1)\big)\,(-\Delta_2+\Delta_1)\cdot(2y_1+1)$$

Be $(2u_2+1) = (-\Delta_2+\Delta_1)\cdot 2\cdot(2x_1y_1+x_1+y_1) + z(2x_1+1)$

$$(2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}\,\frac{(-\Delta_2+\Delta_1)\cdot 2\,(\overbrace{2x_1y_1+x_1+y_1}^{\cancel{(2x_1+1)(2y_1+1)-1\frac{1}{2}}}) + z(2x_1+1)}{2x_1+1} \; +(-\Delta_2+\Delta_1)$$

$$= (2u_2+1) + 2\Delta x_{1,2}\,\cancel{(-\Delta_2+\Delta_1)(2x_1+1)(2y_1+1)}$$

$$(-\Delta_2+\Delta_1)\,\frac{\cancel{2\cdot((2x_1+1)(2y_1+1)-1)\frac{1}{2}} + 1 \cancel{+2(\ldots)}}{2\cancel{x_1}+1}$$

$$+\, 2\Delta x_{1,2}\,\frac{\cancel{z(2x_1+1)}}{\cancel{2x_1+1}}$$

$$= (2u_2+1) + 2\Delta x_{1,2}\,\big((-\Delta_2+\Delta_1)(2y_1+1) + z\big)$$

NR:
$2x_1+1+2\Delta x_{1,2}$
$= 2\cdot(x_1+\Delta x_{1,2})+1$
$= 2x_2+1$ ✓

$$(2u_1+1) = (2u_2+1) + 2\Delta x_{1,2}\,(-\Delta_2+\Delta_1)(2y_1+1) + 2\Delta x_{1,2}\, z$$

$$= (-\Delta_2+\Delta_1)\cdot 2\,(2x_1y_1+x_1+y_1) + z(2x_1+1) + 2\Delta x_{1,2}\,(-\Delta_2+\Delta_1)(2y_1+1) + 2\Delta x_{1,2}\, z$$

$$(2u_1+1) = (-\Delta_2+\Delta_1)\cdot 2\,(2x_2y_1+x_2+y_1) + \underbrace{z\big((2x_1+1)+2\Delta x_{1,2}\big)}_{=z(2x_2+1)}$$