Main Test Series – 2016

Test – 1 Answer Key

Linear Algebra, Calculus & 3-D

1(a) Find the rank of the matrix

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Sol? Let us denote the given matrix by A.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

by interchanging R_1 and R_2

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

by $R_4 \to R_4 - R_3 - R_2 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 5 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

which is in Echelon form.

The no. of non-zeros rows of this matrix is 3.

$\therefore$ The Rank is 3.

i.e. rank $A = 3$.

1(b) If $\alpha_1 = (1,-1)$, $\beta_1 = (1,0)$,
$\alpha_2 = (2,-1)$, $\beta_2 = (0,1)$
$\alpha_3 = (-3,2)$, $\beta_3 = (1,1)$ is there a linear transformation T from $\mathbb{R}^2$ into $\mathbb{R}^2$ such that $T\alpha_i = \beta_i$ for $i = 1, 2$ and 3?

Sol We have $T(1,-1) = (1,0)$,
$T(2,-1) = (0,1)$
$T(-3,2) = (1,1)$

Since the vectors $(1,-1)$, $(2,-1)$ and $(-3,2)$ are not basis vectors.

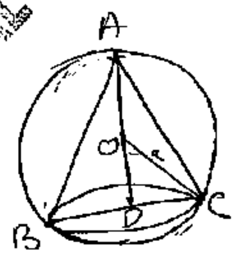
$\therefore$ There exist no linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

1(c) Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere.

Sol: Let x be the radius of the base and y be the height of a cone inscribed in a given sphere of radius a.

Let V be the volume of the cone.

Then $V = \frac{1}{3}\pi x^2 y$ ——— ①



from the figure, we have

$OD = AD - AO = y - a$

$\therefore a^2 = (y-a)^2 + x^2$

$\Rightarrow x^2 = a^2 - (y-a)^2$

$= a^2 - y^2 + 2ay - a^2$

$= y(2a-y)$

putting the value of x^2 in ①, we get

$V = \frac{1}{3}\pi y^2(2a-y)$

$\frac{dV}{dy} = \frac{1}{3}\pi[2y(2a-y) - y^2]$

$= \frac{1}{3}\pi y(4a-3y)$

for a maximum or minimum of V,

$\frac{dV}{dy} = 0$

i.e, $y(4a-3y)=0$

i.e, $y=\left(\frac{2}{3}\right)(2a)$ $\quad$ ($\because y=0$ is inadmissible)

Also $\frac{d^2V}{dy^2}=\frac{1}{3}\pi(4a-3y)+\frac{1}{3}\pi y(-3)$

$=-ve$ when $y=\frac{4a}{3}$.

$\therefore$ V is maximum when $y=\frac{2}{3}(2a)$

i.e, when $3y=2(2a)$.

1(d) In the closed interval $(-1,1)$, let $f(x)$ defined as $x^2\sin\frac{1}{x^2}$ for $x\neq 0$ and $f(0)=0$.

In the given interval,

(i) Is the function bounded? (ii) Is it continuous?

(iii) Is it uniformly continuous?

(iv) Is it absolutely continuous?

Sol^n: Here $f(0)=0$

and $f(0+)=f(0+h)$

$$=\lim_{h\to 0} h^2\sin\frac{1}{h^2}=0$$

and $f(0-)=\lim_{h\to 0} f(0-h)$

$$=\lim_{h\to 0} h^2\sin\frac{1}{h^2}=0$$

$\therefore f(0+)=f(0-)=f(0)=0$.

$\therefore f(x)$ is continuous at $x=0$

Again $f(\alpha) = \alpha^2 \sin\frac{1}{\alpha^2}$.

$\therefore f(\alpha+0) = \alpha^2 \sin\frac{1}{\alpha^2} = f(\alpha-0)$.

where $\cancel{-1<\alpha}<0$ or $0<\alpha\le 1$.

So, the function is continuous at $x=\alpha\neq 0$.

Thus the function is continuous ~~for~~ all points in the interval $(-1,1)$.

Since it is continuous, it is bounded and also uniformly continuous in the interval.

Now the fluctuation of $f(x)$ in the interval $\left[\frac{1}{\sqrt{\pi(r+1)}}, \frac{1}{\sqrt{\pi r}}\right]$ is greater than $\frac{2}{(r+1)\pi}$

Therefore the sum of the absolute value of fluctuations in the sequence of intervals $\left[\frac{1}{\sqrt{(r+1)\pi}}, \frac{1}{\sqrt{r\pi}}\right]$ corresponding to $r=n, n+1, \ldots$ is greater than $\frac{2}{\pi}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots\right]$

which exceeds any finite bound

as the series within brackets is divergent, while, as is easily seen that the sum of the infinite set of the intervals tends to zero as r tends to infinite

Hence, however large n may be, and so however, small the neighbourhood of the origin may be, the function is not of bounded total fluctuation, i.e, it is not absolutely continuous although uniformly continuous in this neighbourhood

1(c) Show that the two circles

$x^2+y^2+z^2-y+2z=0,\ x-y+z-2=0$;

$x^2+y^2+z^2+x-3y+z-5=0,\ 2x-y+4z-1=0$

lie on the same sphere and find its equation.

Solⁿ: The equations of any sphere through the first circle is

$x^2+y^2+z^2-y+2z+k(x-y+z-2)=0$ — (1)

Similarly the equation of any sphere through the second circle is

$x^2+y^2+z^2+x-3y+z-5+k'(2x-y+4z-1)=0$ — (2)

It is given circles lies on the same sphere then (1) and (2) should represent the same sphere, so comparing the coefficients of x, y, z and constant terms in (1) and (2), we get

$k=2k'+1$; $-1-k=-k'-3$; $2+k=4k'+1$; $-2k=-k'-5$

The first two equations give $k=3$, $k'=1$, and these values clearly satisfy the remaining two equations also.

Putting $k=3$ in (1), we get the required equation of the sphere as

$x^2+y^2+z^2-y+2z+3(x-y+z-2)=0$

$\Rightarrow x^2+y^2+z^2+3x-y+5z-6=0$

2(B) Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent:

(i) The intersection of the range of T and the null space of T is the zero subspace of V

i.e. $R(T) \cap N(T) = \{0\}$.

(ii) If $T(T(\alpha)) = 0$ then $T(\alpha) = 0$.

Sol (i) $\Rightarrow$ (ii):

Let $R(T) \cap N(T) = \{0\}$

Let $T(\alpha) = \beta$ then $\beta \in R(T)$ ——— (1)

Now we have $T(T(\alpha)) = \bar{0} \Rightarrow T(\beta) = \bar{0}$

$\Rightarrow \beta \in N(T)$ ——— (2)

from (1) & (2), $\beta \in R(T) \cap N(T)$.

But $R(T) \cap N(T) = \{\bar{0}\} \Rightarrow \beta = \bar{0}$

$\Rightarrow T(\alpha) = \bar{0}$

$\therefore T(T(\alpha)) = \bar{0} \Rightarrow T(\alpha) = \bar{0}$

(ii) $\Rightarrow$ (i): Given that $T(T(\alpha)) = \bar{0} \Rightarrow T(\alpha) = \bar{0}$

let $\beta \in R(T) \cap N(T)$ then $\beta \in R(T)$ and $\beta \in N(T)$

Now we have $\beta \in R(T) \Rightarrow T(\alpha) = \beta$ for some $\alpha \in V$

and $\beta \in N(T) \Rightarrow T(\beta) = \bar{0}$

$\Rightarrow T(T(\alpha)) = \bar{0}$

$\Rightarrow T(\alpha) = \bar{0}$

$\therefore R(T) \cap N(T) = \{\bar{0}\}$. $\beta = \bar{0}$ ($\because T(\alpha) = \beta$)

2(b) Note: it is for knowledge !!

Defn: Minimal polynomial for an operator:-

Let T be a linear operator on F.D.V.S. $V(F)$. A polynomial $p(x) \in F[x]$ is called a monic polynomial if the coefficient of the highest power of x in $p(x)$ is unity.

Ex:- $p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in F[x]$ is a monic polynomial.

- We say that T satisfies the monic polynomial $p(x)$ if $p(T) = T^n + a_1 T^{n-1} + \cdots + a_{n-1}T + a_n = 0$.
- If $p(T) = 0$ then we say that the polynomial $p(x)$ annihilates the linear operator.
- Defn: Let T be a linear on a F.D.V.S $V(F)$. A monic polynomial $p(x) \in F[x]$ of lowest degree such that $p(T) = 0$ is called a minimal polynomial for 'T' over 'F'.

Similarly: we can define the minimal polynomial for a square matrix A.

Note(1): Let T be a linear operator on a finite-dimensional vector space $V(F)$. Then the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

Note(2): Let T be a diagonalizable linear operator on V. Let $c_1, c_2, \ldots, c_k$ be the distinct characteristic values of T. Then the minimal polynomial for T is the polynomial,

$$p(x) = (x-c_1)(x-c_2)\cdots(x-c_k).$$

i.e. If T is a diagonalizable linear operator then the minimal polynomial for T is the product of distinct linear factors.

Note(3): The minimal polynomial of a linear operator T divides its characteristic polynomial.

Note(4): The characteristic and minimal polynomials for T are the same if the eigen values of T are distinct.

2(b) Show that the minimal polynomial for the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is $(x-1)(x-2)$

Sol. The characteristic equation of A is $|A - xI| = 0$

$$\Rightarrow \begin{vmatrix} 5-x & -6 & -6 \\ -1 & 4-x & 2 \\ 3 & -6 & -4-x \end{vmatrix} = 0 \Rightarrow x = 1, 2, 2.$$

$\boxed{\text{Note:}}$ (1): Students are requested to find characteristic vectors of the above characteristic roots.

corr. to the characteristic root 1:-

$$X_1 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

corr. to the characteristic root 2:-

$$X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} ; X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

$\because$ X_1, X_2, X_3 are linearly independent

$\therefore$ The given matrix A is diagonalizable.

let $P = [X_1, X_2, X_3] = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$

Then $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D.$

$\therefore$ A is diagonalizable.

$\therefore$ The minimal polynomial for A is

$p(x) = (x-1)(x-2)$.

Note(1): It can be verified that

$(A-I)(A-2I) = 0$.

Note(2): If the matrix A is not diagonalizable then $p(x) = (x-1)(x-2)$ is not a minimal polynomial for A.

Hence the minimal for A is

$p(x) = (x-1)(x-2)^2$, which is same as the characteristic polynomial of A.

$\boxed{\text{It is for knowledge only}}$

2(c) Find the volume of the solid bounded below by the paraboloid $z = x^2+y^2$ and above by the plane $z = 2y$.

Sol^n. The required volume is

$$\iint_A \int_{x^2+y^2}^{2y} dx\, dy\, dz \quad \text{—— (1)}$$

where A is the area above which the volume stands. Eliminating z between the equations $z = x^2+y^2$ and $z = 2y$, we get

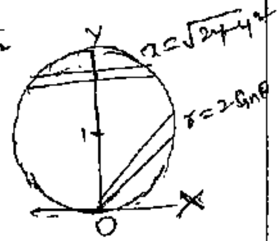
$$x^2+y^2 = 2y \quad \text{—— (2)}$$

This is the equation of the cylinder through the curve of intersection of the plane and the paraboloid, whose generators are parallel to z-axis. The section of this cylinder by the xy-plane gives the area A.

Taking a strip parallel to x-axis, the area A can be considered as enclosed by the curves

$x = -\sqrt{2y-y^2}$, $x = \sqrt{2y-y^2}$

$y = 0$, $y = 2$



So the integral (1) may be written as

$$= \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dy\, dx\, dz$$

$$= 2\int_0^2 \int_0^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dy\,dx\,dz$$, as the volume is symmetrical about yoz plane

$$= 2\int_0^2 \int_0^{\sqrt{2y-y^2}} (2y-x^2-y^2)\, dy\, dx.$$

we now change the variables to r, θ. Then

eqn (2) becomes $r^2 = 2r\sin\theta \Rightarrow r = 2\sin\theta$.

and half the area A lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required volume

$$= 2\int_0^{\pi/2} \int_0^{2\sin\theta} (2r\sin\theta - r^2)\, r\, d\theta\, dr$$

$$= 2\int_0^{\pi/2} \left(\frac{2}{3}r^3 \sin\theta - \frac{1}{4}r^4\right)_0^{2\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{16}{3}\sin^4\theta - 4\sin^4\theta\right) d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= \frac{8}{3}\, \frac{\Gamma 5/2\, \Gamma 1/2}{2\,\Gamma(3)}$$

$$= \frac{8}{3} \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot 2\cdot 1} = \frac{\pi}{2}$$

26) Prove that the equation to the two planes inclined at an angle α to xy-plane and containing the line $y=0$, $z\cos\beta = x\sin\beta$ is $(x^2+y^2)\tan^2\beta + z^2 - 2zx\tan\beta = y^2\tan^2\alpha$.

Sol^n: The equation of the plane containing the line $y=0$, $x\sin\beta = z\cos\beta$ is

$$(x\sin\beta - z\cos\beta) + \lambda y = 0$$

$$x\sin\beta + \lambda y - z\cos\beta = 0 \quad \text{--- (1)}$$

The other plane is xy-plane i.e. $z=0$.

i.e, $0 \cdot x + 0 \cdot y + 1 \cdot z = 0$ --- (2)

It is given that the angle between plane (1) and (2) is α, so we have

$$\cos\alpha = \frac{0 \cdot \sin\beta + 0 \cdot \lambda + 1(-\cos\beta)}{\sqrt{\sin^2\beta + \lambda^2 + \cos^2\beta}\,\sqrt{0^2+0^2+1^2}}$$

Squaring and cross multiplying, we have

$$(\lambda^2+1)\cos^2\alpha = \cos^2\beta$$

$$\Rightarrow \lambda^2 = \frac{\cos^2\beta - \cos^2\alpha}{\cos^2\alpha}$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{\cos^2\beta - \cos^2\alpha}{\cos^2\alpha}} = \pm\mu \text{ (say)}.$$

which gives two values of λ which are equal in magnitude but opposite in sign.

$\therefore$ from (1) combined equation of two required planes is

$$\left[(x\sin\beta - z\cos\beta) + \mu y\right]\left[(x\sin\beta - z\cos\beta) - \mu y\right] = 0$$

$$\Rightarrow (x\sin\beta - z\cos\beta)^2 - \mu^2 y^2 = 0$$

$$\Rightarrow (x\sin\beta - z\cos\beta)^2 = \mu^2 y^2$$

$$\Rightarrow (x\sin\beta - z\cos\beta)^2 = \left(\frac{\cos^2\beta - \cos^2\alpha}{\cos^2\alpha}\right) y^2 \quad \text{(from (3))}$$

$$\Rightarrow (x\sin\beta - z\cos\beta)^2 \cos^2\alpha = (\cos^2\beta - \cos^2\alpha) y^2$$

$$\Rightarrow \left[x^2\sin^2\beta + z^2\cos^2\beta - 2xz\sin\beta\cos\beta\right]\cos^2\alpha = (\cos^2\beta - \cos^2\alpha) y^2$$

dividing on both sides by $\cos^2\alpha\cos^2\beta$

$$x^2\tan^2\beta + z^2 - 2xz\tan\beta = (\sec^2\alpha - \sec^2\beta) y^2$$

$$\Rightarrow x^2\tan^2\beta + z^2 - 2xz\tan\beta = (1 + \tan^2\alpha - 1 - \tan^2\beta) y^2$$

$$\Rightarrow (x^2 + y^2)\tan^2\beta + z^2 - 2xz\tan\beta = y^2\tan^2\alpha.$$

3(e) Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by
$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

(i) If β is the standard ordered basis for $\mathbb{R}^3$ and β' is the standard ordered basis for $\mathbb{R}^2$, what is the matrix of T relative the pair β, β'?

(ii) If $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\beta' = \{\beta_1, \beta_2\}$ where $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (1, 0, 0)$, $\beta_1 = (0, 1)$, $\beta_2 = (1, 0)$.
What is the matrix of T relative to the pair β, β'?

Sol. (i) Let $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $\mathbb{R}^3$.
and $\beta' = \{(1,0), (0,1)\}$ be the standard basis of $\mathbb{R}^2$.

Then $T(1,0,0) = (1,-1) = 1(1,0) + (-1)(0,1)$
$T(0,1,0) = (1,0) = 1(1,0) + 0(0,1)$
$T(0,0,1) = (0,2) = 0(1,0) + 2(0,1)$.

$\therefore$ The matrix of T is

$$[T: \beta, \beta'] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

(ii) try yourself.

30(b) (i) Evaluate $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx$

(ii) If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

so that $\frac{\partial^2 u}{\partial x\partial y} = \frac{x^2-y^2}{x^2+y^2}$ and $\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$.

Sol: (i) Given that

$$I = \int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx.$$

Let $x = \tan\theta$. $\Rightarrow dx = \sec^2\theta\, d\theta$

Limits of θ: If $x = \infty$, $\theta = \pi/2$

$x = 0$, $\theta = 0$.

$\therefore \theta$ varies from 0 to $\pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\log(1+\tan^2\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \log\sec^2\theta\, d\theta = 2\int_0^{\pi/2} \log\sec\theta\, d\theta$$

$$= -2\int_0^{\pi/2} \log\cos\theta\, d\theta = 2I_1 \text{ (say)} \quad \text{—(1)}$$

Now let $I_1 = \int_0^{\pi/2} \log\cos\theta\, d\theta = \int_0^{\pi/2} \log\cos\left(\frac{\pi}{2}-\theta\right)d\theta$

$$= \int_0^{\pi/2} \log\sin\theta\, d\theta = I_1$$

$$\therefore 2I_1 = \int_0^{\pi/2} (\log\cos\theta + \log\sin\theta)\, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin\theta\cos\theta}{2}\right) d\theta$$

$$= \int_0^{\pi/2} \log\frac{\sin 2\theta}{2}\, d\theta.$$

$$= \int_0^{\pi/2} \log \sin 2\theta \, d\theta - \int_0^{\pi/2} \log 2 \, d\theta$$

$$= \int_0^{\pi/2} \log \sin 2\theta \, d\theta - \log 2 \int_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \log \sin 2\theta - \frac{\pi}{2} \log 2$$

$$= I' - \frac{\pi}{2} \log 2 \quad \text{--- (2)}$$

where $I' = \int_0^{\pi/2} \log \sin 2\theta \, d\theta$

put $2x = t$ so that $2dx = dt$

when $x = 0$, $t = 0$

when $x = \pi/2$, $t = \pi$

$$\therefore I' = \int_0^{\pi} \log \sin t \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin \theta \, d\theta$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin \theta \, d\theta$$

$$= \int_0^{\pi/2} \log \sin \theta \, d\theta = I_1$$

$\therefore$ from (2)

$$2I_1 = I' - \frac{\pi}{2} \log 2$$

$$= I_1 - \frac{\pi}{2} \log 2$$

$$\therefore I_1 = -\frac{\pi}{2} \log 2$$

$$\therefore \text{from (1)} \quad I = -2 I_1 = -2 \left[-\frac{\pi}{2} \log 2 \right] = \pi \log 2$$

(ii) Given $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ —— ①

Differentiating ① partially w.r.t y.

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= x - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \quad \text{—— ②}$$

Differentiating ① partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\frac{y}{x} + x^2\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 2x\tan^{-1}\frac{y}{x} - \frac{x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\frac{y}{x} - y.$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2x\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2} \quad \text{—— ③}$$

$\therefore$ From ② and ③, we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

3(c) prove that the area of the section of the cone $bcx^2+cay^2+abz^2=0$ by the plane $lx+my+nz=p$ is $\dfrac{\pi p^2\sqrt{abc}}{(al^2+bm^2+cn^2)^{3/2}}$.

Sol: Let the equation of the given cone be

$Ax^2+By^2+Cz^2=0$, where
$A=bc,\ B=ca,\ C=ab$. —— (A)

If (α,β,γ) be the centre, then the plane is

$T=S_1$
$\Rightarrow A\alpha x+B\beta y+C\gamma z = A\alpha^2+B\beta^2+C\gamma^2$ —— (1)

which is $lx+my+nz=p$

comparing $l=a\alpha,\ m=b\beta,\ n=c\gamma,$
$p=A\alpha^2+B\beta^2+C\gamma^2$.

Also the equation of the plane section is given by $lx+my+nz=p$ —— (2)

Comparing (1) and (2), we have

$$\frac{A\alpha}{l}=\frac{B\beta}{m}=\frac{C\gamma}{n}=\frac{A\alpha^2+B\beta^2+C\gamma^2}{p}=\lambda \text{ (say)} \quad \text{—— (3)}$$

$$\Rightarrow \alpha=\frac{\lambda l}{A},\ \beta=\frac{m\lambda}{B},\ \gamma=\frac{n\lambda}{C}.$$

and so $A\alpha^2+B\beta^2+C\gamma^2=p\lambda$ gives

$$\lambda^2\left(\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}\right)=p\lambda \Rightarrow \lambda=\frac{p}{\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}}.$$

$\lambda=\dfrac{p}{p_0^2}$ where $p_0^2=\dfrac{l^2}{A}+\dfrac{m^2}{B}+\dfrac{n^2}{C}=A\alpha^2+B\beta^2+C\gamma^2$.

we can prove that the axes of the section are given by $\frac{l^2}{Ap_0^2r^2+p^2}+\frac{m^2}{Bp_0^2r^2+p^2}+\frac{n^2}{Cp_0^2r^2+p^2}=0.$

$$\Rightarrow \sum l^2(Bp_0^2r^2+p^2)(Cp_0^2r^2+p^2)=0$$

$$\Rightarrow \sum\{l^2(BCp_0^4r^4+p_0^2p^2r^2(B+C)+p^4)\}=0$$

$$\Rightarrow p_0^4r^4(BCl^2+CAm^2+ABn^2)+p_0^2p^2r^2[l^2(B+C)+m^2(C+A)+n^2(A+B)]+p^4(l^2+m^2+n^2)=0$$

If r_1^2 and r_2^2 be its roots, then

$$r_1^2r_2^2=\frac{p^4(l^2+m^2+n^2)}{p_0^4(BCl^2+CAm^2+ABn^2)}$$

$$=\frac{p^4(l^2+m^2+n^2)}{p_0^4(abc)(al^2+bm^2+cn^2)} \quad \text{(from (A) substitute the values of } A,B,C) \quad \text{--- (4)}$$

where $p_0^2=\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}$

$$=\frac{l^2}{bc}+\frac{m^2}{ca}+\frac{n^2}{ab}=\frac{1}{abc}(al^2+bm^2+cn^2) \quad \text{--- (5)}$$

$\therefore$ Required area $=\pi r_1r_2$

$$=\frac{\pi p^2(l^2+m^2+n^2)}{p_0^2\sqrt{abc}\sqrt{al^2+bm^2+cn^2}}$$

$$=\frac{\pi p^2(1)}{\left(\frac{1}{abc}\right)(al^2+bm^2+cn^2)\sqrt{abc(al^2+bm^2+cn^2)}} \quad \text{(from 5)}$$

$$=\frac{\pi p^2\sqrt{abc}}{(al^2+bm^2+cn^2)^{3/2}}$$

Hence proved

4(a). Let V be the vectorspace over the complex numbers of all functions from $\mathbb{R}$ in C
i.e the vectorspace all complex-valued functions on the real line. let
$f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$
(i) prove that f_1, f_2, f_3 are L.I.
(ii) let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$
find an invertible 3×3 matrix P such that $g_j = \sum_{i=1} P_{ij} f_i$.

Sol let $V = \{ f / f : \mathbb{R} \to C \}$ be the given vectorspace over 'C'.

let $c_1, c_2, c_3 \in C$. s.t

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

$\Rightarrow c_1(1) + c_2 e^{ix} + c_3 e^{-ix} = 0$ ——— (1)

diff. (1) w.r.t 'x' we get

$i c_2 e^{ix} - i c_3 e^{-ix} = 0$

$\Rightarrow c_2 e^{ix} - c_3 e^{-ix} = 0$ ——— (2)

Again diff. w.r.t 'x' we get

$-c_2 e^{ix} - c_3 e^{-ix} = 0$

$\Rightarrow c_2 e^{ix} + c_3 e^{-ix} = 0$ ——— (3)

Again diff. w.r.t 'x', we get

$$c_2 e^{ix} - c_3 e^{-ix} = 0 \quad \text{--- (4)}$$

from (3) & (4) we have

$$2c_2 e^{ix} = 0 \Rightarrow \boxed{c_2 = 0} \left(\because e^{ix} \neq 0 \; \forall x\right)$$

$$\text{Sly } \boxed{c_2 = c_1 = 0}$$

$\therefore c_1 = c_2 = c_3 = 0.$

$\therefore f_1, f_2, f_3$ are linearly independent

(ii) $g_1 = p_{11} f_1 + p_{21} f_2 + p_{31} f_3$

$g_2 = p_{12} f_1 + p_{22} f_2 + p_{32} f_3.$

$g_3 = p_{13} f_1 + p_{23} f_2 + p_{33} f_3.$

$\Rightarrow 1 = 1(1) + 0e^{ix} + 0\, e^{-ix}$

$\cos x = 0(1) + \frac{1}{2} e^{ix} + \frac{1}{2} e^{-ix}.$

$\sin x = 0(1) + \frac{1}{2i} e^{ix} + \frac{-1}{2i} e^{-ix}$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2i} & -\frac{1}{2i} \end{bmatrix}$$ is the required invertible matrix

4(i) prove or disprove $f_{xy}(0,0) = f_{yx}(0,0)$

where $f(x,y) = \dfrac{(x^2y + xy^2)\sin(x-y)}{x^2+y^2}$, $(x,y) \neq (0,0)$

$f(0,0) = 0$

(ii) prove that the function

$f(x,y) = x^2 - 2xy + y^2 + x^4 + y^4$ has a minima at the origin.

Solⁿ: $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,0+k) - f(0,0)}{k}$

$= \lim\limits_{k\to 0} \dfrac{0-0}{k} = 0$

$f_y(h,0) = \lim\limits_{k\to 0} \dfrac{f(h,0+k) - f(h,0)}{k}$

$= \lim\limits_{k\to 0} \dfrac{(h^2k + hk^2)\sin(h-k) - 0}{k(k^2+h^2)}$

$= \lim\limits_{k\to 0} \dfrac{(h^2+hk)\sin(h-k)}{h^2+k^2} = \sin h.$

$f_{xy}(0,0) = \lim\limits_{h\to 0} \dfrac{f_y(0+h,0) - f_y(0,0)}{h}$

$= \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$

$f_x(0,k) = \lim\limits_{h\to 0} \dfrac{f(0+h,k) - f(0,k)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(h^2k + hk^2)\sin(h-k)}{h(h^2+k^2)} = -\sin k.$

$f_{yx}(0,0) = \lim\limits_{k\to 0} \dfrac{f_x(0,0+k) - f_x(0,0)}{k}$

$= \lim_{k \to 0} \frac{-\sin k}{k} = -1$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$

(ii) $f(x,y) = x^2 - 2xy + y^2 + x^4 + y^4$.

$f_x = 2x - 2y + 4x^3$; $f_y = -2x + 2y + 4y^3$

$f_{xx} = 2 + 12x^2$ | $f_{yy} = 2 + 12y^2$.

At the origin

$f_{xx} f_{yy} - f_{xy}^2 = 4 - 4 = 0$

Thus it is doubtful case and so requires further investigation

we have $f(x,y) = (x-y)^2 + x^4 + y^4$

Also $f(0,0) = 0$

Clearly $f(x,y) > f(0,0)$ for all values of (x,y) in a deleted nbd of $(0,0)$.

Hence the given function has a minimum at the origin.

4(c) Reduce the equation

$2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$

to the standard form. What does it represent?

Sol^n: Comparing the given equation $F(x,y,z)=0$ with the equation

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

we have $a=2,\ b=-7,\ c=2,\ f=-5,\ g=-4,\ h=-5,\ u=3,\ v=6,\ w=-3,\ d=5.$

Now coordinates of the centre (x_1,y_1,z_1) of the given surface are given by

$\frac{\partial F}{\partial x}=0,\quad \frac{\partial F}{\partial y}=0$ and $\frac{\partial F}{\partial z}=0$

$4x_1-8z_1-10y_1+6=0 \Rightarrow 2x_1-5y_1-4z_1+3=0$ — (1)

$-14y_1-10z_1-10x_1+12=0 \Rightarrow 5x_1+7y_1+5z_1-6=0$ — (2)

$4z_1-10y_1-8x_1-6=0 \Rightarrow 4x_1+5y_1-2z_1+3=0$ — (3)

Solving (1), (2) and (3) we get

$x_1=\frac{1}{3},\ y_1=-\frac{1}{3},\ z_1=\frac{4}{3}.$

$\therefore$ centre of the given surface is $(\frac{1}{3},-\frac{1}{3},\frac{4}{3})$

Also $d'=ux_1+vy_1+wz_1+d$

$=3(\frac{1}{3})+6(-\frac{1}{3})+(-3)(\frac{4}{3})+5$

$=1-2-4+5=0$ — (4)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix}=0$$

$\Rightarrow (2-\lambda)\left[-(7+\lambda)(2-\lambda)-25\right]+5\left[-5(2-\lambda)-20\right]$

$-4\left[25-4(7+\lambda)\right]=0$

$\Rightarrow \lambda^3+3\lambda^2-90\lambda+216=0$

$\Rightarrow (\lambda-3)(\lambda^2+6\lambda-72)=0$

$\Rightarrow (\lambda-3)(\lambda-6)(\lambda+12)=0$

$\Rightarrow \lambda = 3, 6, -12.$

$\therefore$ Let $\lambda_1=3,\ \lambda_2=6,\ \lambda_3=-12$

$\therefore$ By rotation of axes the given equation transforms to

$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$

substituting the values of $\lambda_1, \lambda_2, \lambda_3, d'$

$\Rightarrow 3x^2+6y^2-12z^2+0=0$

$\Rightarrow x^2+2y^2-4z^2=0$

which is the required standard form and represents a cone.

Also the vertex of the cone is

$\left(\frac{1}{3}, -\frac{1}{3}, \frac{4}{3}\right)$.

5(A) What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices and all lower triangular matrices? What is the largest subspace that contained in both of those subspaces?

Let $V = \{[a_{ij}]_{3\times3} / a_{ij} \in F\}$ be the given vector space.

Sol let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3\times3}$ be the symmetric matrix

then $A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$+ b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$\therefore W_1 = L(S_1)$ is smallest subspace of V spanned by S that contains all symmetric matrices ~~lower triangular matrices~~.

where $S_1 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$.

let $B = \begin{bmatrix} a & 0 & 0 \\ h & b & 0 \\ g & f & c \end{bmatrix}$ be a lower triangular matrix

then $B = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$+ h\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$+ f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\therefore W_2 = L(S_2)$ is smallest subspace of V spanned by S_2 that contains all lower triangular matrices

where $S_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

Now we have

$W_3 = L(S_3)$ is largest subspace of V that contains both of those subspaces.

where $S_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$.

5(b) Is there a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^2$ such that $T(1,-1,1) = (1,0)$ and $T(1,1,1) = (0,1)$?

Sol. clearly T is not a linear transformation.

Since T is defined only on the subspace W of $\mathbb{R}^3$ spanned by $(1,-1,1)$, $(1,1,1)$, where $\dim W = 2$, 5(c) Evaluate $\lim_{x\to\infty} \frac{a^{1/x}-b^{1/x}}{\log(x/x-1)}$

Sol^n: we have $\lim_{x\to\infty} \frac{a^{1/x}-b^{1/x}}{[\log(x/x-1)]}$

$= \lim_{x\to\infty} \frac{a^{1/x}-b^{1/x}}{\log(x/x\{1-(1/x)\}]}$

$= \lim_{x\to\infty} \frac{a^{1/x}-b^{1/x}}{\log\{1/\{1-(1/x)\}\}]}$

$= \lim_{x\to\infty} \frac{a^{1/x}-b^{1/x}}{-\log\{1-(1/x)\}}$ [form 0/0]

$= \lim_{x\to\infty} \frac{(a^{1/x}\log a)(-1/x^2)-(b^{1/x}\log b)(-1/x^2)}{-1/\{1-(1/x)\}\cdot(1/x^2)}$ by Hospital's rule

$= \lim_{x\to\infty} \frac{b^{1/x}\log b - a^{1/x}\log a}{-1/\{1-(1/x)\}}$ cancelling $1/x^2$ from the Nr. and the Dr.

$= \frac{b^0 \log b - a^0 \log a}{-1/(1-0)}$

$= \frac{\log b - \log a}{-1}$

$= \log a - \log b$

$= \log(a/b)$

5(d) How far is the Point $(4,1,1)$ from the line of intersection of
$x+y+z-4=0=x-2y-z-4$?

The equations of the line AB (say) are
$(x-4)+y+z=0$ and $(x-4)-2y-z=0$

$\therefore$ The equations of the line in the symmetric form are

$$\frac{x-4}{-1+2}=\frac{y}{1+1}=\frac{z}{-2-1} \text{ (or) } \frac{x-4}{1}=\frac{y}{2}=\frac{z}{-3} \quad \text{---(i)}$$

A $(4,0,0)$ is any point on this line given by (i)
Here P is $(4,1,1)$ and let N, the foot of the perpendicular from P on the line (i), be at a distance r from A.

Then the coordinates of N are $(4+r, 2r, -3r)$ ---(ii)

$\therefore$ The d.r.'s of the line PN are
$(4+r)-4, 2r-1, 3r-1$ (or) $r, 2r-1, -3r-1$

Also from (i) we find that the d.c.'s of the line AB are $1, 2, -3$

Since PN is perpendicular to AB, so we have
$1 \cdot r + 2(2r-1) - 3(-3r-1) = 0$ (or) $14r+1=0$ (or) $r = -1/14$

$\therefore$ from (ii) the coordinates of N are $\left[\frac{55}{14}, -\frac{1}{7}, \frac{3}{14}\right]$

$\therefore$ The required distance $= PN$

$$= \sqrt{\left[\left\{4-\frac{55}{14}\right\}^2+\left\{1+\frac{1}{7}\right\}^2+\left\{1-\frac{3}{14}\right\}^2\right]}$$

$$= \sqrt{\left[\left(\frac{1}{14}\right)^2+\left(\frac{16}{14}\right)^2+\left(\frac{11}{14}\right)^2\right]}$$

$$= \frac{1}{14}\sqrt{[1+256+121]}$$

$$= (1/14)\sqrt{(378)} = (3/14)\sqrt{(42)}$$

5(e) Find the equation of the sphere whose centre is the point $(1,2,3)$ and which touches the plane $3x+2y+z+4=0$. Find also the radius of the circle in which the sphere is cut by the plane $x+y+z=0$

Sol^n: Since the sphere touches the plane $3x+2y+z+4=0$ ——— ①

So its radius = length of $\perp$lar from its centre $(1,2,3)$ on ①.

$$= \frac{3(1)+2(2)+1(3)+4}{\sqrt{3^2+2^2+1^2}}$$

$$= \frac{14}{\sqrt{14}} = \sqrt{14}$$

$\therefore$ The required equation of the sphere is

$$(x-1)^2+(y-2)^2+(z-3)^2 = (\sqrt{14})^2$$

$$\Rightarrow x^2+y^2+z^2-2x-4y-6z=0$$

The co-ordinates of C, the centre of the sphere are given as $(1,2,3)$ and OC being the perpendicular to the plane $x+y+z=0$, its direction ratios are the same as those of the normal to this plane i.e., $1,1,1$ the coefficients of x, y, z in the equation $x+y+z=0$.

$\therefore$ The equation of the line OC are

$$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1} = r \text{ (say)}$$

~~Let OC = r~~, then the co-ordinates of C are $(r+1, r+2, r+3)$ and C lies on the plane $x+y+z=0$.

$\therefore (r+1)+(r+2)+(r+3)=0$

$\Rightarrow r=-2$.

$\therefore$ The co-ordinates of C are

$(-2+1, -2+2, -2+3)$ (or) $(-1, 0, 1)$

6(a) Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}.$$

(i) Find a basis for V

(ii) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V

(iii) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its co-ordinates in the basis chosen for part (i) ?

<u>Sol</u>. we have

$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 3 & 21 & 0 & 9 & 0 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix} \quad R_2 \to \tfrac{1}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 5R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

∴ clearly it is in echelon form.

∴ the basis of V is

$\{(1,7,-1,-2,-1), (0,0,1,5,1), (0,0,0,0,1)\}$.

(ii) We have

$(x_1, x_2, x_3, x_4, x_5) = \alpha(1, 7, -1, -2, -1) + \beta(0, 0, 1, 5, 1) + \gamma(0, 0, 0, 0, 1)$ —— (1)

$\Rightarrow \boxed{x_1 = \alpha}, \boxed{x_2 = 7\alpha},$

$x_3 = -\alpha + \beta$, $x_4 = -2\alpha + 5\beta$

$x_5 = -\alpha + \beta + \gamma$.

$\Rightarrow (x_1, x_2, x_3, x_4, x_5) = (\alpha, 7\alpha, -\alpha + \beta, -2\alpha + 5\beta, -\alpha + \beta + \gamma) \in V$

(iii) $\therefore (x_1, x_2, x_3, x_4, x_5) = x_1(1, 7, -1, -2, -1) + (x_3 + x_1)(0, 0, 1, 5, 1) + (x_5 - x_3)(0, 0, 0, 0, 1)$.

$\therefore$ The co-ordinates of the vector $(x_1, x_2, x_3, x_4, x_5)$ w.r.t basis of V are x_1, $x_1 + x_3$, $-x_3 + x_5$

6(b) Let F be the field of complex numbers and $T: F^3 \to F^3$ be given as
$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$
(i) Show that T is a linear transformation.
(ii) Find its rank and nullity
(iii) find conditions on a, b, c such that $(a, b, c) \in$ Range T and $(a, b, c) \in$ null space of T.

Sol: Let $T: F^3 \to F^3$ defined by
$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$ — (1)
and F is a subfield of complex numbers

(i) Let $\alpha, \beta \in F^3$ such that $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$
and $a, b \in F$ Then we have
$a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$
$= (ax_1 + by_1,\ ax_2 + by_2,\ ax_3 + by_3)$
Now we have
$T(a\alpha + b\beta) = T(ax_1 + by_1,\ ax_2 + by_2,\ ax_3 + by_3)$
$= (ax_1 + by_1 - ax_2 - by_2 + 2ax_3 + 2by_3,$
$2ax_1 + 2by_1 + ax_2 + by_2,$
$-ax_1 - by_1 - 2ax_2 - 2by_2 + 2ax_3 + 2by_3)$
$= \big(a(x_1 - x_2 + 2x_3) + b(y_1 - y_2 + 2y_3),$
$a(2x_1 + x_2) + b(2y_1 + y_2)$
$+ a(x_1 - 2x_2 + 2x_3) + b(-y_1 - 2y_2 + 2y_3)\big)$
$= a(x_1 - x_2 + 2x_3,\ \cancel{2x_1 + x_2},\ 2x_1 + x_2,\ 2y_1 + y_2)$
$+ b(y_1 - y_2 + 2y_3,\ 2y_1 + y_2,\ -y_1 - 2y_2 + 2y_3)$
$= aT(\alpha) + bT(\beta).$
$\therefore T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$\therefore T: F^3 \to F^3$ is a linear transformation.

Let $R(T) = \{T(\alpha) / \alpha \in F^3\}$

$= \{(x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3) / x_1, x_2, x_3 \in F\}$

be the range space.

Let $(a, b, c) \in R(T)$ if $(a, b, c) \in T(x_1, x_2, x_3)$

for some $(x_1, x_2, x_3) \in F^3$

then
$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= a \\ 2x_1 + x_2 &= b \\ -x_1 - 2x_2 + 2x_3 &= c \end{aligned}\right\} \quad (1)$$

$\Rightarrow AX = B$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c+a \end{bmatrix} \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ b+c-a \end{bmatrix} \quad R_3 \to R_3 + R_2$$

Hence $b + c - a = 0 \Rightarrow a = b + c$ is the required condition such that $(a, b, c) \in$ Range T.

Let $(x_1, x_2, x_3) \in \operatorname{Ker} T$ then

$T(x_1, x_2, x_3) = (0, 0, 0)$

Using (1), we obtain
$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}\right\} \quad (2)$$

$\Rightarrow AX = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad \text{--- (3)}$$

Since the rank of the coefficient matrix is 2, the no. of L.I. solutions of the system (2) is $n - r = 3 - 2 = 1$

from (3), we have

$$\left.\begin{matrix} x_1 - x_2 + 2x_3 = 0 \\ 3x_2 - 4x_3 = 0 \end{matrix}\right\} \text{--- (4)}$$

we can give an arbitrary value to x_3

Taking $x_3 = 3$, we get $x_2 = 4$ and $x_1 = -2$

$(x_1, x_2, x_3) = (-2, 4, 3)$ is the only L.I. element of $\ker T$ and so

$\ker T = \{a(-2, 4, 3) \mid a \in \mathbb{R}\}$

We have $\dim \ker T = 1$.

Since $\{(-2, 4, 3)\}$ is a basis of $\ker T$.

$\therefore$ Nullity $T = 1$.

Since Rank T + Nullity T = $\dim \mathbb{R}^3$

$\Rightarrow$ Rank + 1 = 3

$\Rightarrow$ Rank T = 2

Let $(a,b,c) \in \ker T$ if $T(a,b,c) = (0,0,0)$

Then $a-b+2c = 0$

$2a+b = 0$

$-a-2b+2c = 0$

$\Rightarrow a-2b+2c = 0, \quad 3b-4c = 0$

Hence $a = -2k, \; b = 4k, \; c = 3k$

where k is arbitrary.

are the required conditions if $(a,b,c) \in \ker T$.

$\Rightarrow$ 7(a)

6(c) Find the characteristic values and bases of the corresponding characteristic spaces of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$$

Is A similar to a diagonal spaces of the matrix

Solution: The characteristic equation of A is

$$\begin{vmatrix} 2-x & 1 & 0 \\ 0 & 1-x & -1 \\ 0 & 2 & 4-x \end{vmatrix} = 0$$

or $(2-x)\{(1-x)(4-x)+2\}=0$

or $(2-x)\{x^2-5x+6\}=0$ or $(2-x)(x-2)(x-3)=0$

Hence the characteristic values are 2, 2, 3.

The characteristic vector corresponding to $x=2$ is given by

$$(A-2I)x=0$$

or $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, by $R_3 \to R_3 + 2R_2$

or $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, by $R_2 \to R_2 + R_1$

$\Rightarrow x_2 = 0, x_3 = 0$ and x_1 can be given any value. We take $x_1 = 1, x_2 = 0, x_3 = 0$. Clearly, there is only one L.I. vector corresponding to the characteristic value 2 viz.

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The characteristic vector corresponding to $\lambda = 3$ is given by

$$(A - 3I)\, x = 0$$

$$\text{or} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_2$$

$\therefore -x_1 + x_2 = 0$ and $2x_2 + x_3 = 0$

These equations are satisfied by $x_1 = 1, x_2 = 1, x_3 = 2$.

A characteristic vector corresponding to the characteristic value $x = 3$ is

$$x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

The characteristic space W_2 corresponding to the characteristic value $x = 2$ is spanned by x_1. Hence $\{x_1\}$ is a basis of W_2. Similarly, $\{x_2\}$ is a basis of W_3, the characteristic space corresponding to the characteristic value $x = 3$. We have obtained two L.I. characteristic vectors x_1 and x_2 corresponding to the characteristic values 2, 2, 3 of A, so A is not diagonalizable i.e., there does not exist any invertible matrix P such that

$$P^{-1}AP = \text{diag}(2, 2, 3)$$

Hence A is not similar to a diagonal matrix.

7(a) Let $f(x) = x\left\{1+\frac{1}{3}\sin\log(x^2)\right\}$ for $x \neq 0$, $f(x) = 0$ for $x = 0$. show that $f(x)$ is continuous and monotone.

Sol: Here $f(0) = 0$

$f(0+) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} h\left[1+\frac{1}{3}\sin\log(h^2)\right]$

$= 0 \quad (\because -1 < \sin\log h^2 < 1$

Also $f(0-) = \lim_{h\to 0} -h\left[1+\frac{1}{3}\sin\log(h^2)\right]$

$= 0$

$\therefore$ the function is continuous at $x = 0$.

we shall show that even when $x \neq 0$.

$\lim_{h\to 0}\left[f(x+h) - f(x)\right] = 0$

i.e, we shall prove $f(x)$ to be continuous for all non-zero values of x and also that $f(x)$ is monotone.

Now, $f(x+h) - f(x)$ (taking h as +ve)

$= (x+h)\left[1+\frac{1}{3}\sin\log(h^2)\right]$

$-x\left[1+\frac{1}{3}\sin\log(x^2)\right]$

$= h + \frac{x}{3}\left[\sin\log(x+h)^2 - \sin\log(x^2)\right]$

$\frac{h}{3}\left[\sin\log(x+h)^2\right]$

$$= h\left[1+\frac{1}{3}\sin\log(x+h)^2\right] + \frac{2x}{3}\left[\cos\log\{x(x+h)\} \times \sin\log\left(1+\frac{h}{x}\right)\right]$$

$$= h\left[1+\frac{1}{3}\sin\log(x+h)^2\right] + \frac{2x}{3}\left[\cos\left\{\log x^2+\log\left(1+\frac{h}{x}\right)\right\} \times \sin\left(\frac{h}{x}-\frac{h^2}{2x^2}+\cdots\right)\right]$$

$$= h\left[1+\frac{1}{3}\sin\log(x+h)^2\right] + \frac{2x}{3}\cos\left(\log x^2+\frac{h}{x}-\frac{h^2}{2x^2}+\cdots\right)\left[\frac{h}{x}-\frac{h^2}{2x^2}+\cdots\right]$$

$$= h\left[1+\frac{1}{3}\sin\log(x+h)^2\right] + \frac{2x}{3}\left(\frac{h}{x}\right)\cos\left(\log x^2+\frac{h}{x}\right)$$

leaving out higher powers of h.

Taking -1 as the least value of the sine and the cosine.

So $f(x)$ is an ever increasing function.

i.e., monotonic and continuous

7(b) Show that $\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}\,dx$

Find the values of the two integrals.

Sol: $$LHS = \int_0^1 dx\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,dy$$

$$= \int_0^1 dx\int_0^1 \left\{\frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}\right\} dy$$

$$= \int_0^1 \left[\frac{-x}{(x+y)^2}+\frac{1}{x+y}\right]_0^1 dx.$$

$$= \int_0^1 \left[\frac{-x}{(1+x)^2} + \frac{1}{x} + \frac{1}{1+x} - \frac{1}{x} \right] dx$$

$$= \int_0^1 \frac{dx}{(1+x)^2} = \left[-\frac{1}{1+x} \right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

$$RHS = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$$

$$= \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx$$

$$= \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left(-\frac{1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y} \right) dy$$

$$= -\int_0^1 \frac{dy}{(1+y)^2}$$

$$= \left[\frac{1}{1+y} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}.$$

LHS $\neq$ RHS

The two integrals are not equal.

7(c) Find the minimum value of $x^4 + y^4 + z^4$, where $xyz = c^3$

Sol^n. Let $u = x^4 + y^4 + z^4$ —— (1)

where the variables x, y, z are connected by the relation $xyz = c^3$ —— (2)

for a maximum or a minimum of u, we have

$du = 0 \Rightarrow 4x^3 dx + 4y^3 dy + 4z^3 dz = 0$

$\Rightarrow x^3 dx + y^3 dy + z^3 dz = 0$ —— (3)

Also from the given relation (2), we have

$\log x + \log y + \log z = \log c^3$.

Differentiating this, we get

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ —— (4)

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$x^3 + \frac{\lambda}{x} = 0$, $y^3 + \frac{\lambda}{y} = 0$, $z^3 + \frac{\lambda}{z} = 0$

$\Rightarrow x^4 = y^4 = z^4 = -\lambda$

Now from (2), $x^4 y^4 z^4 = c^{12}$.

$\Rightarrow (-\lambda)^3 = c^{12}$

$\lambda = -c^4$.

$\therefore$ u is stationary when $x^4 = y^4 = z^4 = c^4$

i.e., when $x = y = z = c$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have $\frac{\partial u}{\partial x} = 4x^3 + 4z^3 \frac{\partial z}{\partial x}$.

Now from (2), we have $\log x + \log y + \log z = \log c^3$

Differentiating this partially w.r.t. x taking y as constant, we get

$\frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}$.

$\therefore \frac{\partial u}{\partial x} = 4x^3 - 4z^3 \cdot \frac{z}{x} = 4x^3 - 4\frac{z^4}{x}$.

$\frac{\partial^2 u}{\partial x^2} = 12x^2 + \frac{4z^4}{x^2} - \frac{16}{x} z^3 \frac{\partial z}{\partial x}$.

$= 12x^2 + \frac{4z^4}{x^2} - \frac{16 z^3}{x}\left(-\frac{z}{x}\right) = 12x^2 + \frac{4z^4}{x^2} + \frac{16z^4}{x^2}$

At the stationary point (c, c, c) found above, we have $\frac{\partial^2 u}{\partial x^2} = 12c^2 + 4c^2 + 16c^2 = 32c^2$.

which is +ve.

$\therefore$ u is minimum at the point $x = y = z = c$

and the minimum value of u is $c^4 + c^4 + c^4 = 3c^4$.

Ex: If $0 < x < 1$, show that $2x < \log \frac{1+x}{1-x} < 2x\left(1 + \frac{1}{3}\frac{x^2}{1-x^2}\right)$

Deduce that $e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e \cdot e^{\frac{1}{12n(n+1)}}$

Sol: To show that $2x < \log \frac{1+x}{1-x}$

consider the function f defined on $[0,1)$

by setting $f(x) = 2x - \log \frac{1+x}{1-x}$, $\forall\, x \in [0,1)$

If c be any real number $[0,1)$, then f is

continuous in $[0, c]$ and derivable in $(0, c)$.

Also $f'(x) = \frac{-2x^2}{1-x^2}$

so that $f'(x) < 0$ in $(0, c)$.

This shows that f is strictly decreasing

in $[0, c]$.

In particular, $f(c) < f(0)$

since $f(0) = 0$, this means that

$2x - \log \frac{1+x}{1-x} < 0$ when $x = c$.

Since c is any real number in $[0,1)$, it follows that $2x < \log\frac{1+x}{1-x}$, whenever $0<x<1$ —— (1)

To show that

$$\log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)$$

consider the function g defined by setting

$$g(x) = \log\frac{1+x}{1-x} - 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)$$

Then $g'(x) = -\frac{4}{3}\frac{x^4}{(1-x^2)^2}$.

Since $g'(x) < 0 \ \forall x \in [0,1)$ and since $g(0)=0$

therefore, it follows in the same manner as above that $g(x) < 0$ whenever $0<x<1$

that is $\log\left(\frac{1+x}{1-x}\right) < 2x\left(1+\frac{1}{3}\frac{x^2}{1-x^2}\right)$ —— (2)

whenever $0<x<1$

putting $x = \frac{1}{2n+1}$ in (1), (2) we have

$$\frac{2}{2n+1} < \log\frac{n+1}{n} < \frac{2}{2n+1}\left(1+\frac{1}{12(n+1)}\right)$$

$$1 < \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) < 1+\frac{1}{12n(n+1)}$$

so that $e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$

which is the desired inequality.

81(a) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C are $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

$\therefore$ The d.c.'s of sides AB and AC of the ΔABC are

$$\frac{a}{\sqrt{a^2+b^2}}, \frac{-b}{\sqrt{a^2+b^2}}, 0 \quad \text{and} \quad \frac{a}{\sqrt{a^2+c^2}}, 0, \frac{-c}{\sqrt{a^2+c^2}}$$ respectively.

Also we know that the d.r.'s of the internal bisector of the two lines whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2 are given by

$$\frac{1}{2}(l_1+l_2), \frac{1}{2}(m_1+m_2), \frac{1}{2}(n_1+n_2)$$

$\therefore$ The d.c's of the internal bisector of AB and AC are

$$\frac{1}{2}\left[\frac{a}{\sqrt{a^2+b^2}} - \frac{a}{\sqrt{a^2+c^2}}\right], \frac{1}{2}\left[-\frac{b}{\sqrt{a^2+b^2}}-0\right], \frac{1}{2}\left(0+\frac{c}{\sqrt{a^2+c^2}}\right)$$

$$\Rightarrow \frac{1}{2}a\left[\frac{1}{\sqrt{a^2+b^2}}-\frac{1}{\sqrt{a^2+c^2}}\right], -\frac{1}{2}\frac{b}{\sqrt{a^2+b^2}}, \frac{1}{2}\frac{c}{\sqrt{a^2+c^2}}$$

l, m, n (say) —— (1)

Any plane through x-axis i.e., $y=0, z=0$ is $y+\lambda z=0$ —— (2)

If the internal bisector of AB and AC whose d.c's l, m, n are given by (1) above lies on the plane (2) then we have

$l.0 + m.1 + n.\lambda = 0$

$\Rightarrow \lambda = -\frac{m}{n}$.

$\therefore$ from (2), the equation of the plane through x-axis and the internal bisector of AB and AC is $y - \frac{m}{n}z = 0$

$\Rightarrow \frac{y}{m} = \frac{z}{n}$.

$\Rightarrow \frac{2y\sqrt{a^2+b^2}}{b} = \frac{2z\sqrt{a^2+c^2}}{c}$ (by putting the values of m, n)

$\Rightarrow \frac{y}{b\sqrt{a^2+c^2}} = \frac{z}{c\sqrt{a^2+b^2}}$ — (3)

Similarly the equations of the other planes are $\frac{z}{c\sqrt{c^2+a^2}} = \frac{x}{a\sqrt{b^2+c^2}}$ — (4)

and $\frac{x}{a\sqrt{b^2+c^2}} = \frac{y}{b\sqrt{a^2+c^2}}$ — (5)

clearly the line of intersection of the planes (3), (4) and (5) is

$$\frac{x}{a\sqrt{b^2+c^2}} = \frac{y}{b\sqrt{a^2+c^2}} = \frac{z}{c\sqrt{a^2+b^2}}.$$

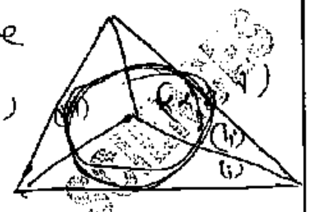
Hence proved.

8(3) Find the equation of the sphere inscribed in the tetrahedron formed by the planes whose equations are
$y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=1$

Sol: The given planes are

$y+z=0$ — (i) $z+x=0$ — (ii)

$x+y=0$ — (iii), $x+y+z=1$ — (iv)



Let (α, β, γ) be the centre of sphere inscribed in the tetrahedron then the perpendicular distances of the centre from all the four planes (i), (ii), (iii) & (iv) (faces) are equal, and each equal to radius (r)

$$\therefore \frac{|\beta+\gamma|}{\sqrt{2}} = \frac{|\gamma+\alpha|}{\sqrt{2}} = \frac{|\alpha+\beta|}{\sqrt{2}} = \frac{|\alpha+\beta+\gamma-1|}{\sqrt{3}} = r$$

$$\Rightarrow \left.\begin{array}{l} \alpha+\beta = \sqrt{2}r \\ \alpha+\gamma = \sqrt{2}r \\ \beta+\gamma = \sqrt{2}r \end{array}\right\} \text{— (1)} \quad \text{and} \quad \alpha+\beta+\gamma-1 = \sqrt{3}r \text{ — (2)}$$

solving (1), we get

$$\alpha = \beta = \gamma = \frac{r}{\sqrt{2}}$$

Eliminating α, β, γ, we get

$$\frac{3r}{\sqrt{2}} - 1 = \sqrt{3}r \Rightarrow r(3+\sqrt{6}) = \sqrt{2}$$

$$r = \frac{\sqrt{2}}{3+\sqrt{6}}$$

$\therefore \alpha = \beta = \gamma = \frac{1}{3+\sqrt{6}}$.

Hence the centre (α, β, γ) is

$\left(\frac{1}{3+\sqrt{6}}, \frac{1}{3+\sqrt{6}}, \frac{1}{3+\sqrt{6}}\right)$ and radius r is $\frac{\sqrt{2}}{3+\sqrt{6}}$

Thus the equation of the sphere is given by

$$\left(x-\frac{1}{3+\sqrt{6}}\right)^2+\left(y-\frac{1}{3+\sqrt{6}}\right)^2+\left(z-\frac{1}{\sqrt{6}+3}\right)^2=\left(\frac{\sqrt{2}}{3+\sqrt{6}}\right)^2$$

$$x^2+y^2+z^2-\frac{2}{3+\sqrt{6}}(x+y+z)+\frac{3}{(3+\sqrt{6})^2}=\frac{2}{(3+\sqrt{6})^2}$$

$$\Rightarrow x^2+y^2+z^2-\frac{2}{3+\sqrt{6}}(x+y+z)+\frac{1}{(3+\sqrt{6})^2}=0$$

which is the required result

8(c) The generators through P of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meets the principal elliptic section of A and B. If the median of the triangle APB through P is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, show that P lies on the surface $z(\alpha x+\beta y)+\gamma(c^2+z^2)=0$.

Sol^n:- Let the co-ordinates of P, A and B be (x_1, y_1, z_1), $(a\cos\theta, b\sin\theta, 0)$ and $(a\cos\phi, b\sin\phi, 0)$ respectively

Also the co-ordinates of F, the mid point of AB are $\left(\frac{1}{2}(a\cos\theta+a\cos\phi), \frac{1}{2}(b\sin\theta+b\sin\phi), 0\right)$

$= \left(a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, 0\right)$

$\therefore$ Direction ratios of the median PF through P are

$x_1-a\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right), \; y_1-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, \; z_1-0.$ —— (1)

[The values of x_1, y_1, z_1 can be found as follows.

The equation of ~~the~~ tangent to the given hyperboloid at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$ and it meets the plane $z=0$ in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0 \quad \text{--- (i)}$$

which is the same as the line joining the points A and B.

i.e, $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2},\ z=0$ --- (ii)

Comparing (i) & (ii), we get

$$\frac{x_1/a^2}{\frac{1}{a}\cos\frac{\theta+\phi}{2}}=\frac{y_1/b^2}{\frac{1}{b}\sin\frac{\theta+\phi}{2}}=\frac{1}{\cos\frac{\theta-\phi}{2}}$$

$$\Rightarrow \frac{x_1}{a}=\frac{\cos(\theta+\phi)/2}{\cos\frac{\theta-\phi}{2}},\quad \frac{y_1}{b}=\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \quad \text{--- (iii)}$$

Again $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1$

$$\Rightarrow \frac{1}{\cos^2\frac{\theta-\phi}{2}}-\frac{z_1^2}{c^2}=1 \Rightarrow \frac{z_1^2}{c^2}=\sec^2\left(\frac{\theta-\phi}{2}\right)-1$$

$$=\tan^2\frac{\theta-\phi}{2}$$

$$\Rightarrow \frac{z_1}{c}=\pm\frac{\sin\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\therefore P(x_1,y_1,z_1)\equiv\left(\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ \frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ \frac{\pm c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}\right)$$

$\therefore$ from (1), we have.

$$\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}-a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\quad \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

$\Rightarrow \dfrac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right), \dfrac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right),$

$$\dfrac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

$\Rightarrow a\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},$

$c\tan\frac{\theta-\phi}{2}\operatorname{cosec}\frac{\theta-\phi}{2}.$

$\Rightarrow x_1, y_1, z_1 \operatorname{cosec}^2\frac{\theta-\phi}{2}$

$\Rightarrow x_1, y_1, z_1\left(1+\cot^2\frac{\theta-\phi}{2}\right)$ $\quad \frac{z_1}{c} = \tan\frac{\theta-\phi}{2}$

$\Rightarrow x_1, y_1, z_1\left(1+\frac{c^2}{z_1^2}\right)$

As PF is parallel to the plane

$\alpha x+\beta y+\gamma z=0$

$\therefore \alpha x_1+\beta y_1+\gamma z_1\left(1+\frac{c^2}{z_1^2}\right)=0$

$\Rightarrow (\alpha x_1+\beta y_1)z_1+\gamma(z_1^2+c^2)=0$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$

is $z(\alpha x+\beta y)+\gamma(z^2+c^2)=0$

— Hence proved

Main Test Series - 2016
Test - 2 - Answer key
paper-II - MA, Real Analysis, CA & LPP

1(a) Let $G = \{a \in \mathbb{R} / -1 < a < 1\}$. Define $*$ on G by $a*b = \frac{a+b}{1+ab}$ $\forall a,b \in G$. Show that $*$ is a binary operation on G. Hence prove that $(G,*)$ is a group

Sol: Let $G = \{a \in \mathbb{R} / -1 < a < 1\}$

observe that $-1 < x < 1$ iff $x^2 < 1$ $\forall x \in \mathbb{R}$

Let $a, b \in G$.

First let us show that $a*b \in G$.

Now $a^2 < 1$ and $b^2 < 1 \Rightarrow 1-a^2 > 0$, & $1-b^2 > 0$

$\therefore (1-a^2)(1-b^2) > 0$

$\Rightarrow 1-a^2-b^2+a^2b^2 > 0$. — ①

Now $(1+ab)^2-(a+b)^2 = 1+a^2b^2+2ab-a^2-b^2-2ab$
$= 1-a^2-b^2+a^2b^2 > 0$ ($\because$ from ①)

$\Rightarrow \frac{(a+b)^2}{(1+ab)^2} < 1$

$\therefore a*b \in G$

Hence G is closed under $*$.

we now show that $*$ is well defined.

Let $a, b, c, d \in G$, $a = c$, and $b = d$

Thus $a*b = \frac{a+b}{1+ab} = \frac{c+d}{1+cd} = c*d$.

and hence $*$ is well defined.

To show the associativity of $*$.

Let $a, b, c \in G$.

Now $(a*b)*c = \frac{a+b}{1+ab} * c$

$$= \frac{\frac{a+b}{1+ab} + c}{1+\left(\frac{a+b}{1+ab}\right)c}$$

$$= \frac{a+b+c+abc}{1+ab+bc+ac}.$$

Similarly $a*(b*c) = \frac{a+b+c+abc}{1+ab+bc+ac}$.

$\therefore (a*b)*c = a*(b*c)$

and so $*$ is associative.

Hence $(G, *)$ is a semigroup.

Note that $0 \in G$ so that

$0*a = \frac{0+a}{1+0(a)} = a \ \forall a \in G$

This shows that the semigroup $(G,*)$ has a left identity.

Let $a \in G$. Then $-a \in G$

and $(-a)*a = \frac{-a+a}{1+(-a)(a)} = 0$

Thus in $(G,*)$ for each $a \in G$, $\exists$ a left inverse. $(G,*)$ is a group

11(b) Show that x^2+1 is irreducible over the integers mod 7

Sol: we have $F = Z_7 = \{0,1,2,3,4,5,6\}$.

Let $x^2+1 = (x+a)(x+b)$, $a,b \in F$

comparing the coefficients of x and constants on both the sides, we get

$0 = a+b$ — ①

$1 = ab$ — ②

① is satisfied when $(a,b) = (0,0), (1,6),$ $(2,5), (3,4), (4,3), (5,2), (6,1)$.

For these values of a and b,

$ab = 0, 6, 3, 5, 5, 3, 6$

Thus ② is not satisfied and

so x^2+1 is irreducible over F.

Aliter: Since $f(x) = x^2+1$ is not satisfied by the elements of F. (i.e, $f(\alpha) \neq 0 \ \forall \alpha \in F$).

$\therefore f(x)$ has no linear factors in $F[x]$.

This shows that $f(x) = x^2+1$ is irreducible over $F = Z_7$.

1(c) Show that the series

$$\sum_{n=1}^{\infty} \frac{3\cdot6\cdot9\cdots3n}{7\cdot10\cdot13\cdots(3n+4)}\, x^n,\quad x>0$$

converges for $x\le1$, and diverges for $x>1$.

Sol^n: Let $u_n = \dfrac{3\cdot6\cdot9\cdots3n}{7\cdot10\cdot13\cdots(3n+4)}$

$\Rightarrow u_{n+1} = \dfrac{3\cdot6\cdot9\cdots(3n)(3n+3)}{7\cdot10\cdot13\cdots(3n+4)(3n+7)}$

$\therefore \dfrac{u_n}{u_{n+1}} = \dfrac{3n+7}{3n+3}\cdot\dfrac{1}{x}$

$= \dfrac{1+\frac{7}{3n}}{1+\frac{1}{n}}\cdot\dfrac{1}{x} \to \dfrac{1}{x}$ as $n\to\infty$

$\therefore$ By Ratio test $\sum u_n$ converges if $\frac{1}{x}>1$ i.e. $x<1$ and diverges if $\frac{1}{x}<1$ i.e. $x>1$.

If $x=1$ then the ratio test fails.

when $x=1$, $\dfrac{u_n}{u_{n+1}} = \dfrac{3n+7}{3n+3}$

$\therefore n\left(\dfrac{u_n}{u_{n+1}}-1\right) = n\left(\dfrac{3n+7}{3n+3}-1\right)$

$= n\left(\dfrac{4}{3n+3}\right) = \dfrac{4n}{3n+3}$

$\therefore \lim_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\dfrac{4n}{3n+3} = \lim_{n\to\infty}\dfrac{4}{3+\frac{3}{n}}$

$= \frac{4}{3} > 1.$

$\therefore$ By Raabe's test, the series converges.

Hence the given series cgs if $x\le1$ and diverges for $x>1$.

1(d) prove that the function

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

satisfies Laplaces equation and determine corresponding analytic function u+iv.

Sol^n: Here $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \phi_1(x,y)$ say

$\frac{\partial u}{\partial y} = -6xy - 6y = \phi_2(x,y)$ say

$\frac{\partial^2 u}{\partial x^2} = 6x + 6$ and $\frac{\partial^2 u}{\partial y^2} = -6x - 6$.

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0$.

i.e. u satisfies Laplaces equation. Hence u is a harmonic function.

By Milne's Method we have

$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$

$= 3z^2 + 6z$.

Integrating, we get

$f(z) = z^3 + 3z^2 + c$.

1(e) A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents N_1, N_2, N_3 is necessary to buy two additional products P_1 and P_2. One unit of P_1 contains 36 units of N_1, 3 units of N_2 and 20 units of N_3. One unit of product P_2 contains 6 units of N_1, 12 units of N_2 and 10 units of N_3. The minimum requirement of N_1, N_2 and N_3 is 108 units, 36 units and 100 units respectively. Product P_1 costs ₹20 per unit and P_2 costs ₹40 per unit. Formulate this Diet problem as an LP model and solve it graphically.

Sol: On the basis of the information, the appropriate mathematical formulation of LP model is

Minimize $Z = 20x_1 + 40x_2$

Subject to

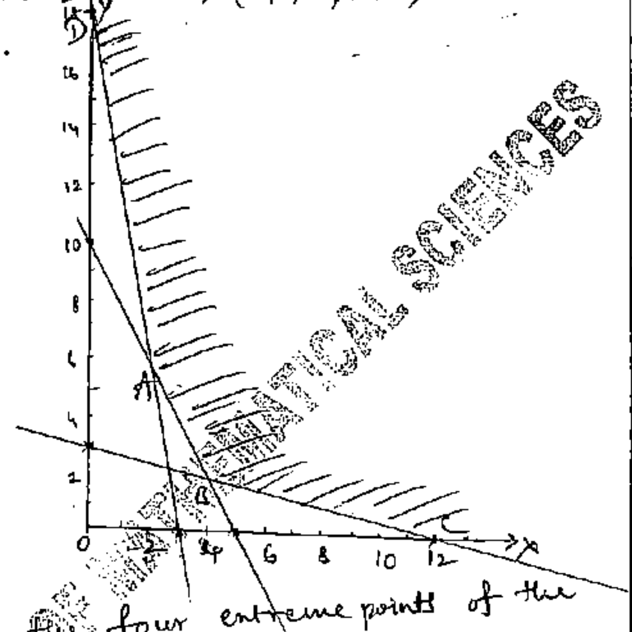
$$36x_1 + 6x_2 \geq 108$$
$$3x_1 + 12x_2 \geq 36$$
$$20x_1 + 10x_2 \geq 100$$
$$x_1, x_2 \geq 0.$$

Graphical Solution:

$36x_1 + 6x_2 = 108 \Rightarrow (3,0),\ (0,18)$

$3x_1 + 12x_2 = 36 \Rightarrow (0,3),\ (12,0)$

$20x_1 + 10x_2 = 100 \Rightarrow (5,0),\ (0,10)$



ABCD are the four extreme points of the unbounded feasible region satisfying the constraints simultaneously.

Coordinates of extreme points in the shaded region:

$A(2,6) \longrightarrow$ point of intersection of $36x_1+6x_2=108$ and $20x_1+10x_2=100$.

$B(4,2) \longleftrightarrow$ point of intersection of $3x_1+12x_2=36$ and $20x_1+10x_2=100$.

$C(12,0)$

$D(0,18)$.

values of the objective function at extreme points $z_A = 280$, $z_B = 160$, $z_C = 240$, $z_D = 720$.

Since the problem is a minimisation problem, the optimum solution admits at any one corner in the finite region of unbounded feasible space. Here minimum occurs at the point $B(4,2)$.

Therefore, the required optimum solution is $x_1 = 4$

$x_2 = 2$

$Z_{min} = 160$

2(a) (i) Let R be the ring of all real valued continuous functions on $[0,1]$. Show that set $S = \{f \in R / f(\frac{1}{2}) = 0\}$ is an ideal of R.

(ii) Give an example of two ideals A and B of R such that $A \subseteq B \subseteq R$, where A is an ideal of B, B is an ideal of R, but A is not an ideal of R.

Sol: (i) Let $f, g \in S$. Then $f(\frac{1}{2}) = 0$ and $g(\frac{1}{2}) = 0$ —— (1)

Consider $(f-g)(\frac{1}{2}) = f(\frac{1}{2}) - g(\frac{1}{2}) = 0 - 0 = 0$

$\therefore (f-g)\left(\frac{1}{2}\right)=0 \Rightarrow f-g \in S$

Let $f \in S$ and $h \in R$. Then

$$(fh)\left(\tfrac{1}{2}\right)=f\left(\tfrac{1}{2}\right)h\left(\tfrac{1}{2}\right)=0\cdot h\left(\tfrac{1}{2}\right)=0$$

and $(hf)\left(\frac{1}{2}\right)=h\left(\frac{1}{2}\right)f\left(\frac{1}{2}\right)=h\left(\frac{1}{2}\right)\cdot 0=0$

Thus $fh, hf \in S$ $\forall f \in S$ and $h \in R$

Hence S is an ideal of R.

(ii) Hint:

Let $R=\left\{\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & z\end{pmatrix} \middle/ x_i, y_i, z \text{ are integers}\right\}$

$B=\left\{\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0\end{pmatrix} \middle/ x, y \text{ are integers}\right\}$

$A=\left\{\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} \middle/ x \text{ is an integer}\right\}$

Then A is an ideal of B, B is an ideal of R. But A is not an ideal of R.

Since $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix} \notin A$

2(b)(i) Show that the function f defined by

$$f(x)=x\,\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} \quad \text{if } x\neq 0$$
$$=0 \quad \text{if } x=0$$

is continuous at $x=0$.

(ii) Test for convergence $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$.

Sol^n: (i) we have

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h), \quad h>0$$

$$= \lim_{h\to 0} (-h)\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}} = 0$$

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} (h)\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}} = 0$$

$\therefore f(0-) = f(0+) = f(0) = 0$

$\therefore f$ is continuous at $x=0$.

(ii) Here $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$

1 is the only point of infinite discontinuity of f on $[0,1]$

Take $g(x) = \frac{1}{\sqrt{1-x}}$ then

$\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}}$ which is non zero and finite

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converge or diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ converges $(\because n = \frac{1}{2} < 1)$

$\therefore \int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{\sqrt{1-x^3}}$ is convergent

2(c) Find different developments of $\frac{1}{(z-1)(z-3)}$ in powers of z; according to the position of the point z in the plane. Expand the function in a Taylor's series about $z=2$ and indicate the circle of convergence.

Solⁿ: Here $f(z) = \frac{1}{(z-1)(z-3)}$

$$= \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

Thus function is regular every where except at $z=1$, $z=3$. We give below different expansion of $f(z)$ according to the positions of z in plane.

(i) when $0<|z|<1$

we have $f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$

$$= -\frac{1}{6}\left[1-\frac{z}{3}\right]^{-1} + \frac{1}{2}(1-z)^{-1}$$

arranged suitably in order to make binomial expansions valid for $0<|z|<1$.

$$= -\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n + \frac{1}{2}(z)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{2}\left(1-\frac{1}{3^{n+1}}\right)z^n$$

which is obviously Taylor's expansion of $f(z)$ valid in $0<|z|<1$.

(ii) $1<|z|<3$

Then $f(z) = \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$

$= -\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$

arranged suitably

$= -\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$

which is obviously a Laurent's series in the positive and the negative powers of z within the annulus $1<|z|<3$.

(iii) When $|z|>3$

then $f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$

$= \frac{1}{2z}\left[1-\frac{3}{z}\right]^{-1} - \frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1}$

$= \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$

$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(3^n-1)}{z^{n+1}}$

which is obviously a Laurent's series, in negative power of z, with annulus $3<|z|<R$.

If the centre of a circle is at $z=2$ then distance of both the singularities $z=1$ and $z=3$ from this centre is 1.

Hence if a circle is drawn with centre

at $z=2$ and radius 1, then within this circle i.e, with in the circle $|z-2|=1$ the function $f(z)$ is regular, hence it can be expanded in a Taylor's series within the circle $|z-2|=1$, which is therefore the circle of convergence.

Now, $f(z) = \dfrac{1}{(z-1)(z-3)} = \dfrac{1}{z^2-4z+3}$

$$= \frac{1}{(z-2)^2 - 1}$$

$$= -\left[1-(z-2)^2\right]^{-1}$$

expansion is now valid for $|z-2|<1$

$$= -\sum_{n=0}^{\infty} (z-2)^{2n}$$

which is obviously a Taylor's series in positive powers of $(z-2)$ within a circle $|z-2|=1$

3(a) (i) Find all the homomorphisms of the group $(\mathbb{Z},+)$ to the group $(\mathbb{Z},+)$.

(ii) Let G be a group. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Sol^n:

(i) Let n be an integer

Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(t) = nt \ \forall t \in \mathbb{Z}$

Let $r, s \in \mathbb{Z}$. Then $f(r+s) = n(r+s)$
$= nr + ns$

Hence f is a homomorphism.

we denote this homomorphism by f_n.

we show that any homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is one of these f_n, $n \in \mathbb{Z}$

To show this, let us consider an integer $m \in \mathbb{Z}$.

Now if f is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$,

then $f(m) = f(m1) = mf(1)$.

So we find that f is completely determined if we know $f(1)$.

If $f(1) = n$, then $f(m) = nm = f_n(m)$.

Hence $f = f_n$ and all the homomorphism of $\mathbb{Z}$ into $\mathbb{Z}$ are given by f_n, $n = 0, \pm1, \pm2, \cdots$

(ii) Let us write $Z(G) = N$. Then $\frac{G}{N}$ is cyclic.

Suppose it is generated by Ng.

Let $a, b \in G$ be any two elements

then $Na, Nb \in \frac{G}{N}$

$\Rightarrow Na = (Ng)^n, Nb = (Ng)^m$ for some n, m

$\Rightarrow Na = Ng \cdot Ng \cdots \cdots Ng = Ng^n$
and $Nb = Ng^m$

$\Rightarrow ag^{-n} \in N,\ bg^{-m} \in N$

$\Rightarrow ag^{-n} = x,\ bg^{-m} = y$ for some $x, y \in N$.

$\Rightarrow a = xg^n,\ b = yg^m$

$\Rightarrow ab = (xg^n)(yg^m) = x(g^n y)g^m$

$= x(yg^n)g^m$ as $g \in N = Z(G)$

$= xy g^n g^m$

$= xy g^{n+m}$

Similarly, $ba = (yg^m)(xg^n) = y(g^m x)g^n$

$= y(xg^m)g^n$

$= yx\, g^{m+n}$

$\Rightarrow ab = ba$ as $xy = yx$ as $x, y \in Z(G)$.

$\therefore G$ is abelian

3(b)(i) prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$

(ii) Show that $\frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \le \frac{2}{\pi}$.

Sol^n: Let $\epsilon = \frac{1}{2}$ and δ be any positive number.

we can choose a +ve integer n such that $n > \pi/\delta^2$ — ②

Let $x_1 = \sqrt{\frac{n\pi}{2}}$, $x_2 = \sqrt{(n+1)\frac{\pi}{2}} \in [0,\infty)$

Then $|f(x_2) - f(x_1)| = |\sin x_2^2 - \sin x_1^2|$

$$= \left|\sin (n+1)\frac{\pi}{2} - \sin \frac{n\pi}{2}\right|$$

$$= \begin{cases} |0-(\pm 1)| = 1, & \text{if } n \text{ is odd} \\ |\pm 1 - 0| = 1, & \text{if } n \text{ is even} \end{cases}$$

$\therefore |f(x_2) - f(x_1)| = 1 > \epsilon.$

and $|x_2 - x_1| = \left|\frac{x_2^2 - x_1^2}{x_2 + x_1}\right| = \frac{\pi/2}{\sqrt{(n+1)\frac{\pi}{2}} + \sqrt{\frac{n\pi}{2}}}$

$$< \frac{\pi}{2\left[2\sqrt{\frac{n\pi}{2}}\right]} < \frac{\pi}{\sqrt{n\pi}}$$

Thus $|x_2 - x_1| = \sqrt{\pi/n} < \delta$, by ①

$\therefore |f(x_2) - f(x_1)| > \epsilon$, when $|x_2 - x_1| < \delta$.

Hence $f(x) = \sin x^2$ is not uniformly continuous on $[0,\infty)$

(ii) Let $f(x) = \frac{1}{1+x^2}$ and $g(x) = \sin \pi x$.

then f, g are continuous on $[0,1]$ and hence integrable on $[0,1]$.

Also $g(x) = \sin \pi x \geq 0$ on $[0,1]$

Since f is decreasing on $[0,1]$,

$\inf f = f(1) = \frac{1}{2}$ and $\sup f = f(0) = 1$.

$\therefore$ By the first mean value theorem, $\exists\ \mu \in [\frac{1}{2}, 1]$ such that

$$\int_0^1 f(x)\, g(x)\, dx = \mu \int_0^1 g(x)\, dx$$

i.e., $$\int_0^1 \frac{\sin \pi x}{1+x^2}\, dx = \mu \int_0^1 \sin \pi x\, dx$$

But $$\int_0^1 \sin \pi x = -\frac{\cos \pi x}{\pi}\Big]_0^1 = \frac{2}{\pi}$$

$$\therefore \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx = \mu \frac{2}{\pi} \quad \text{—— (1)}$$

Since f is continuous on $[0,1]$, it attains every value between its bound $\frac{1}{2}$ and 1.

$\therefore \mu \in [\frac{1}{2}, 1] \Rightarrow \exists$ a number $c \in [0,1]$ such that $f(c) = \mu$.

from (1), $$f(c) = \mu = \frac{\pi}{2} \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx.$$

But $0 \le c \le 1$ and f is decreasing on $[0,1]$.

$$\Rightarrow f(0) \geq f(c) \geq f(1)$$

$$\Rightarrow \frac{1}{2} \le \frac{\pi}{2} \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \le 1$$

$$\therefore \frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \le \frac{2}{\pi}$$

3(c) Solve the following by Simplex Method

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Sol: The objective function of the given is of maximization type.

Now we write the given LPP in standard form.

$\text{Max } Z = x_1 + x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to

$x_1 + 2x_2 + 3x_3 + A_1 = 15$

$2x_1 + x_2 + 5x_3 + A_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4, A_1, A_2 \geq 0.$

Where A_1, A_2 are artificial variables.

Now the initial basic feasible solution is

$x_1 = x_2 = x_3 = 0$ (Non-basic)

$A_1 = 15, A_2 = 20, x_4 = 10.$

Note that the decision variable $x_4 \geq 0$ will provide a column $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ of initial basic matrix and that will reduce the addition of no. of artificial variables.

for which $Z = -35M - 10$

now we put the above information in the simplex tableau.

	c_j	1	2	3	-1	-M	-M		
C_B	Basis	x_1	x_2	x_3	x_4	A_1	A_2	b	θ
-M	A_1	1	2	3	0	1	0	15	5
-M	A_2	2	1	(5)	0	0	1	20	4 →
-1	x_4	1	2	1	1	0	0	10	10
$z_j = \Sigma c_B a_{ij}$		-3M	-3M	-8M	-1	-M	-M	-35M-10	
$C_j = c_j - z_j$		1+3M	2+3M	4+8M ↑	0	0	0		

from the above table, x_3 is the entering variable, A_2 is the out going variable and omit the column for this variable in new table. (5) is the key element make it unity and all other elements in its column equal to zero.

Then the revised simplex table is:

	c_j	1	2	3	-1	-M		
C_B	Basis	x_1	x_2	x_3	x_4	A_1	b	
-M	A_1	$-1/5$	(7/5)	0	0	1	3	$\frac{15}{7}$ →
2	x_3	$2/5$	$1/5$	1	0	0	4	20
-1	x_4	$3/5$	$9/5$	0	1	0	6	$\frac{30}{9}$
$z_j = \Sigma a_{ij} c_B$		$+\frac{M}{5}+\frac{3}{5}$	$-\frac{7M}{5}-\frac{6}{5}$	3	-1	-M	-3M+6	
$C_j = c_j - z_j$		$-\frac{M}{5}+\frac{2}{5}$	$\frac{7M}{5}+\frac{16}{5}$ ↑	0	0	0		

x_2 is the entering variable, A_1 is the outgoing variable and omit the column for this variable in the next table; $\frac{7}{5}$ is the key element.

	c_j	1	2	3	-1	
C_B	Basis	x_1	x_2	x_3	x_4	b
2	x_2	$-1/7$	1	0	0	$15/7$
3	x_3	$3/7$	0	1	0	$25/7$
-1	x_4	$(6/7)$	0	0	1	$15/7$
$z_j = \sum a_{ij} c_B$		$1/7$	2	3	-1	$90/7$
$C_j = c_j - z_j$		$6/7$	0	0		

Ratios: $\frac{25}{3}$, $\frac{15}{2}$ →

Introduce x_1, drop x_4, $\frac{6}{7}$ is the key element

	c_j	1	2	3	-1	
C_B	Basis	x_1	x_2	x_3	x_4	b
2	x_2	0	1	0	$1/6$	$5/2$
3	x_3	0	0	1	$-1/2$	$5/2$
1	x_1	1	0	0	$7/6$	$5/2$
$z_j = \sum a_{ij} c_B$		1	2	3	0	15
$C_j = c_j - z_j$		0	0	0	-1	

Since all $C_j \leq 0$, an optimal solution has been reached.

$\therefore$ The optimum basic feasible solution is

$x_1 = 5/2,\ x_2 = 5/2,\ x_3 = 5/2,\ x_4 = 0,\ Z_{max} = 15$

4(a) Prove that $Z[\sqrt{-5}]$ is not a UFD.

Sol^n: We see that $9 = 9 + 0\sqrt{-5} \in Z[\sqrt{-5}]$

and $9 = 3 \cdot 3$
$9 = (2+\sqrt{-5})(2-\sqrt{-5})$ $\Big\}$ —— (1)

We shall prove that $3, 2\pm\sqrt{5}$, are irreducible elements of $Z[\sqrt{-5}]$

Let $3 = (a+b\sqrt{5}i)(c+d\sqrt{5}i)$; $a,b,c,d \in Z$

$\Rightarrow \bar{3} = 3 = (a-b\sqrt{5}i)(c-d\sqrt{5}i)$

$\Rightarrow 9 = (a^2+5b^2)(c^2+5d^2)$

which gives the following possibilities.

(i) $a^2+5b^2=1$ and $c^2+5d^2=9$

(ii) $a^2+5b^2=9$ and $c^2+5d^2=1$

(iii) $a^2+5b^2=3$ and $c^2+5d^2=3$

It is clear that case (iii) is not possible in Z.

case (i) is possible when $a=\pm1$, $b=0$

$\Rightarrow a+b\sqrt{5}i = \pm1$, which are units in $Z[\sqrt{-5}]$

Similarly case (ii) yields that $c+d\sqrt{5}i = \pm1$

which are units in $Z[\sqrt{-5}]$.

Hence 3 is an irreducible element of $Z[\sqrt{-5}]$

Now let

$2+\sqrt{-5} = (a+b\sqrt{-5})(c+d\sqrt{-5})$

$\Rightarrow 2-\sqrt{-5} = (a-b\sqrt{-5})(c-d\sqrt{-5})$; $a,b,c,d \in Z$.

On multiplying respective sides of the above equations, we get

$9 = (a^2+5b^2)(c^2+5d^2)$

which gives the following possibilities

(i) $a^2+5b^2 = 1$ and $c^2+5d^2=9$

(ii) $a^2+5b^2 = 9$ and $c^2+5d^2=1$

(iii) $a^2+5b^2 = 3$ and $c^2+5d^2=3$

The first two possibilities imply

$a=\pm 1,\ b=0$ or $c=\pm 1, d=0$

$\Rightarrow a+b\sqrt{-5} = \pm 1$ (or) $c+d\sqrt{-5} = \pm 1$

Hence $2\pm\sqrt{-5}$ are irreducible elements of $Z[\sqrt{-5}]$.

$\therefore$ from ①, we see that $9 \in Z[\sqrt{-5}]$ has two distinct expressions as products of irreducible elements of $Z[\sqrt{-5}]$.

Hence $Z[\sqrt{-5}]$ is not a UFD.

4(B) Give examples of each of the following (Justify your answers)

(i) a set having no limit point

(ii) every point of the set is its limit point

(iii) an infinite no. of limit points

(iv) exactly one limit point

(v) exactly two limit points

(vi) finite no. of limit points.

Sol: (i) The set N of all natural numbers is an infinite set having no limit point.

(ii) Every point of the closed interval $[1,2]$ is a limit point.

(iii) The set $(1,2)$ has an infinite no. of limit points.

(iv) The set $\{\frac{1}{n} / n \in N\}$ is an infinite set having only one limit point.

(v). The set $\{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \cdots \frac{n}{n+1} \cdots\}$ has two limit points 1 and -1.

(vi). The set $\{\frac{1}{n} / n \in N\}$ is an infinite set having only one i.e., finitely many limit point 0.

4(c) What kind of singularity have the following functions

(i) $\frac{1}{1-e^z}$ at $z = 2\pi i$ (ii) $\frac{1}{\sin z - \cos z}$ at $z = \pi/4$

(iii) $\frac{\cot \pi z}{(z-a)^2}$ at $z=0$ and $z=\infty$

Sol^n: (i) $f(z) = \frac{1}{1-e^z}$.

poles of $f(z)$ are given by putting the denominator equal to zero. i.e, by $1-e^z=0$
$\Rightarrow e^z = 1 = e^{2n\pi i}$.
$\therefore z = 2n\pi i$, $n = 0, \pm 1, \pm 2, \cdots$

obviously $z = 2\pi i$, is a simple pole

(ii) Here $f(z) = \frac{1}{\sin z - \cos z}$

poles of $f(z)$ are given by putting the denominator equal to zero.

i.e, by $\sin z - \cos z = 0 \Rightarrow \tan z = 1$.

$z = n\pi + \frac{\pi}{4}, (n = 0, 1, 2, \ldots)$

obviously $z = \frac{\pi}{4}$ is a simple pole.

(iii) $f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{(z-a)^2 \sin \pi z}$

poles of $f(z)$ are given by putting the denominator equal to zero.

i.e $\sin \pi z \, (z-a)^2 = 0$

by $\sin \pi z = 0$ and $(z-a)^2 = 0$

$\sin \pi z = 0$ gives $\pi z = n\pi$.

$\Rightarrow z = n$; $n = 0, \pm 1, \pm 2, \ldots$

obviously $z = \infty$ is the limit point of these poles, hence $z = \infty$ is a non isolated essential singularity.

~~and~~ and $(z-a)^2 = 0$ gives $z = a$ repeated twice.

Hence $z = a$ is a double pole

4(d) Find the optimal (maximisation) solution of the following transportation problem apply VAM to find the initial basic feasible solution.

sale agencies

	S_1	S_2	S_3	S_4	S_5	a_i Capacity
F_1	15	17	12	11	11	140
F_2	5	9	7	15	7	190
F_3	14	15	16	20	10	115
Demand b_j	74	94	69	39	119	

Sol: $\Sigma a_i \neq \Sigma b_j$.

It is an unbalanced problem.

we first make it a balanced problem by creating a dummy sale agency s_6 to take up the excess capacity of 50 units. The associated unit profit is zero in each cell. Next the profit matrix so obtained by introducing dummy column converted into loss matrix by changing the signs of all the cell values as given in the following table.

-15	-17	-12	-11	-11	0	140
-5	-9	-7	-15	-7	0	190
-14	-15	-16	-20	-10	0	115
74	94	69	39	119	50	

By using VAM find the initial basic feasible solution. Then test the optimality by u-v method. The second basic feasible solution give the unique optimal solution and is given by

$x_{11} = 46$, $x_{12} = 94$, $x_{24} = 21$, $x_{25} = 119$,

$x_{26} = 50$, $x_{31} = 28$, $x_{33} = 69$, $x_{34} = 18$.

and optimum profit (use original profit max) is Rs 5292

Note that the production capacity of F_2 remains unused by 50 units and yields no profit

5(a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$; $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 3 & 5 & 2 \end{pmatrix}$

be elements of S_7.

(i) write α as a product of disjoint cycles

(ii) write β as a product of 2-cycles

(iii) Is β an even permutation

(iv) Is α^{-1} an even permutation.

Sol:

Here α is such that $\alpha(1) = 6$, $\alpha(6) = 3$, $\alpha(3) = 7$, $\alpha(7) = 1$.

Now, we define $\alpha_1 : I_7 \to I_7$ by

$$\alpha_1 : \begin{array}{l} 1 \to 6 \\ 6 \to 3 \\ 3 \to 7 \\ 7 \to 1 \\ x \to x \end{array}$$

when $x \neq 1, 6, 3, 7$

Hence we have a cycle $\alpha_1 = (1\ 6\ 3\ 7)$.

Now, $2 \notin \{1, 6, 3, 7\}$ and we see that

$\alpha(2) = 4$, $\alpha(4) = 5$, $\alpha(5) = 2$

So define, $\alpha_2 : I_7 \to I_7$ by

$$\alpha_2 : \begin{array}{l} 2 \to 4 \\ 4 \to 5 \\ 5 \to 2 \\ x \to x \end{array}$$

when $x \neq 2, 4, 5$.

Hence we have a cycle $\alpha_2 = (2\ 4\ 5)$.

As we see that $\{1, 6, 3, 7\} \cup \{2, 4, 5\} = I_7$.

we conclude that

$$\therefore \alpha = \alpha_1 \alpha_2 = (1\,6\,3\,7)(2\,4\,5)$$

(ii) we see that the permutation β can be written as

$$\beta : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 4 \\ 4 \longrightarrow 7 \\ 7 \longrightarrow 2 \\ 3 \longrightarrow 6 \\ 6 \longrightarrow 5 \\ 5 \longrightarrow 3 \end{array}$$

Hence $\beta = (1)(2\,4\,7)(3\,6\,5)$

$= (2\,4\,7)(3\,6\,5)$.

$= (2\,7)(2\,4)(3\,5)(3\,6)$

(iii) Since β is a product of an even number of 2-cycles, β is an even.

(iv) From the given permutation α, we may calculate α^{-1} as follows.

$$\alpha : \begin{array}{l} 1 \rightarrow 6 \\ 2 \rightarrow 4 \\ 3 \rightarrow 7 \\ 4 \rightarrow 5 \\ 5 \rightarrow 2 \\ 6 \rightarrow 3 \\ 7 \rightarrow 1 \end{array} \quad \text{so that} \quad \alpha^{-1} = \begin{array}{l} 1 \rightarrow 7 \\ 2 \rightarrow 5 \\ 3 \rightarrow 6 \\ 4 \rightarrow 2 \\ 5 \rightarrow 4 \\ 6 \rightarrow 1 \\ 7 \rightarrow 3 \end{array}$$

Hence $\alpha^{-1} = (1\,7\,3\,6)(2\,5\,4) = (1\,6)(1\,3)(1\,7)(2\,4)(2\,5)$

As α^{-1} is a product of an odd no. of 2-cycles, we conclude that α^{-1} is not an even permutation

5(b) → show that a finite integral domain is a field.

Sol'n: Let F be the finite integral domain.

Let $F = \{a_1, a_2, \ldots a_n\}$ and F contains 'n' distinct elements.

To Prove that F is a field.

For this we are enough to Prove that the non-zero elements of F have multiplicative inverses.

Let $a(\neq 0) \in F$

$\therefore aa_1, aa_2, \ldots aa_n \in F$ (by closure prop.)

All these elements are distinct.

because, if possible

let $aa_i = aa_j$, $a_i, a_j \in F$

$\Rightarrow a(a_i - a_j) = 0$

$\Rightarrow (a_i - a_j) = 0$ ($\because a \neq 0$ & F is an ID in F

$\Rightarrow a_i = a_j$ doesnot contain zero divisors)

This is a contradiction to hypothesis that F contains n distinct elements.

$\therefore$ our assumption that $aa_i = aa_j$ is wrong

$\therefore aa_1, aa_2, \ldots aa_n$ are all distinct elements in F. which has exactly 'n' elements. By the Pigeon-hole Principle, one of these products must be equal to one

Let $aa_r = 1$ for some $a_r \in F$ ($\because$ F is an ID)

$\therefore a^{-1} = a_r$

$\therefore$ Every non-zero element of F has multiplicative inverse. $\therefore$ F is a field.

For example, $\mathbb{Z}$ (set of all integers) is an infinite integral domain, which is not a field.

($\because$ Inverse property is not satisfied w.r.t $\times^n$).

5(c) Show that the series.

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots$$

Converges uniformly in $0 < a \le x \le b < 2\pi$.

Sol^n: Take $u_n(x) = \sin nx$, $v_n(x) = \frac{1}{n}$

Here $f_n(x) = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$

$$= \frac{\sin\left(x + \frac{n-1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$= \frac{\sin\frac{n+1}{2}x \sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore |f_n(x)| = \frac{\left|\sin\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\left|\sin\frac{x}{2}\right|}$$

i.e. $|f_n(x)| \le \left|\operatorname{cosec}\frac{x}{2}\right|$

But $\operatorname{cosec}\frac{x}{2}$ is bounded for all values of x in $0 < a \le x \le b < 2\pi$. If k be its least upper bound in this interval $|f_n(x)| < k$ for all values of x in this interval.

Also the sequence $\langle\frac{1}{n}\rangle$ is a ~~positive~~ monotonic decreasing sequence converging to zero. Hence by

Dirichlet's test the given series is uniformly convergent in $0 < a \le x \le b < 2\pi$.

5(d). Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy - Riemann equations are satisfied at the point.

Sol^n: Let the function be $f(z) = u(x,y) + iv(x,y)$

Here $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Hence Cauchy Riemann equations are satisfied at the origin.

Again, $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$

$$= \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x+imx} \quad \text{letting } z\to 0 \text{ along } y=mx$$

$$= \frac{\sqrt{|m|}}{1+im}$$

this depends on m, i.e. $f'(0)$ is not unique.

Hence ~~f(z) is~~ not analytic at the origin although Cauchy Riemann equations are satisfied there.

5(e) If $x_1=2, x_2=3, x_3=1$ is a feasible solution of the LPP

Maximize $Z = x_1+2x_2+4x_3$

subject to $2x_1+x_2+4x_3=11$

$3x_1+x_2+5x_3=14$

$x_1, x_2, x_3 \geq 0$.

find a basic feasible solution of the problem

Sol^n: The given system of equations may be put in matrix notations as $\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$

$\Rightarrow AX=B$

where $\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}$; $B=\begin{pmatrix} 11 \\ 14 \end{pmatrix}$; $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Let the columns of A be denoted by

$A_1=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $A_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $A_3=\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Here $\rho(A)=2$.

$\therefore$ A basic solution to the given system of equations exist with not more than two variable different from zero

Also, the column vectors A_1, A_2, A_3 are linear dependent (we can easily verify)

$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 = 0$

$\Rightarrow 2\lambda_1+\lambda_2+4\lambda_3=0$

$3\lambda_1+\lambda_2+5\lambda_3=0$

Clearly this is a system of two equations in three unknowns $\lambda_1, \lambda_2, \lambda_3$:

Let us choose one of the λs arbitrarily say $\lambda_1 = 1$.

$\therefore \lambda_2 + 4\lambda_3 = -2$

$\lambda_2 + 5\lambda_3 = -3$

Solving, we get $\lambda_2 = 2$, $\lambda_3 = -1$

To reduce the no. of +ve variables, the variable to be driven to zero is found by choosing r for which

$$\frac{x_r}{\lambda_r} = \text{Min}\left\{\frac{x_i}{\lambda_i} \,/\, \lambda_i > 0\right\}$$

$$= \text{Min}\left\{\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3}\right\} = \text{Min}\left\{\frac{2}{1}, \frac{3}{2}, \frac{1}{-1}\right\} = \frac{3}{2}$$

Thus, we can remove vector A_2 for which $\frac{x_2}{\lambda_2} = \frac{3}{2}$ and obtain new solution with not more than two non-negative (non-zero) variables

The variables of new are given by

$\hat{x}_1 = x_1 - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2}$

$\hat{x}_2 = x_2 - \frac{3}{2}(2) = 3 - 3 = 0$

$\hat{x}_3 = x_3 - \frac{3}{2}(-1) = 1 + \frac{3}{2} = \frac{5}{2}$.

$\therefore$ The basic feasible solution is

$x_1 = \frac{1}{2}$, $x_3 = \frac{5}{2}$ with $x_2 = 0$ (non-basic)

6(a) Show that there does not exist any isomorphism from the group $(\mathbb{R},+)$ to the group $(\mathbb{R}^*,\cdot)$

Sol^n: Observe that '0' is the identity element of the group $(\mathbb{R},+)$. Hence no non-zero element of $\mathbb{R}$ is of finite order. Now for the group $(\mathbb{R}^*,\cdot)$, 1 is the identity element and -1 is an element of order 2.

So, if there exists an isomorphism $f:\mathbb{R}\to\mathbb{R}^*$, then should exist $a\in\mathbb{R}$ such that $O(a)=2$. But this is not the case.

Hence there is no isomorphism from the group $(\mathbb{R},+)$ to the group $(\mathbb{R}^*,\cdot)$.

6(b) Give an example that the field with nine elements and write its composition tables.

Sol^n: Let $\mathbb{Z}_3[i] = \{a+bi \,/\, a,b\in\mathbb{Z}_3\}$

$= \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$

Where $i^2=-1$, This is the ring of Gaussian Integers modulo 3. Elements are added and multiplied as in the complex numbers, except that the coefficients are reduced modulo 3. In particular, $-1=2$.

multiplication table for the non zero elements of $Z_3[i]$ are given below.

$\times$	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
1	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
2	2	1	$2i$	$2+2i$	$1+2i$	i	$2+i$	$1+i$
i	i	$2i$	2	$2+i$	$2+2i$	1	$1+i$	$1+2i$
$1+i$	$1+i$	$2+2i$	$2+i$	$2i$	1	$1+2i$	2	[illegible]
$2+i$	$2+i$	$1+2i$	$2+2i$	1	i	$1+i$	$2i$	2
$2i$	$2i$	i	1	$1+2i$	$1+i$	2	$2+2i$	$2+i$
$1+2i$	$1+2i$	$2+i$	$1+i$	2	$2i$	$2+2i$	i	1
$2+2i$	$2+2i$	$1+i$	$1+2i$	i	2	$2+i$	1	$2i$

6(c) Let $R = M_2(\mathbb{C})$, the ring of all 2×2 complex matrices and let S be a subset of R consisting of the matrices of the form $B = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. Show that S is a subring of R. Find the center of S.

Sol: By defn of S is nonempty. Let

$$A = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix},\ B = \begin{bmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{bmatrix} \in S$$

Then $A - B = \begin{bmatrix} \alpha-\gamma & \beta-\delta \\ -(\overline{\beta-\delta}) & (\overline{\alpha-\gamma}) \end{bmatrix}$ and $AB = \begin{pmatrix} \mu & \nu \\ -\bar{\nu} & \bar{\mu} \end{pmatrix}$

Where $\mu = \alpha\gamma - \beta\bar{\delta}$ and $\nu = \alpha\delta + \beta\bar{\gamma}$.

Hence $A-B,\ AB \in S$ and so S is a subring of R.

now let $A = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \in C(R)$ and

$B = \begin{bmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{bmatrix}$ be any element of R

where $C(R)$ is the center of R.

Then $AB = BA$ which implies that

$$\begin{pmatrix} \alpha\gamma - \beta\bar{\delta} & \alpha\delta + \beta\bar{\gamma} \\ -\bar{\alpha}\bar{\delta} - \bar{\beta}\gamma & \bar{\alpha}\bar{\gamma} - \bar{\beta}\delta \end{pmatrix} = \begin{pmatrix} \gamma\alpha - \delta\bar{\beta} & \delta\bar{\alpha} + \gamma\beta \\ -\bar{\delta}\alpha - \bar{\gamma}\bar{\beta} & \bar{\gamma}\bar{\alpha} - \bar{\delta}\beta \end{pmatrix}$$

Thus we have $\beta\bar{z} = z\bar{\beta}$ $\forall z \in C$

putting $z=1$, we get $\beta = \bar{\beta}$ which implies β is real and putting $z = i$, we get $-i\beta = i\beta$ which implies $\beta = 0$.

Also if $\beta = 0$ then clearly $AB = BA$.

Therefore

$$C(R) = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix} \Big/ \alpha \in C \right\}$$

6(d) In the ring $Z[i]$, show that $I = \{a+bi \in Z[i] / a, b$ are both even$\}$ is an ideal of $Z[i]$, but not a maximal ideal of $Z[i]$.

Sol^n: Let $x = a+bi$, $y = c+di \in I$ and $u = r+si \in Z[i]$. Then exist $r_1, r_2, r_3, r_4 \in Z$ such that $a = 2r_1$, $b = 2r_2$, $c = 2r_3$, $d = 2r_4$.

Then $x-y = (a+ib)-(c+id) = (a-c)+(b-d)i$
$= 2(r_1-r_3)+2(r_2-r_4)i \in I$

and $ux = (r+si)(a+bi) = (ra-sb)+(rb+sa)i$
$= 2(rr_1-sr_2)+2(rr_2+sr_1)i \in I$

Since $Z[i]$ is a commutative ring, it follows that I is an ideal of $Z[i]$.

Let $J = \{a+bi \in Z[i] \,/\, 2 \text{ divides } a^2+b^2\}$

Since $0+0i \in J$, $J \neq \phi$.

Let $x = a+ib$, $y = c+id \in J$ and $u = r+si \in Z[i]$.

Then $2 / a^2+b^2$ and $2 / c^2+d^2$

Now $x-y = (a-c)+(b-d)i$

Since $(a-c)^2+(b-d)^2 = a^2+b^2+c^2+d^2-2(ac+bd)$,

it follows that $2/(a-c)^2+(b-d)^2$

Again $ux = (r+si)(a+bi) = (ra-sb)+(rb+sa)i$

and $(ra-sb)^2+(rb+sa)^2 = r^2(a^2+b^2)+s^2(a^2+b^2)$

Hence $2/(ra-sb)^2+(rb+sa)^2$ implies that

$ux = xu \in J$.

Consequently J is an ideal.

Now $1+2i \in Z[i]$, but $1+2i \notin J$

Hence $J \neq Z[i]$.

Again $1+i \in J$ but $1+i \notin I$.

Hence we find that $I \subset J \subset Z[i]$.

Consequently, I is not a maximal ideal

7(a) Find the derived sets of the following sets:

(i) $\left\{1+\frac{(-1)^n}{n} : n\in N\right\}$ (ii) $\left\{\frac{1+(-1)^n}{n} : n\in N\right\}$

(iii) $\left\{(-1)^n+\frac{1}{n} : n\in N\right\}$ (iv) $\left\{2^n+\frac{1}{2^n} : n\in N\right\}$

Sol'n: Let

(i) $S=\left\{1+\frac{(-1)^n}{n} / n\in N\right\}$

$\therefore$ 1 is the only limit point of S

$\therefore$ derived set $S' = (1)$.

(ii) Let $A=\left\{\frac{1+(-1)^n}{n} / n\in N\right\}$ ($\because$ when n is odd $\frac{1+(-1)^n}{n}=0$; n is even, $\frac{1+(-1)^n}{n}=\frac{2}{n}$; $S=\{0\}\cup\{\frac{2}{n}/n\in N\}$ $\Rightarrow$ 0 is the limit point.)

0 is the limit point of A

$\therefore$ derived set $A' = \{0\}$

(iii) Let $B=\left\{(-1)^n+\frac{1}{n} / n\in N\right\}$

Derived set $B' = \{-1, 1\}$

(iv) Let $C = \left\{2^n+\frac{1}{2^n} / n\in N\right\}$

Derived set $C' = \phi$

7(b) Show that the series

$$2xe^{-x^2} = \sum_1^\infty 2x\left[\frac{1}{n^2}e^{-x^2/n^2} - \frac{1}{(n+1)^2}e^{-x^2/(n+1)^2}\right]$$ can be

integrated term by term between any two finite limits.
Can the function defined by the series be integrated between the limits 0 and ∞? If so, what is the value of this

integral given by integrating the series term by term between these limits?

Sol'n: Here $u_n(x) = 2x\left[\frac{e^{-x^2/n^2}}{n^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right]$

$\therefore f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$

$$= 2x\left[e^{-x^2} - \frac{e^{-x^2/2^2}}{2^2}\right] + 2x\left[\frac{e^{-x^2/2^2}}{2^2} - \frac{e^{-x^2/3^2}}{3^2}\right] + \cdots + 2x\left[\frac{e^{-x^2/n^2}}{n^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right]$$

$$= 2x\left[e^{-x^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right]$$

Then $f(x) = \lim_{n\to\infty} f_n(x) = 2xe^{-x^2}$ for all values of x.

Now $\int_a^b f(x)\,dx = \int_a^b 2xe^{-x^2}dx = e^{-a^2} - e^{-b^2}$

and $\int_a^b f_n(x)\,dx = e^{-a^2} - e^{-b^2} + \left[e^{-b^2/(n+1)^2} - e^{-a^2/(n+1)^2}\right]$

$$\to e^{-a^2} - e^{-b^2} \text{ as } n\to\infty$$

Since $\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx$,

the series can be integrated term by term between any two finite limits.

Again $\int_0^\infty f(x)\,dx = \int_0^\infty 2xe^{-x^2}dx = 1$

and $\int_0^\infty f_n(x)\,dx = \left[-e^{-x^2} + e^{-x^2/(n+1)^2}\right]_0^\infty = 0$.

Since $\int_0^\infty f(x)\,dx \neq \lim_{n\to\infty}\int_0^\infty f_n(x)\,dx$, the series cannot be integrated term by term b/w the limits 0 and ∞.

7(C) A sequence $\langle a_n \rangle$ is defined as $a_1 = 1$, $a_{n+1} = (4+3a_n)/(3+2a_n)$, $n \geq 1$. show that $\langle a_n \rangle$ converges and find its limit.

Sol: Here $a_1 = 1$ and $a_2 = (4+3a_1)/3+2a_1 = 7/5 > 1 \Rightarrow a_2 > a_1$

Let us assume that $a_{n+1} > a_n$ ——— (1)

then, $a_{n+2} - a_{n+1} = \dfrac{4+3a_{n+1}}{3+2a_{n+1}} - \dfrac{4+3a_n}{3+2a_n}$

$$= \frac{a_{n+1} - a_n}{(3+2a_{n+1})(3+2a_n)} \quad \text{——— (2)}$$

Since $a_{n+1} > a_n$ by (1) and $a_n > 0 \ \forall n \in N$, So from (3)

$$a_{n+2} > a_{n+1}$$

Hence by mathematical induction, $a_{n+1} > a_n \ \forall n \in N$ and so $\langle a_n \rangle$ is monotonically increasing sequence.

Again $a_{n+1} = \dfrac{4+3a_n}{3+2a_n} = \dfrac{3}{2} - \dfrac{1}{2(2a_n+3)} \Rightarrow a_{n+1} < \dfrac{3}{2} \ \forall n \in N$

Also $a_1 = 1 < 3/2$. Thus $a_n < 3/2 \ \forall n \in N$ and hence $\langle a_n \rangle$ is bounded above.

Since $\langle a_n \rangle$ is monotonically increasing and bounded above, it is convergent.

Let $\lim a_n = l$, then $\lim a_{n+1} = l$.

Now, $a_{n+1} = \dfrac{4+3a_n}{3+2a_n} \Rightarrow \lim a_{n+1} = \dfrac{4+3\lim a_n}{3+2\lim a_n}$

$\Rightarrow l = (4+3l)/(3+2l) \Rightarrow l^2 = 2 \Rightarrow l = \pm\sqrt{2}$

Since all the terms of the given sequence are +ve, it follows that l cannot be $-$ve.

Since $l = \sqrt{2}$ (or) $\underline{\underline{a_n = \sqrt{2}}}$.

4(d). Show that $\lim\{I_n\}$, where $I_n = \int_0^h \frac{\sin nx}{x} dx$, $n \in I$ exists and that the limit is equal to $\pi/2$.

Sol^n: The integrand becomes continuous for every value of x, if we assign to it the value n for $x=0$.

I. Firstly, it will be proved that $\{I_n\}$ is convergent.

Putting $nx = t$, we have

$$I_n = \int_0^{nh} \frac{\sin t}{t} dt.$$

$$\Rightarrow \left| I_{n+p} - I_n \right| = \left| \int_{nh}^{(n+p)h} \frac{\sin t}{t} dt \right|$$

As $1/t$ is +ve and monotonically decreasing when $t \in]nh, (n+p)h[$, we have, by Bonnet's form of the second mean value theorem,

$$\left| I_{n+p} - I_n \right| = \frac{1}{nh} \left| \int_{nh}^{\xi} \sin t \, dt \right| \le \frac{2}{nh} < \varepsilon \ \forall\, n \in \frac{2}{\varepsilon h}$$

Hence by, Cauchy's principle of convergence, $\{I_n\}$ converges.

II. It will now be proved that, when $n \to \infty$

$$\lim I_n = \lim \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx$$

we write $\int_0^{\pi/2} \frac{\sin nx}{x} dx = \int_0^h \frac{\sin nx}{x} dx + \int_h^{\pi/2} \frac{\sin nx}{x} dx$

As proved in the preceding example,

$$\int_0^{\pi/2} \frac{\sin nx}{x} dx \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\lim I_n = \lim \int_0^{\pi/2} \frac{\sin nx}{x} dx$$

Again, taking $f(x) = \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ in the preceding example,

$$\lim \int_0^{\pi/2} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \sin nx \, dx = 0$$

If f is continuous in $[0, \pi/2]$, if we set $f(0) = 0$.

It follows that

$$\lim \int_0^{\pi/2} \frac{\sin nx}{x} dx = \lim \int_0^{\pi/2} \frac{\sin nx}{x} dx$$

To determine the actual value of the limit, we proceed by making $n \to \infty$ through odd integer values.

M. we have, it is easily shown

$$\frac{\sin(2n+1)x}{\sin x} = 2\left[\frac{1}{2} + \cos 2x + \cos 4x + \cdots + \cos 2nx\right]$$

So that $\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx = \frac{\pi}{2}$

Hence the result.

8(a). Prove that all the roots $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$.

Sol^n: Consider the circle $C_1: |z|=1$

Let $f(z)=12$, $g(z)=z^7-5z^3$

Both $f(z)$ and $g(z)$ being polynomial are analytic within and on C_1. (Here $f(z)=$ constant, i.e. constant polynomial)

we have

$$|g(z)| = |z^7-5z^3|$$
$$\leq |z^7|+|5z^3|$$
$$\leq 1+5=6<12=|f(z)|$$

Hence by Rouche's theorem [If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)|<|f(z)|$ on C, then $f(z)+g(z)$ and $f(z)$ have the same number of zeros inside C]

$f(z)+g(z)=z^7-5z^3+12$ has the same number of zeros inside $|z|=1$ as $f(z)=12$

i.e. there are no zero's inside C_1.

Consider the circle $C_2: |z|=2$

Let $f(z)=z^7$, $g(z)=12-5z^3$

Both $f(z)$ and $g(z)$ being polynomial are analytic within and on C_2.

on C_2, we have

$$|g(z)| = |12-5z^3|$$
$$\leq |12|+|5z^3|$$
$$\leq 52<2^7=|f(z)|$$

i.e. $|g(z)|<|f(z)|$

Hence by Rouche's theorem
$f(z)+g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z|=2$ as $f(z) = z^7$.
i.e. all the zeros are inside C_2.
Hence all the roots lie inside $|z|=2$
but outside $|z|=1$
i.e. all the roots lie between $|z|=1$ and $|z|=2$.

8(b) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.
(i) $\int_{|z|=1} \frac{\cos z}{z(z-4)} dz$ (ii) $\int_{|z-a|=1} \frac{e^{2\pi z}}{(z-a)} dz$

Sol: (i) Given that $\int_{|z|=1} \frac{\cos z}{z(z-4)} dz$.
comparing the given integral with $\int_C \frac{f(z)}{z-z_0} dz$
where $C: |z|=1$.
Since $f(z) = \frac{\cos z}{z-4}$ and $z_0 = 0$ is a point inside $|z|=1$.
$\therefore$ we can apply Cauchy's integral formula

$$\int_{|z|=1} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$
$$= 2\pi i f(0) = 2\pi i \left(\frac{\cos 0}{0-4}\right) = \frac{2\pi i}{-4} = -\frac{\pi i}{2}$$

(ii) Given that $\int_{|z-a|=1} \frac{e^{2\pi z}}{(z-a)} dz$.
Here $z_0 = a$ & $C: |z-a|=1$.

Since $f(z) = e^{2\pi z}$ and $z_0 = a$ is a point inside $|z-a|=1$.

$\therefore$ we can apply Cauchy's integral formula.

$$\int_{|z-a|=1} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$
$$= 2\pi i\, f(a)$$
$$= 2\pi i\, e^{2\pi a}.$$
$$\therefore \int_{|z-a|=1} \frac{e^{2\pi z}}{z-a}\,dz = 2\pi i\, e^{2\pi a}.$$

8(c) Show that $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

Sol: Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

$$= \int_0^{2\pi} \frac{2\,d\theta}{4+e^{i\theta}+e^{-i\theta}}$$

$$= \frac{1}{i}\int_C \frac{2\,dz}{z\left(4+z+\frac{1}{z}\right)}$$

Thus $I = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$

writing $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$ where C is the unit circle $|z|=1$.

poles of $z^2+4z+1=0$ are given by

$$z = \frac{-4\pm\sqrt{16-4}}{2} = -2\pm\sqrt{3}.$$

Only $z=-2+\sqrt{3}$ lies inside C.

the residue at $z = -2+\sqrt{3}$

$$= \frac{2}{i} \frac{1}{\frac{d}{dz}\left[(z^2+4z+1)\right]}$$

$$= \frac{2}{i}\left[\frac{1}{2z+4}\right] = \frac{2}{i}\left[\frac{1}{-4+2\sqrt{3}+4}\right] = \frac{1}{i\sqrt{3}}$$

Hence by Cauchy's residue Theorem

$$I = 2\pi i \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

8(d) from the optimal table of the following primal problem, find the optimal solution to the associated dual problem

Maximize $Z = 30x_1 + 23x_2 + 29x_3$

subject to

$6x_1 + 5x_2 + 3x_3 \le 26$

$4x_1 + 2x_2 + 5x_3 \le 7$

$x_1, x_2, x_3 \ge 0$

Sol: Standard form of primal

Maximize $Z = 30x_1 + 23x_2 + 29x_3 + 0s_1 + 0s_2$

subject to

$6x_1 + 5x_2 + 3x_3 + s_1 = 26$

$4x_1 + 2x_2 + 5x_3 + s_2 = 7$

$x_1, x_2, x_3, s_1, s_2 \ge 0$

where s_1, s_2 are slack variables

Initial basic feasible solution is given by

	c_j	30	23	29	0	0		
c_B	Basis	x_1	x_2	x_3	s_1	s_2	b	θ
0	s_1	6	5	3	1	0	26	$\frac{26}{6}=4.3$
0	s_2	(4)	2	5	0	1	7	$\frac{7}{4}=1 \rightarrow$
$z_j=\Sigma a_{ij}c_B$		0	0	0	0	0	0	
$C_j=c_j-z_j$		30 ↑	23	29	0	0		

from the above table as $C_j(=30)$ is maximum and the corresponding column is known as key column and s_2 is the outgoing variable and (4) is the key element.

now convert the key element to unity and all other elements in its column to zero. Then we obtain a new iterated simplex tableau

	c_j	30	23	29	0	0		
c_B	Basis	x_1	x_2	x_3	s_1	s_2	b	θ
0	s_1	0	2	$-\frac{9}{2}$	1	$-\frac{3}{2}$	$\frac{31}{2}$	$31/4$
30	x_1	1	$(1/2)$	$5/4$	0	$1/4$	$7/4$	$7/2 \rightarrow$
$z_j=\Sigma a_{ij}c_B$		30	15	$\frac{75}{2}$	0	$\frac{15}{2}$	$\frac{105}{2}$	
$C_j=c_j-z_j$		0	8 ↑	$\frac{-17}{2}$	0	$\frac{-15}{2}$		

Introduce x_2, drop x_1, $\frac{1}{2}$ is the key element.

$15/2$
1.5

Main Test series − 2016
Test − 3 (Answer Key)
ODE, statics & Dynamics and VA

1(a) Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$.

Sol^n: Given that $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x.$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\int P\,dy = \int \frac{1}{1+y^2}\,dy = \tan^{-1}y.$$

$$\therefore I.F = e^{\int P\,dy} = e^{\tan^{-1}y}$$

Hence the required solution is

$$x e^{\tan^{-1}y} = \int e^{\tan^{-1}y}\cdot\frac{\tan^{-1}y}{1+y^2}\,dy + C.$$

put $\tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}dy = dt.$

$$x e^{\tan^{-1}y} = \int e^t\, t\, dt + C$$

$$x e^{\tan^{-1}y} = te^t - e^t + c$$

$$x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$\boxed{x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}}$$

1(b) Using Laplace transform, solve $ty'' + 2y' + ty = \cos t$ given that $y(0) = 1$.

Sol^n: Taking Laplace transform of both sides of the equation and noting that

$L\{t f(t)\} = -\frac{d}{ds}\left[L\{f(t)\}\right]$, we get

$L\{t y''\} + 2L\{y(t)\} + L(t y(t)) = L\{\cos t\}$

$\Rightarrow -\frac{d}{ds}\left[L(y''(t))\right] + 2L(y(t)) - \frac{d}{ds}\left[L\, y(t)\right] = \frac{s}{s^2+1}$

$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y} - sy(0) - y'(0)\right] + 2\left[s\bar{y} - y(0)\right] - \frac{d}{ds}\bar{y} = \frac{s}{s^2+1}$

$\Rightarrow \left[-s^2\frac{d\bar{y}}{ds} - 2s\bar{y}\right] + y(0) + 0 + 2s\bar{y} - 2(1) - \frac{d\bar{y}}{ds} = \frac{s}{s^2+1}$

$\Rightarrow -s^2\frac{d\bar{y}}{ds} + 1 - 2s\bar{y} + 2s\bar{y} - 2 - \frac{d\bar{y}}{ds} = \frac{s}{s^2+1}$

$\Rightarrow (s^2+1)\frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2+1}$

$\Rightarrow \frac{d\bar{y}}{ds} = -\frac{s}{(s^2+1)^2} - \frac{1}{s^2+1}$

$L^{-1}\left(\frac{d\bar{y}}{ds}\right) = -L^{-1}\left(\frac{1}{s^2+1}\right) - L^{-1}\frac{s}{(s^2+1)^2}$

$\left(\because L^{-1}\left[\bar{f}'(s)\right] = -t f(t)\right)$

$-t\, y(t) = -\sin t - \frac{1}{2}t\sin t$

$y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t.$

which is the desired solution

1(c) The extremities of a heavy string of length $2l$ and weight $2lw$, are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to lw. Show that the distance apart of the rings is, $2l \log(1+\sqrt{2})$.

the length of string $ACB = 2l$ and its weight $= 2lw$

$\therefore$ the weight per unit length of the string $= w$

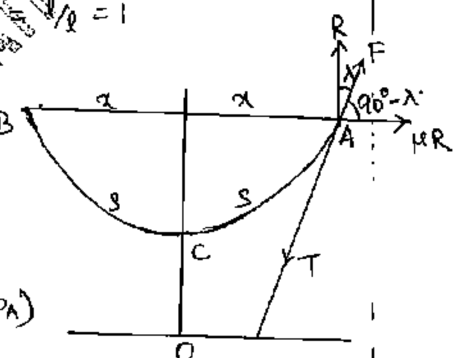
Here it is given that the small ring at A is acted on by a horizontal force lw. For the equilibrium of the small ring at A, the horizontal force lw must balance the horizontal component of the tension at A. But the horizontal component of the tension at any point of the string is equal to wc. So we must have

$$lw = wc \Rightarrow l = c$$

Since arc $CA = l$, therefore for the point A, $s = l$. So using the formula $s = c\tan\psi$ for the point A, we have

$$l = c\tan\psi \Rightarrow \tan\psi = l/c = l/l = 1$$

$$\therefore \text{for } A,\ \psi = \psi_A = 45^\circ$$



If (x_A, y_A) are the coordinates of the point A, then the distance between the rings $= AB = 2x_A$

$$= 2c\log(\tan\psi_A + \sec\psi_A)$$

$$= 2l\log(\tan 45^\circ + \sec 45^\circ)$$

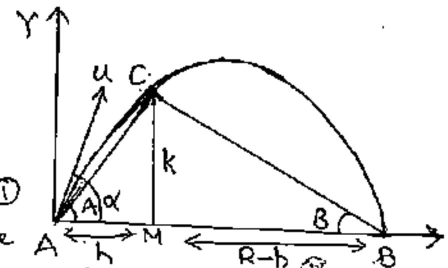
$$= 2l\log(1+\sqrt{2})$$

1(d). A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, Prove that

$$\tan\alpha = \tan A + \tan B$$

Sol^n: Let A be the point of projection, u the velocity of

projection and α the angle of projection
the particle while grazing over the vertex C falls at the point B.
If $AB = R$, then $R = \frac{2u^2 \sin\alpha \cos\alpha}{g}$ —(1)



Take the horizontal the AB as the x-axis and the vertical line AY as the y-axis. Let the coordinates of the vertex C be (h, k). Then the point (h, k) lies on the trajectory whose equation is

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\therefore k = h\tan\alpha - \frac{1}{2}g\frac{h^2}{u^2\cos^2\alpha} = h\tan\alpha\left[1 - \frac{gh}{2u^2\sin\alpha\cos\alpha}\right].$$

$$= h\tan\alpha\left[1 - \frac{h}{R}\right]$$

$$\therefore \frac{k}{h} = \tan\alpha\left(\frac{R-h}{R}\right)$$

$$\Rightarrow \tan A = \tan\alpha\left(\frac{R-h}{R}\right) \quad \left[\because \text{ from } \Delta CAM,\ \tan A = \frac{k}{h}\right]$$

$$\therefore \tan\alpha = \tan A\left(\frac{R}{R-h}\right) = \tan A\left[\frac{(R-h)+h}{R-h}\right]$$

$$= \tan A\left[1 + \frac{h}{R-h}\right] = \tan A + \tan A\frac{h}{R-h}$$

$$= \tan A + \frac{k}{h}\cdot\frac{h}{R-h} \quad \left[\because \tan A = \frac{k}{h}\right]$$

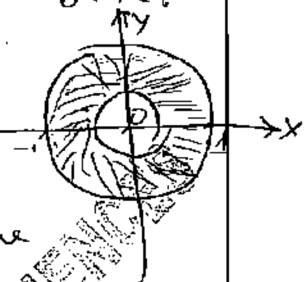
$$= \tan A + \frac{k}{(R-h)}$$

But from the ΔCMB, $\tan B = \frac{k}{(R-h)}$

$$\therefore \tan\alpha = \tan A + \tan B$$

11) Verify the Green's theorem for $M = \frac{-y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$. $R = \{(x,y) / h^2 \le x^2+y^2 \le 1\}$, where $0<h<1$

Sol^n: Green's theorem states that



$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\,dy$$

The boundary of R consists of the circle C_1:

$x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$

around which we shall integrate counter clockwise, and the circle C_h:

$x = h\cos\phi,\ y = h\sin\phi,\ 2\pi \ge \phi \ge 0$.

around which we shall integrate in the clockwise direction.

Note that the origin is not included in R because h is positive.

for all $(x,y) \ne (0,0)$, the functions M and N and their partial derivatives are continuous.

now $\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0 \quad \text{--- (1)}$$

The line integral is

$$\int_C M\,dx + N\,dy = \oint_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2} + \oint_{C_h} \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\, d\theta + \int_{2\pi}^{0} \frac{h^2(\cos^2\phi + \sin^2\phi)}{h^2}$$

$$= 2\pi - 2\pi = 0 \quad \text{---} (2)$$

$\therefore$ from (1) & (2)

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\,dy$$

$\Rightarrow$ Green's theorem is verified.

2(a) (i) Find the orthogonal trajectories of the family of $r^n = a^n \sin n\theta$.

(ii) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol^n: Given equation of family of curves is

$r^n = a^n \sin n\theta$, a is a parameter. --- (1)

$n \log r = \log a^n + \log \sin n\theta$. --- (2)

Differentiating (2) w.r.t θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{1}{\sin n\theta} \cdot \cos n\theta \cdot n$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta = 0 \quad \text{---} (3)$$

which is the differential equation of the given family of curves (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), the differential equation of the required orthogonal trajectories is $\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) - \cot n\theta = 0$

$$\therefore r\frac{d\theta}{dr} + \cot n\theta = 0$$

$$\Rightarrow \frac{dr}{r} + \tan n\theta\, d\theta = 0$$

Integrating, we get

$$\log r + \frac{\log \cos n\theta}{n} = \log c.$$

$$\Rightarrow n\log r + \log \cos n\theta = n\log c.$$

$$\Rightarrow \boxed{r^n = c^n \cos n\theta}$$

which is the required equation of orthogonal trajectories, c being parameter

(ii) Given equation can be written as

$$(D^2-2D+1)y = xe^x\sin x.$$

A.E is $D^2-2D+1=0$

$$\Rightarrow (D-1)^2 = 0$$

$$\Rightarrow D = 1, 1$$

$$\therefore C.F = (c_1+c_2x)e^x.$$

To find P.I:

$$P.I = \frac{1}{(D-1)^2} e^x x\sin x = e^x \frac{1}{(D+1-1)^2} x\sin x$$

$$= e^x \frac{1}{D^2} x\sin x$$

$$= e^x \frac{1}{D}\int x\sin x$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1(-\cos x)dx\right]$$

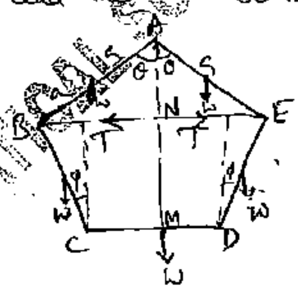
$$= e^x \int [-x\cos x + \sin x]\, dx$$

$$= e^x\left[-\{x\sin x - \int 1 \sin x dx - \cos x\right]$$

$$= e^x\left[-x\sin x - \cos x - \cos x\right] = -e^x(x\sin x + 2\cos x)$$

2(b) Five equal uniform rods, freely jointed at their ends, form a regular pentagon ABCDE and BE is joined by a weightless bar. The system is suspended from A in a vertical plane. Prove that the thrust in BE is $W\cot\frac{1}{10}\pi$, where W is the weight of the rod.

Solⁿ: ABCDE is a pentagon formed of five equal rods each of weight W and say of length 2a. It is suspended from A and BE is jointed by a weightless bar. Let T be the thrust in the bar BE. The line AM joining A to the middle point M of CD is vertical and the line BE is horizontal. The weights of the rods AB, BC, CD, DE and EA act at their respective middle points.



In the position of equilibrium the pentagon is a regular one so that each of its interior angles is $180°-72°$ i.e. $108°$ (or) $\frac{3}{5}\pi$ radians. Let θ be the angle which the two upper slant rods AB and AE make with the vertical and ϕ be the angle which the two lower slant rods CB and DE make with the vertical. Replace the rod BE by two equal and opposite forces T. Give the system a small symmetrical displacement about the vertical AM in which θ changes to $\theta+\delta\theta$ and $\phi \to \phi+\delta\phi$.

The point A remains fixed. The lengths of the rods AB, BC etc remains fixed, the length BE changes and the middle points of the rods AB, BC are slightly displaced. The $\angle ANB$ remains $90°$.

We have $BE = 2BN = 2 \cdot 2a\sin\theta = 4a\sin\theta$

the depth of the middle point of AB (or) AE below

$$A = a\cos\theta$$

the depth of the middle point of BC (or) ED below A

$$= 2a\cos\theta + a\cos\phi$$

and the depth of the middle point M of CD below A

$$= 2a\cos\theta + 2a\cos\phi$$

The equation of virtual work is

$$T\delta(4a\sin\theta) + 2W\delta(a\cos\theta) + 2W\delta(2a\cos\theta + a\cos\phi) + W\delta(2a\cos\theta + 2a\cos\phi) = 0$$

$$\Rightarrow 4aT\cos\theta\,\delta\theta - 2aW\sin\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta - 2aW\sin\phi\,\delta\phi - 2aW\sin\theta\,\delta\theta - 2aW\sin\phi\,\delta\phi = 0$$

$$\Rightarrow 4a(T\cos\theta - 2W\sin\theta)\delta\theta = 4aW\sin\phi\,\delta\phi$$

$$\Rightarrow (T\cos\theta - 2W\sin\theta)\delta\theta = W\sin\phi\,\delta\phi \quad \text{--- (1)}$$

from fig., finding the length BE in two ways i.e, from the upper portion ABE & from the lower portion BCDE, we have

$$4a\sin\theta = 2a + 4a\sin\phi$$

Differentiating, we get $4a\cos\theta\,\delta\theta = 4a\cos\phi\,\delta\phi$

$$\Rightarrow \cos\theta\,\delta\theta = \cos\phi\,\delta\phi \quad \text{--- (2)}$$

Dividing (1) by (2) we get

$$\frac{T\cos\theta - 2W\sin\theta}{\cos\theta} = \frac{W\sin\phi}{\cos\phi}$$

$$\Rightarrow T - 2W\tan\theta = 2W\tan\phi \Rightarrow T = W(\tan\phi + 2\tan\theta)$$

But in the position of equilibrium

$$\theta = \frac{1}{2}\cdot\frac{3}{5}\pi = \frac{3}{10}\pi, \quad \phi = \frac{3}{5}\pi - \frac{1}{2}\pi = \frac{1}{10}\pi$$

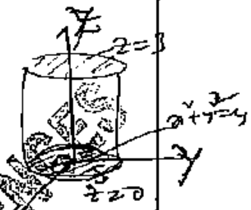
$$\therefore T = W\left(\tan\tfrac{1}{10}\pi + 2\tan\tfrac{3}{10}\pi\right) = W\left[\tan\tfrac{1}{10}\pi + 2\cot\tfrac{2}{10}\pi\right]$$

$$\left[\because \tan\tfrac{3}{10}\pi = \left(\cot\tfrac{1}{2}\pi - \tfrac{3}{10}\pi\right) = \cot\tfrac{2}{10}\pi\right] \quad = W\left[\tan\tfrac{1}{10}\pi + 2\,\frac{1-\tan^2(\pi/10)}{2\tan\pi/10}\right]$$

$$\left[\because \cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1-\tan^2\alpha}{2\tan\alpha}\right] \quad = W\cot(\pi/10)$$

2(c) Evaluate $\int_S F\cdot ds$ where $F = 4xi - 2y^2 j + z^2 k$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$ using the divergence theorem.

Sol: By divergence theorem,



$$\int_S F\cdot ds = \int_V \operatorname{div} F\, dV$$

$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint (4-4y+2z)\, dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\, dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z-4yz+z^2\right)_0^3 dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\, dy\, dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\, dy\, dx$$

$$= \int_{-2}^{2} \left(21y-6y^2\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 42\int_{-2}^{2} \sqrt{4-x^2}\, dx = 84\int_0^2 \sqrt{4-x^2}\, dx$$

$$= 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\pi.$$

36) (i) Solve $(D^2-1)y = x\sin 3x + \cos x$

(ii) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Sol: (i) $(D^2-1)y = x\sin 3x + \cos x$

$\therefore$ It's A.E is $D^2-1=0$

$\Rightarrow D = \pm 1.$

$C.F = C_1e^{x} + C_2e^{-x}.$

$P.I = \frac{1}{D^2-1}(x\sin 3x + \cos x) = \frac{1}{D^2-1} x(I.P. \text{ of } e^{3ix}) + \frac{1}{D^2-1}\cos x$

$= I.P. \text{ of } \frac{1}{D^2-1} e^{i3x}(x) + \frac{1}{(-1)^2-1}\cos x$

$= I.P. \text{ of } \left[e^{i3x}\frac{1}{(D+3i)^2-1}x\right] + \frac{1}{-2}\cos x$

$= I.P. \text{ of } e^{i3x}\frac{1}{D^2+6iD-10}x - \frac{\cos x}{2}$

$= I.P. \text{ of } e^{3ix}\left[-\frac{1}{10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}(x)\right] - \frac{\cos x}{2}$

$= I.P. \text{ of } e^{3ix}\left[-\frac{1}{10}\left(1+\frac{3iD}{5}+\dots\right)x\right] - \frac{\cos x}{2}$

$= I.P. \text{ of } \left[-\frac{1}{10}(\cos 3x + i\sin 3x)\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$

$= -\frac{1}{10}\left[x\sin 3x + \frac{3}{5}\cos 3x\right] - \frac{\cos x}{2}$

$\therefore y = C.F + P.I$

$y = C_1e^{x} + C_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$ which is the required solution

(ii) Let $1+x = v \Rightarrow \frac{dy}{dx} = \frac{dy}{dv},\quad \frac{d^2y}{dx^2} = \frac{d^2y}{dv^2}$

Given equation reduces to

$$v^2\frac{d^2y}{dv^2}+v\frac{dy}{dv}+y = 2\sin\log v \quad \text{——} (1)$$

now put $v=e^z \Rightarrow z=\log v$ and let $D_1=\frac{d}{dz}$.

$$\therefore (1) \equiv [D_1(D_1-1)+D_1+1]y = 2\sin z.$$

$$(D_1^2+1)y = 2\sin z \quad \text{——} (2)$$

A.E. of (2) is $D_1^2+1=0$

$\Rightarrow D_1 = \pm i$

$C.F = C_1\cos z + C_2\sin z$

$C.F = C_1\cos\log v + C_2\sin\log v$

$= C_1\cos\log(1+x) + C_2\sin\log(1+x)$

$$P.I = \frac{1}{D_1^2+1}(2\sin z) = 2\frac{1}{D_1^2+1}\sin z$$

$= 2\left(-\frac{z}{2}\right)\cos z$

$= -z\cos z$

$= -\log v \cos(\log v)$

$= -\log(1+x)\cos\log(1+x).$

$y = C.F + P.I$

$y = C_1\cos[\log(1+x)] + C_2\sin[\log(1+x)]$
$\quad -\log(1+x)\cos[\log(1+x)]$

which is the required solution

3(b) A particle moves under a central force $m\lambda(3a^3u^4+8au^2)$. It is projected from an apse at a distance a from the centre of force with velocity $\sqrt{10\lambda}$. Show that the second apsidal distance is half of the first and that the equation to the path is $2r=a[1+\operatorname{sech}(\theta/\sqrt{5})]$

Sol^n: Here the particle moves under the central force $m\lambda(3a^3u^4+8au^2)$. Therefore the central acceleration P is given by $P=\lambda(3a^3u^4+8au^2)$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\lambda}{u^2}(3a^3u^4+8au^2)$$

$$\Rightarrow h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\lambda(3a^3u^2+8a)$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[2\cdot\frac{u^2}{2}+\left(\frac{du}{d\theta}\right)^2\right]=2\lambda\cdot(a^3u^3+8au)+A$$

$$\Rightarrow v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda(2a^3u^3+16au)+A \quad \text{---(1)}$$

(A = constant)

But initially at apse, $r=a$, $u=1/a$, $du/d\theta=0$ & $v=\sqrt{10\lambda}$.

from (1), we have

$$10\lambda=h^2\left[\frac{1}{a^2}\right]=\lambda\left(2a^3\cdot\frac{1}{a^3}+16a\cdot\frac{1}{a}\right)+A$$

$\therefore$ $h^2=10a^2\lambda$ and $A=10\lambda-18\lambda=-8\lambda$

Substituting the values of h^2 and A in (1), we have

$$10a^2\lambda\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda(2a^3u^3+16au)-8\lambda$$

$$\Rightarrow 10a^2\left(\frac{du}{d\theta}\right)^2 = 2a^3u^3 - 10a^2u^2 + 16au - 8$$

$$\Rightarrow 5a^2\left(\frac{du}{d\theta}\right)^2 = \left[a^3u^3 - 5a^2u^2 + 8au - 4\right]$$

$$= a^2u^2(au-1) - 4au(au-1) + 4(au-1)$$

$$= (au-1)(a^2u^2 - 4au + 4)$$

$$= (au-1)(au-2)^2 \quad \text{——— (2)}$$

To find the second apsidal distance: At an apse, we have $du/d\theta = 0$

$\therefore$ from (2), $u = \frac{1}{a}$ & $\frac{2}{a} \Rightarrow r = a$, and $\frac{a}{2}$.

But $r = a$ is the first apsidal distance. therefore the second apsidal distance is $a/2$ which is half of the first.

<u>To find the equation of path</u>:

from equ (1), we have $\sqrt{5}\,a\frac{du}{d\theta} = -(au-2)\sqrt{(au-1)}$

$$\therefore \frac{d\theta}{\sqrt{5}} = \frac{-a\,du}{(au-2)(\sqrt{au-1})}$$

Substituting $au - 1 = z^2$, so that $a\,du = 2z\,dz$, we have

$$\frac{d\theta}{\sqrt{5}} = \frac{-2z\,dz}{(z^2-1)z}$$

$$\Rightarrow \frac{d\theta}{2\sqrt{5}} = \frac{dz}{1-z^2}$$

Integrating $\frac{\theta}{2\sqrt{5}} + B = \tanh^{-1} z$, where B is a constant.

$$\Rightarrow \frac{\theta}{2\sqrt{5}} + B = \tanh^{-1}\sqrt{(au-1)} \quad \text{——— (3)}$$

But initially when $u = 1/a$, $\theta = 0$

$\therefore$ from (3), $B = 0$

Putting $B=0$ in (3), we get

$$\frac{\theta}{2\sqrt{5}} = \tanh^{-1}\left(\sqrt{au-1}\right)$$

$$\tanh\left(\frac{\theta}{2\sqrt{5}}\right) = \sqrt{au-1}$$

$$\therefore \cosh\left(\frac{\theta}{\sqrt{5}}\right) = \frac{1+\tanh^2(\theta/2\sqrt{5})}{1-\tanh^2(\theta/2\sqrt{5})} = \frac{1+(au-1)}{1-(au-1)} = \frac{au}{2-au}$$

$\Rightarrow 2-au = au\,\text{sech}(\theta/\sqrt{5})$

$\Rightarrow 2 = au\left[1+\text{sech}(\theta/\sqrt{5})\right] = \frac{a}{r}\left[1+\text{sech}(\theta/\sqrt{5})\right]$

$\Rightarrow 2r = a\left[1+\text{sech}(\theta/\sqrt{5})\right]$ which is the required equation of path.

3(c) If $F = (y^2+z^2-x^2)i + (z^2+x^2-y^2)j + (x^2+y^2-z^2)k$, evaluate $\iint \text{Curl}\,F\cdot\hat{n}\,dS$ taken over the portion of the surface $x^2+y^2+z^2-2ax+az=0$ above the plane $z=0$, and verify Stoke's theorem.

Sol^n: The surface $x^2+y^2+z^2-2ax+az=0$ meets the plane $z=0$ in the circle C given by $x^2+y^2-2ax=0$, $z=0$. The polar equation of the circle C lying in the xy-plane is $r=2a\cos\theta$, $0\le\theta<\pi$. Also the equation $x^2+y^2-2ax=0$ can be written as $(x-a)^2+y^2=a^2$. Therefore the parametric equations of the circle C can be taken as

$$x = a + a\cos t,\quad y = a\sin t,\quad z=0,\quad 0\le t<2\pi,$$

$x = a + a\cos t,\ y = a\sin t,\ z = 0,\ 0 \le t < 2\pi$.

Let S denote the portion of the surface $x^2+y^2+z^2-2ax+az=0$ lying above the plane $z=0$ and S_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have.

$$\iint_S \text{curl } F.n\,ds = \iint_{S_1} \text{curl } F.k\,ds$$

Now $\text{Curl } F.k = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix} . k$

$$= \left[\frac{\partial}{\partial x}(z^2+x^2-y^2) - \frac{\partial}{\partial y}(y^2+z^2-x^2)\right]k.k$$

$$[\because i.k = j.k = 0]$$

$$= 2(x-y)$$

$$\therefore \iint_S \text{Curl } F.n\,ds = \iint_{S_1} \text{Curl } F.k\,ds = \iint_{S_1} 2(x-y)\,ds$$

$$= 2\int_{\theta=0}^{\pi} (\cos\theta - \sin\theta) \int_{r=0}^{2a\cos\theta} (r\cos\theta - r\sin\theta)\, r\,d\theta\,dr$$

Changing to polars

$$= 2\int_{\theta=0}^{\pi} (\cos\theta - \sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= 2\times\frac{8a^3}{3}\int_0^{\pi} (\cos\theta - \sin\theta)\cos^3\theta\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi} \cos^4\theta\,d\theta \quad \left[\because \int_0^{\pi} \cos^3\theta \sin\theta\,d\theta = 0\right]$$

$$= 2\times\frac{16a^3}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 2 \times \frac{16a^3}{3} \times \frac{3\times 1}{4\times 2} \qquad = 2\pi a^3 \quad \text{——} (1)$$

Also $\int_C F\cdot dr = \int_C (y^2+z^2-x^2)dx + (z^2+x^2-y^2)dy + (x^2+y^2-z^2)dz$

$$= \int_C (y^2-x^2)dx + (x^2-y^2)dy \quad [\because \text{ C}, z=0 \ \& \ dz=0]$$

$$= \int_0^{2\pi} (x^2-y^2)\left(\frac{dy}{dt} - \frac{dx}{dt}\right)dt$$

$$= \int_0^{2\pi} \left[(a+a\cos t)^2 - a^2\sin^2 t\right](a\cos t + a\sin t)dt$$

$$= a^3\int_0^{2\pi} (1+\cos^2 t + 2\cos t - \sin^2 t)(\cos t + \sin t)dt$$

$$= a^3\int_0^{2\pi} 2\cos^2 t \, dt \quad \text{the other integrals vanish}$$

$$= 2a^3 \times 4\int_0^{\pi/2} \cos^2 t \, dt = 8a^3 \times \frac{1}{2} \times \frac{\pi}{2} = 2\pi a^3 \quad \text{——} (2)$$

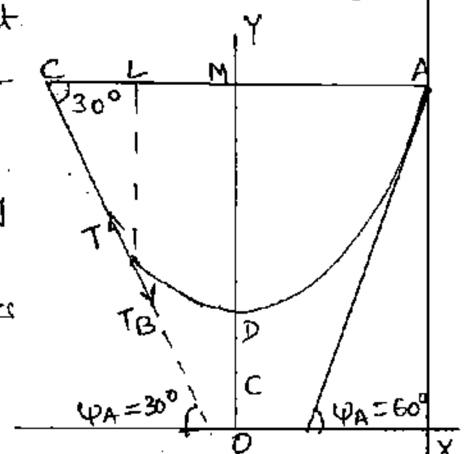
Comparing (1) & (2), we see that

$$\iint_S \text{curl } F\cdot n \, dS = \int_C F\cdot dr$$

Hence Stoke's theorem is verified.

4(b) A heavy uniform chain AB hangs freely under gravity with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angles 60° & 30° respectively with the horizontal. Prove that the ratio of these lengths is $(\sqrt{3}-1):1$

Sol'n : Let the lengths of the heavy uniform chain AB and light string BC be l and a respectively. The chain AB being heavy will hang in the form of an arc of a catenary while the string BC being light will hang in the form of a straight line. Since the tension T_B of the chain at B will be balanced by the tension T in the string, therefore the string BC will be along the tangent at the point B of the chain. Let D be the lowest point i.e. the vertex of the catenary and OX the directrix such that OD = c. If the tangents at A and B are inclined at angles ψ_A and ψ_B to the horizontal, then given that



$$\psi_A = 60^\circ \quad \& \quad \psi_B = 30^\circ$$

Let y_A and y_B be the coordinates of the points A and B. Then from $y = c\sec\psi$, we have

$y_B = c\sec\psi_B = c\sec 30^\circ = 2c/\sqrt{3}$

and $y_A = c\sec\psi_A = c\sec 60^\circ = 2c$

Let BL be the ⊥lar from B on AC

Then $BL = BC\sin 30^\circ = \frac{1}{2}a$ $[\because BC = a]$

$\therefore a = 2BL = 2(y_A - y_B) = 2\left(2c - \frac{2c}{\sqrt{3}}\right) = \frac{4c}{\sqrt{3}}(\sqrt{3}-1).$

If the length of the arc DA be s_1 and that of the arc DB be s_2, then from $s=c\tan\psi$, we have

$s_1 = c\tan 60^\circ = c\sqrt{3}$ and $s_2 = c\tan 30^\circ = c/\sqrt{3}$

$\therefore$ l = the length of the chain ADB

$= s_1 + s_2 = c\sqrt{3} + c/\sqrt{3} = 4c/\sqrt{3}$

Hence the ratio of the lengths of the string and the chain

$$= a/l = \frac{(4c/\sqrt{3})(\sqrt{3}-1)}{(4c/\sqrt{3})} = \frac{\sqrt{3}-1}{1} = \sqrt{3}-1 : 1$$

4(a) Apply the method of variation of parameters

Solve $y_2 + 4y = \sec 2x$

Sol: Given $y_2 + 4y = \sec 2x$ ——— (1)

consider $y_2 + 4y = 0$

A.E. of (2) is $(D^2+2^2)y = 0$ ——— (2)

$D^2+2^2=0$

$\Rightarrow D = \pm 2i$.

C.F of (1) is $= C_1\cos 2x + C_2 \sin 2x$

C_1, C_2 being arbitrary constants

Let $u = \cos 2x$, $v = \sin 2x$ and $R = \sec 2x$

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2 \neq 0$ ——— (3)

P.I of (1) $= u f(x) + v g(x)$

where $f(x) = -\int \frac{vR}{W}dx = -\int \frac{\sin 2x \sec 2x}{2} dx$

$= -\frac{1}{2}\int \tan 2x\, dx$

$= \frac{1}{4}\log(\cos 2x)$

and $g(x) = \int \frac{uR}{W} = \int \frac{\cos 2x \cdot \sec 2x}{2} = \int \frac{1}{2}dx = \frac{x}{2}$.

$\therefore$ P.I. of (1) $= \cos 2x \cdot \frac{\log(\cos 2x)}{4} + \sin 2x\left(\frac{x}{2}\right)$

$\therefore y = C.F + P.I$

$= C_1\cos 2x + C_2 \sin 2x + \frac{1}{4}\cos 2x \log(\cos 2x) + \frac{x}{2}\sin 2x$.

which is the required solution of (1)

4(c). A particle starts from rest at a distance a from the Centre of force which attracts inversely as the distance. Prove that the time of arriving at the Centre is $a\sqrt{\pi/2\mu}$.

Sol'n: If x is the distance of the particle from the Centre of force at time t, then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x}$$

Multiplying bothsides by $2(dx/dt)$ & then integrating w.r.t 't', we have $(dx/dt)^2 = -2\mu \log x + A$, where A = constant

But initially at $x=a$, $dx/dt=0$

$\therefore 0 = -2\mu \log a + A \Rightarrow A = 2\mu \log a$

$\therefore \left(\frac{dx}{dt}\right)^2 = 2\mu(\log a - \log x)$

$\Rightarrow dx/dt = -\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}}$

where the $-$ve sign has been taken since the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\frac{1}{\sqrt{(2\mu)}}\frac{dx}{\sqrt{\{\log(a/x)\}}}$$

Integrating from $x=a$ to $x=0$, the required time t_1 to reach the Centre is given by

$$t_1 = -\frac{1}{\sqrt{(2\mu)}}\int_{x=a}^{0}\frac{dx}{\sqrt{\{\log(a/x)\}}}$$

Put $\log(a/x) = u^2$ i.e. $x = ae^{-u^2} \Rightarrow dx = -2ae^{-u^2}u\,du$

when $x=a$, $u=0$ and when $x\to 0$, $u\to\infty$

$\therefore t_1 = \frac{2}{\sqrt{(2\mu)}} \int_0^\infty e^{-u^2} du.$

But $\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2}$

$\therefore t_1 = \frac{2a}{\sqrt{(2\mu)}} \cdot \frac{\sqrt{\pi}}{2} = a\sqrt{\frac{\pi}{2\mu}}.$

4(d) (i) If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.

(ii) If $F = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$, If find a, b, c such that $\text{curl} F = 0$, then find ϕ such that $F = \nabla\phi$.

Sol^n: (i) Given that ϕ is the solution of the Laplace equation.

i.e. $\nabla^2\phi = 0$

$\Rightarrow \nabla\cdot(\nabla\phi) = 0$

i.e. $\text{div}(\nabla\phi) = 0$

$\therefore \nabla\phi$ is solenoidal.

(ii) we have $\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$\therefore \text{curl}\,\nabla\phi = \nabla\times\nabla\phi$

$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) + j\left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + k\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$$= 0i + 0j + 0k = 0$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

$\therefore \nabla \times \nabla \phi = 0$

$\therefore \nabla \phi$ is irrotational.

(ii) $f = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$.

Given curl $f = 0$ i.e. $\nabla \times f = 0$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix} = 0$$

$\Rightarrow i(c+1) + j(a-1) + k(b-1) = 0i + 0j + 0k$

$\Rightarrow c+1=0,\ a-1=0,\ b-1=0$

$\Rightarrow \boxed{c=-1,\ a=1,\ b=1}$

5(a) Solve the following differential equation

$$x^4 \frac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0 \quad \text{---}(1)$$

Sol^n:- Let $xy = t$

$$x\frac{dy}{dx} + y = \frac{dt}{dx} \quad \text{---}(2)$$

Equation (1) can be written as

$$x^3\left(x\frac{dy}{dx} + y\right) + \operatorname{cosec} xy = 0$$

$$\Rightarrow x^3 \frac{dt}{dx} + \operatorname{cosec} t = 0 \quad (\text{from } (2))$$

$$\Rightarrow \sin t \, dt + \frac{dx}{x^3} = 0$$

Integrating, we get

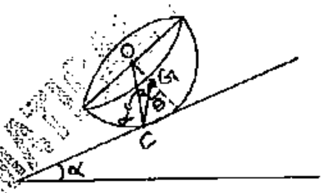
$$\Rightarrow -\cos t - \frac{1}{2x^2} = +c_1$$

$$\Rightarrow \cos t + \frac{1}{2x^2} = c \quad (\because -c_1 = c)$$

which is the required solution

5(b). A solid sphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1} 3/8$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.

Sol^n: Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then OC = r. Let G be the centre of gravity of the hemisphere. Then OG = 3r/8. In the position of equilibrium the line CG must be vertical.



Since OC is ⊥lar to the inclined plane and CG is ⊥lar to the horizontal,

$\therefore \angle OCG = \alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From ΔOGC, we have

$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin\theta} \Rightarrow \frac{3r/8}{\sin\alpha} = \frac{r}{\sin\theta}$$

$$\therefore \sin\theta = \frac{8}{3}\sin\alpha \Rightarrow \theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right)$$

Giving the position of equilibrium of the hemisphere

Since $\sin\theta < 1$, therefore $\frac{8}{3}\sin\alpha < 1$

$$\Rightarrow \sin\alpha < \frac{8}{3} \Rightarrow \alpha < \sin^{-1} 3/8$$

Thus for the equilibrium to exist, we must have

$$\alpha < \sin^{-1} 3/8.$$

Now let CG = h, Then

$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha} \text{ so that } h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha}$$

Here $\rho_1 = r$ & $\rho_2 = \infty$

The equilibrium will be stable if

$$h < \frac{\rho_1 \rho_2 \cos\alpha}{\rho_1 + \rho_2}$$

$$\Rightarrow \frac{1}{h} > \frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \sec\alpha \Rightarrow \frac{1}{h} > \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\sec\alpha$$

$$\Rightarrow \frac{1}{h} > \frac{1}{r}\sec\alpha \qquad (\because \rho_1 = r,\ \rho_2 = \infty)$$

$$\Rightarrow h < r\cos\alpha$$

$$\Rightarrow \frac{3r\sin(\theta - \alpha)}{8\sin\alpha} < r\cos\alpha$$

$$\Rightarrow 3\sin(\theta - \alpha) < 8\sin\alpha\cos\alpha$$

$$\Rightarrow 3\sin\theta\cos\alpha - 3\cos\theta\sin\alpha < 8\sin\alpha\cos\alpha$$

$$\Rightarrow 8\sin\alpha\cos\alpha - 3\sin\alpha\sqrt{\left(1 - \frac{64}{9}\sin^2\alpha\right)} < 8\sin\alpha\cos\alpha \qquad \left[\because \sin\theta = \frac{8}{3}\sin\alpha\right]$$

$$\Rightarrow -\sin\alpha\sqrt{(9 - 64\sin^2\alpha)} < 0 \quad \text{——} \ (2)$$

$$\Rightarrow \sin\alpha\sqrt{(9 - 64\sin^2\alpha)} > 0$$

But from (1),

$\sin\alpha < 3/8$ i.e, $64\sin^2\alpha < 9$ i.e, $\sqrt{9 - 64\sin^2\alpha}$ is a +ve real number. Therefore the relation (2) is true. Hence the equilibrium is stable.

5(c) Show that in a simple Harmonic motion of amplitude a and period T, the velocity v at a distance x from the centre is given by the relation $v^2T^2 = 4\pi^2(a^2 - x^2)$. Find the new amplitude if the velocity were doubled when the particle is at a distance $\frac{1}{2}a$ from the centre, the period remaining unaltered.

Sol^n: Let the equation of S.H.M with Centre as origin be $\frac{d^2x}{dt^2} = -\mu x$.

The time period $T = 2\pi/\sqrt{\mu}$ ——— ①

Let a be the amplitude. Then the velocity v at a distance x from the Centre is given by

$$v^2 = \mu(a^2 - x^2) \text{ ——— } ②$$

from ① $\mu = 4\pi^2/T^2$. Putting this value of μ in (2) we have $v^2 = \frac{4\pi^2}{T^2}(a^2 - x^2) \Rightarrow v^2T^2 = 4\pi^2(a^2 - x^2)$ ——— ③

Let v_1 be the velocity at a distance $\frac{1}{2}a$ from the Centre. Then putting $x = \frac{1}{2}a$ and $v = v_1$ in ③, we get

$$v_1^2T^2 = 4\pi^2(a^2 - \tfrac{1}{4}a^2) = 3\pi^2a^2 \text{ ——— } ④$$

Let a_1 be the new amplitude when the velocity at the point $x = \frac{1}{2}a_1$ is doubled i.e. when the velocity at the point $x = \frac{1}{2}a$ is any how made $2v_1$. Since the period remains unchanged, therefore putting $v = 2v_1$, $a = a_1$ and $x = \frac{1}{2}a$ in (3), we get

$$4v_1^2T^2 = 4\pi^2(a_1^2 - \tfrac{1}{4}a^2)$$

$$\Rightarrow 4 \times 3\pi^2a^2 = 4\pi^2(a_1^2 - \tfrac{1}{4}a^2) \quad [\because \text{from (4)}\ v_1^2T^2 = 3\pi^2a^2]$$

$\Rightarrow a_1^2 = 3a^2 + \frac{1}{4}a^2 = \frac{13a^2}{4}$. Hence the new amplitude

$$a_1 = \frac{a\sqrt{13}}{2}.$$

5(d) If $a = \sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta k$, $b = \cos\theta\,\hat{i} - \sin\theta\, j - 3k$ and $c = 2i + 3j - 3k$, find $\frac{d}{d\theta}\{a\times(b\times c)\}$ at $\theta = \pi/2$.

Solⁿ: We have

$$b\times c = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix}$$

$$= (3\sin\theta+9)\hat{i} + (3\cos\theta - 6)\hat{j} + (3\cos\theta + 2\sin\theta)k.$$

$$\therefore\ a\times(b\times c) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta+9 & 3\cos\theta-6 & 3\cos\theta+2\sin\theta \end{vmatrix}$$

$$= (3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta)\hat{i}$$
$$+ (3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta)j$$
$$+ (-6\sin\theta - 9\cos\theta)k$$

$$\therefore\ \frac{d}{d\theta}\{a\times(b\times c)\} = \big(-6\cos\theta\sin\theta + 2\cos^2\theta - 2\sin^2\theta - 3\cos\theta + 3\theta\sin\theta + 6\big)\hat{i}$$
$$+ \big(3\sin\theta + 3\theta\cos\theta + 9 - 3\cos^2\theta + 3\sin^2\theta - 4\sin\theta\cos\theta\big)j$$
$$+ (-6\cos\theta + 9\sin\theta)k$$

putting $\theta = \pi/2$, we get

$$\frac{d}{d\theta}\left(a\times(b\times c)\right) = \left(4 + \tfrac{3}{2}\pi\right)i + 15j + 9k.$$

5(e) Find the constant a so that $\overrightarrow{V}$ is a conservative vector field, where
$V = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)az^2k.$

Solⁿ: Given that

$$V = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)az^2k$$

The field V is conservative force if $\nabla \times V = 0$

$$\therefore \nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$\Rightarrow \hat{i}(0-0) + \hat{j}(0 - 3z^2) + k\{(a-2)(2x) - ax\} = 0$

$\Rightarrow \hat{i}(0) - 3z^2\hat{j} + (ax - 4x)k = 0$

$\Rightarrow 0\hat{i} - 3z^2\hat{j} + (a-4)xk = 0$

$\Rightarrow z = 0$ and $a - 4 = 0 \Rightarrow a = 4$

$\therefore$ The value of $a = 4$

6(a) Find the inverse Laplace transforms of the following: (i) $\log \frac{s^2+1}{s(s+1)}$ (ii) $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Sol: (i) Let $f(s) = \log \frac{s^2+1}{s(s+1)}$

$= \log(s^2+1) - \log s - \log(s+1)$

$f'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$

$\therefore L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{2s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$

$(-1)^1 t L^{-1}\{f(s)\} = 2\cos t - 1 - e^{-t}$

$L^{-1}\{f(s)\} = \frac{-2\cos t + 1 + e^{-t}}{t}$

$\therefore L^{-1}\left\{\log \frac{s^2+1}{s(s+1)}\right\} = \frac{1}{t}\left[-2\cos t + 1 + e^{-t}\right]$.

(ii) Let $f(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$$f'(s) = \frac{1}{1+\frac{4}{s^4}}\left(\frac{-4}{s^3}\right) = \frac{-4s}{s^4+4}.$$

$$L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{-4s}{s^4+4}\right\}$$

$$(-t)^1 L^{-1}\{f(s)\} = L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\}$$

$$t\,L^{-1}\{f(s)\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)\,(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= e^{t}\sin t - e^{-t}\sin t = (e^{t}-e^{-t})\sin t$$

$$L^{-1}\{f(s)\} = \frac{1}{t}\left[2\sinh t \sin t\right]$$

6(5) Obtain the singular solution of the equation

$p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$ —(1)

directly from the equation and also from its complete primitive, explaining the geometrical significance of the irrelevant factors that present themselves.

Sol^n: Since the given equation is quadratic in p, the p-disc. relation is

$$4x^2y^2\sin^4\alpha - 4y^2\cos^2\alpha\,(y^2-x^2\sin^2\alpha) = 0.$$

$$\Rightarrow y^2\left[x^2\sin^2\alpha\,(\sin^2\alpha+\cos^2\alpha) - y^2\cos^2\alpha\right] = 0$$

$$\Rightarrow y^2\left(x^2\sin^2\alpha - y^2\cos^2\alpha\right) = 0 \qquad (\because \sin^2\alpha+\cos^2\alpha=1)$$

$\Rightarrow y^2\cos^2\alpha\,(x^2\tan^2\alpha - y^2) = 0.$

$\Rightarrow y^2(x\tan\alpha - y)(x\tan\alpha + y) = 0$ ——— (2)

To find the general solution of (1):
Given equation (1) can be written as

$(py)^2 - 2(py)x\tan^2\alpha + (y^2\sec^2\alpha - x^2\tan^2\alpha) = 0$

$\therefore py = \dfrac{2x\tan^2\alpha \pm \sqrt{4x^2\tan^4\alpha - 4(y^2\sec^2\alpha - x^2\tan^2\alpha)}}{2}$

$py = x\tan^2\alpha \pm \sqrt{x^2\tan^2\alpha\,(\tan^2\alpha+1) - y^2\sec^2\alpha}$

$y\dfrac{dy}{dx} = x\tan^2\alpha \pm \sec\alpha\,.\sqrt{x^2\tan^2\alpha - y^2}$

$\Rightarrow y\,dy - x\tan^2\alpha\,dx = \pm\sec\alpha\sqrt{x^2\tan^2\alpha - y^2}\,dx$

$\Rightarrow \dfrac{\pm\, x\tan^2\alpha\,dx - y\,dy}{\sqrt{x^2\tan^2\alpha - y^2}} = -\sec\alpha\,dx$

Integrating,

$\pm\sqrt{x^2\tan^2\alpha - y^2} = c - x\sec\alpha.$

$x^2\tan^2\alpha - y^2 = (c - x\sec\alpha)^2$

$= c^2 + x^2\sec^2\alpha - 2cx\sec\alpha.$

$(1+\tan^2 = \sec^2)$

$\Rightarrow x^2(\tan^2\alpha - \sec^2) - y^2 - c^2 + 2cx\sec\alpha = 0$

$\Rightarrow x^2(-1) - y^2 - c^2 + 2cx\sec\alpha = 0$

$\Rightarrow x^2 + y^2 + c^2 - 2cx\sec\alpha = 0$ ——— (3)

which is the general solution of (1)

$\therefore$ The c-disc. relation is

$4x^2\sec^2\alpha - 4(x^2+y^2) = 0$

$\Rightarrow x^2(\sec^2\alpha - 1) - y^2 = 0$

$\Rightarrow x^2\tan^2\alpha - y^2 = 0$

$\Rightarrow (x\tan\alpha - y)(x\tan\alpha + y) = 0$ ——— (4)

$\Rightarrow x\tan\alpha - y = 0$ and $x\tan\alpha + y = 0$

$\Rightarrow$ i.e, the lines $y = \pm x\tan\alpha$ are singular solutions and $y=0$ is a tac-locus.

Now the general solution (3) represents a family of circles all having their centres on x-axis. The circles of the system touch one another on x-axis and $y=0$. i.e, x-axis passes through the points of contact of non-consecutive circles which touch on x-axis. The family of circles is being touched by $y = \pm x\tan\alpha$, which are equally inclined to the line of centres of the circles. i.e, x-axis and pass through the origin

6(c) Solve $y'' + (1-\cot x)y' - y\cot x = \sin^2 x$

Sol: Given that $y'' + (1-\cot x)y' - y\cot x = \sin^2 x$ ——— (1)

comparing (1) with $y'' + Py' + Qy = R$

Here $P = 1 - \cot x$, $Q = -\cot x$, $R = \sin^2 x$

Here $1-P+Q = 1-(1-\cot x)-\cot x = 0$.

and so e^{-x} is an integral of C.F.

$\therefore$ $y = u = e^{-x}$ is a part of C.F. of the given equation. —— (2)

Let the required general solution be $y = uv$ —— (3)

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$\Rightarrow \frac{d^2v}{dx^2} + \left(1-\cot x + \frac{2}{e^{-x}}(-e^{-x})\right)\frac{dv}{dx} = \frac{\sin^2 x}{e^{-x}}$

$\Rightarrow \frac{d^2v}{dx^2} - (\cot x + 1)\frac{dv}{dx} = e^{x}\sin^2 x$

Let $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$ —— (4)

$\therefore \frac{dq}{dx} - (1+\cot x)q = e^{x}\sin^2 x$.

which is linear in q and x

I.F. $= e^{-\int(1+\cot x)dx} = e^{-x-\log\sin x} = e^{-x}e^{-\log\sin x} = e^{-x}/\sin x$.

$\therefore q\left(e^{-x}/\sin x\right) = \int e^{x}\sin^2 x \cdot \frac{e^{-x}}{\sin x}dx + C$

$= \int \sin x\,dx + C$

$= -\cos x + C$.

$\therefore q = e^{x}(-\cos x\sin x) + C\sin x$

$v = \int\left(-\frac{e^{x}}{2}\sin 2x + e^{x}C\sin x\right)dx$

$= -\frac{1}{2}\left[\frac{1}{5}e^{x}(\sin 2x - 2\cos 2x)\right] + \frac{C}{2}e^{x}(\sin x - \cos x) + C_1$

$= -\frac{1}{10}e^{x}(\sin 2x - 2\cos 2x) + \frac{Ce^{x}}{2}(\sin x - \cos x) + C_1$

$\therefore$ The required general solution is

$y = uv = e^{-x}\left[-\frac{1}{10}e^{x}(\sin 2x - 2\cos 2x) + \frac{Ce^{x}}{2}(\sin x - \cos x) + C_1\right]$

$$\therefore y = c_1 e^{-x} - \frac{1}{10}(\sin 2x - 2\cos 2x) + \frac{c}{2}(\sin x - \cos x)$$

$$= c_1 e^{-x} - \frac{1}{10}(\sin 2x - 2\cos 2x) + c_2(\sin x - \cos x)$$

$$\left[\because c_2 = \frac{c}{2}\right]$$

6(d) Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x \sin(\log x) + 1}{x}$

Sol^n: Given $(x^2D^2 - 3xD + 1)y = x^{-1}[1 + \log x(\sin \log x)]$ —— (1)

Let $x = e^z$ so that $z = \log x$ and

let $D_1 = d/dz$

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$

$\therefore$ (1) $\equiv$ $[D_1(D_1-1) - 3D_1 + 1]y = e^{-z}[1 + z\sin z]$

$\Rightarrow (D_1^2 - 4D_1 + 1)y = e^{-z} + e^{-z}z\sin z$ —— (2)

A.E. of (2) is $D_1^2 - 4D_1 + 1 = 0$

$\Rightarrow D_1 = 2 \pm \sqrt{3}$.

$\therefore$ C.F $= e^{2z}[c_1 \cosh\sqrt{3}z + c_2 \sinh\sqrt{3}z]$

$= x^2[c_1 \cosh\sqrt{3}\log x + c_2 \sinh\sqrt{3}\log x]$

P.I corresponding to e^{-z}:

$$= \frac{1}{D_1^2 - 4D_1 + 1}e^{-z} = \frac{1}{1+4+1}e^{-z} = \frac{1}{6}x^{-1} = \frac{1}{6x}$$

P.I corresponding to $e^{-z}z\sin z$:

$$= \frac{1}{D_1^2 - 4D_1 + 1}e^{-z}z\sin z = e^{-z}\frac{1}{(D_1-1)^2 - 4(D_1-1) + 1}z\sin z$$

$$= e^{-z}\frac{1}{D_1^2 - 6D_1 + 6}z\sin z.$$

$$= e^{-z}\left[z\frac{1}{D_1^2-6D_1+6}\sin z - \frac{2D_1-6}{(D_1^2-6D_1+6)^2}\sin z\right]$$

$$\left(\because \frac{1}{F(D)}(xV) = x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V\right)$$

$$= e^{-z}\left[z\frac{1}{-1-6D+6}\sin z - (2D_1-6)\frac{1}{(-1+6D_1+6)^2}\sin z\right]$$

$$= e^{-z}\left[z\frac{1}{5-6D_1}\sin z - \frac{2D_1-6}{(5-6D_1)^2}\sin z\right]$$

$$= e^{-z}\left[z\frac{5+6D_1}{25-36D_1^2}\sin z - (2D_1-6)\frac{1}{25-60D_1+36D_1^2}\sin z\right]$$

$$= e^{-z}\left[z\frac{5+6D_1}{25+36}\sin z - (2D_1-6)\frac{1}{25-60D_1-36}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z + 6\cos z) + \frac{2D_1-6}{11+60D_1}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z + 6\cos z) + \frac{(2D_1-6)(60D_1-11)}{3600D_1^2-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z + 6\cos z) + \frac{120D_1^2-382D_1+66}{-3600-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z + 6\cos z) + \frac{120(-\sin z)-382\cos z+66\sin z}{-3721}\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z + 6\cos z) + \frac{54\sin z+382\cos z}{3721}\right]$$

$$= \frac{1}{x}\left[\frac{\log x}{61}\left(5\sin(\log x) + 6\cos(\log x)\right) + \frac{54\sin(\log x)+382\cos(\log x)}{3721}\right]$$

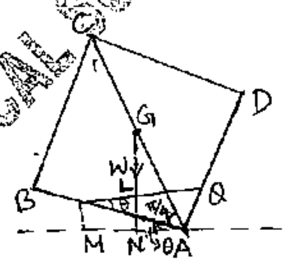
$$\therefore y = x^2\left[c_1\cosh(\sqrt{3}\log x) + c_2\sinh(\sqrt{3}\log x)\right] + \frac{1}{6x}$$

$$+\frac{1}{x}\left[\frac{\log x}{61}\left(5\sin(\log x) + 6\cos(\log x)\right) + \frac{54\sin(\log x)+382\cos(\log x)}{3721}\right]$$

7(a). A Square of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either $\pi/4$ (or) $\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right)$.

Sol'n: The sides AB and AD of the square lamina ABCD rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ = c$ and $AB = 2a$.

The weight W of the lamina acts at G, the middle point of the diagonal AC. Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that



$$\angle PAM = \theta = \angle QPA$$

we have $\angle BAC = \frac{\pi}{4}$ = constant

Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$= LG = NG - NL = NG - MP \quad \left[\because AG = \tfrac{1}{2}AC = \tfrac{1}{2}2a\sqrt{2} = a\sqrt{2} \text{ and } AP = PQ\cos\theta\right]$$

$$= AG\sin\left(\tfrac{\pi}{4}+\theta\right) - AP\sin\theta$$

$$= \tfrac{a}{\sqrt{2}}\sin\left(\tfrac{\pi}{4}+\theta\right) - PQ\cos\theta\sin\theta$$

$= a\sqrt{2}\left(\sin\frac{1}{4}\pi\cos\theta + \cos\frac{1}{4}\pi\sin\theta\right) - C\cos\theta\sin\theta$

$= a(\cos\theta + \sin\theta) - C\cos\theta\sin\theta$

The equation of virtual work is.

$-W\delta(LG) = 0 \Rightarrow \delta(LG) = 0$

$\Rightarrow \delta[a(\cos\theta + \sin\theta) - C\cos\theta\sin\theta] = 0$

$\Rightarrow [a(-\sin\theta + \cos\theta) - C(\cos^2\theta - \sin^2\theta)]\delta\theta = 0$

$\Rightarrow a(\cos\theta - \sin\theta) - C(\cos^2\theta - \sin^2\theta) = 0 \quad [\because \delta\theta \neq 0]$

$\Rightarrow (\cos\theta - \sin\theta)[a - c(\cos\theta + \sin\theta)] = 0$

$\therefore$ either $\cos\theta - \sin\theta = 0$.

$\Rightarrow \sin\theta = \cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{1}{4}\pi$

giving one position of equilibrium in which the lamina rests symetrically on the pegs

$\Rightarrow a - c(\cos\theta + \sin\theta) = 0$

$\Rightarrow c^2(\cos\theta + \sin\theta)^2 = a^2$

$\Rightarrow c^2(1 + \sin 2\theta) = a^2$

$\Rightarrow \sin 2\theta = \frac{a^2}{c^2} - 1 = \frac{a^2 - c^2}{c^2}$

$\Rightarrow \theta = \frac{1}{2}\sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right)$

giving the other position of equilibrium.

7(b). A particle is projected with a velocity u from a point on an inclined plane whose inclination to the horizontal is β, and strikes it at right angles. show that

(i) the time of flight is $\frac{2u}{g\sqrt{(1 + 3\sin^2\beta)}}$

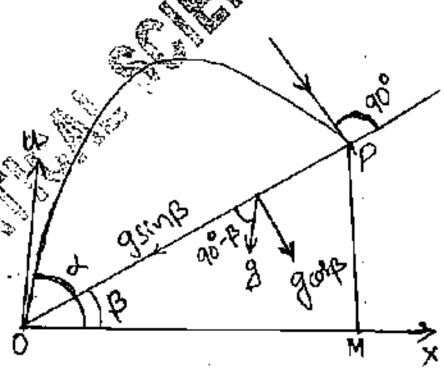
(ii) the range on the inclined plane is $\frac{2u^2}{g} \cdot \frac{\sin\beta}{1 + 3\sin^2\beta}$

and (iii) the vertical height of the point struck, above the point of projection is $\dfrac{2u^2 \sin^2\beta}{g(1+3\sin^2\beta)}$

Sol^n: Let O be the point of projection, u the velocity of projection, α the angle of projection and P the point where the particle strikes the plane at right angles.

Let T be the time of flight from O to P. Then by the time of flight on inclined plane we have

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$



Since the particle strikes the inclined plane at right angles at P, therefore the velocity of the particles at P along the inclined plane is zero. Also the resolved part of the acceleration g along the inclined plane and using the formula $v = u + ft$,

$$0 = u\cos(\alpha-\beta) - g\sin\beta T$$

$$\Rightarrow T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad \text{——— (2)}$$

Equating the values of T from (1) & (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\Rightarrow \tan(\alpha-\beta) = \tfrac{1}{2}\cot\beta \quad \text{——— (3)}$$

as the condition for striking the plane at right angles,

(i) from (2)

$$T = \frac{u}{g\sin\beta\sec(\alpha-\beta)} = \frac{u}{g\sin\beta\sqrt{\{1+\tan^2(\alpha-\beta)\}}}$$

$$= \frac{u}{g\sin\beta\sqrt{(1+\frac{1}{4}\cot^2\beta)}}$$ (from eqn (3). $\tan(\alpha-\beta)=\frac{1}{2}\cot\beta$).

$$= \frac{2u\sin\beta}{g\sin\beta\sqrt{(4\sin^2\beta+\cos^2\beta)}} = \frac{2u}{g\sqrt{\sin^2\beta+\cos^2\beta+3\sin^2\beta}}$$

$$= \frac{2u}{g\sqrt{(1+3\sin^2\beta)}}$$

(ii) Let R be the range on the inclined plane; then $R=OP$. Considering the motion from O to P the inclined plane and using the formula $v^2=u^2+2fs$

we have $0 = u^2\cos^2(\alpha-\beta) - 2g\sin\beta R$

$$\Rightarrow R = \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} = \frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)}$$

$$= \frac{u^2}{2g\sin\beta\{1+\tan^2(\alpha-\beta)\}}$$

$$= \frac{u^2}{2g\sin\beta\{1+\frac{1}{4}\cot^2\beta\}}$$ [from (3)]

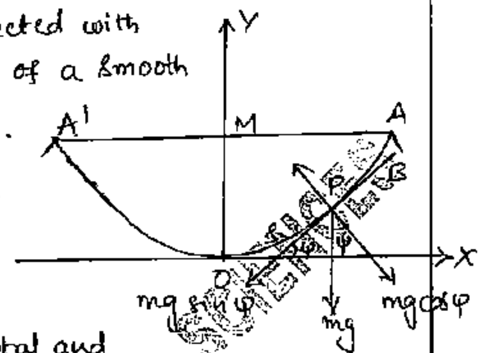
$$= \frac{4u^2\sin^2\beta}{2g\sin\beta(4\sin^2\beta+\cos^2\beta)} = \frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}$$

(iii) The vertical height of P above $O = PM$

$$= OP\sin\beta = R\sin\beta = \frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}.$$

7(c) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{a/g}\tan^{-1}\left[\sqrt{4ag}/V\right]$.

Sol'n: Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP = s$, then the equations of motion of the particle are



$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{--- (1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{--- (2)}$$

For the cycloid, $s = 4a\sin\psi$ --- (3)

from (1) & (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$

Multiplying both sides by $2(ds/dt)$ and integrating,

we have $v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$

But initially at the cusp A, $s = 4a$ and $\left(ds/dt\right)^2 = V^2$

$\therefore V^2 = -(g/4a)\cdot 16a^2 + A$

$\Rightarrow A = V^2 + 4ag$

$$\therefore v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2 = \left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag) - s^2\right]$$

$$\Rightarrow \frac{ds}{dt} = -\frac{1}{2}\sqrt{g/a}\sqrt{\left[\frac{4a}{g}(V^2+4ag) - s^2\right]}$$

[$-$ve sign is taken because the particle is moving in the direction of s decreasing]

$$\Rightarrow dt = -2\sqrt{(a/g)}\ \frac{ds}{\sqrt{[(4a/g)(v^2+4ag)-s^2]}}$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(v^2+4ag)-s^2]}}$$

$$= 2\sqrt{(a/g)}\int_{0}^{4a}\frac{ds}{\sqrt{[(4a/g)(v^2+4ag)-s^2]}}$$

$$= 2\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(v^2+4ag)}}\right]_0^{4a}$$

$$= 2\sqrt{(a/g)}\ \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}\right\}$$

$$= 2\sqrt{a/g}\ \theta \qquad (4)$$

where $\theta = \sin^{-1}\left\{\frac{2\sqrt{ag}}{\sqrt{(v^2+4ag)}}\right\}$

we have $\sin\theta = \frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}$

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{\left[1-\frac{4ag}{v^2+4ag}\right]} = \frac{v}{\sqrt{(v^2+4ag)}}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{ag}}{v} = \frac{\sqrt{(4ag)}}{v}$$

$\Rightarrow \theta = \tan^{-1}\left[\frac{\sqrt{(4ag)}}{v}\right]$, from (4), the time of reaching the vertex is $= 2\sqrt{a/g}\ \tan^{-1}\left[\sqrt{4ag}/v\right]$.

8(a) (i) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$

(ii) If $F = (3x^2y - z)\hat{i} + (xz^2 + y^4)\hat{j} - 2x^3z^2 k$, find $\nabla(\nabla\cdot F)$ at the point $(2,-1,0)$.

Sol (i) The given surfaces are

$$f_1 \equiv ax^2 - byz - (a+2)x = 0 \quad \text{--- (1)}$$

and $$f_2 \equiv 4x^2y + z^3 - 4 = 0 \quad \text{--- (2)}$$

The point $(1,-1,2)$ obviously lies on the surface (2). It will also lie on the surface (1) if

$a + 2b - (a+2) = 0 \Rightarrow 2b - 2 = 0 \Rightarrow \boxed{b=1}$

Now $\text{grad } f_1 = [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$

$\text{grad } f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2 k.$

Then $n_1 = \text{grad } f_1$ at the point $(1,-1,2)$
$= (a-2)\hat{i} - 2b j + bk.$

and $n_2 = \text{grad } f_2$ at the point $(1,-1,2)$
$= -8\hat{i} + 4j + 12k.$

The vectors $\hat{n}_1$ and $\hat{n}_2$ are along the normals to the surfaces (1) and (2) at the point $(1,-1,2)$.
These surfaces will intersect orthogonally at the point $(1,-1,2)$ if the vectors $\hat{n}_1$ and $\hat{n}_2$ are perpendicular i.e, if $n_1 \cdot n_2 = 0$.

$\Rightarrow -8(a-2) - 8b + 12b = 0 \Rightarrow b - 2a + 4 = 0$ --- (3)

$\Rightarrow 1 - 2a + 4 = 0 \; (\because b = 1)$

$\Rightarrow a = 5/2$

$\therefore \boxed{a = 5/2,\ b = 1}$

(iv) $\nabla\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left((3x^2y-z)i + (xz^3+y^4)j - 2x^3z^2k\right)$

$= 6xy + 4y^3 - 4zx^3$.

$\nabla(\nabla\cdot F) = i\frac{\partial}{\partial x}(6xy+4y^3-4zx^3) + j\frac{\partial}{\partial y}(6xy+4y^3-4zx^3) + k\frac{\partial}{\partial z}(6xy+4y^3-4zx^3)$.

$= i(6y-12zx^2) + j(6x-12zx^2) + k(-4x^3)$

$\nabla(\nabla\cdot F)\Big|_{(2,-1,0)} = i(-6-0) + j(12-0) + k(-32)$

$= -6i + 12j - 32k$.

8(b) find the curvature and torsion of the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.

Sol: The position vector for any point on the curve is

$\vec{r} = 3\cos t\, i + 3\sin t\, j + 4t\, k$

$\frac{d\vec{r}}{dt} = -3\sin t\, i + 3\cos t\, j + 4k$, $\frac{d^2r}{dt^2} = -3\cos t\, i - 3\sin t\, j$

and $\frac{d^3r}{dt^3} = 3\sin t\, i - 3\cos t\, j$.

$\left|\frac{dr}{dt}\right| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{9+16} = 5$

$\frac{dr}{dt}\times\frac{d^2r}{dt^2} = 12\sin t\, i - 12\cos t\, j + 9k$

$\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right| = \sqrt{12^2+9^2} = 15$

$\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] = \frac{d^3r}{dt^3}\cdot\left(\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right) = 36\sin^2 t + 36\cos^2 t$
$= 36$.

$\therefore \kappa = \left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right| \Big/ \left|\frac{dr}{dt}\right|^3 = \frac{15}{(5)^3} = \frac{3}{25}$

$\tau = \left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] \Big/ \left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|^2 = \frac{36}{15\times 15} = \frac{4}{25}$

8(c). A particle moves so that its position vector is given by $r = \cos\omega t\, i + \sin\omega t\, j$ where ω is a constant; show that (i) the velocity of the particle is $\perp$lar to r, (ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin,
(iii) $r \times \frac{dr}{dt}$ is a constant vector.

Sol^n: (i) velocity $V = \frac{dr}{dt} = -\omega \sin\omega t\, i + \omega\cos\omega t\, j$

we have $r \cdot \frac{dr}{dt} = (\cos\omega t\, i + \sin\omega t\, j)\cdot(-\omega\sin\omega t\, i + \omega\cos\omega t\, j)$

$= -\omega\cos\omega t \sin\omega t + \omega\sin\omega t\cos\omega t = 0$

$\therefore$ the velocity is $\perp$lar to r.

(ii) Acceleration of the particle

$= a = \frac{d^2r}{dt^2} = -\omega^2\cos\omega t\, i - \omega^2 \sin\omega t\, j$

$= -\omega^2(\cos\omega t\, i + \sin\omega t\, j) = -\omega^2 r$

$\therefore$ Acceleration is a vector opposite to the direction of r i.e, acceleration is directed towards the origin. Also magnitude of acceleration $= |a| = |-\omega^2 r| = \omega^2 r$. which is proportional to r i.e, the distance of the particle from the origin.

(iii) $r \times \frac{dr}{dt} = (\cos\omega t\, i + \sin\omega t\, j) \times (-\omega\sin\omega t\, i + \omega\cos\omega t\, j)$

$= \omega\cos^2\omega t\, i\times j - \omega\sin^2\omega t\, j\times i$ $[\because i\times i = 0,\ j\times j = 0]$

$= \omega\cos^2\omega t\, k + \omega\sin^2\omega t\, k$ $[\because i\times j = k = -j\times i]$

$= \omega(\cos^2\omega t + \sin^2\omega t)k$

$= \omega k$, a constant vector.

8(d). Verify divergence theorem for the function.
$F = yi + xj + z^2k$ over the cylindrical region bounded by $x^2+y^2=a^2$, $z=0$ and $z=h$.

Sol: Let S denote the closed surface bounded by the cylinder $x^2+y^2=a^2$ and the planes $z=0$, $z=h$. Also let V be the volume bounded by the surface S.

By Gauss divergence theorem, we have

$$\iint_S F\cdot n\, ds = \iiint_V \operatorname{div} F\, dV.$$

We have $\iiint \operatorname{div} F\, dV = \iiint_V [\operatorname{div}(yi+xj+z^2k)]\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y)+\frac{\partial}{\partial y}(x)+\frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V 2z\, dV$$

$$= \int_{z=0}^{h}\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z\, dz\, dx\, dy$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} z\, dz\, dx\, dy$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a} z\,[y]_0^{\sqrt{a^2-x^2}}\, dz\, dx$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a} z\sqrt{a^2-x^2}\, dz\, dx$$

$$= 8\int_{z=0}^{h}\int_{0}^{a} z\sqrt{a^2-x^2}\, dz\, dx$$

$$= 8\int_{x=0}^{a} \sqrt{a^2-x^2}\left[\frac{z^2}{2}\right]_0^h dx$$

$$= 8\int_0^a \frac{h^2}{2}\sqrt{a^2-x^2}\,dx = 4h^2\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4h^2\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4h^2\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \pi a^2 h^2 \quad \text{--- (1)}$$

Now we shall evaluate the surface integral

$$\iint_S F\cdot n\,ds$$

The surface S consists of three surfaces:

(i) the surface S_1 of the base of the cylinder. i.e, the plane face $z=0$

(ii) the surface S_2 of the top face of the cylinder i.e. the plane $z=h$ and

(iii) the surface S_3 of the convex portion of the cylinder.

for the surface S_1 i.e, $z=0$, $F = yi + xj$, putting $z=0$ in F.

A unit vector $\hat{n}$ along the outward drawn normal S_1 is obviously $-k$.

$$\therefore \iint_{S_1} F\cdot n\,ds = \iint_{S_1} (yi+xj)\cdot(-k)\,ds = 0$$

for the surface S_2 i.e, $z=h$, $F = yi + xj + h^2k$ putting $z=h$ in F.

A unit vector $\hat{n}$ along the outward drawn normal to S_2 is given by $\hat{n} = k$.

$$\therefore \iint_{S_2} F\cdot\hat{n}\,dS = \iint_{S_2} (yi + xj + h^2 k)\cdot k\,dS$$

$$= \iint_{S_2} h^2 dS = h^2 \iint_{S_2} dS = h^2 \text{ (area of the plane face } S_2 \text{ of the cylinder)}$$

$$= h^2 (\pi a^2) = \pi a^2 h^2$$

For the convex portion S_3 i.e., $x^2+y^2=a^2$, a vector normal to S_3 is given by

$$\nabla(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$$

$\therefore \hat{n} =$ a unit vector along outward drawn normal at any point of S_3.

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{a} \quad (\because x^2+y^2=a^2 \text{ on } S_3)$$

$$\therefore \text{ on } S_3,\ F\cdot\hat{n} = (yi + xj + z^2 k)\cdot\left[\frac{1}{a}(xi+yj)\right]$$

$$= \frac{1}{a}xy + \frac{1}{a}xy = \frac{2}{a}xy$$

Also $ds =$ elementary area on the surface S_3

$= a\,d\theta\,dz$, using cylindrical coordinates r,θ,z.

$$\therefore \iint_{S_3} F\cdot\hat{n}\,dS = \iint_{S_3} \frac{2}{a}xy\,a\,d\theta\,dz, \quad \text{where } x = a\cos\theta,\ y = a\sin\theta.$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi} 2a\cos\theta\, a\sin\theta\,d\theta\,dz = 2a^2\int_{\theta=0}^{2\pi} \cos\theta\sin\theta\,[z]_0^h\,d\theta$$

$$= 2a^2h\int_0^{2\pi} \frac{\sin 2\theta}{2}d\theta = a^2h\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} = 0$$

Hence the total surface integral

$$\iint F\cdot\hat{n}\,dS = 0 + \pi a^2h^2 + 0 = \pi a^2h^2 \quad \text{---(2)}$$

$\therefore$ from (1) & (2), we see that $\iiint \operatorname{div} F\,dV = \iint_S F\cdot\hat{n}\,dS$

This verifies divergence theorem

$T_4 - 2016$, PDE, NA & CP and Mechanics & Fluid Dynamics

1(a) Find a Partial differential equation by eliminating a, b, c from $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Sol'n: Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— (1)

Differentiating (1) w.r.t x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \text{ —— (2)}$$

and $\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \Rightarrow c^2 y + b^2 z \frac{\partial z}{\partial y}$ —— (3)

Differentiating (2) w.r.t x and (3) w.r.t y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \text{ —— (4)}$$

& $c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2}$ —— (5)

from (2), $c^2 = \frac{-a^2 z}{x}\left(\frac{\partial z}{\partial x}\right)$

putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} \text{ —— (6)}$$

Similarly from (3) & (5)

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \text{ —— (7)}$$

Differentiating (2) partially w.r.t y, we get

$$a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + z\frac{\partial^2 z}{\partial x \partial y}\right\} = 0$$

i.e, $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial x \partial y} = 0$ —— (8)

$\therefore$ (6), (7) and (8) are three possible forms of the required partial differential equation.

1(b) Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x+y)$.

Soln: The given equation can be rewritten as

$$(D+D')(D-2D'+2)z = xy + \sin(2x+y) \quad \text{—(1)}$$

$$\therefore \ C.F = \phi_1(y-x) + e^{-2x}\phi_2(y+2x)$$

ϕ_1, ϕ_2 being arbitrary functions.

P.I corresponding to xy

$$= \frac{1}{(D+D')(D-2D'+2)}(xy)$$

$$= \frac{1}{D\left[1+\frac{D'}{D}\right] \times 2\left[1+\left(\frac{D}{2}-D'\right)\right]} xy$$

$$= \frac{1}{2D}\left[1+\frac{D'}{D}\right]^{-1}\left[1+\left(\frac{D}{2}-D'\right)\right]^{-1} xy$$

$$= \frac{1}{2D}\left[1-\frac{D'}{D}+\cdots\right]\left[1-\left(\frac{D}{2}-D'\right)+\left(\frac{D}{2}-D'\right)^2+\cdots\right]xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\cdots\right)\left(1-\frac{D}{2}+D'-DD'+\cdots\right)xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\cdots\right)\left(xy-\frac{y}{2}+x-1\right)$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{1}{D}\left(x-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{x^2}{2}+\frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2y}{2}-\frac{xy}{2}+\frac{x^2}{2}-x-\frac{x^3}{6}+\frac{x^2}{4}\right]$$

$$= \frac{x^2y}{4}+\frac{3x^2}{8}-\frac{xy}{2}-\frac{x}{2}-\frac{x^3}{12}$$

P.I corresponding to $\sin(2x+y)$

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{-2^2 + (2\times 1) - 2(-1)^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2(D+D')} \sin(2x+y)$$

$$= \frac{1}{2}(D-D')\frac{1}{D^2-D'^2} \sin(2x+y)$$

$$= -\frac{1}{6}(D-D') \sin(2x+y)$$

$$= -\frac{1}{6}\left[D \sin(2x+y) - D' \sin(2x+y)\right]$$

$$= -\frac{1}{6}\left[2\cos(2x+y) - \cos(2x+y)\right]$$

$$= -\frac{1}{6}\cos(2x+y)$$

$\therefore$ The required solution is

$$z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) + \frac{1}{4}x^2y + \frac{3}{8}x^2 - \frac{1}{4}xy - \frac{x}{2} - \frac{x^3}{12} - \frac{1}{6}\cos(2x+y)$$

1(c) Apply Newton – Raphson method to determine a root of the equation $\cos x = xe^x$ correct to three decimal places.

Sol^n: Let $f(x) = \cos x - xe^x$.

$f'(x) = -\sin x - xe^x - e^x$.

$f(0) = 1 - 0 = 1 > 0$ & $f(1) = \cos 1 - 1(e) = -ve < 0$

So a root of $f(x)=0$ lies between 0 and 1.

Let us take $x_0 = 0$.

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n + \frac{\cos x_n - x_n e^{x_n}}{(\sin x_n + x_n e^{x_n} + e^{x_n})}$$

put $n=0$, the first approximation is

$$x_1 = x_0 + \frac{\cos x_0 - x_0 e^{x_0}}{\sin x_0 + x_0 e^{x_0} + e^{x_0}} = 1$$

$x_2 = 0.653071$

$x_3 = 0.5313$

$x_4 = 0.5179$

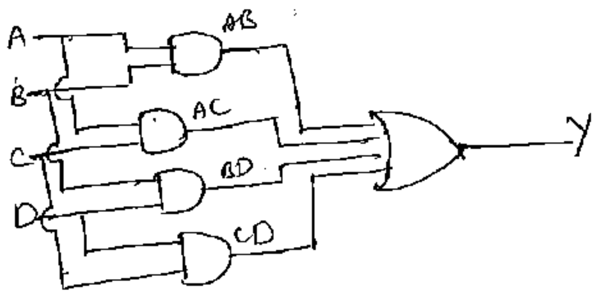
$x_5 = 0.51775$

$\therefore x = 0.517$ is root of $f(x)$ correct upto three decimal places.

1(d) (i) Draw a logic circuit for the Boolean equation $Y = AB + AC + BC + CD$.

(ii) Simplify the expression and draw logic circuit for the simplified expression.

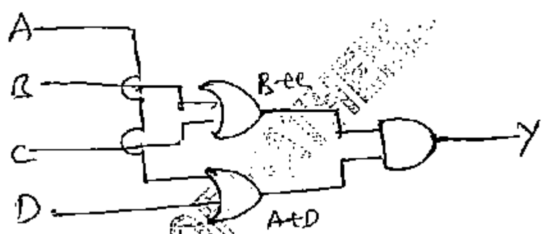
(i) The logic circuit requires 4 AND gates and one OR gate.



(ii) $Y = AB + AC + BD + CD$

$= A(B+C) + D(B+C)$

$= (A+D)(B+C)$.

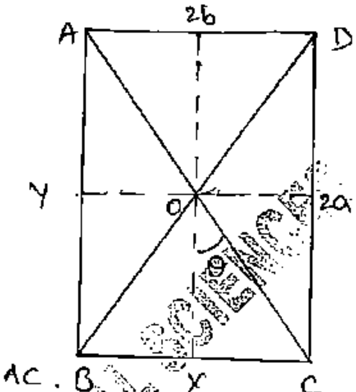


1(e) Show that M.I of a rectangle of mass M and sides $2a, 2b$ about a diagonal is $\frac{2M}{3}\frac{a^2b^2}{a^2+b^2}$.

Sol^n: Let ABCD be a rectangle of mass M and $AB = 2a$, $BC = 2b$ and M.I of rectangle about $OX = A = \frac{1}{3}Mb^2$ and M.I of rectangle about $OY = B = \frac{1}{3}Ma^2$.

P.I of the rectangle about OX and $OY = F = 0$

If diagonal AC make an angle θ with AB, then



$$\cos\theta = \frac{AB}{AC} = \frac{2a}{\sqrt{4a^2+4b^2}}$$

$$= \frac{a}{\sqrt{(a^2+b^2)}}$$

and $\sin\theta = \frac{BC}{AB} = \frac{b}{\sqrt{(a^2+b^2)}}$

$\therefore$ M.I of rectangle about AC.

$$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$$

$$= \frac{1}{3}Mb^2 \cdot \frac{a^2}{a^2+b^2} + \frac{1}{3}Ma^2 \cdot \frac{b^2}{a^2+b^2} - 0$$

$$= \frac{2M}{3} \cdot \frac{a^2b^2}{a^2+b^2}$$

2(a)(i) Form a partial differential equation by eliminating the arbitrary functions f and g from $z = y f(x) + x g(y)$.

(ii) Find the surface which is orthogonal to the one parameter system $z = cxy(x^2+y^2)$ which passes through the parabola $x^2 - y^2 = a^2$, $z = 0$

Sol: (i) Given $z = y f(x) + x g(y)$ ——— ①

Differentiating ① partially w.r.t. x and y, we get

$\frac{\partial z}{\partial x} = y f'(x) + g(y)$ ——— ②

$\frac{\partial z}{\partial y} = f(x) + x g'(y)$ ——— ③

Differentiating ③ w.r.t. x, we get

$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$ ——— ④

From ② and ③, $f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right]$

and $g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$

Substituting these values in ④, we have

$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$

$xy \frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \left[x g(y) + y f(x)\right]$

(ii) The given system of surfaces

$f(x, y, z) = \frac{z}{x^3 y + x y^3} = c.$

$\frac{\partial f}{\partial x} = \frac{-z(3x^2y+y^3)}{(x^3y+xy^3)^2}$, $\frac{\partial f}{\partial y} = \frac{-z(3y^2x+x^3)}{(x^3y+xy^3)^2}$,

$\frac{\partial f}{\partial z} = \frac{1}{x^3y+xy^3}$.

The required orthogonal surface is solution of $p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$.

$$-\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2}p - \frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}q = \frac{1}{x^3y+xy^3}$$

$$\left\{\frac{(3x^2+y^2)}{x}\right\}p + \left[\frac{3y^2+x^2}{y}\right]q = -\frac{(x^2+y^2)}{z} \quad \text{—(2)}$$

Lagrange's auxillary equations for (2) are

$$\frac{dx}{\left(\frac{3x^2+y^2}{x}\right)} = \frac{dy}{\left(\frac{3y^2+x^2}{y}\right)} = \frac{dz}{-\frac{(x^2+y^2)}{z}} \quad \text{—(3)}$$

Taking the first two fractions of (3),

$2x\,dx - 2y\,dy = 0$ so that $x^2-y^2 = c_1$.

Choosing $x, y, 4z$ as multipliers, each fraction of (3) $= \frac{(x\,dx+y\,dy+4z\,dz)}{0}$

$\therefore 2x\,dx + 2y\,dy + 8z\,dz = 0$

$\Rightarrow x^2+y^2+4z^2 = c_2$

Hence any surface which is orthogonal to (1) is of the form

$x^2+y^2+4z^2 = \phi(x^2-y^2)$, ϕ being an arbitrary function. —(4)

For the particular surface passing through the hyperbola $x^2-y^2=a^2,\ z=0$.

we must take $\phi(x^2-y^2) = \dfrac{a^4(x^2+y^2)}{(x^2-y^2)^2}$.

Hence the required surface is given by

$$(x^2+y^2+4z^2)^2(x^2-y^2)^2 = a^4(x^2+y^2).$$

2(b) Solve the following system of equations

$10x-7y+3z+5w=6$

$-6x+8y-z-4w=5$

$3x+y+4z+11w=2$

$5x-9y-2z+4w=7$ by Gauss Seidel Metod.

Sol^n: let $(x^0,y^0,z^0,w^0)=(0,0,0,0)$ be the initial approximation.

let us apply Gauss seidel Metod by using pivotal process.

$$x^{K+1} = \frac{1}{10}(6+7y^K-3z^K-5w^K)$$

$$y^{K+1} = \frac{1}{9}(-7+5x^{K+1}-2z^K+4w^K)$$

$$z^{K+1} = \frac{1}{4}(2-3x^{K+1}-y^{K+1}-11w^K)$$

$$w^{K+1} = \frac{1}{4}(-5-6x^{K+1}+8y^{K+1}-z^{K+1})$$

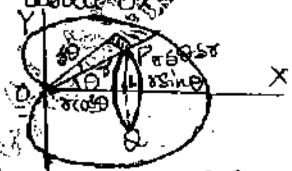
where $K=1,2,3,\ldots$

proceed in this way, we get

Answer: $x=5,\ y=4,\ z=-7,\ w=1$.

8(c) A solid body of density ρ is in the shape of the solid formed by the revolution of the Cardiod $r=a(1+\cos\theta)$ about the initial line, show that its M.I about a straight line through the pole and $\perp$lar to the initial line is $\frac{352}{105}\pi\rho a^5$.

Sol^n: Let OX be the initial line (axis of the cardiod) and OY the line $\perp$lar to it through the pole O. Consider an elementry area $r\delta\theta\delta r$ at the point $P(r,\theta)$. Then the mass of the elementry ring of radius PL obtained by the revolution of element $r\delta\theta\delta r$ about OX



$$\delta m = \rho . 2\pi PL . r\delta\theta\delta r$$
$$= 2\pi\rho r\sin\theta . r\delta\theta\delta r$$
$$= 2\pi\rho r^2\sin\theta\,\delta r\delta\theta$$

where ρ is the mass per unit volume of the solid formed by the revolution of the Cardiod about the initial line OX.

M.I of this elementry ring about OY = its M.I about the diameter PQ + M.I of mass δm at centre L about OY

$$= \frac{1}{2}\delta m . PL^2 + \delta m . OL^2 = \left(\frac{1}{2}PL^2 + OL^2\right)\delta m$$
$$= \left(\frac{1}{2}r^2\sin^2\theta + r^2\cos^2\theta\right) 2\pi\rho r^2\sin\theta\,\delta\theta\delta r.$$
$$= \pi\rho\,(\sin^2\theta + 2\cos^2\theta)\, r^4 \sin\theta\,\delta\theta\delta r$$
$$= \pi\rho\,(1+\cos^2\theta)\, r^4\sin\theta\,\delta\theta\delta r$$

$\therefore$ M.I of the solid of revolution about OY

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} \pi\rho\,(1+\cos^2\theta) r^4 \sin\theta\, d\theta\, dr$$

$$= \frac{1}{5}\pi\rho a^5 \int_0^{\pi} (1+\cos^2\theta)(1+\cos\theta)^5 \sin\theta\, d\theta$$

$$= -\frac{1}{5}\pi\rho a^5 \int_2^0 \{1+(t-1)^2\} . t^5\, dt \qquad \text{Putting } (1+\cos\theta)=t$$

$$= \frac{1}{5}\pi\rho a^5 \int_0^2 (2t^5 - 2t^6 + t^7)\, dt$$

$$= \frac{1}{5}\pi\rho a^5 \left[\frac{1}{3}t^6 - \frac{2}{7}t^7 + \frac{1}{8}t^8\right]_0^2$$

$$= \frac{1}{5}\pi\rho a^5 \left(\frac{352}{21}\right) = \frac{352}{105}\pi\rho a^5$$

3(a) Solve the boundary value problem $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$ satisfying the conditions $u(0,t) = u(l,t) = 0$ and $u(x,0) = x$ when $0 \le x \le \frac{l}{2}$, $u(x,0) = l - x$ when $\frac{l}{2} \le x \le l$.

Sol: Given that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$ (or) $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ —(1)

Boundary conditions: $u(0,t) = u(l,t) = 0$ —(2)

and the initial conditions:

$u(x,0) = x$ when $0 \le x \le \frac{l}{2}$
$u(x,0) = l - x$ when $\frac{l}{2} \le x \le l$. } —(3)

Suppose that (1) has solution of the form

$u(x,t) = X(x)T(t)$ —(4)

where X is a function of x alone and T is a function of t alone

Substituting this value of u in (1), we get

$kX''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{kT} = \mu$ (say) —(5)

$\Rightarrow X'' - \mu X = 0$ —(6) and $T' = \mu k T$ —(7)

Using (2), (4) gives $X(0)T(t) = 0$ and $X(l)T(t) = 0$ —(8)

Since $T(t) = 0$ leads $u = 0$

so suppose that $T(t) \neq 0$.

$\therefore$ from ⑧, $X'(0)=0$ and $X'(l)=0$. —— ⑨

we now solve ⑥ under B.C. ⑨.

Three cases arise.

Case (i): Let $\mu=0$. Then the solution of ⑥ is

$X(x) = Ax+B$.

Using B.C. ⑨, we get $A=B=0$

$\therefore X(x)=0$ so that $u\equiv 0$ which doesn't satisfy ③

so we reject $\mu=0$

Case (ii): Let $\mu=\lambda^2$, $\lambda\neq 0$. Then the solution of ⑥ is

$X(x)=Ae^{\lambda x}+Be^{-\lambda x}$

Using B.C. ⑨, we get $A=B=0$. so that

$X(x)=0$ and hence $u=0$.

which doesnot satisfy ③, so we also reject $\mu=\lambda^2$.

Case (3): Let $\mu=-\lambda^2$, $\lambda\neq 0$. The solution of ⑥ is

$X(x) = A\cos\lambda x + B\sin\lambda x$.

Using B.C. ⑨, we get $A=0$ and $0=A\cos\lambda a+B\sin\lambda a$

$\Rightarrow \sin\lambda a=0$, $B\neq 0$, since otherwise

$X\equiv 0$ so that $u\equiv 0$ which doesnot

satisfy ③.

Solving the trigonometric equation $\sin\lambda a=0$ —— ⑩

we have $\lambda a = n\pi \Rightarrow \lambda = n\pi/a$, $n=1,2,3\ldots$

Hence non solutions of $X_n(x)$ of ⑥ are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$.

Using ⑩, ⑦ reduces to

$$\frac{dT}{T} = -\frac{n^2\pi^2 k}{l^2}\,dt \quad \text{as } \mu = -\lambda^2 = -\frac{n^2\pi^2}{l^2}.$$

$$\frac{1}{T}\,dT = -C_n^2\,dt \quad \text{where } C_n^2 = \frac{n^2\pi^2 k}{l^2} \qquad \text{(A)}$$

whose general solution is

$$T_n(t) = D_n e^{-C_n^2 t}.$$

$$\therefore u_n(x,t) = X_n(x)T_n(t) = E_n \sin\left(\frac{n\pi x}{l}\right) e^{-C_n^2 t}.$$

are solutions of ①, satisfying ②.

Here $E_n (= B_n D_n)$ is another arbitrary constant.

In order to obtain a solution also satisfying ③, we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\, e^{-C_n^2 t} \qquad (11)$$

putting $t=0$ in ⑪ and using ③,

we get $\underset{=f(x)}{u(x,0)} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$.

i.e., where $E_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right)dx \quad n = 1,2,3,\ldots$

$$= \frac{2}{l}\left[\int_0^{l/2} f(x) \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} f(x) \sin\frac{n\pi x}{l}\,dx\right]$$

$$= \int_0^{l/2} \frac{2x}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2}{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \left[\frac{2x}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)}\right] - \left(\frac{2}{l}\right)\left(\frac{-\sin(n\pi x/l)}{(n\pi)^2/l^2}\right)\right]_0^{l/2}$$

$$+ \left[\left(\frac{2(l-x)}{l}\right)\left(\frac{-\cos(n\pi x)/l}{(n\pi/l)}\right) - \left(\frac{2}{l}\right)\left(\frac{-\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_{l/2}^{l}$$

$$= -\frac{l}{n\pi}\cos\frac{n\pi}{2} + \frac{2l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2l}{n^2\pi^2}\right)\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore E_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n = 2m \text{ and } m = 1, 2, \ldots \\ 4l/(2m-1)^2\pi^2 & \text{if } n = 2m-1,\ m = 1, 2, \ldots \end{cases}$$

Then (A) $\Rightarrow \lambda_n^2 = (2m-1)^2\pi^2 k/l^2$

$\therefore$ From (II)

$$u(x,t) = \frac{4l}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\,e^{-\{(2m-1)^2\pi^2 kt\}/l^2}$$

3(b) Using fourth order Runge-Kutta Method, find the solution of $x(dy+dx) = y(dx-dy)$, $y(0)=1$ at $x=0.1$ and 0.2 by taking $h=0.1$.

Sol^n: Given that $x(dy+dx) = y(dx-dy)$

$(x+y)dy = (y-x)dx \Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x}$

$\therefore$ we have $f(x,y) = \frac{dy}{dx} = \frac{y-x}{y+x}$

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$k_1 = h f(x_0, y_0) = (0.1)\left(\frac{1-0}{1+0}\right) = 0.1$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.0909$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.09087$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.083205$

$y(0.1) = y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) =$

$= 1 + 0.091124$

$y(0.1) = 1.091124$

To find $y(0.2)$:

Here $x_1 = 0.1$, $y_1 = 1.091124$

$k_1 = 0.083209$

$k_2 = 0.07676$

$k_3 = 0.07655$

$k_4 = 0.070753$

$\therefore y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.091124 + 0.076763$

$= 1.167887$

3(c) In two dimensional irrotational fluid motion, show that if the stream lines are confocal ellipses

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b^2+\lambda}=1,\quad \psi = A\log\left[\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right]+B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the confocal radii of the point.

Sol^n: The conformal of transformation $z = c\cos w$ — (1) is known to yield the given type of confocal ellipses.

(1) $\Rightarrow x+iy = c\cos(\phi+i\psi) \Rightarrow x = c\cos\phi\cosh\psi,\ y = c\sin\phi\sinh\psi$

Eliminating ϕ, we get

$$\frac{x^2}{c^2\cosh^2\psi}+\frac{y^2}{c^2\sinh^2\psi}=1 \quad \text{—} \ (2)$$

Streamlines are given by $\psi=$ const. By virtue of this, (2) declares that streamlines are confocal ellipses. Comparing (2) with the equation.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,\ \text{we get}\ c\cosh\psi=\sqrt{(a^2+\lambda)},$$
$$c\sinh\psi=\sqrt{(b^2+\lambda)} \quad \text{—}\ (3)$$

This $\Rightarrow c(\cosh\psi+\sinh\psi) = \sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}$

$$\Rightarrow ce^{\psi} = \sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}$$

$$\Rightarrow \psi = \log\left[\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right]-\log c \quad \text{—}\ (4)$$

If $w=\phi+i\psi$ is the complex potential of some fluid motion, then so is Aw. Hence (4) gives

$$\psi = A\log\left[\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right]-B$$

Velocity: (1) $\Rightarrow \frac{dz}{dw} = -c\sin w = -c\sqrt{\left[1-(z^2/c^2)\right]}$

$$q^{-1} = \frac{1}{q} = \left|-\frac{dz}{dw}\right| = \sqrt{|c^2-z^2|} = \sqrt{[|c-z|.|c+z|]} \quad \text{—}\ (5)$$

By (3), $c^2(\cosh^2\varphi - \sinh^2\varphi) = (a^2+\lambda) - (b^2+\lambda) = a^2 - b^2$

$\Rightarrow c^2 = a^2 - a^2(1-e^2)$. For $b^2 = a^2(1-e^2)$

$\Rightarrow c = ae$.

Now (5) becomes $q^{-1} = \sqrt{[|z-ae|.|z+ae|]}$ — (6)

$(\pm ae, 0)$ are coordinates of foci, denoted by S and S'. P is a point z. Then $r_1 = SP = |z-ae|$

$$r_2 = S'P = |z+ae|$$

Now (6) is expressible as

$q^{-1} = \sqrt{(r_1 r_2)} \Rightarrow q = \frac{1}{\sqrt{r_1 r_2}}$ from this the required result follows.

4(a) Prove that for the equation $z+px+qy-1-pqx^2y^2=0$ the characteristic strips are given by

$x=(B+Ce^{-t})^{-1}$, $y=(A+De^{-t})^{-1}$, $z=E-(AC+BD)e^{-t}$

$p=A(B+Ce^{-t})^2$, $q=B(A+De^{-t})^2$.

where A, B, C, D and E are arbitrary constants. Hence, find the integral surface which passes through the line $z=0$, $x=y$.

Sol: Here $f=z+px+qy-1-pqx^2y^2=0$ ——— (1)

The characteristic equations of the given partial differential equation (1) are given by

$\frac{dx}{dt}=\frac{\partial f}{\partial p}=x-qx^2y^2$ ——— (2)

$\frac{dy}{dt}=\frac{\partial f}{\partial q}=y-px^2y^2$ ——— (3)

$\frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}=p(x-qx^2y^2)+q(y-px^2y^2)$
$=px+qy-2pq\,x^2y^2$ ——— (4)

$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z}=-(p-2pq\,xy^2)-p.1=-2p(1-qxy^2)$ ——— (5)

& $\frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}=-(q-2pq\,x^2y)-q.1=-2q(1-px^2y)$ ——— (6)

from (2) & (5), we have

$\frac{1}{x}\frac{dx}{dt}=-\frac{1}{2p}\frac{dp}{dt}\Rightarrow\frac{2}{x}\frac{dx}{dt}+\frac{1}{p}\frac{dp}{dt}=0$

$\Rightarrow 2\log x+\log p=\log A$

$\Rightarrow x^2p=A$ ——— (7)

Again from (3) & (6), we have

$\frac{1}{y}\frac{dy}{dt}=-\frac{1}{2q}\frac{dq}{dt}\Rightarrow\frac{2}{y}\frac{dy}{dt}+\frac{1}{q}\frac{dq}{dt}=0$

$\Rightarrow 2\log y+\log q=B$

$\Rightarrow y^2q=B$ ——— (8)

from ⑦ & ⑧, we have $\frac{dx}{dt} = x - Bx^2$

$\Rightarrow \frac{1}{x^2}\frac{dx}{dt} = \frac{1}{x} - B$

putting $\frac{1}{x} = v \Rightarrow -\frac{1}{x^2}\frac{dx}{dt} = \frac{dv}{dt}$

$\Rightarrow \frac{dv}{dt} + v = B.$

which LDE with

$IF = e^{\int 1 dt} = e^t.$

$\therefore ve^t = C + \int Be^t dt = C + Be^t.$

$\Rightarrow \frac{1}{x}e^t = C + Be^t.$

$\Rightarrow x = (B + Ce^{-t})^{-1}$ ——— ⑨

Again from ③ & ⑦,

we have $\frac{dy}{dt} = y - Ay^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dt} = \frac{1}{y} - A$

putting $\frac{1}{y} = u \Rightarrow -\frac{1}{y^2}\frac{dy}{dt} = \frac{du}{dt}$

$\Rightarrow \frac{du}{dt} + u = A.$ which is LDE

with $IF = e^{\int 1 dt} = e^t,$

$\therefore ue^t = D + \int Ae^t dt = D + Ae^t$

$\Rightarrow \frac{1}{y}e^t = D + Ae^t$

$\Rightarrow y = (A + De^{-t})^{-1}$ ——— ⑩

Using ⑦, ⑧, ⑨ & ⑩ from ⑥, we have

$\frac{dz}{dt} = \frac{A}{x} + \frac{B}{y} - 2AB$

$= A(B + Ce^{-t}) + B(A + De^{-t}) - 2AB$

$= (AC + BD)e^{-t}.$

Integrating, we get $z = E - (AC + BD)e^{-t}.$

Thus, the characteristic strips for equation ① are given by $x = (B + Ce^{-t})^{-1}$, $y = (A + De^{-t})^{-1}$, $z = E - (AC + BD)e^{-t}$

$p = A(B + Ce^{-t})^{2}$, $q = B(A + De^{-t})^{2}$ ——— (4)

The parametric equation of the given line can be taken as

$x = f_1(\lambda) = \lambda$, $y = f_2(\lambda) = \lambda$, $z = 0$, λ being a parameter.

Initial values of x, y, z are $x = x_0 = \lambda$, $y = y_0 = \lambda$, $z = z_0 = 0$ when $t = 0$.

The corresponding initial values of p_0 and q_0 are given by

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$

$\Rightarrow 0 = p_0(1) + q_0(1) \Rightarrow p_0 + q_0 = 0 \Rightarrow q_0 = -p_0$

and $z_0 + p_0 x_0 + q_0 y_0 - 1 - p_0 q_0 x_0^2 y_0^2 = 0$ (from ①)

$\Rightarrow 0 + \lambda(p_0 + q_0) - 1 - p_0 q_0 \lambda^4 = 0$

$\Rightarrow p_0 q_0 \lambda^4 = -1 \Rightarrow p_0^2 = \frac{1}{\lambda^4} \Rightarrow p_0 = \frac{1}{\lambda^2}$, $q_0 = -\frac{1}{\lambda^2}$

Using initial values in characteristic strip given by (4), we have

$x_0 = \lambda = (B + C)^{-1}$, $y_0 = (A + D)^{-1} \Rightarrow B + C = \frac{1}{\lambda}$, $A + D = \frac{1}{\lambda}$.

$\therefore p_0 = \frac{1}{\lambda^2} = A(B + C)^2 = A\left(\frac{1}{\lambda}\right)^2 \Rightarrow A = 1$.

$q_0 = -\frac{1}{\lambda^2} = B(A + D)^2 = B\left(\frac{1}{\lambda}\right)^2 \Rightarrow B = -1$.

$\therefore C = \frac{1}{\lambda} + 1$, $D = \frac{1}{\lambda} - 1$;

$z_0 = 0 = E - (AC + BD) \Rightarrow E = 1\left(\frac{1}{\lambda} + 1\right) - \left(\frac{1}{\lambda} - 1\right) = 2$

$\therefore x = \left\{-1 + \left(\frac{1}{\lambda} + 1\right)e^{-t}\right\}^{-1}$, $y = \left\{1 + \left(\frac{1}{\lambda} - 1\right)e^{-t}\right\}^{-1}$, $z = 2(1 - e^{-t})$

The required integral surface is obtained by eliminating parameters λ and t from x, y and z

Here $\frac{1}{x} + 1 = \left(\frac{1}{\lambda} + 1\right)e^{-t}$, $\frac{1}{y} - 1 = \left(\frac{1}{\lambda} - 1\right)e^{-t} \Rightarrow \frac{1}{x} - \frac{1}{y} + 2 = 2e^{-t}$.

$\therefore z = 2 - \left(\frac{1}{x} - \frac{1}{y} + 2\right) = -\frac{1}{x} + \frac{1}{y} \Rightarrow \boxed{xyz = x - y}$

Which is the required integral surface.

4(c). Test whether the motion specified by $q = \frac{k^2(xj - yi)}{x^2+y^2}$ (k = const), is a possible motion for an incompressible fluid. If so, determine the equations of streamlines. Also tell whether the motion is of the potential kind and if it determines the velocity potential.

Sol'n: Here $u = \frac{-k^2 y}{x^2+y^2}$, $v = \frac{k^2 x}{x^2+y^2}$, $w=0$

I. Equation of continuity for incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

But $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2k^2xy}{(x^2+y^2)^2} - \frac{2k^2xy}{(x^2+y^2)^2} = 0$

Hence equation of continuity is satisfied.

II. Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx\,(x^2+y^2)}{-k^2y} = \frac{(x^2+y^2)\,dy}{k^2x} = \frac{dz}{0}$$

$\Rightarrow x\,dx + y\,dy = 0$, $dz = 0$

$\Rightarrow x^2+y^2 = a^2$, $z = b$

Hence streamlines are circles whose centres lie on z-axis.

III. To test the existence of velocity potential.

$$-d\phi = u\,dx + v\,dy + w\,dz$$

$$= -k^2y\,\frac{dx}{x^2+y^2} + k^2x\,\frac{dy}{x^2+y^2}$$

$$d\phi = k^2\left[\frac{y\,dx}{x^2+y^2} - \frac{x}{x^2+y^2}\,dx\right]$$

$$= k^2(M\,dx + N\,dy), \text{ say}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2+y^2} + y\left[\frac{-2y}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = -\left[\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial M}{\partial y}$$

Hence $M\,dx + N\,dy$ is exact. Therefore its solution given by

$$\phi = \int \frac{k^2 y}{x^2+y^2}\,dx + \int 0\,dy + C = \frac{k^2 y}{y}\tan^{-1}\left(\frac{x}{y}\right) + C$$

Hence ϕ exists and is given by

$$\phi = k^2 \tan^{-1}\left(\frac{x}{y}\right) + C$$

Q) Solve $\cos(x+y)\,p + \sin(x+y)\,q = z$

Sol^n: Here the Lagrange's auxiliary equations are $\dfrac{dx}{\cos(x+y)} = \dfrac{dy}{\sin(x+y)} = \dfrac{dz}{z}$ — ①

Choosing 1, 1, 0 as multipliers, each fraction of ①

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} \quad \text{— ②}$$

Choosing 1, −1, 0 as multipliers, each fraction of

$$① = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} \quad \text{— ③}$$

From ①, ② and ③

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} \quad \text{— ④}$$

Taking first two fractions of ④

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} \quad \text{— ⑤}$$

putting $x+y=t \Rightarrow d(x+y)=dt$, ⑤ reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right]}$$

$$\Rightarrow \frac{dz}{z} = \frac{dt}{\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)}$$

$$\therefore \left(\frac{\sqrt{2}}{z}\right)dz = \operatorname{cosec}\left(t+\frac{\pi}{4}\right)$$

Integrating

$\sqrt{2}\log z = \log \tan \frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1$

$\Rightarrow z^{\sqrt{2}} = c_1 \tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$

$z^{\sqrt{2}} \cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = c_1 \quad (\because t = x+y)$ —— (6)

Taking the last two fractions of (4),

$dx - dy = \frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)} d(x+y)$ —— (7)

put $x+y = t \Rightarrow d(x+y) = dt$

$\therefore$ (7) reduces to

$dx - dy = \frac{\cos t - \sin t}{\cos t + \sin t} dt$

$\Rightarrow x - y = \log(\sin t + \cos t) - \log c_2$

$(\sin t + \cos t)e^{-(x-y)} = c_2$.

$\Rightarrow e^{y-x}[\sin(x+y) + \cos(x+y)] = c_2$ —— (8)

as $t = x+y$

from (6) and (8), the required solution is

$\phi\left[z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right),\ e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right] = 0$

where ϕ is an arbitrary function

5(b) Solve $(D^2 - 6DD' + 9D'^2)z = \tan(y+3x)$.

Sol:- Here auxiliary equation is $(m-3)^2 = 0$

$\Rightarrow m = 3, 3$.

$\therefore$ C.F $= \phi_1(y+3x) + x\phi_2(y+3x)$

ϕ_1, ϕ_2 being arbitrary functions

$$P.I = \frac{1}{(D-3D')^2} \tan(y+3x) = \frac{x^2}{1^2 \times 2!} \tan(y+3x)$$

$$= \frac{x^2}{2} \tan(y+3x)$$

$\therefore$ The required solution is

$$z = \phi_1(y+3x) + x\phi_2(y+3x) + \frac{x^2}{2}\tan(y+3x)$$

5(c) Find Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$. Hence find $y(5)$.

Sol: Using Lagrange's interpolation formula we have

$$y(x) = f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132).$$

$$= \frac{x^3-13x^2+54x-72}{-30}(-3) + \frac{x^3-11x^2+34x-24}{6}(0)$$

$$+ \frac{x^3-10x^2+27x-18}{-6}(30) + \frac{x^3-8x^2+19x-12}{30}(132)$$

$$y(x) = \frac{1}{10}(-5x^3+135x^2-460x+300)$$

$$= \frac{1}{2}(-x^3+27x^2-92x+60).$$

which is the required Lagrange's interpolation formula.

Now, $y(5) = 75$

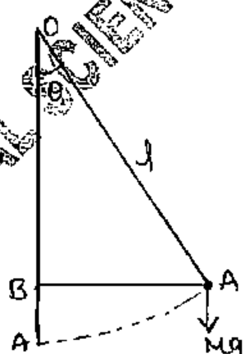
5(d) For a simple pendulum (i) find the Lagrangian function and (ii) obtain an equation describing its motion.

Sol'n: Let l be the length of the simple pendulum and θ be angle made by the string with the vertical at time t. Thus θ is the only generalised coordinate. Then the velocity of mass M at A will be $v = l\dot{\theta}$.

$\therefore$ Total K.E. $T = \frac{1}{2}Mv^2 = \frac{1}{2}Ml^2\dot{\theta}^2$

and the potential function

$$V = Mg(A'B) = Mg(l - l\cos\theta)$$
$$= Mgl(1-\cos\theta)$$



(i) $\therefore$ the Lagrangian function

$$L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$$

(ii) Lagrange's θ-equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(Ml^2\dot{\theta}) + Mgl\sin\theta = 0$$

$$\Rightarrow Ml^2\ddot{\theta} + Mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} = \left(-\frac{g}{l}\right)\sin\theta \Rightarrow \ddot{\theta} \simeq -\left(\frac{g}{l}\right)\theta$$

5(e) Find the stream function ψ for given velocity potential $\phi = cx$, where c is constant. Also, draw a set of streamlines and equipotential lines.

Sol'n: The velocity potential $\phi = cx$ represents fluid flow because it satisfies Laplace equation $\nabla^2\phi = 0$

Since $-\frac{\partial\phi}{\partial x} = -c = u$ and $u = -\frac{\partial\psi}{\partial y}$

$\therefore \frac{\partial\psi}{\partial y} = c \Rightarrow \psi = cy + f(x)$ —— (1)

Differentiating with regard to x, we have

$$\frac{\partial\varphi}{\partial x} = f'(x)$$

But $\frac{\partial\psi}{\partial x} = v = -\frac{\partial\phi}{\partial y} \Rightarrow \frac{\partial\psi}{\partial x} = 0$, as $\frac{\partial\phi}{\partial y} = 0$.

$$\Rightarrow f'(x) = 0, \Rightarrow f(x) = \text{const.} \quad \text{—— (2)}$$

The stream function φ is given as

$$\varphi = \text{const} + Cy.$$

which represents parallel flow in which stream lines are parallel to x-axis.

6(c) Find a surface satisfying the equation $D^2z = 6x+2$ and touching $z = x^3+y^3$ along its section by the plane $x+y+1=0$.

Sol: Given that $D^2z = 6x+2$

i.e. $\frac{\partial^2 z}{\partial x^2} = 6x+2$ —— (1)

Integrating (1) w.r.t x, we get

$$\frac{\partial z}{\partial x} = 3x^2+2x+f(y) \quad \text{—— (2)}$$

Integrating (2) w.r.t x, we get

$$z = x^3+x^2+xf(y)+g(y) \quad \text{—— (3)}$$

where $f(y)$ and $g(y)$ are arbitrary functions

The given surface $z = x^3+y^3$ —— (4)

and the given plane is $x+y+1=0$ —— (5)

Since (3) and (4) touch each other along their section by (5), the values of p and q at any point on (5) must be equal. Thus we must have

$$3x^2+2x+f(y) = 3x^2$$

$\left[\because (3) = \frac{\partial z}{\partial x} = 3x^2;\ (4) = \partial z/\partial x = 3x^2+2x+f(y)\right]$

$\Rightarrow f(y) = -2x$ —— (6)

and $x f'(y) + g'(y) = 3y^2$ —— (7) $\left[\because (1) = \frac{\partial z}{\partial y} = 3y^2 \right.$
$\left. (3) = \frac{\partial z}{\partial y} = x f'(y) + g'(y) \right]$

from (5) and (6),

$f(y) = -2x = 2(y+1)$ —— (8) $[\because x+y+1=0 \Rightarrow x = -(y+1)]$

$\Rightarrow f'(y) = 2$

$\therefore (7) \equiv 2x + g'(y) = 3y^2$

$\Rightarrow g'(y) = 3y^2 - 2x = 3y^2 + 2(y+1)$ [$\because$ by eqn (8)]

$\Rightarrow g'(y) = 3y^2 + 2y + 2$

Integrating it,

$g(y) = y^3 + y^2 + 2y + c$ —— (9)

where c is an arbitrary constant.

using (8) and (9), (3) gives

$z = x^3 + x^2 + x[2(y+1)] + y^3 + y^2 + 2y + c$ —— (10)

Now at the point of contact of (4) and (10) values of z must be the same and hence we have

$x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + c = x^3 + y^3$

$\Rightarrow x^2 + 2x(y+1) + y^2 + 2y + c = 0$

$\Rightarrow x^2 + 2x(-x) + (x+1)^2 - 2(x+1) + c = 0$

$\Rightarrow -x^2 + x^2 + 2x + 1 - 2x - 2 + c = 0$ $[\because x+y+1=0$

$\Rightarrow \boxed{c = 1}$ $\Rightarrow y+1 = -x$

$\Rightarrow y = -(x+1)]$

Putting this in (10), the required surface is

$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + 1$

$\Rightarrow z = x^3 + y^3 + (x+y+1)^2$

6(b) Reduce $x^2\left(\frac{\partial^2 z}{\partial x^2}\right) - y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$ to canonical form and hence solve it.

Solⁿ: Rewriting the given equation $x^2 r - y^2 t = 0$ —①

Comparing ① with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2y^2 > 0$ for $x \neq 0$, $y \neq 0$ and hence ① is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$

So that $\lambda = y/x, -y/x$ and hence the corresponding characteristic equations become

$$\frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{y}{x} = 0$$

Integrating these $xy = c_1$ and $x/y = c_2$

In order to reduce ① to its canonical form, we choose $u = xy$ and $v = x/y$ —— ②

$$\therefore\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}, \text{ using (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}, \text{ using (2)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2} \times y + \frac{\partial^2 z}{\partial v \partial u} \times \frac{1}{y}\right) + \frac{1}{y}\left(\frac{\partial^2 z}{\partial v \partial u} \times y + \frac{\partial^2 z}{\partial v^2} \times \frac{1}{y}\right),$$

$$\therefore\ r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \left[-\frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}\times x + \frac{\partial^2 z}{\partial v \partial u}\times\left(\frac{-x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u \partial v}\times x + \frac{\partial^2 z}{\partial v^2}\times\left(\frac{-x}{y^2}\right)\right]$$

$$\therefore\ t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r & t in (1), we get

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

$$\Rightarrow 4x^2\frac{\partial^2 z}{\partial u \partial v} - \frac{2x}{y}\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad 2xy\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$$

$$\Rightarrow 2u\left(\frac{\partial^2 z}{\partial u \partial v}\right) - \left(\frac{\partial z}{\partial v}\right) = 0, \text{ using (2)} \quad \text{—— (3)}$$

This is the required canonical form of (1)

Now we find the solution of (1), Multiplying bothsides of (3) by v, we get

$$2uv\frac{\partial^2 z}{\partial u \partial v} - v\frac{\partial z}{\partial v} = 0 \quad (\text{or}) \ (2uv\,DD' - vD')z = 0 \quad \text{—— (4)}$$

where $D \equiv \partial/\partial u$ and $D' = \partial/\partial v$. We now reduce (4) to a linear equation with constant coefficients by usual method.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ & $Y = \log v$ —— (5)

Let $D_1 \equiv \frac{\partial}{\partial X}$ and $D_1' = \frac{\partial}{\partial Y}$. Then (4) reduces to

$$(2D_1D_1' - D_1')z = 0 \Rightarrow D_1'(2D_1 - 1)z = 0$$

Its general solution is given by

$$z = e^{X/2}\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u), \text{ using (5)}$$

$$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x\left(y/x\right)^{1/2}\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy), \text{ using (2)}$$

where f and ψ_2 are arbitrary functions.

6(c). Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length whose one end is kept at $10°c$ and the other end is insulated. Further it is given that $Y(x,0)=1-x,\ 0<x<1$.

Solⁿ: Suppose the bar be placed along the x-axis with its one end (which is at $10°c$) at origin and other end at $x=1$ (which is insulated so that flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity). Then we are to solve heat equation,

$$\frac{\partial y}{\partial t}=k\left(\frac{\partial^2 y}{\partial x^2}\right) \quad \text{—} (1)$$

with boundary conditions $y_x(1,t)=0,\ y(0,t)=10$ — (2)

and initial conditions $y(x,0)=1-x,\ 0<x<1$ — (3)

Let $y(x,t)=u(x,t)+10$ — (4)

i.e, $u(x,t)=y(x,t)-10$ — (5)

using (4) or (5), (1), (2) and (3) reduces to

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2} \quad \text{—} (6)$$

$u_x(1,t)=0,\ u(0,t)=0$ — (7)

and $u(x,0)=y(x,0)-10=-(x+9)$ — (8)

Suppose that (6) has solutions of the form $u(x,t)=X(x)T(t)$ — (9)

Substituting this value of u in (6), we get

$$XT'=kX''T \Rightarrow \frac{X''}{X}=\frac{T'}{kT} \quad \text{—} (10)$$

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (10) gives $X''-\mu X=0$ — (11)

and $T'=\mu kT$ — (12)

Using (7), (9) gives $X'(1)T(t)=0$ and $X(0)T(t)=0$ — (13)

Since $T(t)=0$ leads to $u\equiv 0$, so we suppose that $T(t)\neq 0$.

Then, from (13), we get $X'(1)=0$ and $X(0)=0$ — (14)

We now solve (11) under B.C (14), Three cases arise.

Case I. Let $\mu=0$. Then solution of (11) is

$$X(x) = Ax+B \quad \text{——— (15)}$$

from (15), $X'(x) = A$ ——— (15)(i)

Using B.C (14), (15) and (15)(i) give $0=A$ and $0=B$

So from (15), $X(x)\equiv 0$, which leads to $u\equiv 0$, so reject $\mu=0$.

Case II. Let $\mu=\lambda^2$, $\lambda\neq 0$. Then solution of (11) is

$$X(x) = Ae^{\lambda x}+Be^{-\lambda x} \quad \text{——— (16)}$$

So that $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ——— (16)(i)

Using B.C (14), (16) and (16)(i) give $0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda}$ and $0 = A+B$

These give $A=B=0$ so that $X(x)\equiv 0$ and hence $u(x)\equiv 0$

So we reject $\mu=\lambda^2$ and hence $u(x)\equiv 0$.

Case III: Let $\mu=-\lambda^2$, $\lambda\neq 0$. Then solution of (12) is

$$X(x) = A\cos\lambda x + B\sin\lambda x. \quad \text{——— (17)}$$

So that $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ ——— (17)(i)

Using B.C (14), (17) and (17)(i) give $0=-A\lambda\sin\lambda + B\lambda\cos\lambda$ and $0=A$.

These give $A=0$ and $\cos\lambda=0$ ——— (18)

where we have taken $B\neq 0$. Since otherwise $X(x)\equiv 0$ and hence $u\equiv 0$.

Now, $\cos\lambda=0 \Rightarrow \lambda = (2n-1)\times\frac{\pi}{2} = \frac{1}{2}\times(2n-1)\pi$, $n=1,2,3\ldots$

So, $\mu=-\lambda^2 = -\frac{1}{4}\times(2n-1)^2\pi^2$ ——— (19)

Hence non-zero solutions $X_n(x)$ of (17) are given by

$$X_n(x) = B_n\sin\{(2n-1)\pi x/2\}.$$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4}T \Rightarrow \frac{dT}{T} = -C_n^2\,dt \quad \text{--- (20)}$$

where $C_n^2 = \frac{1}{4}\times(2n-1)^2\pi^2 k$ ——— (21)

Solving (20), $T_n(t) = D_n e^{-C_n^2 t}$ ——— (22)

Thus, $u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$

are solutions of (6), satisfying (7), Here $E_n (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \quad \text{--- (23)}$$

putting $n=0$ in (23) and using (8),

we have $-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$ ——— (24)

Multiplying both sides of (24) by $\sin\frac{(2m-1)\pi x}{2}$ and then integrating w.r.t x from 0 to 1, we get

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx \quad \text{--- (25)}$$

But $\int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx = \begin{cases} 0, & \text{if } m\neq n \\ 1, & \text{if } m=n \end{cases}$ ——— (26)

using (26), (25) gives $-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = E_m$

$$\therefore E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}\,dx$$

$$\Rightarrow E_n = -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{(2n-1)\pi/2}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{(2n-1)^2\pi^2/4}\right\}\right]_0^1$$

$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \quad \left[\because \cos(2n-1)\pi/2 = 0 \text{ and } \sin(2n-1)\pi/2 = (-1)^{n-1}\right] \quad \text{— (27)}$$

Using (23) and (4), the required solution is given by

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$$

where C_n and E_n are given by (21) & (27) respectively.

7(a) A missile is launched from a ground station. The acceleration during its first 80 seconds of flight, as recorded, is given in the following table:

t sec	0	10	20	30	40	50	60	70	80
a (m/s²)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

compute the velocity of the missile when t = 80s. using Simpson's 1/3rd rule.

Sol: Since acceleration is defined as the rate of change of velocity.

we have $\frac{dv}{dt} = a$ or $v = \int_0^{80} a\,dt$

Using Simpson's $\frac{1}{3}$-rule, we have

$$v = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$= \frac{10}{3}\Big[(30+50.67) + 4(31.63+35.47+40.33+46.69)$
$+2(33.34+37.75+43.25)\Big]$

$= 3086.1$ m/s

Therefore the required velocity is given by $v = 3.0861$ km/s.

7(3) Using modified Euler's method, obtain the solution of the differential equation

$$\frac{dy}{dt} = t + \sqrt{y} = f(t, y)$$

with the initial condition $y_0 = 1$ at $t_0 = 0$ for the range $0 \le t \le 0.6$ in steps of 0.2

Sol^n: At first, we use Euler's method to get

$y_1^{(1)} = y_0 + h f(t_0, y_0) = 1 + (0.2)(0+1) = 1.2$

then, we use modified Euler's method to find

$y(0.2) = y_1 = y_0 + h\,\frac{f(t_0, y_0) + f(t_1, y_1^{(1)})}{2}$

$= 1.0 + 0.2\left(\frac{1 + 0.2 + \sqrt{1.2}}{2}\right) = 1.2295$

Similarly, proceeding, we have from Euler's method

$y_2^{(1)} = y_1 + h f(t_1, y_1) = 1.2295 + 0.2\left(0.2 + \sqrt{1.2295}\right)$

$= 1.4913$

Using modified Euler's method, we get

$y_2 = y_1 + h\left(\frac{f(t_1, y_1) + f(t_2, y_2^{(1)})}{2}\right)$

$$y_2 = 1.2295 + (0.2)\frac{\left(0.2+\sqrt{1.2295}\right)+\left(0.4+\sqrt{1.493}\right)}{2}$$

$$= 1.5225$$

Finally,

$$y_3^{(1)} = y_2 + h\,f(t_2, y_2) = 1.5225 + (0.2)\left[0.4+\sqrt{1.5225}\right]$$

$$= 1.8493$$

Now, Modified Euler's method gives

$$y(0.6) = y_3 = y_2 + h\left[\frac{f(t_2, y_2) + f(t_3, y_3^{(1)})}{2}\right]$$

$$= 1.5225 + (0.1)\left[0.4+\sqrt{1.5225}\right.$$

$$\left.+ \left(0.6+\sqrt{1.8493}\right)\right]$$

$$y(0.6) = 1.8819.$$

Hence, the solution to the given problem is given by

t	0.2	0.4	0.6
y	1.2295	1.5225	1.8819

7(c)) Obtain the principal disjunctive and Conjunctive normal forms of

$p \to [(p\to q) \wedge \sim(\sim q \vee \sim p)$.

Sol^n: (i) $p \to [(p\to q)\wedge \sim(\sim q \vee \sim p)]$

$\Leftrightarrow \sim p \wedge [(\sim p \vee q) \wedge (q \wedge p)]$

(using Demorgan's law and equivalence $p\to q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow \sim p \vee [\sim p \wedge (q \wedge p)] \vee [q \wedge (q \wedge p)]$

$\Leftrightarrow \sim p \vee (q \wedge p)$

$\Leftrightarrow [\sim p \wedge (q \vee \sim q)] \vee (q \wedge p)$

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p)$

This is the desired pdnf.

(ii) $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$

$\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$

$\Leftrightarrow [\sim p \vee (\sim p \vee q)] \wedge (\sim p \vee (q \wedge p))$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee q)$

$\Leftrightarrow \sim p \vee q.$

This is the desired pcnf.

7(d) In Boolean algebra $[B, +, \cdot, ']$, show that $(x \cdot y' + y \cdot z)(x \cdot z + y \cdot z') = x \cdot z$.

Sol: $\{(x \vee y') \wedge (y \vee z)\} \vee \{(x \vee z) \wedge (y \vee z')\}$

$= \{(x \vee y') \wedge y\} \vee \{(x \vee y') \wedge z\} \vee \{(x \wedge z) \wedge y\} \vee \{(x \vee z) \wedge z'\}$

$= \{(x \wedge y) \vee (y' \wedge y)\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee (z \wedge z')\}$

$= \{(x \wedge y) \vee 0\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee 0\}$

$= \{(x \wedge y) \vee (x \wedge y)\} \vee \{(x \wedge z) \vee (x \wedge z')\} \vee \{(z \wedge y) \vee (z \wedge y')\}$

$= (x \wedge y) \vee (x \wedge 1) \vee (z \wedge 1) = (x \wedge y) \vee x \vee z$

$= (x \vee x \vee z) \wedge (y \vee x \vee z) = (x \vee z) \wedge (y \vee x \vee z)$

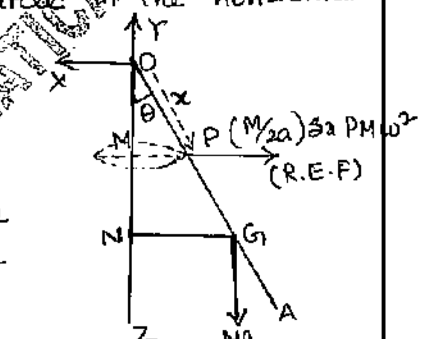
$= (x \vee z) \wedge (1 \vee y) = (x \vee z) \wedge 1$

$= x \vee z.$

8(a). A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or $\cos^{-1}(3g/4a\omega^2)$

Sol^n: Let the rod OA of length 2a and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ. Let PQ $= \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}\delta x$

This element PQ will make a circle in the horizontal plane with radius PM$(= x\sin\alpha)$ and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\frac{M}{2a}\delta x . PM . \omega^2$ along PM. Then the reversed effective force on the element PQ is $\frac{M}{2a}\delta x . x\sin\alpha . \omega^2$ along MP.



Now By D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and reaction at O are in equilibrium. To avoid reaction at O, taking moment about O, we get

$$\Sigma\left(\frac{M}{2a}\delta x . \omega^2 . \sin\alpha\right) . OM - Mg . NG = 0$$

$\Rightarrow \int_0^{2a} \frac{M}{2a} \omega^2 x^2 \sin\alpha \cos\alpha \, dx - Mg \cdot a \sin\alpha = 0$ ($\because OM = x\cos\alpha$)

$\Rightarrow \frac{M}{2a} \omega^2 \left\{ \frac{1}{3}(2a)^3 \right\} \cdot \sin\alpha \cos\alpha - Mga \sin\alpha = 0$

$\Rightarrow Mga \sin\alpha \left(\frac{4a}{3g} \omega^2 \cos\alpha - 1 \right) = 0$

$\therefore$ either $\sin\alpha = 0 \Rightarrow \alpha = 0$.

$\Rightarrow \frac{4a}{3g} \omega^2 \cos\alpha - 1 = 0 \Rightarrow \cos\alpha = \frac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero (or) $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

8(4). A uniform rod, of length $2a$, which has one end attached to a fixed point by a light inextensible string of length $5a/12$, is performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time, and show that the period of its principal oscillations are $2\pi\sqrt{(5a/3g)}$ & $\pi\sqrt{(a/3g)}$

Sol'n: Let OA be the string of length $\frac{5}{12}a$ and AB the rod of mass M and length $2a$. Let O be fixed point. At time t, let the string and the rod make angles θ and ϕ, to the vertical respectively.

Referred to O as origin, horizontal and vertical lines OX and OY as axes, the coordinates of the C.G. G of the rod are given by

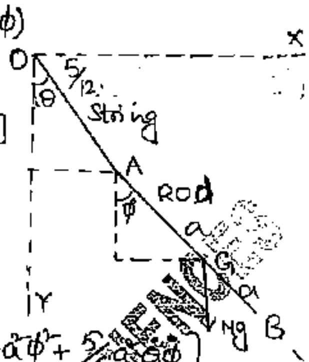
$$x_G = \frac{5}{12} a \sin\theta + a \sin\phi$$

$$y_G = \frac{5}{12} a \cos\theta + a \cos\phi$$

$$\therefore V_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = \left(\tfrac{5}{12} a\cos\theta\,\dot\theta + a\cos\phi\,\dot\phi\right)^2 + \left(-\tfrac{5}{12} a\sin\theta\,\dot\theta - a\sin\phi\,\dot\phi\right)^2$$

$$= \frac{25}{144}a^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5}{6}a^2\dot\theta\dot\phi\cos(\theta-\phi)$$

$$= \frac{25}{144}a^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5}{6}a^2\dot\theta\dot\phi \quad [\because \theta \,\&\, \phi \text{ are small}]$$



If T be the total K.E and W the workfunction of the system, then

$$T = \tfrac{1}{2}M\cdot\tfrac{1}{3}a^2\dot\phi^2 + \tfrac{1}{2}M\cdot V_G^2$$

$$= \tfrac{1}{2}M\left[\tfrac{1}{3}a^2\dot\phi^2 + \tfrac{25}{144}a^2\dot\theta^2 + a^2\dot\phi^2 + \tfrac{5}{6}a^2\dot\theta\dot\phi\right]$$

$$\Rightarrow T = \tfrac{1}{2}Ma^2\left[\tfrac{25}{144}\dot\theta^2 + \tfrac{4}{3}\dot\phi^2 + \tfrac{5}{6}\dot\theta\dot\phi\right]$$

and $W = Mg\,y_G + C = Mga\left(\tfrac{5}{12}\cos\theta + \cos\phi\right) + C$

Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$$

$$\Rightarrow \frac{d}{dt}\left[\tfrac{1}{2}Ma^2\left(\tfrac{25}{72}\dot\theta + \tfrac{5}{6}\dot\phi\right)\right] - 0 = -\tfrac{5}{12}Mga\sin\theta$$

$$\Rightarrow 5\ddot\theta + 12\ddot\phi = -12c\theta, \quad (\because \theta \text{ is small}) \text{ Taking } (g/a) = c \quad \text{—(1)}$$

And Lagrange's ϕ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot\phi}\right) - \dfrac{\partial T}{\partial\phi} = \dfrac{\partial W}{\partial\phi}$

i.e. $\dfrac{d}{dt}\left[\tfrac{1}{2}Ma^2\left(\tfrac{8}{3}\dot\phi + \tfrac{5}{6}\dot\theta\right)\right] - 0 = -Mga\sin\phi$

$$\Rightarrow 5\ddot\theta + 16\ddot\phi = -12c\phi, \quad [\because \phi \text{ is small} \,\&\, (g/a) = c] \quad \text{—(2)}$$

Equations (1) & (2), can be written as

$$(5D^2+12c)\theta + 12D^2\phi = 0 \text{ and } 5D^2\theta + (16D^2+12c)\phi = 0$$

Eliminating ϕ b/w these two equations, we have

$$\left[(5D^2+12c)(16D^2+12c) - 60D^4\right]\theta = 0 \quad \text{—(3)}$$

Eliminating ϕ between these two equations, we have

$$\left[(5D^2+12c)(16D^2+12c)-60D^4\right]\theta=0$$

$$\Rightarrow 5D^4+63cD^2+36c^2)\theta=0$$

Let the solution of (3) be

$\theta=A\cos(pt+B)$, $\therefore$ $D^2\theta=-p^2\theta$ and $D^4\theta=p^4\theta$

Substituting in (3), we get

$$(5p^4-63cp^2+36c^2)\theta=0$$

$$\Rightarrow 5p^4-63cp^2+36c^2=0 \quad \because \theta\neq 0.$$

$$\Rightarrow (5p^2-3c)(p^2-12c)=0$$

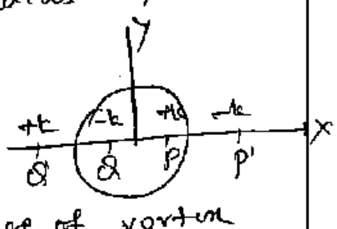
$$\therefore p_1^2=\tfrac{3}{5}c=\frac{3g}{5a} \;\&\; p_2^2=12c=\frac{12g}{a} \quad \because c=g/a$$

Hence period of oscillations are $\frac{2\pi}{p_1}$ & $\frac{2\pi}{p_2}$.

$$\Rightarrow 2\pi\sqrt{\left(\frac{5a}{3g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{12g}\right)}$$

$$\Rightarrow 2\pi\sqrt{\left(\frac{5a}{3g}\right)} \text{ and } \pi\sqrt{\left(\frac{a}{3g}\right)}$$

8(c) If a vortex pair is situated within a cylinder show that it will remain at rest if the distance of either from the centre is given by $a(\sqrt{5}-2)^{1/2}$ where a is the radius of the cylinder.



Sol^n: The vortex pair PQ consists of vortex +k at P and vortex −k at Q. The image of vortex +k at P is a vortex −k at P', the inverse point of P.

Similarly, the image of vortex $-k$ at Q is a vortex $+k$ at Q'.

Let $OP = OQ = r$.

Then $OP \cdot OP' = a^2 = OQ \cdot OQ'$

Hence $OP' = \frac{a^2}{r} = OQ'$.

Thus $z_p = r$, $z_Q = -r$

$z_Q' = -a^2/r$, $z_p' = a^2/r$

The complex potential for the motion is

$$W = \frac{ik}{2\pi}\left[\log(z-z_p) - \log(z-z_{p'}) - \log(z-z_Q) + \log(z-z_{Q'})\right]$$

The motion of P is due to other vortices

for the motion of P,

$$W_1 = W - \frac{ik}{2\pi}\log(z-z_p)$$

$$\frac{dW_1}{dz} = \frac{-ik}{2\pi}\left[\frac{1}{z-z_{p'}} + \frac{1}{z-z_Q} - \frac{1}{z-z_{Q'}}\right]$$

$$u_p - iv_p = \left(\frac{-dW_1}{dz}\right)_{z=z_p}$$

$$= \frac{ik}{2\pi}\left[\frac{1}{z_p - z_{p'}} + \frac{1}{z_p - z_Q} - \frac{1}{z_p - z_{Q'}}\right]$$

This implies $u_p = 0$, $v_p = \frac{-k}{2\pi}\left[\frac{r}{a^2-r^2} - \frac{1}{2r} + \frac{r}{r^2+a^2}\right]$

The vortex at P will be at rest if $q_p = 0$

i.e, $\sqrt{u_p^2+v_p^2}=0$

(or) $v_p=0$ (or) $\frac{r}{a^2-r^2}-\frac{1}{2r}+\frac{r}{r^2+a^2}=0$

$2r^2(r^2+a^2)-(a^2-r^2)(a^2+r^2)+2r^2(a^2-r^2)=0$

$\Rightarrow r^4+4r^2a^2-a^4=0$

$\Rightarrow \left(\frac{r^2}{a^2}\right)^2+4\left(\frac{r^2}{a^2}\right)-1=0$

$\Rightarrow \frac{r^2}{a^2}=\frac{-4\pm\sqrt{20}}{2}=-2\pm\sqrt{5}$

$r=a(-2\pm\sqrt{5})^{1/2}$

The value $(-2-\sqrt{5})^{1/2}$ is not admissible because this root gives imaginary value of r.

Hence $r=a(-2+\sqrt{5})^{1/2}$

Main Test Series – 2016
Test – 05 – paper-I

11a) Find a basis and dimension of the subspace W of V spanned by the polynomials
$v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = 2t^3 - 3t^2 + 9t - 1$, $v_3 = t^3 + 6t - 5$
$v_4 = 2t^3 - 5t^2 + 7t + 5$.

Sol^n: Since W is spanned by polynomials of degree 3.

$\therefore$ W is a subspace of the space $V_3(\mathbb{R})$.
(the space of all real polynomials of degree ≤ 3)

W.K.T $\{1, t, t^2, t^3\}$ is a basis for $V_3(\mathbb{R})$

$\therefore$ The co-ordinate vectors of v_1, v_2, v_3, v_4 w.r.t the above basis are

$(1, 4, -2, 1)$, $(-1, 9, -3, 2)$, $(-5, 6, 0, 1)$ and $(5, 7, -5, 2)$

Now form the matrix A whose rows are these co-ordinate vectors and reduce it to an echelon form.

$$A = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 9 & -3 & 2 \\ -5 & 6 & 0 & 1 \\ 5 & 7 & -5 & 2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 26 & -10 & 6 \\ 0 & -13 & 5 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 5R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + R_2 \end{matrix}$$

which is in the echelon form

The non-zero rows of the echelon form of A form a basis of the subspace W.

i.e, the vectors $(1,4,-2,1)$, $(0,13,-5,3)$ form a basis for W.

$\therefore$ A basis for W consists of polynomials t^3-2t^2+4t+1 and $3t^3-5t^2+13t$.

$\therefore \dim W = 2$

1(c) For what values of η the equations

$x+y+z=1$

$x+2y+4z=\eta$

$x+4y+10z=\eta^2$

have a solution and solve them completely in each case.

Sol^n:- The matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$$

performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta-1 \\ \eta^2-1 \end{bmatrix}$$

performing $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta-1 \\ \eta^2-3\eta+2 \end{bmatrix} \quad \text{---} (1)$$

The given equations will be consistent if

$\eta^2 - 3\eta + 2 = 0$

i.e. $(\eta-2)(\eta-1) = 0$

i.e. $\eta = 2$ or $\eta = 1$.

case(i): If $\eta = 2$, the equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow y + 3z = 1$, $x + y + z = 1$.

$\therefore y = 1 - 3z$, $x = 2z$.

Thus $x = 2k$, $y = 1 - 3k$, $z = k$ constitute the general solution where k is an arbitrary constant.

case(ii): If $\eta = 1$, the equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The above system of equations is equivalent to

$y + 3z = 0$, $x + y + z = 1$ $\Rightarrow y = -3z$, $x = 1 + 2z$.

Thus $x = 1 + 2c$, $y = -3c$, $z = c$ constitute the general solution, where c is arbitrary constant.

1(d) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Sol^n: Volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$ is

$$V=\iiint dx\,dy\,dz.$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{4-y} dx\,dy\,dz.$$

$$=2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}(4-y)\,dx\,dy$$

$$=2\int_{-2}^{2}\left(4y-\frac{y^2}{2}\right)_0^{\sqrt{4-x^2}}dx$$

$$=8\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$=8\left[\frac{x}{2}\sqrt{4-x^2}-2\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$=8\left[2\sin^{-1}(1)-2\sin^{-1}(-1)\right]$$

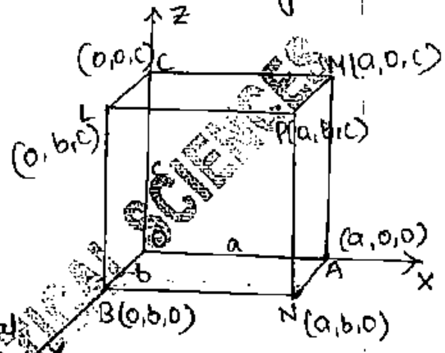
$$=8\left[2(\pi/2)-2(-\pi/2)\right]$$

$$=8[\pi+\pi]$$

$$=16\pi$$

1(e) If the edges of a rectangular parallepiped be a, b, c show that the angles between the four diagonals are given by $\cos^{-1}\left[\dfrac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right]$

Sol: Take 'O', a corner of the rectangular parallelopiped as the origin and three edges OA, OB, OC through it (i.e. 'O') as the axes.



Then the coordinates of the various corners are O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c), L(0,b,c), M(a,0,c), N(a,b,0), P(a,b,c). The four diagonals are AL, BM, CN and OP.

The d.c's of AL are proportional to $0-a, b-0, c-0 \Rightarrow -a, b, c$ (using $(x_2-x_1, y_2-y_1, z_2-z_1)$)

Similarly d.c's of BM are proportional to $a, -b, c$

d.c's of CN are proportional to $a, b, -c$

d.c's of OP are proportional to a, b, c

If α is the angle b/w the diagonals OP and AL then

$$\cos\alpha = \frac{a(-a)+b(b)+c(c)}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right) \quad \text{——— (1)}$$

Similarly angle b/w OP & BM $= \cos^{-1}\left(\dfrac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$

" " OP & CN $= \cos^{-1}\left(\dfrac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$

" " AL & BM $= \cos^{-1}\left(\dfrac{-a^2-b^2+c^2}{a^2+b^2+c^2}\right)$

$= \cos^{-1}\left[\dfrac{-(a^2+b^2-c^2)}{a^2+b^2+c^2}\right]$

$= \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$ (when acute angle is taken)

This angle is the same as the angle b/w OP and CN

Similarly we can show that all other angle b/w other two diagonals are repeated and we get only three different angles as given by (1), (2) and (3)

Hence the angles between four diagonals are given by $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$

2(a) Find the range, rank, kernel and nullity of the linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$T(x_1, x_2, x_3, x_4) = (x_1 - x_4,\ x_2 + x_3,\ x_3 - x_4)$.

Sol^n: Let $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ be the standard basis set of $\mathbb{R}^4$.

$\therefore$ The transformation T on S will be

$T(1,0,0,0) = (1,0,0)$, $T(0,1,0,0) = (0,1,0)$

$T(0,0,1,0) = (0,1,1)$, $T(0,0,0,1) = (-1,0,-1)$

Now we verify whether S_1 is LI or not, If not, we find the least LI set by forming the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_4 \to R_4 + R_1 + R_3 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_3 - R_2$$

Clearly which is in echelon form.

$\therefore$ The non zero rows of vectors $\{(1,0,0), (0,1,0), (0,1,1)\}$ constitute the L.I. set by forming the basis of $R(T)$.

$\therefore \dim\{R(T)\} = 3$.

$\therefore \boxed{\text{Rank of } T = 3}$

Basis for null space of T:

$N(T) = \{\alpha \in V_4 / T(\alpha) = 0\}$

Let $\alpha \in N(T) \Rightarrow T(\alpha) = 0$

$\therefore T(x_1, x_2, x_3, x_4) = 0$
where $0 = (0,0,0) \in R^3$

$\Rightarrow (x_1 - x_4,\ x_2 + x_3,\ x_3 - x_4) = (0,0,0)$

$\Rightarrow x_1 - x_4 = 0,\ x_2 + x_3 = 0,\ x_3 - x_4 = 0$

We have to solve these for x_1, x_2, x_3, x_4

$\therefore x_1 = x_4,\ x_3 = x_4,\ x_2 = -x_4,$

$N(T) = \{(x_4, -x_4, x_4, x_4) / x_4 \in R\}$
$= \{x_4(1,-1,1,1) / x_4 \in R\}$

$\therefore$ The no. of free variables is 1 namely x_4 and the values x_1, x_2, x_3 depend on these.

and hence

$N(T) = \dim N(T) = 1$

$\therefore \{(1,-1,1,1)\}$ constitute a basis of $N(T)$.

2(b)(i) Is the vector $(3,-1,0,-1)$ in the subspace of R^4 spanned by the vectors $\alpha_1 = (2,-1,3,2)$, $\alpha_2 = (-1,1,1,-3)$ and $\alpha_3 = (1,1,9,-5)$?

(ii) If α is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is a characteristic root of Adj A.

Sol: (i) Suppose $(3,-1,0,-1)$ is spanned by α_1, α_2 and α_3.

Then $(3,-1,0,-1) = x(2,-1,3,2) + y(-1,1,1,-3) + z(1,1,9,-5)$, $x, y, z \in R$

$\therefore$

$$2x - y + z = 3 \quad \text{---} (1)$$
$$-x + y + z = -1 \quad \text{---} (2)$$
$$3x + y + 9z = 0 \quad \text{---} (3)$$
$$2x - 3y - 5z = -1 \quad \text{---} (4)$$

from (1) & (2) $x + 2z = 2$ ——— (5)

from (3) & (4) $11x + 22z = -1 \Rightarrow x + 2z = -1$ ——— (6)

from (5) & (6), we see that the given system of equations has no solution. Hence $(3,-1,0,-1)$ is not in the subspace spanned by the vectors α_1, α_2 and α_3.

(ii) Since α is a characteristic root of a non-singular matrix, therefore $\alpha \neq 0$. Also α is characteristic root of A implies that there exists a non-zero vector X such that

$$AX = \alpha X$$

$\Rightarrow (\text{Adj } A)(AX) = (\text{Adj } A)(\alpha X)$

$\Rightarrow [(\text{Adj } A)A]X = \alpha(\text{Adj } A)X$

$\Rightarrow |A| IX = \alpha(\text{Adj } A)X \quad [\because (\text{Adj } A)A = |A|I]$

$\Rightarrow |A|X = \alpha(\text{Adj } A)X \quad [\because IX = X]$

$\Rightarrow \frac{|A|}{\alpha} X = (\text{Adj} A)X \quad [\because \alpha \neq 0]$

$\Rightarrow (\text{Adj } A)X = \frac{|A|}{\alpha} X$

Since X is a non-zero vector, therefore from the relation (1) it is obvious that $\frac{|A|}{\alpha}$ is a characteristic root of the matrix Adj A.

2(c) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

Sol^n: The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow [(1-\lambda)(3-\lambda)+2] = 0$

$\Rightarrow 5 - 4\lambda + \lambda^2 = 0$ —— (1)

$\therefore$ By Caley-Hamilton theorem, the matrix A must satisfy its characteristic equation (1).

So we must have $A^2 - 4A + 5I = 0$ —— (2)

Now dividing the polynomial $\lambda^6 - 4\lambda^5 + 8\lambda^4 - 12\lambda^3 + 14\lambda^2$ by the polynomial $\lambda^2 - 4\lambda + 5$, we get

$$\lambda^6-4\lambda^5+8\lambda^4-12\lambda^3+14\lambda^2 = (\lambda^2-4\lambda+5)(\lambda^4+3\lambda^2-1) + (-4\lambda+5I)$$

$$\therefore A^6-4A^5+8A^4-12A^3+14A^2 = (A^2-4A+5I)(A^4+3A^2-I) - 4A+5I$$

But $A^2-4A+5I=0$.

$\therefore$ we have

$$A^6-4A^5+8A^4-12A^3+14A^2 = 0(A^4+3A^2-I) - 4A+5I$$

$$= -4A+5I$$

which is a linear polynomial in A.

2(d) Let $B=\begin{bmatrix}1 & -3 & 2\\ -3 & 7 & -5\\ 2 & -5 & 8\end{bmatrix}$, a symmetric matrix, Find a non-singular matrix P such that P^TBP is a diagonal and find the diagonal matrix P^TBP.

Sol^n: First form the block matrix (B, I):

$$(B,I)=\left[\begin{array}{ccc|ccc}1 & -3 & 2 & 1 & 0 & 0\\ -3 & 7 & -5 & 0 & 1 & 0\\ 2 & -5 & 8 & 0 & 0 & 1\end{array}\right]$$

Apply the row operations $R_2\to 3R_1+R_2$ and $R_3\to -2R_3$ to (B, I) and then the corresponding column operations $C_2\to 3C_1+C_2$ and $C_3\to -2C_1+C_3$ to B to obtain

$$\left[\begin{array}{ccc|ccc}1 & -3 & 2 & 1 & 0 & 0\\ 0 & -2 & 1 & 3 & 1 & 0\\ 0 & 1 & 4 & -2 & 0 & 1\end{array}\right] \text{ and then } \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & -2 & 0 & 3 & 1 & 0\\ 0 & 1 & 4 & -2 & 0 & 1\end{array}\right]$$

Next apply the row operation $R_3\to R_2+2R_3$ and then the corresponding operation $C_3\to C_2+2C_3$ to obtain

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & -2 & 1 & 3 & 1 & 0\\ 0 & 0 & 9 & -1 & 1 & 2\end{array}\right] \text{ and then } \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & -2 & 0 & 3 & 1 & 0\\ 0 & 0 & 18 & -1 & 1 & 2\end{array}\right]$$

Now B has been diagonalized.

Set $P=\begin{bmatrix}1 & 3 & -1\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix}$; then $P^TBP=\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 18\end{bmatrix}$

3(a). Show that the function $f: R^2 \to R$ defined by setting

$$f(x,y) = \begin{cases} x\sin \frac{1}{x} + y\sin\left(\frac{1}{y}\right), & \text{when } xy \neq 0 \\ x\sin\frac{1}{x}, & \text{when } x \neq 0, y = 0 \\ y\sin\frac{1}{y}, & \text{when } x = 0, y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$

is continuous but not differentiable at $(0,0)$.

Sol^n: we have

$$|f(x,y) - f(0,0)| = \left|x\sin\frac{1}{x} + y\sin\frac{1}{y}\right|$$

$$\leq |x|\left|\sin\frac{1}{x}\right| + |y|\left|\sin\frac{1}{y}\right|$$

$$\leq |x| + |y| \quad (\because |\sin\theta| \leq 1)$$

Let $\epsilon > 0$ be given, choose $\delta = \epsilon/2$

Then $|f(x,y) - f(0,0)| < \epsilon$ if $|x| < \delta, |y| < \delta$

Hence the given function is continuous at $(0,0)$

Now $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{h\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \sin\frac{1}{h}$$ which does not exist.

Similarly $f_y(0,0)$ does not exist.

$\therefore$ f is not differentiable. ($\because$ if either of f_x, f_y does not exist at (a,b), then f is not differentiable at (a,b))

3(b). (i) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(ii) Determine $\lim \left(\frac{\pi}{2}-x\right)^{\tan x}$ as $x \to \left(\frac{\pi}{2}-0\right)$.

Sol'n: (ii) Let $y = \left(\frac{\pi}{2}-x\right)^{\tan x}$

$$\log y = \tan x \log\left(\frac{\pi}{2}-x\right)$$

$$\lim_{x\to(\pi/2-0)} \log y = \lim_{x\to(\pi/2-0)} \tan x \log\left(\frac{\pi}{2}-x\right)$$

$$= \lim_{x\to\left(\frac{\pi}{2}-0\right)} \frac{\log\left(\frac{\pi}{2}-x\right)}{\cot x}$$

$$= \lim_{x\to\left(\frac{\pi}{2}-0\right)} \frac{-\sin^2 x}{\left(\frac{\pi}{2}-x\right)}$$

$$= \lim_{x\to\left(\frac{\pi}{2}-0\right)} \frac{-2\sin x\cos x}{1}$$

$$= 0$$

$$\log \lim_{x\to(\pi/2-0)} y = 0.$$

$$\therefore \lim_{x\to(\pi/2-0)} y = e^0 = 1$$

(i) $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$. —①

Diff ① partially w.r.t. x, y, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right)\right]$$

$$+ y\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)\right]$$

$$= \cancel{\frac{x}{y}\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}} - \cancel{\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{y}{x}\right)} - \cancel{\frac{x}{y}\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}}$$

$$+ \cancel{\frac{y}{x}\frac{1}{\left(1+\frac{y^2}{x^2}\right)}}$$

$$= 0$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

3(e) Evaluate the following integral by changing to polar co-ordinates. $\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$

Ans: $a\pi/4$

3(d). Evaluate $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx$

Sol'n: Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

then $I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}+\sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}\,dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

Adding $2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}+\sqrt{(\cos x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx$

$= \int_0^{\pi/2} dx = |x|_0^{\pi/2} = \pi/2$

Hence $I = \pi/4$

4(a) Find the incentre of the tetrahedron formed by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=a$.

Sol^n: Evidently the planes $x=0$, $y=0$ and $z=0$ meet in $(0,0,0)$. Hence the incentre lies on the $\perp$lar from $(0,0,0)$ to the plane $x+y+z=a$ and divides it in the ratio $3:1$ [3 from the vertex $(0,0,0)$ and 1 from the plane $x+y+z=a$].

The equations of the $\perp$lar from $(0,0,0)$ to the plane $x+y+z=a$ is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=r$

Any point on this $\perp$lar is (r,r,r). If it lies on the plane $x+y+z=a$, then we have $r+r+r=a \Rightarrow r=a/3$.

$\therefore$ The $\perp$lar from $(0,0,0)$ meets the plane $x+y+z=a$ in (r,r,r) i.e. $(a/3, a/3, a/3)$. Also the incentre divides the join of $(0,0,0)$ and $a/3, a/3, a/3$ in the ratio $3:1$, therefore if (x_1, y_1, z_1) be the required incentre, we have $x_1 = \frac{3.\frac{1}{3}a+1.0}{3+1} = \frac{a}{4}$. Similarly $y_1=\frac{1}{4}a=z_1$

$\therefore$ The required incentre is $\left(\frac{1}{4}a, \frac{1}{4}a, \frac{1}{4}a\right)$.

4(b) Find the equation of the sphere that passes through the points $(4,1,0)$, $(2,-3,4)$, $(1,0,0)$ and touches the plane $2x+2y-z=11$.

Sol^n: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—— (1)}$$

Its centre is $(-u,-v,-w)$ and radius $=\sqrt{(u^2+v^2+w^2-d)}$

Since the sphere (1) passes through the point $(4,1,0)$, so

$$4^2+1^2+0^2+2u(4)+2v(0)+2w(0)+d=0$$
$$8u+2v+d+17=0 \quad \text{—— (2)}$$

Similarly if the sphere (1) passes through $(2,-3,4)$ and $(1,0,0)$, then we have $4u-6v+8w+d+29=0$ — (3)

and $\quad 2u+d+1=0$ — (4)

Also as the sphere (1) touches the given plane, so the length of ⊥ar from its centre $(-u,-v,-w)$ to the given plane $2x+2y-z-11=0$ must be equal to radius $\sqrt{(u^2+v^2+w^2-d)}$ of the sphere (1)

i.e. $$\frac{2(-u)+2(-v)-(-w)-11}{\sqrt{[2^2+2^2+(-1)^2]}} = \sqrt{u^2+v^2+w^2-d}$$

$$\Rightarrow (-2u-2v+w-11)^2 = 9(u^2+v^2+w^2-d)$$

$$\Rightarrow 5u^2+5v^2+8w^2-8uv+4vw+4wu+44u+44v-22w-9d-121=0 \quad \text{— (5)}$$

from (4), $u=\frac{1}{2}(-d-1) = -\frac{1}{2}(d+1)$ — (6)

from (3), $2v = -8u... = 4d+4-d-17 = 3d-13$

$$v=\frac{1}{2}(3d-13) \quad \text{— (7)}$$

from (3), $8w = -4u+6v-d...$

$$=(2d+2)+(9d-39)-d-29, \text{ from (6),(7)}$$

$$\Rightarrow 8w=10d-66 \Rightarrow w=\frac{1}{4}(5d-33) \quad \text{— (8)}$$

Substituting values of u,v,w from (6),(7),(8) in (5) and simplifying we get $72d^2-747d+1935=0$ which gives $d=5$

∴ from (6),(7) and (8) we get $u=-3, v=1, w=-2$

∴ from (1) the required equation is

$$x^2+y^2+z^2-6x+2y-4z+5=0$$

4(c) Show that the feet of the normals from the point (α,β,γ) on the paraboloid $x^2+y^2=2az$ lie on a sphere.

Sol^n: Let (x_1, y_1, z_1) be any point on the given paraboloid

then $x_1^2 + y_1^2 = 2az_1$ —— ①

the tangent plane to this paraboloid at (x_1, y_1, z_1) is

$$xx_1 + yy_1 = a(z+z_1) \Rightarrow xx_1 + yy_1 - az = az_1$$

∴ The equation of the normal to the given paraboloid at (x_1, y_1, z_1) i.e. the line through (x_1, y_1, z_1) at right angles to the above tangent plane is

$$\frac{x-x_1}{x_1} = \frac{y-y_1}{y_1} = \frac{z-z_1}{-a}$$

If this normal passes through the fixed point (α, β, γ) then we have

$$\frac{\alpha-x_1}{x_1} = \frac{\beta-y_1}{y_1} = \frac{\gamma-z_1}{-a} \quad \text{—— ②}$$

$$\frac{\alpha-x_1}{x_1} = \frac{\beta-y_1}{y_1} = \frac{\gamma-z_1}{-a} = \frac{x_1(\alpha-x_1) + y_1(\beta-y_1)}{x_1(x_1)+y_1(y_1)}$$

$$= \frac{z_1(\gamma-z_1)}{z_1(-a)}$$

from the last two fractions we have

$$\frac{\alpha x_1 - x_1^2 + \beta y_1 - y_1^2}{x_1^2+y_1^2} = \frac{\gamma z_1 - z_1^2}{-az_1}$$

$$\Rightarrow \frac{(\alpha x_1 + \beta y_1) - (x_1^2+y_1^2)}{2az_1} = \frac{2(\gamma z_1 - z_1^2)}{-2az_1}, \text{ from ①}$$

$$\Rightarrow x_1^2 + y_1^2 - (\alpha x_1 + \beta y_1) - 2\gamma z_1 + 2z_1^2 = 0$$

$$\Rightarrow x_1^2 + y_1^2 + 2z_1^2 - 2\gamma z_1 = \alpha x_1 + \beta y_1 \quad \text{—— ③}$$

Also from ② we have $\frac{\alpha}{x_1} - 1 = \frac{\beta}{y_1} - 1$

$\Rightarrow \frac{\alpha}{x_1} = \frac{\beta}{y_1} \Rightarrow \alpha y_1 = \beta x_1$ —— (4)

Now from (3) we have $x_1^2+y_1^2+2z_1^2-2\gamma z_1 = (\alpha\beta x_1+\beta^2 y_1)/\beta$

$= \frac{\alpha^2 y_1+\beta^2 y_1}{\beta}$, from (4)

$\Rightarrow x_1^2+y_1^2+2z_1^2-2\gamma z_1 = (\alpha^2+\beta^2)y_1/\beta$ —— (5).

Adding (1) and (5) we get

$2x_1^2+2y_1^2+2z_1^2-2\gamma z_1 = 2az_1+[\{2(\alpha^2+\beta^2)y_1\}/(2\beta)]$

$\Rightarrow x_1^2+y_1^2+z_1^2-(\gamma+a)z_1-\{(\alpha^2+\beta^2)y_1/(2\beta)\}=0$

$\therefore$ The locus of the foot (x_1, y_1, z_1) of the normal from (α, β, γ) is

$x^2+y^2+z^2-(\gamma+a)z-\{(\alpha^2+\beta^2)/(2\beta)\}y=0$. Hence proved.

4(d) Prove that the projections of the generators of a hyperboloid on coordinate plane are tangents to the section of the hyperboloid by that plane.

Sol'n: Let the equation of the hyperboloid be

$$\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right)=1 \quad \text{—— (1)}.$$

we know that a generator of the hyperboloid (1) is

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{c} \quad \text{—— (2)}$$

Now consider the coordinate plane $z=0$. The section of the hyperboloid (1) by this plane $z=0$ is given by

$$x^2/a^2 + y^2/b^2 = 1,\ z=0 \quad \text{—— (3)}$$

The projection of the generator (2) on the plane $z=0$ is given by

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta},\ z=0$$

which is a plane through the generator ⊥ar to the plane $z=0$

on simplyfying it reduces to $\frac{x}{a\sin\theta}-\frac{\cos\theta}{\sin\theta}=\frac{y}{-b\cos\theta}+\frac{\sin\theta}{\cos\theta}$, $z=0$

i.e. $\frac{x}{a\sin\theta}+\frac{y}{b\cos\theta}=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{1}{\sin\theta\cos\theta}$, $z=0$

$\Rightarrow \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,\ z=0$

which is evidently a tangent to the section ③ of the hyperboloid ① by the plane $z=0$ at the point $(a\cos\theta, b\sin\theta, 0)$

Again consider the coordinate plane $x=0$. The section of the hyperboloid ① by this plane $x=0$ is given by

$$\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right)=1,\ x=0 \quad \text{------} \quad (4)$$

the projection of the generator ② on the plane $x=0$ is given by $\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}$, $x=0$ which is a plane through the generator ⊥ar to the plane $x=0$.

On simplifying it reduces to $\frac{y}{-b\cos\theta}+\frac{\sin\theta}{\cos\theta}=\frac{z}{c}$, $x=0$

$\Rightarrow \frac{y}{b}+\frac{z\cos\theta}{c}=\sin\theta,\ x=0 \Rightarrow \frac{y}{b}\operatorname{cosec}\theta+\frac{z}{c}\cot\theta=1, x=0$

which is evidently a tangent to the section ④ of the hyperboloid ① by the plane $x=0$ at the point $(0, b\operatorname{cosec}\theta, -c\cot\theta)$.

5(a) Solve $\sin x\left(\frac{dy}{dx}\right)+3y=\cos x$

Sol^n: Rewriting, we have $\frac{dy}{dx}+(3\operatorname{cosec}x)y=\cot x$

Here $\int P\,dx = 3\int \operatorname{cosec}x\,dx = 3\log\tan(x/2)$

So I.F $= e^{\int P dx} = \tan^3 x/2$

So the required solution is

$$y\tan^3(x/2) = \int \cot x \tan^3(x/2)\,dx + C$$

$$\Rightarrow y\tan^3 x/2 = \int \frac{1-\tan^2 x/2}{2\tan x/2}\tan^3 x/2\,dx + C$$

$$\Rightarrow y\tan^3 x/2 = \frac{1}{2}\int (1-\tan^2 x/2)\tan^2 x/2\,dx + C$$

$$\Rightarrow y\tan^3 x/2 = \frac{1}{2}\int (1-t^2)t^2 \times \frac{2dt}{1+t^2} + C \quad \left[\text{Put } \tan x/2 = t\right.$$

$$\Rightarrow \frac{1}{2}\sec^2 x/2\,dx = dt$$

$$\Rightarrow dx = \frac{2dt}{\sec^2 x/2} = \frac{2dt}{1+\tan^2 x/2} = \left.\frac{2dt}{1+t^2}\right]$$

$$\Rightarrow y\tan^3 x/2 = \int \frac{t^2-t^4}{1+t^2}\,dt$$

$$\Rightarrow y\tan^3 x/2 = \int\left[-t^2+2-\frac{2}{t^2+1}\right]dt$$

$$\Rightarrow y\tan^3 x/2 = -\left(\frac{1}{3}\right)t^3 + 2t - 2\tan^{-1}t + C$$

$$\Rightarrow y\tan^3 x/2 = -\frac{1}{3}\tan^3 x/2 + 2\tan x/2 - 2\tan^{-1}(\tan x/2) + C$$

$$\Rightarrow \left(y+\frac{1}{3}\right)\tan^3(x/2) = 2\tan(x/2) - x + C$$

5(b) Solve $(x^2-4)p^2-2xyp-x^2=0$ and examine for singular solutions and extraneous loci.

Sol^n: The given equation is

$$(x^2-4)p^2-2xyp-x^2=0 \quad \text{—}(1).$$

Solving for y

$$2y = xp - \frac{y}{x}p - \frac{x}{p} \quad \text{—}(2).$$

Differentiating ② partially w.r.t x, we have.

$$2p = p + x\frac{dp}{dx} - \frac{y}{x}\frac{dp}{dx} + \frac{4p}{x^2} - \frac{1}{p} + \frac{x}{p^2}\frac{dp}{dx}$$

$$\Rightarrow (p^2x^2 - 4p^2 + x^2)\left(p - x\frac{dp}{dx}\right) = 0$$

from $p - x\frac{dp}{dx} = 0$ (omitting the 2nd factor)

$$\Rightarrow \frac{dx}{x} - \frac{dp}{p} = 0$$

$$\Rightarrow p = xc$$

putting $p = xc$ in ①, we get

$$x^2c^2(x^2-4) - 2xy(xc) - x^2 = 0$$

$$\Rightarrow x^2\left[c^2(x^2-4) - 2yc - 1\right] = 0$$

$$\Rightarrow (x^2-4)c^2 - 2cy - 1 = 0 \quad \text{—③}$$

which is the general solution of ①

the c-discriminant relation is

$$(-2y)^2 - 4(x^2-4)(-1) = 0$$

$$\Rightarrow y^2 + x^2 - 4 = 0$$

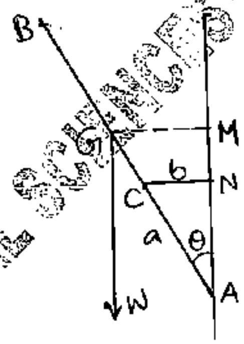
and p-disc. relation is $x^2(x^2+y^2-4) = 0$.

Now $\boxed{x^2+y^2=4}$ occurs once in both the discriminant relations, and satisfies the given differential equation and therefore it is a singular solution. Also $x=0$ occurs twice in the p-discriminant relation doesnot occur in the c-disc. relation and doesnot satisfy the differential equation,

$\therefore x=0$ is a tac locus.

5(C) A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$.

Sol^n: A uniform beam AB of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg C whose distance CN from the wall is b. Suppose the rod makes an angle θ with the wall. i.e. $\angle BAM = \theta$. The weight W of the rod acts at its middle point G. Give the rod a small displacement in which θ changes to $\theta + \delta\theta$. The peg C remains fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and C do not work.



We have, the height of G above the fixed point C.

$$= NM = AM - AN = AG\cos\theta - CN\cot\theta$$

$$= a\cos\theta - b\cot\theta$$

the equation of virtual work is

$$-W\delta(a\cos\theta - b\cot\theta) = 0$$

$$\Rightarrow \delta(a\cos\theta - b\cot\theta) = 0$$

$$\Rightarrow -a\sin\theta\,\delta\theta + b\operatorname{cosec}^2\theta\,\delta\theta = 0$$

$$\Rightarrow (-a\sin\theta + b\operatorname{cosec}^2\theta) = 0 \qquad [\because \delta\theta \neq 0]$$

$$\Rightarrow a\sin\theta = b\operatorname{cosec}^2\theta$$

$$\Rightarrow \sin^3\theta = b/a$$

$$\Rightarrow \theta = \sin^{-1}(b/a)^{1/3}$$ giving the inclination of the rod to the vertical in the position of equilibrium.

5(d) A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is $\frac{gm^2}{\mu^2}\left[e^{-(\mu/m)t} - 1 + \frac{\mu t}{m}\right]$.

Sol^n: Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards i.e., in the direction of x decreasing. The weight mg of the particle acts vertically downwards i.e. in the direction of x decreasing.

∴ The equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

$$\Rightarrow \frac{dv}{dt} = g - \frac{\mu}{m}v \Rightarrow dt = \frac{dv}{g - (\mu/m)v} \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$$

Integrating we have

$$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + A, \text{ where A is constant}$$

But initially when $t=0, v=0$; ∴ $A = (m/\mu)\log g$.

$$\therefore t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + \frac{m}{\mu}\log g.$$

$$\Rightarrow t = -\frac{m}{\mu}\log\left(\frac{g - \frac{\mu}{m}v}{g}\right)$$

$$\Rightarrow -\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm}v\right) \Rightarrow 1 - \frac{\mu}{gm}v = e^{-\mu t/m}$$

$$\Rightarrow v = \frac{dx}{dt} = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right) \Rightarrow dx = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)dt.$$

Integrating we have $x = \frac{gm}{\mu}\left[t + \frac{m}{\mu}e^{-\mu t/m}\right] + B$ —(1)

B is a constant

But initially when $t=0, x=0$

$$\therefore 0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right]+B \quad \text{——} \; (2)$$

Subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu} + t\right] = \frac{gm^2}{\mu^2}\left\{e^{-(\mu t/m)} - 1 + \frac{\mu t}{m}\right\}$$

5(e) Show that for the curve $R = a(3t-t^3)I + 3at^2 J + a(3t+t^2)K$, the curvature equals torsion.

Sol^n:

6(a) Find the orthogonal trajectories of the family of co-axial circles $x^2+y^2+2gx+c=0$, where g is a parameter.

Sol^n: The given family of curves is

$$x^2+y^2+2gx+c=0, \text{ with } g \text{ as parameter.} \quad \text{—— (1)}$$

Differentiating (1) with respect to x, we get

$$2x+2y\frac{dy}{dx}+2g=0 \Rightarrow g=-\left(x+y\frac{dy}{dx}\right) \quad \text{—— (2)}$$

Eliminating g from (1) and (2), we get

$$x^2+y^2+2x\left(-x-y\frac{dy}{dx}\right)+c=0 \Rightarrow y^2-x^2-2xy\frac{dy}{dx}+c=0 \quad \text{—— (3)}$$

which is the differential equation of the given family of circles (1). Replacing dy/dx by $-dx/dy$ in (3) the differential equation of the required orthogonal trajectories is

$$y^2-x^2+2xy\frac{dx}{dy}+c=0 \Rightarrow 2x\frac{dx}{dy}-\frac{1}{y}x^2=-\frac{c}{y}-y \quad \text{—— (4)}$$

which can be reduced to linear differential equation as follows.

Putting $x^2=v$ so that $2x\frac{dx}{dy}=\frac{dv}{dy}$

(4) gives $\frac{dv}{dy}-\frac{1}{y}v=-\frac{c}{y}-y$ —— (5)

whose $I.F = e^{\int(-1/y)dy} = e^{-\log y} = (1/y)$

$\therefore$ Solution of (5) is $v(I.F) = \int\left(-\frac{c}{y}-y\right)\cdot(I.F)\,dy+d$

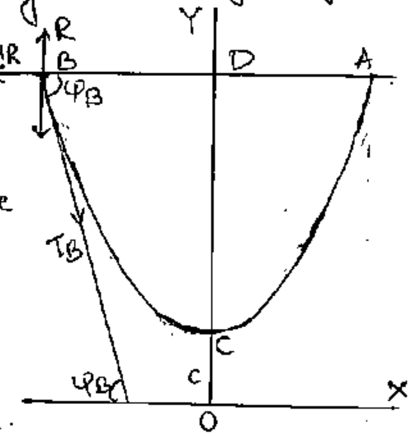
$$\Rightarrow v\cdot\frac{1}{y}=-\int\left(\frac{c}{y}+y\right)\frac{1}{y}dy+d=-\int(cy^{-2}+1)dy+d$$

$$\Rightarrow \frac{v}{y}=cy^{-1}-y+d \Rightarrow \frac{x^2}{y}=\frac{c}{y}-y+d \quad (\because v=x^2)$$

$$\Rightarrow x^2+y^2-dy-c=0, \; d \text{ being parameter.}$$

(b) A heavy chain, of length $2l$, has one end tied at A and other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\left\{\lambda+\sqrt{1+\lambda^2}\right\}$, where $\frac{1}{\lambda}=\mu(2n+1)$ and μ is the coefficient of friction.

Sol'n: Let one end of a heavy chain of length $2l$ be fixed at A and the other end be attached to a small heavy ring which can slide on a rough horizontal rod ADB through A. Let B be the position of limiting equilibrium of the ring when it is at greatest possible distance from A. In this position of limiting equilibrium the forces acting on the ring are: (i) the weight $2nlw$ of the ring acting vertically downwards, (ii) the normal reaction R of the rod, the force of limiting friction μR of the rod acting in the direction AB and (iv) the tension T_B in the string at B acting along the tangent to the string at B.



For the equilibrium of the ring at B, resolving the forces acting on it horizontally & vertically, we have

$$\mu R = T_B\cos\varphi_B \quad \text{—(1)}$$

and $$R = 2nlw + T_B\sin\varphi_B \quad \text{—(2)}$$

where φ_B is the angle of inclination of tangent at B to the horizontal.

Let C be the lowest point of catenary formed by the chain, OX be the directrix and $OC=c$ be the parameter. We have arc $CB = s_B = l$. By the formula $T\cos\varphi = wc$, we have $T_B \cos\varphi_B = wc$. Also by the formula $T\sin\varphi = ws$. we have $T_B \sin\varphi_B = ws_B = wl$.

Putting these values in (1) and (2), we have

$$\mu R = wc \text{ and } R = 2nlw + wl = (2n+1)wl$$

$$\therefore \mu(2n+1)wl = wc \Rightarrow \mu(2n+1)l = c$$

But it is given that $\mu(2n+1) = 1/\lambda$ —— (3)

Using the formula $s = c\tan\varphi$ for the point B, we have

$$l = c\tan\varphi_B$$

$$\therefore \tan\varphi_B = l/c = \lambda \quad \text{——} \quad (4)$$

Now the required greatest possible distance of the ring from A = AB = 2DB = $2x_B$

$$= 2c\log(\sec\varphi_B + \tan\varphi_B) \quad [\because x = c\log(\sec\varphi + \tan\varphi)]$$

$$= 2c\log[\tan\varphi_B + \sqrt{(1+\tan^2\varphi_B)}]$$

$$= \frac{2l}{\lambda}\log[\lambda + \sqrt{(1+\lambda^2)}]$$

$[\because$ from (3), $c = \frac{l}{\lambda}$ & from (4), $\tan\varphi_B = \lambda]$

6(c) (i) A person going eastwards with a velocity of 4 km per hour, finds the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.

(ii) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 = -4$ at $(-1,2,1)$?

Sol: (i) Let the actual velocity of the wind be

$xi + yj$, where i, j represents velocities of 1 km. per hour towards the east and north respectively. As the person is going eastwards with a velocity of 4 km per hour, his actual velocity is $4i$.

Then the velocity of the wind relative to the man is $(xi+yj) - 4i$, which is parallel to $-j$, as it appears

to blow from the north. Hence $x=4$.
when the velocity of the person becomes $8i$, the velocity of the wind relative to man is $(xi+yj)-8i$,
But this is parallel to $-(i+j)$.

$\therefore \quad (x-8)/y = 1$, which by ① gives $y=-4$

Hence the actual velocity of the wind is $4(i-j)$ i.e. $4\sqrt{2}$ km. per hour towards the south-east.

(ii) we have $\phi = xy^2 + yz^3$.

$$\Rightarrow \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = y^2 i + (2xy+z^3)j + 3yz^2 k.$$

$$\nabla\phi\Big|_{(2,-1,1)} = i - 3j - 3k \quad \text{—①}$$

and also $\nabla(x\log z - y^2) = i(\log z) - 2yj + \frac{x}{z}$

$$\nabla(x\log z - y^2 + 4)\Big|_{(-1,2,1)} = -4j - k. \quad \text{—②}$$

But ① is normal to surface $x\log z - y^2 = -4$

$$\therefore \hat{a} = \frac{\nabla(x\log z - y^2 + 4)}{|\nabla(x\log z - y^2 + 4)|}$$

$$= \frac{-4j-k}{|-4j-k|} = \frac{-4j-k}{\sqrt{16+1}} = \frac{-4j-k}{\sqrt{17}}$$

$\therefore$ The directional derivative of ϕ at $(2,-1,1)$ in the direction of normal to the surface =

$$= \nabla\phi\cdot\hat{a} = (i-3j-3k)\cdot\left(\frac{-4j-k}{\sqrt{17}}\right) = 15/\sqrt{17}.$$

7(a) Solve $\left[(x+1)^2D^2+(x+1)D-1\right]y = \ln(x+1)^2+x-1$

Sol^n: Let $1+x=v \Rightarrow \frac{dy}{dx} = \frac{dy}{dv}$.

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dv^2}$$

$\therefore$ $v^2\frac{d^2y}{dv^2}+v\frac{dy}{dv}-y = \ln v^2+v-2$

$v^2\frac{d^2y}{dv^2}+v\frac{dy}{dv}-y = 2\log v+v-2$ ——(1)

Now put $v=e^z \Rightarrow z=\log v$

and let $D_1 = d/dz$.

$\therefore$ (1) gives

$\left[D_1(D_1-1)+D_1-1\right]y = 2z+e^z-2$

i.e. $(D_1^2-1)y = 2z+e^z-2$ ——(2)

A.E. of (2) is $D_1^2-1=0$

$\Rightarrow D_1 = \pm 1$

$\therefore$ C.F $= c_1e^z+c_2e^{-z}$

$= c_1v+c_2v^{-1} = c_1(1+x)+c_2(1+x)^{-1}$.

$P.I = \frac{1}{D_1^2-1}(2z+e^z-2) = \frac{1}{(D_1+1)(D_1-1)}e^z+\frac{1}{D_1^2-1}2(z-1)$

$= \frac{1}{2}\frac{1}{D_1-1}e^z-(1-D_1^2)^{-1}2[z-1]$

$= \frac{1}{2}ze^z-(1+D_1^2+\cdots)2(z-1)$

$= \frac{1}{2}ze^z-2(z-1) = \frac{ze^z}{2}+(2-2z)$

$= \frac{v\log v}{2}+2(1-\log v)$

$\therefore$ $y =$ C.F $+$ P.I

$= c_1(1+x)+c_2(1+x)^{-1}+\frac{(1+x)\log(1+x)}{2}+2\left[1-\log(1+x)\right]$

which is the required solution.

7(b). Solve by using the method of variation of parameters $d^2y/dx^2 - 2(dy/dx) = e^x \sin x$.

Sol'n: Given $(D^2 - 2D)y = e^x \sin x$, where $D \equiv d/dx$ — (1)

Consider $(D^2-2D)y = 0$ — (2)

Auxiliary equation of (2) is $D^2-2D=0$ so that $D=0,2$

$$\text{C.F. of (1)} = C_1 + C_2 e^{2x} \quad \text{— (3)}$$

Let $u=1$, $v=e^{2x}$ and $R = e^x \sin x$ — (4)

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$ — (5)

Then P.I of (1) $= u f(x) + v g(x)$ — (6)

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{e^{2x} e^x \sin x}{2e^{2x}} = -\frac{1}{2}\int e^x \sin x \, dx$

$= -\frac{1}{2} \frac{e^x}{1^2+1^2}(\sin x - \cos x)$ $\quad \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$

$= -\left(\frac{1}{4}\right) e^x (\sin x - \cos x)$

and $g(x) = \int \frac{uR}{W} dx = \int \frac{e^x \sin x}{2e^{2x}} dx = \frac{1}{2}\int e^{-x} \sin x \, dx$

$= \frac{1}{2} \frac{e^{-x}}{(-1)^2+1^2}\left\{(-1)\sin x - \cos x\right\} = -\frac{e^{-x}}{4}(\sin x + \cos x)$

$\therefore$ P.I of (1) $= \left(-\frac{1}{4}\right) e^x (\sin x - \cos x) + e^{2x}\left(-\frac{1}{4}\right)e^{-x}(\sin x + \cos x)$, by (6)

$= -\frac{1}{4} e^x \left\{(\sin x - \cos x) + (\sin x + \cos x)\right\} = -\frac{1}{2} e^x \sin x$

Hence the required general solution is $y = \text{C.F} + \text{P.I}$

i.e, $y = C_1 + C_2 e^{2x} - \frac{1}{2}e^x \sin x$, C_1, C_2 being arbitrary constants.

7(c) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$

Sol: Here the central acceleration varies inversely as the cube of the distance i.e., $P = \mu/r^3 = \mu u^3$, where μ is a constant. If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

$$V = \sqrt{(\mu/a^2)}$$

$\therefore$ the velocity of projection $v = \sqrt{2}V = \sqrt{(2\mu/a^2)}$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying bothsides by $2(du/d\theta)$ and integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \quad \text{— (1)}$$

where A is a constant.

But initially when $r = a$, i.e. $u = 1/a$, $du/d\theta = 0$ (at an apse)

and $v = \sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$\therefore h^2 = 2\mu$ and $A = \mu/a^2$

substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2u^2}{a^2}$$

$$\Rightarrow \sqrt{2}\,a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)} \Rightarrow \frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}$$

Integrating, $(\theta/\sqrt{2}) + B = \sin^{-1}(au)$, where B is a constant.

But initially, when $u = 1/a$, $\theta = 0$, $\therefore$ $B = \sin^{-1} 1 = \frac{1}{2}\pi$

$\therefore (\theta/\sqrt{2}) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au) \Rightarrow au = a/r = \sin\left\{\frac{1}{2}\pi + (\theta/\sqrt{2})\right\}$

$\Rightarrow a = r\cos(\theta/\sqrt{2})$, which is required equation of the path.

8(a) By using Laplace transform method

Solve $(D^2+2D+5)y = e^{-t}\sin t$, $y(0)=0$, $y'(0)=1$.

Sol'n: Rewriting the given equation and conditions, we have $y''+2y'+5y = e^{-t}\sin t$ —— ①

with initial conditions: $y(0)=0$, and $y'(0)=1$ —— ②

Taking Laplace transform of bothsides of ① we get

$$L(y'')+2L(y')+5(y) = L(e^{-t}\sin t)$$

$$\Rightarrow s^2L(y)-sy(0)-y'(0)+2[sL(y)-y(0)]+5L(y) = f(s+1)$$

[by using shifting theorem taking $f(s)=L\{\sin t\}=\frac{1}{s^2+1}$]

$$\Rightarrow (s^2+2s+5)L(y)-0-1-(2\times 0) = \frac{1}{[(s+1)^2+1]}, \text{ by } ②$$

$$\Rightarrow L(y) = \frac{1}{(s^2+2s+5)} + \frac{1}{(s^2+2s+5)[(s+1)^2+1]}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{(s+1)^2+4} + \frac{1}{((s+1)^2+4)((s+1)^2+1)}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{(s^2+4)} + \frac{1}{(s^2+4)(s^2+1)}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{s^2+4} + \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{3}\times\frac{1}{s^2+1} + \frac{2}{3}\times\frac{1}{s^2+4}\right\}$$

$$= e^{-t}\left[\frac{1}{3}\sin t + \frac{2}{3}\times\frac{1}{2}\sin 2t\right]$$

$\therefore$ $y = \frac{1}{3}\times e^{-t}(\sin t+\sin 2t)$ is required solution.

8(b) A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Sol^n: Let a particle start from rest from the cusp A of the cycloid. The velocity v of the particle at any point P, at time t, given by

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}\left(16a^2 - s^2\right)$$

$\Rightarrow \frac{ds}{dt} = -\frac{1}{2}\sqrt{g/a}\sqrt{16a^2-s^2}$, the $-$ve sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore dt = -2\sqrt{a/g}\,\frac{ds}{\sqrt{16a^2-s^2}} \quad \text{—— (1)}$$

The vertical height of the cycloid 2a. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y = a$, putting $y = a$ in the equation $s^2 = 8ay$, we get $s^2 = 8a^2 \Rightarrow s = 2\sqrt{2}a$

$\therefore$ Integrating (1) from $s = 4a$ to $s = 2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1 = -2\sqrt{a/g}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{16a^2-s^2}} = 2\sqrt{a/g}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2}a}$$

$$= 2\sqrt{a/g}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a} - \cos^{-1}1\right] = 2\sqrt{a/g}\left[\cos^{-1}\frac{1}{\sqrt{2}} - \cos^{-1}1\right]$$

$$= 2\sqrt{a/g}\left[\tfrac{1}{4}\pi - 0\right] = \tfrac{1}{2}\pi\sqrt{a/g}.$$

Again integrating (1) from $s = 2\sqrt{2}a$ to $s = 0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2 = -2\sqrt{a/g}\int_{s=2\sqrt{2}a}^{0}\frac{ds}{\sqrt{16a^2-s^2}}$$

$= 2\sqrt{(a/g)} \cdot \left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0} = 2\sqrt{(a/g)}\left[\cos^{-1}0 - \cos^{-1}\frac{1}{\sqrt{2}}\right]$

$= 2\sqrt{(a/g)}\left[\frac{1}{2}\pi - \frac{1}{4}\pi\right] = \frac{1}{2}\pi\sqrt{(a/g)}$

Hence $t_1 = t_2$ i.e. the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Verify Stoke's theorem for $F=(x^2+y-4)i+3xyj+(2xz+z^2)k$ where S is the upper half of the sphere $x^2+y^2+z^2=16$ and C is its boundary.

Sol^n: The boundary C of S is the circle $x^2+y^2=16$, $z=0$ lying in the xy-plane. Suppose $x=4\cos t$, $y=4\sin t$, $z=0$, $0\le t\le 2\pi$ are parametric equations of C. Then

$$\oint_C F\cdot dr = \oint_C [(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)k]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$=\oint_C [(x^2+y-4)\,dx+3xy\,dy+(2xz+z^2)\,dz]$$

$$=\oint_C (x^2+y-4)dx+3xy\,dy \quad (\because \text{ on } C\ dz=0)$$

$$=\int_0^{2\pi}\left[(x^2+y-4)\frac{dx}{dt}+3xy\frac{dy}{dt}\right]dt$$

$$=\int_0^{2\pi}[(16\cos^2 t+4\sin t-4)(-4\sin t) + 3\cdot 16\sin t\cos t\cdot 4\cos t]\,dt$$

$$=128\int_0^{2\pi}\cos^2 t\sin t\,dt-16\int_0^{2\pi}\sin^2 t\,dt+\int_0^{2\pi}\sin t\,dt$$

$$=128\cdot(0)-16\cdot(4)\int_0^{\pi/2}\sin^2 t\,dt+16(0)$$

$$=-64\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)=-16\pi \quad \text{---} (1)$$

Now let us evaluate $\iint_S \text{curl} F\cdot \hat{n}\, ds$.

$$\text{curl} F=\nabla\times F=\begin{vmatrix} i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}$$

$$=-2z\hat{j}+(3y-1)k$$

If S_1 is the plane region bounded by the circle C, then by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl}F\cdot\hat{n}\,ds=\iiint_V \text{div}(\text{curl}F)\,dV=0$$

$(\because \text{div}(\text{curl} F)=0)$

[Here S' is the surface consisting of S and S_1. The S' is closed surface and let V be the volume bounded by S']

$$\therefore \iint_S \text{curl}\, F\cdot\vec{n}\, ds + \iint_{S_1} (\text{curl}\, F\cdot\vec{n})\, ds = 0.$$

$$\Rightarrow \iint_S \text{curl}\, F\cdot\vec{n}\, ds = \iint_{S_1} \text{curl}\, f\cdot k\, ds \qquad \left(\because \text{on } S_1,\ \vec{n} = -\hat{k}\right)$$

$$\therefore \qquad = \iint_{S_1} (3y-1)\, ds$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} (3r\sin\theta - 1)\, r\, d\theta\, dr$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} 3r^2\sin\theta\, d\theta\, dr - \int_0^{2\pi}\int_0^4 r\, d\theta\, dr$$

$$= 0 - \int_{\theta=0}^{2\pi} \left(\frac{r^2}{2}\right)_0^4 d\theta \qquad \left(\because \int_0^{2\pi} \sin\theta = 0\right)$$

$$= -8\,[\theta]_0^{2\pi}$$

$$= -8(2\pi)$$

$$= -16\pi$$

Main Test Series – 2016
Test – 06 – Answer key
Paper – II

1(a) (i) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \in S_4$. Find the smallest positive integer k such that $\alpha^k = e$.

(ii) In S_6, let $\rho = (123)$ and $\sigma = (456)$. Find a permutation x in S_6 such that $x\rho x^{-1} = \sigma$.

Sol^n: (i) $\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$\alpha^3 = \alpha \cdot \alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$

$\alpha^4 = \alpha \cdot \alpha^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$\therefore \alpha^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = e$.

$\therefore$ The smallest positive integer $k = 4$.

(ii) $\because x\rho x^{-1} = \sigma$

$\Rightarrow x(123)x^{-1} = \sigma$

$\Rightarrow (x(1)\ x(2)\ x(3)) = (4\ 5\ 6)$

$\Rightarrow x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$ (on comparison of both sides same order cycles)

$= (14)(25)(36)$

Check: $x(123)x^{-1} = (x(1)\ x(2)\ x(3))$

$= (4\ 5\ 6)$

There are fewmore possibilites to get x, 1(5) Is $Z[\sqrt{-6}] = \{a+b\sqrt{-6} / a, b \in Z\}$ Euclidean domain? Justify your answer.

Sol^n: Let, if possible $Z[\sqrt{-6}]$ be a Euclidean domain. Consequently an element in $Z[\sqrt{-6}]$ is prime if and only if it is irreducible.

For example: $5 \in Z[\sqrt{-6}]$ is irreducible but not prime.

To show that 5 is an irreducible element:

Let $5 = (a+b\sqrt{6}i)(c+d\sqrt{6}i)$ — ①; $a, b, c, d \in Z[\sqrt{-6}]$

Taking conjugates on both sides, we get

$5 = (a-b\sqrt{6}i)(c-d\sqrt{6}i)$ — ②

Multiplying ① and ②, we get

$25 = (a^2+6b^2)(c^2+6d^2)$

we have the following three cases

(1) $a^2+6b^2=1$ and $c^2+6d^2=25$

(2) $a^2+6b^2=25$ and $c^2+6d^2=1$

(3) $a^2+6b^2=5$ and $c^2+6d^2=5$

It is clear that case (3) is not possible in Z.

case(1) is possible when $a=\pm 1$, $b=0 \Rightarrow a+b\sqrt{6}i=\pm 1$

similarly, case(2) yields that $c+d\sqrt{6}i=\pm 1$

which are units in $Z[\sqrt{-6}]$.

Hence 5 is an irreducible element of $Z[\sqrt{-6}]$.

To show that 5 is not a prime element of $Z[\sqrt{-6}]$:-

we know that
$Z[\sqrt{-6}]=\{a+b\sqrt{6}i \,/\, a,b\in Z$ and $i=\sqrt{-1}\}$
is an integral domain with unity.

$(2+\sqrt{6}i)$, $(2-\sqrt{6}i)\in Z[\sqrt{-6}]$ and

$(2+\sqrt{6}i)(2-\sqrt{6}i)=10$.

obviously, 5 divides $(2+\sqrt{6}i)(2-\sqrt{6}i)=10$ but doesnot divide $2+\sqrt{6}i$, $2-\sqrt{6}i$.

For if 5 divides $2+\sqrt{6}i$, then $2+\sqrt{6}i=5(a+b\sqrt{6}i)$ for some $a,b\in Z$.

Consequently, $5a=2$ $(a\in Z)$ which is impossible.

Similarly, 5 doesnot divide $2-\sqrt{6}i$.

Hence 5 is not prime in $Z[\sqrt{-6}]$

$5\in Z[\sqrt{-6}]$ is an irreducible but not prime.

$\therefore Z[\sqrt{-6}]$ is not an Euclidean domain

1(c) (i) Give an example of an infinite set which is not bounded and having limit point.

(ii) Give an example to show that the intersection of an infinite collection of open sets is not necessarily an open set.

Sol^n: (i) The set of Rational numbers $\mathbb{Q}$ is infinite which is unbounded and having the limit points $Q' = \mathbb{R}$

i.e., every real number is a limit point of the set $\mathbb{Q}$.

(ii) Let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$

Then $\{I_n\}_{n\in\mathbb{N}}$ is an infinite family of open sets.

$\bigcap\limits_{n\in\mathbb{N}} I_n = \{0\}$, which being a non-empty finite set is not an open set.

1(d) If $f(z) = u+iv$ is an analytic function of $z = x+iy$ and $u-v = e^x(\cos y - \sin y)$. Find $f(z)$ in terms of z.

Sol^n: $f(z) = u+iv$.

$\Rightarrow if(z) = iu - v$.

Adding, we have

$(u-v) + i(u+v) = (1+i)f(z) = F(z)$ say

put $u-v = U$ and $u+v = V$.

then $F(z) = U + iV$.

Now $U = e^x(\cos y - \sin y)$

$\therefore \frac{\partial u}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$

and $\frac{\partial u}{\partial y} = e^x(-\sin y - \cos y) = \phi_2(x,y)$

By Milne's method, we have

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

$= (e^z + ie^z) = (1+i)e^z$.

Integrating, we get

$f(z) = \int (1+i)\, e^z\, dz + C$

i.e., $(1+i) f(z) = (1+i)e^z + C$

$f(z) = e^z + \frac{C}{1+i}$

$= e^z + C_1$, where $C_1 = \frac{C}{1+i}$.

$\therefore \boxed{f(z) = e^z + C_1}$

(e) Obtain the dual of the LP problem:

Min $z = x_1 + x_2 + x_3$

subject to ~~the~~ constraints:

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \le 3$

$2x_2 - x_3 \ge 4$

$x_1, x_2 \ge 0$, and x_3 is unrestricted.

Sol: Since the problem is of minimization type, all constraints should be of ($\ge$) type. multiply the second constraint throughout by -1, so that $-x_1 + 2x_2 \ge -3$.

and we write the first equality constraint in the form of two inequalities of $\ge$ type.

$\therefore$ The given problem can be written as

Minimize $Z = x_1 + x_2 + x_3$

subject to

$$\left.\begin{aligned} x_1 - 3x_2 + 4x_3 &\geq 5 \\ -x_1 + 3x_2 - 4x_3 &\geq -5 \\ -x_1 + 2x_2 \quad &\geq -3 \\ 2x_2 - x_3 &\geq 4. \end{aligned}\right\} \longrightarrow (1)$$

$x_1, x_2 \geq 0$, x_3 is unrestricted.

Since x_3 is unrestricted.

put $x_3 = x_3' - x_3''$

The equation (1) can be written as

Min $Z = x_1 + x_2 + x_3' - x_3''$

subject to

$$\begin{aligned} x_1 - 3x_2 + 4x_3' - 4x_3'' &\geq 5 \\ -x_1 + 3x_2 - 4x_3' + 4x_3'' &\geq -5 \\ -x_1 + 2x_2 \quad &\geq -3 \\ 2x_2 - x_3' + x_3'' &\geq 4. \end{aligned}$$

$x_1, x_2, x_3', x_3'' \geq 0.$

Let y_1, y_2, y_3 and y_4 be the dual variables associated with the above 4 constraints.

Then the dual is given by

Maximize $W = 5y_1 - 5y_2 - 3y_3 + 4y_4$

subject to

$$\begin{aligned} y_1 - y_2 - y_3 + 0y_4 &\leq 1 \\ -3y_1 + 3y_2 + 2y_3 + 2y_4 &\leq 1 \\ 4y_1 - 4y_2 \quad - y_4 &\leq 1 \\ -4y_1 + 4y_2 \quad + y_4 &\leq -1 \end{aligned}$$

$y_i \geq 0.$

This dual can be written in more compact form as:

Max $w = 5y' - 3y_3 + 4y_4$

Subject to

$y' - y_3 \le 1$

$-3y' + 2y_3 + 2y_4 \le 1$

$4y' - \quad - y_4 \le 1$

$-4y' \quad + y_4 \le -1$

$y', y_3, y_4 \ge 0$ and $y' (= y_1 - y_2)$ unrestricted

(or)

Max $w = 5y' - 3y_3 + 4y_4$

subject to

$y' - y_3 \le 1$

$-3y' + 2y_3 + 2y_4 \le 1$

$4y' - y_4 = 1$

$y_3, y_4 \ge 0$ and y' is unrestricted.

2(a) If R is a ring with identity such that $(xy)^2 = x^2y^2$ $\forall x, y \in R$ then show that R is commutative. Set an example to show that the above result may be false if R does not have an identity.

Sol: Let R be a ring with identity such that

$(xy)^2 = x^2y^2 \;\forall x, y \in R$ — (1)

Replacing y by $y+1$ in (1), we get

$[x(y+1)]^2 = x^2(y+1)^2$

$\Rightarrow (xy+x)^2 = x^2(y^2+2y+1)$

$\Rightarrow (xy)^2 + (xy)x + x(xy) + x^2 = x^2y^2 + 2x^2y + x^2.$

Using (1) and cancellation laws of $(R,+)$ in (2), we get

$xyx + x^2y = 2x^2y$

$\Rightarrow xyx = x^2y \;\; \forall x,y \in R$ —— (3)

Replacing x by $x+1$ in (3),

$(x+1)y(x+1) = (x+1)^2y$

$\Rightarrow (x+1)(yx+y) = (x+1)(xy+y)$

$\Rightarrow xyx + xy + yx + y = x^2y + xy + xy + y$ —— (4)

Using (3) and cancellation laws of $(R,+)$ in (4), we get

$yx = xy \;\; \forall x,y \in R.$

Hence R is a commutative ring.

Consider the ring R of 2×2 matrices:

$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} / a, b \text{ are integers} \right\}$

Clearly R is not commutative,

Since $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

The possible unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin R$.

Let $X = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R$ be arbitrary

Then $XY = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix}$, $X^2 = \begin{pmatrix} a^2 & ab \\ 0 & 0 \end{pmatrix}$, $Y^2 = \begin{pmatrix} c^2 & cd \\ 0 & 0 \end{pmatrix}$

and $(XY)^2 = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix}\begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^2c^2 & a^2cd \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^2c^2 & a^2cd \\ 0 & 0 \end{pmatrix} = X^2Y^2.$

2(b) Give an example of a homomorphism $f: R \to R'$ such that 1 is the unity of R, but $f(1)$ is not the unity of R'.

Sol: $f: Z \to Z$ defined as
$f(x) = 0 \ \forall x \in Z$ is a homomorphism.

Since, for any $x, y \in Z$, we see that
$f(x+y) = 0 = 0 + 0 = f(x) + f(y)$
and $f(xy) = 0 = 0 \cdot 0 = f(x) \cdot f(y)$
Thus f is homomorphism
and 1 is the unity of Z, but
$f(1) = 0$ is not the unity of Z.
observe that f is not onto.

2(c) Let the function f be defined on $[0,1]$ as follows:
$f(x) = 2rx$ when $\frac{1}{r+1} < x \le \frac{1}{r}$, $r = 1, 2, 3 \ldots$
Prove that f is R-integrable on $[0,1]$ and evaluate $\int_0^1 f(x)\,dx$.

Sol: Here $f(x) = 2x$, when $\frac{1}{2} < x \le 1$
$f(x) = 4x$, when $\frac{1}{3} < x \le \frac{1}{2}$
$= 6x$, when $\frac{1}{4} < x \le \frac{1}{3}$
$\vdots$
$= 2(n-1)x$, when $\frac{1}{n} < x \le \frac{1}{n-1}$

Thus we notice that f is bounded and continuous on $[0,1]$ except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$

The set of points of discontinuity of f on $[0,1]$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on $[0,1]$ has a finite no. of limit points, therefore, f is integrable on $[0,1]$.

$$\text{Now} \int_{1/n}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + \cdots + \int_{1/n}^{1/(n-1)} f(x)\,dx.$$

$$= \sum_{r=1}^{n-1} \int_{1/(r+1)}^{1/r} 2rx\,dx = \sum_{r=1}^{n-1} \left[rx^2\right]_{1/(r+1)}^{1/r}$$

$$= \sum_{r=1}^{n-1} r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$= \sum_{r=1}^{n-1} \frac{2(r+1)}{r(r+1)^2}$$

$$= \sum_{r=1}^{n-1} \left[\frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}\right]$$

$$= \sum_{r=1}^{n-1} \left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n-1} \frac{1}{(r+1)^2}$$

$$= \left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] + \left[\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}\right]$$

$$= \left(1 - \frac{1}{n}\right) + \frac{\pi^2}{6} - 1 \quad \left(\because \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = \frac{\pi^2}{6}\right)$$

proceeding to the limit as $p \to \infty$, we get

$$\int_0^1 f(x)\,dx = \frac{\pi^2}{6}.$$

2(d) Use the method of contour integration to prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^3(b+c)^2} \quad \text{where } b>0,\ c>0$$

Sol: Consider the integral $\int_C f(z)\,dz$

where $f(z) = \frac{1}{(z^2+b^2)(z^2+c^2)^2}$

taken round a closed contour C, consisting of the upper half of a large circle $|z|=R$ and the part of the real axis from $-R$ to R.

poles of $f(z)$ are given by $(z^2+b^2)(z^2+c^2)^2=0$

i.e. $z=\pm ib$ are two simple poles

$z=\pm ic$ are the two double poles

The poles which lie within the contour are: a simple pole at $z=ib$ and a double pole at $z=ic$.

Residue at the simple pole $z=ib$ is

$$\lim_{z\to ib}(z-ib)\,f(z) = \lim_{z\to ib}(z-ib)\frac{1}{(z-ib)(z+ib)(z^2+c^2)^2}$$

$$= \frac{1}{2ib(c^2-b^2)^2} = \frac{-i}{2b(b^2-c^2)^2}$$

To find the residue at the double pole $z=ic$, put $z-ic=t \Rightarrow z=t+ic$ in $f(z)$

then it becomes $\dfrac{1}{\{(ic+t)^2+b^2\}\{(ic+t)^2+c^2\}^2}$

$$= \frac{1}{(b^2-c^2+2ict+t^2)(2ict+t^2)^2}$$

$$= \frac{\left\{1+\frac{2ict+t^2}{b^2-c^2}\right\}^{-1}\left\{1+\frac{t}{2ic}\right\}^{-2}}{(b^2-c^2)(-4c^2t^2)}$$

$$= \frac{\left\{1-\frac{2ict+t^2}{b^2-c^2}+\cdots\right\}\left\{1-\frac{t}{ic}+\cdots\right\}}{-4c^2(b^2-c^2)t^2}$$

in which the coefficient of $\frac{1}{t}$ is

$$= \frac{1}{-4c^2(b^2-c^2)}\left\{\frac{c}{c}\cdots\frac{2ic}{b^2-c^2}\right\}$$

i.e. $\dfrac{(3c^2-b^2)i}{4c^3(b^2-c^2)^2}$

which is the residue at $z=ic$.

$\therefore$ sum of the residues

$$= \frac{-i}{2b(b^2-c^2)} + \frac{(3c^2-b^2)i}{4c^3(b^2-c^2)^2}$$

$$= \frac{-i(b+2c)}{4bc^3(b+c)^2}$$

Hence by Cauchy Residue theorem,

we have $\int f(z)\,dz = 2\pi i \times$ sum of the residues within the contour

i.e., $\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i$ (sum of the residues)

i.e., $\displaystyle\int_{-R}^{R}\frac{1}{(x^2+b^2)(x^2+c^2)^2}dx + \int_{C_R}\frac{1}{(z^2+b^2)(z^2+c^2)^2}dz = 2\pi i\,\frac{-i(b+2c)}{4bc^3(b+c)^2}$; (1)

Now $\left|\int_{C_R} \frac{1}{(z^2+b^2)(z^2+c^2)^2}dz\right| \le \int_{C_R} \frac{|dz|}{|z^2+b^2||z^2+c^2|^2}$

$$\le \int_{C_R} \frac{|dz|}{(|z|^2-b^2)(|z|^2-c^2)^2}$$

$$= \frac{1}{(R^2-b^2)(R^2-c^2)^2}\int_0^{\pi} R\,d\theta \quad \text{since } z = Re^{i\theta}$$

$$= \frac{\pi R}{(R^2-b^2)(R^2-c^2)^2} \text{ which} \to 0 \text{ as } R\to\infty$$

Hence by making $R\to\infty$ relation ① becomes

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{\pi(b+2c)}{2bc^3(b+c)^2}$$

3(a) Let G be the group $\left\{\begin{bmatrix}1 & a\\ 0 & b\end{bmatrix} \middle/ a, b\in\mathbb{R}, b\neq 0\right\}$ and $H = \left\{\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} \middle/ x\in\mathbb{R}\right\}$. Show that H is a subgroup of G. Is H a normal subgroup of G? Justify.

Sol: To show that H is a subgroup of G.

Let $A = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\in H$, $B = \begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\in H$.

$$A\cdot B = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}$$

$\therefore AB\in H$.

$\Rightarrow$ $\therefore H$ is closed.

Also $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}$ which is in H.

Thus H is a subgroup of G.

Since

$$\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}^{-1} = \begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -ab^{-1}\\ 0 & b^{-1}\end{pmatrix}$$

$$= \begin{pmatrix}1 & b^{-1}x\\ 0 & 1\end{pmatrix} \in H$$

$\Rightarrow$ H is normal in G.

3(b)(i) Discuss the convergence of the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right)$.

(ii) If $x>0$, show that $\frac{x}{1+x} < \log(1+x) \le x$.

Sol^n: Here $1+a_n = 1+\frac{x^n}{x^{2n}+1}$

so that $a_n = \frac{x^n}{x^{2n}+1}$

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{x^n}{x^{2n}+1}\cdot\frac{x^{2n+2}+1}{x^{n+1}}\right|$$

$$= \frac{x^{2n+2}+1}{x(x^{2n}+1)}$$

$\therefore$ If $|x|<1$, $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{|x|} > 1$

By ratio test, $\sum|a_n|$ converges and hence $\prod(1+a_n)$ converges absolutely.

If $|x|>1$, $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}+1}{x^{2n+1}+x}\right|$

$$= \lim_{n\to\infty}\left|\frac{x+\frac{1}{x^{2n+1}}}{1+\frac{1}{x^{2n}}}\right| = |x| > 1$$

By ratio test, $\sum|a_n|$ converges and hence $\prod(1+a_n)$ converges absolutely.

If $x=1$, $a_n = \frac{1}{2}$ so that the series Σa_n is divergent and hence $\Pi(1+a_n)$ is also divergent

If $x=-1$, the product becomes

$(1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{2})(1+\frac{1}{2})\cdots\cdots$

which diverges to zero.

(iii) Let $f(t) = \log(1+t)$ $\forall t \in [0,x]$.
where $x>0$.

It is continuous and differentiable in $[0,x]$.

and $f'(t) = \frac{1}{1+t}$ $\forall t \in (0,x)$

By Lagrange's Mean value theorem $\exists$ $c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - 0}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{---(1)}$$

Since $c \in (0,x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$

$\Rightarrow \frac{x}{1+x} < \log(1+x) < x$

3(c) Solve the following LPP by simplex method.

$\text{Max } Z = -2x_1 - x_2$

subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Sol^n: Introducing slack, surplus and artificial variables, the system of constraint equations become:

$3x_1 + x_2 \quad + A_1 = 3$

$4x_1 + 3x_2 - S_1 \quad + A_2 = 6$

$x_1 + 2x_2 + S_2 \quad = 4$

$S_i, x_i, A_i \geq 0.$

Assign the large negative when place $-M$ to the artificial variables A_1 and A_2, the objective function becomes

$\text{Max } Z^* = -2x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2.$

Now the IBFS is given by

setting $x_1 = x_2 = S_1 = 0$, (non-basic)

$A_1 = 3,\ A_2 = 6,\ S_2 = 4$ (basic)

and $Z^* = -9M$.

Now put the above information in the simplex tableau.

	C_j	-2	-1	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	A_1	A_2	b_i	θ
$-M$	A_1	(3)	1	0	0	1	0	3	$\frac{3}{3}=1 \rightarrow$
$-M$	A_2	4	3	-1	0	0	1	6	$\frac{6}{4}=\frac{3}{2}$
0	S_2	1	2	0	1	0	0	4	$4/1=4$
$Z_j = \Sigma C_B a_{ij}$		$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$	
$C_j = c_j - Z_j$		$-2+7M$ $\uparrow$	$-1+4M$	$-M$	0	0	0		

from the above table, x_1 is the entering variable, and A_1 is outgoing variable and omit its column in the next table and (3) is the key element, make it unity and all other elements in its column equal to zero.

Then the revised simplex table is

	C_j	-2	-1	0	0	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	A_2	b	
-2	x_1	1	$1/3$	0	0	0	1	$\frac{1}{1/3}=3$
$-M$	A_2	0	$(5/3)$	1	0	1	2	$\frac{2}{5/3}=\frac{6}{5} \rightarrow$
0	S_2	0	$5/3$	0	1	0	3	$\frac{3}{5/3}=\frac{9}{5}$
$Z_j=\sum C_B a_{ij}$		-2	$\frac{-2}{3}-\frac{5M}{3}$	$-M$	0	$-M$		
$C_j=c_j-Z_j$		0	$\frac{-1+5M}{3}$ $\uparrow$	M	0	0		

$-1+\frac{5}{3}$

from the above table, x_2 is the entering variable and A_2 is outgoing variable and omit its column in the next table and $(5/3)$ is the key element, make it unity and all other elements in its column equal to zero. Then the revised simplex table is

	C_j	-2	-1	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	b
-2	x_1	1	0	$1/5$	0	$3/5$
-1	x_2	0	1	$-3/5$	0	$6/5$
0	S_2	0	0	1	1	1
$Z_j=\sum C_B a_{ij}$		-2	-1	$+\frac{1}{5}$	0	
$C_j=c_j-Z_j$		0	0	$-\frac{1}{5}$	0	

As C_j either zero or negative (ie, ≤ 0) under all columns, the above table gives the optimal basic feasible solution

$\therefore$ The optimal solution is

$x_1 = 3/5$, $x_2 = 6/5$. Max $z = -12/5$.

4(a)(i) Let R denote the ring of all real-valued continuous functions on the closed interval $[0,1]$. Is (0) a prime ideal of R? Justify

(ii) In $Z/(8)$, the ring of integers modulo 8, is the ideal generated by $\bar{2} = 2+(8)$ a prime ideal? Is it also maximal?

Sol: (i) we know that R is a commutative ring with unity.

It may be remarked that R is not an integral domain as shown below:

Let $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$ & $g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$

Then $f \in R$ and $g \in R$, where $f \neq 0$ and $g \neq 0$. but $fg = 0$.

Let if possible, (0) be a prime ideal of R

Let $f, g \in R$ be such that $fg = 0$. Then $fg \in (0)$ and so $f \in (0)$ or $g \in (0)$, since (0) is a prime ideal of R. Thus $f = 0$ or $g = 0$, which means that R is an integral domain, a contradiction. Hence (0) is not a prime ideal of R.

(ii) Let $R = \mathbb{Z}/8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

Then R is a commutative ring with unity. The ideal $(\bar{2}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is a maximal ideal of R, since there doesnot exist any proper ideal between $(\bar{2})$ and R. Since $(\bar{2})$ is a maximal ideal of R, a commutative ring with unity, $(\bar{2})$ is also a prime ideal of R.

4(b) prove that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

Solⁿ: Let $f_n(x) = \frac{x^n}{n^2}$

$|f_n(x)| = \left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2} = M_n$ for $0 \le x \le 1$

Since $\sum_{n=1}^{\infty} M_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore$ By weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent for $0 \le x \le 1$.

The series can be integrated term by term.

$$\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2} dx$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

$$\therefore \int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

4(c) find the Taylor's or Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$.
(i) when $|z|<1$ (ii) when $1<|z|<2$ (iii) when $|z|>2$

Sol: we have $f(z) = \frac{1}{(1+z^2)(z+2)}$
$$= \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{1+z^2}\right]$$

Now, we find below expansions of $f(z)$ under specified values of z:

(i) when $|z|<1$

we have
$$f(z) = \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{1+z^2}\right]$$
since binomial expansion of the form $(1+z)^{-1}$ is valid if $|z|<1$, therefore partial fractions of $f(z)$ are to be so arranged that the binomial expansions involved may be valid for $|z|<1$.
$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} + \frac{(2-z)}{5}(1+z^2)^{-1}$$
obviously binomial expansions are valid for $|z|<1$.
$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n + \frac{(2-z)}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n}.$$
This represents series in positive powers of z, in other words, it is an expansion of $f(z)$ in a Taylor's series within the circle $|z|=1$.

(ii) when $1<|z|<2$
we have $f(z) = \frac{1}{5}\left(\frac{1}{z+2} - \frac{z-2}{1+z^2}\right)$
[In order that the binomial expansions of the form $(1+z)^{-1}$ be valid, it is necessary

that $|x|<1$]

Hence the partial fractions of $f(z)$ are to be so arranged that the binomial expansions may be valid for $1<|z|<2$.

Thus $f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{z-2}{5}\cdot\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$

which are obviously valid for $1<|z|<2$.

$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n + \frac{z-2}{5z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n$$

These are the expansion in positive and negative powers of z i.e, it is a Laurent's expansion of $f(z)$ with in the annulus $1<|z|<2$.

(iii) when $|z|>2$.

$f(z) = \frac{1}{5}\frac{1}{z+2} - \frac{1}{5}\frac{z-2}{z^2+1}$

$= \frac{1}{5}\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{1}{5}(z-2)\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$ arranged suitably to make the binomial expansions valid for $|z|>2$.

$$= \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{1}{5}\left(\frac{1}{z}-\frac{2}{z^2}\right)\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}}$$

This is also Laurent's expansion within the annulus $2<|z|<R$ where R is large.

4(d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in

the matrix form in adjacent table:
find the optimum assignment that will result in minimum manhours needed.

Job / Man	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Step(1):
(i) After subtracting the minimum of each row from all elements of that row, the reduced matrix is given by

3	1	0	6
5	7	0	4
2	0	1	3
0	2	2	3

(ii) Subtracting the minimum elements of column from elements of that column, we get

3	1	0	3
5	7	0	1
2	0	1	0
0	2	2	0

Step(2): Cover all the zeros by minimum no. of horizontal and vertical lines. A symmetric approach for this is to look for a row or column containing maximum no. of zeros we can cover all the zeros by 3 lines only

so, $r = 3 < 4 = n$

so, go to step (3).

$$\begin{bmatrix} 3 & 1 & 6 & 3 \\ 5 & 7 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

Step (3):

1 is the least uncovered element. Subtract 1 from all the uncovered elements. Add 1 to element at intersection of the covering element namely 1 at position (3,3) and 2 at (4,3). Leave other uncovered elements unchanged. and the reduced matrix so obtained is

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 4 & 6 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

Again, cover the zeros by minimum no. of horizontal and vertical lines. we required exactly 4 lines to cover all the zeros. As $r = 4 = n$, optimal assignment can be made at this stage.

$$\begin{bmatrix} 2 & 0 & \cancel{0} & 2 \\ 4 & 6 & 0 & \cancel{0} \\ 2 & \cancel{0} & 2 & 0 \\ 0 & 2 & 3 & \cancel{0} \end{bmatrix}$$

It may be noted that an assignment problem can have more than one optimum solution.

optimum solution I

$$\begin{bmatrix} 2 & \cancel{0} & 0 & 2 \\ 4 & 6 & \cancel{0} & 0 \\ 2 & 0 & 2 & \cancel{0} \\ 0 & 2 & 3 & \cancel{0} \end{bmatrix}$$

Optimum Solution-I

Man	Job	Man hours
1	B	3
2	C	2
3	D	7
4	A	5

Optimum solution II

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

5(a) solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Sol^n: Here the Lagrange's auxiliary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad \text{---} (1)$$

Choosing $1,-1,0$ and $0,1,-1$ as multipliers in turn, each fraction of (1)

So that $\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)} \Rightarrow \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$

Integrating, $\log(x-y) - \log(y-z) = \log c_1 \Rightarrow (x-y)/(y-z) = c_1$ — (2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{---} (3)$$

Again, choosing $1,1,1$ as multipliers, each fraction of (1),

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{---} (4)$$

from (3) and (4)

$2(x+y+z)\,d(x+y+z) - (2xdx+2ydy+2zdz) = 0$

Integrating $(x+y+z)^2 - (x^2+y^2+z^2) = 2c_2$

$\Rightarrow (x^2+y^2+z^2+2xy+2yz+2zx) - (x^2+y^2+z^2) = 2c_2$

$\Rightarrow xy+yz+zx = c_2$, c_2 being an arbitrary constant — (5)

from (2) and (5), the required general solution is given by

$\phi\left[xy+yz+zx, (x-y)/(y-z)\right] = 0$, ϕ being an arbitrary function.

5(b) Solve the following Partial Differential equation

$(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y)$

Sol^n: Here the auxiliary equation is $m^3-4m^2+4m=0 \Rightarrow m(m^2-4m+4)=0$

$\Rightarrow m(m-2)^2=0$ so that $m=0,2,2$

$\therefore$ C.F $= \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, $P.I = \frac{1}{D^3-4D^2D'+4DD'^2} 4\sin(2x+y)$

$= 4\frac{1}{(D-2D')^2}\left\{\frac{1}{D}\sin(2x+y)\right\}$

$= 4\frac{1}{(D-2D')^2}\left\{-\frac{1}{2}\cos(2x+y)\right\}$, Since $\frac{1}{D}$ stands for integration w.r.t x treating y as constant.

$= -2\frac{1}{(D-2D')^2}\cos(2x+y) = -2\frac{x^2}{1^2\times 2!}\cos(2x+y)$.

So the required solution is

$$y = \phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x) - x^2\cos(2x+y)$$

5(c) The area A of a circle of diameter d is given for the following values.

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

Sol: By using Lagrange's interpolation formula.

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1) + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$$

$$f(105) = \frac{(105-85)(105-90)(105-95)(105-100)}{(80-85)(80-90)(80-95)(80-100)}(5026) + \frac{(105-80)(105-90)(105-95)(105-100)}{(85-80)(85-90)(85-95)(85-100)}(5674) + \frac{(105-80)(105-85)(105-95)(105-100)}{(90-80)(90-85)(90-95)(90-100)}(6362) + \frac{(105-80)(105-85)(105-90)(105-100)}{(95-80)(95-85)(95-90)(95-100)}(7088) + \frac{(105-80)(105-85)(105-90)(105-95)}{(100-80)(100-85)(100-90)(100-95)}(7854)$$

$= 5026 - 28370 + 63620 - 70880 + 39270$

$= 8666$

5(d). Show that $\phi = (x-t)(y-t)$ represents the velocity potential of an incompressible two dimensional fluid. Show that the stream lines at time t are the curves $(x-t)^2-(y-t)^2=$ constant.

Sol^n: Given $\phi = (x-t)(y-t)$ —— (1)

(i) To show that the liquid motion is possible.

$$\frac{\partial\phi}{\partial x} = y-t,\ \frac{\partial\phi}{\partial y} = x-t$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = 0,\ \frac{\partial^2\phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\Rightarrow \nabla^2\phi = 0$, which is the equation of continuity.

Hence (1) represents velocity potential of an incompressible two dimensional fluid.

(ii) To determine streamlines.

$$u = -\frac{\partial\phi}{\partial x} = -(y-t)$$

$$v = -\frac{\partial\phi}{\partial y} = -(x-t)$$

Streamlines are given by $\frac{dx}{u} = \frac{dy}{v}$

$$\frac{dx}{-(y-t)} = \frac{dy}{-(x-t)}$$

$$\Rightarrow (x-t)dx = (y-t)dy$$

Integrating, $\frac{x^2}{2} - tx = \frac{y^2}{2} - ty + \text{constant}$.

$$\Rightarrow x^2 - 2tx = y^2 - 2ty + \text{constant}$$

$$\Rightarrow (x-t)^2 = (y-t)^2 + \text{constant} \Rightarrow (x-t)^2 - (y-t)^2 = \text{Constant}$$

which represents stream lines.

5(e). Use Hamilton's equations to find the equation of motion of the simple pendulum.

Sol^n: Let l be the length of the pendulum and M the mass of the bob. At time t, let θ be the inclination of the string to the downward vertical. Then, if T and V

are the kinetic and potential energies of the pendulum,

then $T = \frac{1}{2}M(l\dot{\theta})^2 = \frac{1}{2}Ml^2\dot{\theta}^2$

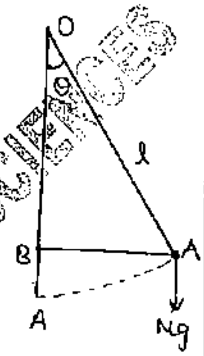
and V = work done against $Mg = Mg\,A'B$

$= Mgl(1-\cos\theta)$

$\therefore L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$ —— (1)

Here θ is the only generalised coordinate

$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Ml^2\dot{\theta}$ —— (2)



Since L does not contain t explicity,

$\therefore H = T + V = \frac{1}{2}Ml^2\dot{\theta}^2 + Mgl(1-\cos\theta)$

$\Rightarrow H = p_\theta^2/(2Ml^2) + Mgl(1-\cos\theta)$ [from (2)]

Here the two Hamilton's equations are

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$ i.e. $\dot{p}_\theta = -Mgl\sin\theta$ —— (3)

and $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$ i.e. $\dot{\theta} = \frac{p_\theta}{Ml^2}$ —— (4).

Differentiating (4), we get

$\ddot{\theta} = \dot{p}_\theta/Ml^2 = -(Mgl\sin\theta)/Ml^2$ from (3)

$\Rightarrow \ddot{\theta} = -(g/l)\sin\theta$

which is the equation of motion of a simple pendulum.

6(a). Find the general integral of the partial differential equation $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ and also the particular integral which passes through the line $x=1, y=0$

Sol'n: Given $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ —— (1)

Given line is given by $x=1$ and $y=0$ —— (2)

Lagrange's auxiliary equations of (1) are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2x-2yz} \quad \text{—— (3)}$$

Taking $z, 1, x$ as multipliers, each fraction of (3) $= (z\,dx + dy + x\,dz)/0$

So that $z\,dx + dy + x\,dz = 0$ (or) $d(xz) + dy = 0$

Integrating $xz + y = c_1$ —— (4)

Again, taking $x, y, \frac{1}{2}$ as multipliers, each fraction of (3) $= \{x\,dx + y\,dy + \frac{1}{2}dz\}/0$

So that $x\,dx + y\,dy + \frac{1}{2}dz = 0 \Rightarrow 2x\,dx + 2y\,dy + dz = 0$

Integrating $x^2 + y^2 + z = c_2$ —— (5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x=1$ and $y=0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2$ —— (6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required surface is given by $1 + xz + y = x^2 + y^2 + z$

$\Rightarrow x^2 + y^2 + z - xz - y = 1$

6(b) Reduce $\frac{\partial^2 z}{\partial x^2} + y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = y$ to canonical form.

Sol^n: Re-writing the given equation, we get

$r + y^2 t - y = 0$ —— (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R=1$, $S=0$ and $T = y^2$ so that $S^2 - 4RT = -4y^2 < 0$ for $y \neq 0$, showing that (1) is elliptic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + y^2 = 0 \Rightarrow \lambda = iy, -iy$

The corresponding characteristic equations are given by

$dy/dx + iy = 0$ and $dy/dx - iy = 0$

Integrating these $\log y + ix = c_1$ and $\log y - ix = c_2$

Choose $u = \log y + ix = \alpha + i\beta$ and $v = \log y - ix = \alpha - i\beta$
where $\alpha = \log y$ and $\beta = x$ —— (2)
are now two independent variables.

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \beta}$, using (2) — (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial \alpha}$, using (2) — (4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2}$, by (3) — (5)

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right) = -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$

$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)\right\}$

$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left(\frac{\partial^2 z}{\partial \alpha^2}\cdot\frac{1}{y}\right) = \frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha}\right)$ — (6)

using (5) and (6) in (1), the required canonical form is

$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} - y^2 = 0$ (or) $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha} + e^{\alpha}$, using (2)

6(c) Form the partial Differential equation by eliminating the arbitrary constants a and b from $\log(az-1) = x + ay + b$

Sol'n: Given $\log(az-1) = x + ay + b$ —— (1).

Differentiating (1) partially w.r.t 'x', we get $\frac{a}{az-1}\frac{\partial z}{\partial x} = 1$ — (2)

Differentiating (1) partially w.r.t 'y', we get $\frac{a}{az-1}\frac{\partial z}{\partial y} = a$ — (3)

from (3), $az - 1 = \frac{\partial z}{\partial y}$ so that $a = \frac{1 + \partial z/\partial y}{z}$ —— (4)

Putting the above values of $az-1$ and a in (2), we have

$\frac{1 + (\partial z/\partial y)}{z(\partial z/\partial y)}\frac{\partial z}{\partial x} = 1$ (or) $\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}$

6(d). The deflection of a vibrating string of length l, is governed by the partial differential equation $y_{tt} = c^2 y_{xx}$. The initial velocity is zero. The initial velocity is zero. The initial displacement is given by

$$y(x,0) = \begin{cases} x/l, & 0 < x < l/2 \\ (l-x)/l, & l/2 < x < l \end{cases}$$ Here $y_{tt} = \partial^2 y/\partial t^2$ and

$y_{xx} = \partial y^2/\partial x^2$. Find the deflection of the string at any instant of time.

Sol'n: The required deflection $y(x,t)$ of the string is the solution of the one-dimensional wave equation.

$$y_{tt} = c^2 y_{xx} \text{ i.e. } \frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{c^2}\right) \times \frac{\partial^2 y}{\partial t^2} \quad \text{---} (1)$$

Subject to the boundary conditions

$$y(0,t) = y(l,t) = 0, \text{ for all } t \quad \text{---} (2).$$

and the given initial conditions, namely.

$$\text{initial displacement} = y(x,0) = f(x) = 0,\ 0 \le x \le l \quad \text{---} 3(a)$$

$$\text{and initial velocity} = y_t(x,t) = g(x) = \begin{cases} x/l, & 0 < x < l/2 \\ (l-x)/l, & l/2 < x < l \end{cases} \quad \text{---} 3(b)$$

Let the solution of (1) be of the form:

$$y(x,t) = X(x)\, T(t) \quad \text{---} (4).$$

Substituting this value of y in (1), we have

$$X''T = \frac{1}{c^2} X T'' \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu \text{ (say)} \quad (5)$$

$$\Rightarrow X'' - \mu X = 0 \ (6) \quad \text{and} \quad T'' - \mu c^2 T = 0 \ (7)$$

Using (2), (4) gives $X(0)\,T(t) = 0$ and $X(l)\,T(t) = 0$ (8)

$$\Rightarrow X(0) = 0 \ \& \ X(l) = 0 \quad (\because T(t) \neq 0) \quad (9)$$

we now solve (6) under B.C. (9).

Three cases arise.

Case (1): Let $\mu = 0$. Then solution of (6) is given by.

$$X(x) = Ax + B \quad \text{---} (10)$$

Using B.C. (9), (10) gives $B=0$, and $Al+B=0$
$\Rightarrow A=0=B$.
$\Rightarrow X(x)=0$.
This leads to $y=0$.
which doesnot satisfy 3(a) & 3(b).
So we reject $\mu=0$.

Case(2):
Let $\mu=\lambda^2$, $\lambda\neq 0$.
Then solution of (6) is $X(x)=Ae^{\lambda x}+Be^{-\lambda x}$ — (11)
Using B.C (9), (11) gives $A+B=0$ and $Ae^{\lambda l}+Be^{-\lambda l}=0$
$\Rightarrow A=B=0$ so that $X(x)=0$.
This leads to $y=0$
which doesnot satisfy 3(a) & 3(b).
So we reject $\mu=\lambda^2$.

Case(3):
Let $\mu=-\lambda^2$, $\lambda\neq 0$.
Then solution of (6) is $X(x)=A\cos\lambda x+B\sin\lambda x$ — (12)
Using B.C(9), (12) gives $A=0$ and $A\cos\lambda l+B\sin\lambda l=0$
$\Rightarrow A=0$ and $\sin\lambda l=0$ $(B\neq 0)$ otherwise so that $y=0$)
Now $\sin\lambda l=0 \Rightarrow \lambda l=n\pi$
$\Rightarrow \lambda=\frac{n\pi}{l}$, $n=1,2,\ldots$ — (13)
Hence non-zero solutions $X_n(x)$ of (6) are given by
$X_n(x)=B_n\sin\left(\frac{n\pi x}{l}\right)$
Using (13), (7) reduces to
$T''+\left(\frac{n^2\pi^2 c^2}{l^2}\right)T=0$ $\left(\because \mu=-\lambda^2=-\frac{n^2\pi^2}{l^2}\right)$
whose general solution is
$T_n(t)=C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l}$ — (14)
$\therefore y_n(x,t)=X_n(t)\,T_n(t)$
$y_n(x,t)=\sum_{n=1}^{\infty}\left(E_n\cos\left(\frac{n\pi ct}{l}\right)+F_n\sin\frac{n\pi ct}{l}\right)\cdot\sin\frac{n\pi x}{l}$ — (15)
where $E_n=B_nC_n$ and $F_n=B_nD_n$

Differentiating (15) partially w.r.t t, we get
$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\left(-\frac{n\pi c E_n}{l}\left(\sin\frac{n\pi ct}{l}\right)+\frac{n\pi c F_n}{l}\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}. \quad (16)$$

putting $t=0$ in (15) and (16) and using 3(a) & 3(b)
we get
$$f(x)=\sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{l} \quad \text{and} \quad g(x)=\sum_{n=1}^{\infty}\frac{n\pi c F_n}{l}\sin\frac{n\pi x}{l}$$
where $E_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx=0 \quad (\because f(x)=0)$

and $\frac{n\pi c F_n}{l}=\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$

$\Rightarrow F_n=\frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$

$$\Rightarrow F_n=\frac{2}{n\pi c}\int_0^{l/2} g(x)\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l} g(x)\sin\frac{n\pi x}{l}dx$$
(from 3(b))
$$=\frac{2}{n\pi c}\int_0^{l/2}\frac{x}{l}\sin\frac{n\pi x}{l}dx+\frac{2}{n\pi c}\int_{l/2}^{l}\frac{l-x}{l}\sin\frac{n\pi x}{l}dx$$
$$=\frac{2}{n\pi c l}\left[(x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-(1)\left[\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]\right]_0^{l/2}$$
$$+\frac{2}{n\pi c l}\left[(l-x)\left[\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right]-(-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$
$$=\frac{2}{n\pi c l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]+\frac{2}{n\pi c l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$
$$=\frac{4l}{cn^3\pi^3}\sin\frac{n\pi}{2}=\begin{cases}0 & \text{if } n=2m,\ m=1,2,\dots\\ \frac{(-1)^{m+1}4l}{c\pi^3(2m-1)^3} & \text{if } n=2m-1 \text{ and } m=1,2,\dots\end{cases}$$
$\left[\because n=2m-1 \Rightarrow \sin\frac{n\pi}{2}=\sin(2m-1)\frac{\pi}{2}=\sin\left(m\pi-\frac{\pi}{2}\right)\right.$
$\left.=\sin m\pi\cos\frac{\pi}{2}-\cos m\pi\sin\frac{\pi}{2}=0-(-1)^m=(-1)^{m+1}\right]$

substituting the above value of E_n and F_n in (15), we get
$\therefore$ the required deflection is given by
$$y(x,t)=\frac{4l}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}.$$

7(a) Solve the following equations by Gauss Seidel method.

$5x_1+x_2+x_3+x_4=4$, $x_1+7x_2+x_3+x_4=12$, $x_1+x_2+6x_3+x_4=-5$,

$x_1+x_2+x_3+4x_4=-6$.

Sol'n: Let $(x_1,x_2,x_3,x_4)=(0,0,0,0)$ be the initial approximation. We write the given equations in the form

$$x_1^{(k+1)}=\frac{1}{5}\left(4-x_2^{(k)}-x_3^{(k)}-x_4^{(k)}\right)$$

$$x_2^{(k+1)}=\frac{1}{7}\left(12-x_1^{(k+1)}-x_3^{(k)}-x_4^{(k)}\right)$$

$$x_3^{(k+1)}=\frac{1}{6}\left(-5-x_1^{(k+1)}-x_2^{(k+1)}-x_4^{(k)}\right)$$

$$x_4^{(k+1)}=\frac{1}{4}\left(-6-x_1^{(k+1)}-x_2^{(k+1)}-x_3^{(k+1)}\right)$$ where $k=0,1,2,3,\ldots$

$k=0$: $x_1^{(1)}=0.8$, $x_2^{(1)}=1.6$, $x_3^{(1)}=-1.2333$, $x_4^{(1)}=-1.791675$

$k=1$: $x_1^{(2)}=1.084995$, $x_2^{(2)}=1.9913$, $x_3^{(2)}=-1.0474$, $x_4^{(2)}=-2.0072$

$k=2$: $x_1^{(3)}=1.01264$, $x_2^{(3)}=2.00599$, $x_3^{(3)}=-1.0019$, $x_4^{(3)}=-2.0042$

$k=3$: $x_1^{(4)}=1.000$, $x_2^{(4)}=2.0009$, $x_3^{(4)}=-0.99945$, $x_4^{(4)}=-2.0004$

$k=4$: $x_1^{(5)}=0.9998$, $x_2^{(5)}=2.000$, $x_3^{(5)}=-0.9999$, $x_4^{(5)}=-1.9999$.

$\therefore$ The solution is $x_1=1$, $x_2=2$, $x_3=-1$, $x_4=-2$.

7(b) The velocity v (km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:

t:	2	4	6	8	10	12	14	16	18	20
v:	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Sol^n: If s(km) be the distance covered in t (min), then $\frac{ds}{dt}=v$

$$\therefore \quad |s|_{t=0}^{20} = \int_0^{20} v\,dt = \frac{h}{3}\left[X+4.O+2.E\right], \text{ by Simpson's rule.}$$

Here $h=2,\ v_0=0,\ v_1=10,\ v_2=18,\ v_3=25$ etc.

$\therefore\ X = v_0 + v_{10} = 0+0=0$

$O = v_1+v_3+v_5+v_7+v_9 = 10+25+32+11+2=80$

$E = v_2+v_4+v_6+v_8 = 18+29+20+5=72$

Hence the required distance $= |s|_{t=0}^{20}$

$$=\frac{2}{3}\left(0+4\times 80+2\times 72\right)=309.33\text{ km}$$

7(c) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2, 0.4$.

Sol^n: We have $f(x,y)=\frac{y^2-x^2}{y^2+x^2}$

To find $y(0.2)$:

Here $x_0=0,\ y_0=1,\ h=0.2$

$k_1 = hf(x_0,y_0) = 0.2f(0,1) = 0.2000$

$k_2 = hf(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_1) = 0.2f(0.1,1.1) = 0.19672$

$k_3 = hf(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_2) = 0.2f(0.1, 1.09836) = 0.1967$

$k_4 = hf(x_0+h,\ y_0+k_3) = 0.2f(0.2, 1.1967) = 0.1891$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}\left[0.2+2(0.19672)+2(0.1967)+0.1891\right] = 0.19599$

Hence $y(0.2) = y_0+k = 1.196$

To find $y(0.4)$:

Here $x_1=0.2,\ y_1=1.196,\ h=0.2$

$k_1 = hf(x_1, y_1) = 0.1891$

$k_2 = hf(x_1+\frac{1}{2}h, y_1+\frac{1}{2}k_1) = 0.2f(0.3, 1.2906) = 0.1795$

$k_3 = hf(x_1+\frac{1}{2}h, y_1+\frac{1}{2}k_2) = 0.2f(0.3, 1.2858) = 0.1793$

$k_4 = hf(x_1+h, y_1+k_3) = 0.2f(0.4, 1.3753) = 0.1688$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}(0.1891 + 2(0.1795) + 2(0.1793) + 0.1688)$

$= 0.1792$

Hence $y(0.4) = y_1+k = 1.196+0.1792 = 1.3752$

7(d) Simplify the following.

(i) $(x+y) \cdot x' \cdot y'$ (ii) $x \vee y \wedge y \vee z \wedge y \vee z'$

(iii) $x \vee y \wedge [(x \wedge y') \vee y]'$

(i) $(x+y)\cdot x'\cdot y' = (x\wedge y) \vee x' \vee y' = (x \wedge y) \vee (x' \vee y')$ (by Demorgan law)

$= (x\wedge y) \vee (x\wedge y)'$

$= 1$ $(\because p \vee p' =)$

(ii) $x\vee y \wedge y\vee z \wedge y \vee z' = (x\vee y)\wedge(y \vee z)\wedge(y \vee z')$

$= (y\vee x)\wedge(y\vee z)\wedge(y\vee z')$ (by commutative)

$= y \vee (x\wedge z) \wedge (y \vee z')$ (by distributive)

$= y\vee[x\wedge z\wedge z']$

$= y\vee[x\wedge(z\wedge z')]$ $(\because z\wedge z' = 0$, $x\wedge 0 = 0)$

$= y\vee(x\wedge 0) = y\vee 0 = y$

(iii) $x\vee y\wedge[(x\wedge y')\vee y]' = x\vee y\wedge[y\vee(x\wedge y')]'$ (by commutative)

$= x\vee y\wedge[(y\vee x)\wedge(y\vee y')]'$ (by distributive)

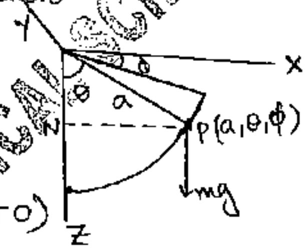
$= (x\vee y)\wedge[(x\vee y)\wedge 1]'$

$= (x\vee y)\wedge(x\vee y)' = 0$

8(a) → Determine the motion, of a spherical pendulum, by using Hamilton's equations.

Sol^n: Let m be the mass of the bob suspended by a light rod of length a, In a spherical pendulum of length a, the path of the motion of the bob is the surface of a sphere of radius a and centre at the fixed point O.

At time t, let $P(a,\theta,\phi)$ be the position of the bob. If (x,y,z) are the cartesian coordinates of P then



$x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$

$\therefore$ K.E, $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

and potential $V = -mgz = -mga\cos\theta$

(since m is below the fixed point O)

$\therefore L = T-V = \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)+mga\cos\theta$

Here θ and ϕ are the generalised coordinates.

$\therefore p_\theta = -\frac{\partial L}{\partial\theta} = ma^2\dot{\theta}$ and $p_\phi = -\frac{\partial L}{\partial\phi} = ma^2\dot{\phi}\sin^2\phi$

Since L does not contain t explicity.

$\therefore H = T+V = \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) - mga\cos\theta$.

Substituting the values of $\dot{\theta}$ and $\dot{\phi}$ from relations (1), we get

$H = \frac{1}{2ma^2}(p_\theta^2+\operatorname{cosec}^2\theta\, p_\phi^2) - mg\cos\theta$

Hence the four Hamilton's equations are.

$\dot{p}_\theta = -\frac{\partial H}{\partial\theta} = \frac{1}{ma^2}\operatorname{cosec}^2\theta\cot\theta\, p_\phi^2 - mga\sin\theta$ —— (H_1)

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{ma^2}p_\theta$ —— (H_2)

$\dot{p}_\phi = -\frac{\partial H}{\partial\phi} = 0$ —— (H_3)

and $\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{1}{ma^2} \operatorname{cosec}^2\theta . p_\phi$ —— (H_4)

From (H_3), Integrating, $p_\phi = C$ (const).

$\therefore$ from (H_4), we have

$\dot{\phi} = \frac{1}{ma^2} C \operatorname{cosec}^2\theta = A/\sin^2\theta$ (where $A = C/ma^2$) —(2)

Also from (H_1) and (H_2), we have

$\ddot{\phi} = \frac{1}{ma^2} \dot{p}_\theta = \frac{1}{ma^2}\left[\frac{1}{ma^2} \frac{\cos\theta}{\sin^3\theta} p_\phi^2 - mga \sin\theta\right]$

$= \frac{1}{(ma^2)^2} C^2 \frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta$ $\because p_\phi = C$

$= A^2 \frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta$ $(\because A = C/ma^2)$

Multiplying bothsides by $2\dot{\theta}$ and integrating, we get

$\dot{\theta}^2 = -\frac{A^2}{\sin^2\theta} + \frac{2g}{a}\cos\theta + B$, (B is a const.) —(3).

Equations (2) and (3) determine the required motion.

8(b). A uniform straight rod of length $2a$ is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string of length a to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{a/g}$.

Sol'n: Let M be the mass of the rod AB of length $2a$, BC the string and M/3 the mass at C.

At time t, let the rod and the string make angles θ and ϕ to the vertical respectively.

Referred to the middle point O of the rod AB as origin, horizontal and vertical lines OX and OY through O as axes, the coordinates of C are given by $x_C = a(\sin\theta + \sin\phi)$

$y_C = a(\cos\theta + \sin\phi)$

$\therefore v_C^2 = \dot{x}_C^2 + \dot{y}_C^2$

$= a^2(\cos\theta\dot\theta + \cos\phi\dot\phi)^2 + a^2(-\sin\theta\dot\theta - \sin\phi\dot\phi)^2$

$= a^2[\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi\cos(\theta-\phi)] = a^2(\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi)$ ($\because \theta, \phi$ are small)

If T be the total kinetic energy and W the work function of the system, then

T = K.E of the rod + K.E of the particle at C.

$= [\frac{1}{2}M\cdot\frac{1}{3}a^2\dot\theta^2 + \frac{1}{2}Mv_0^2] + \frac{1}{2}(\frac{1}{3}M)v_C^2$

$= \frac{1}{6}Ma^2\dot\theta^2 + \frac{1}{6}Ma^2(\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi) = \frac{1}{6}Ma^2(2\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi)$ $(\because v_0 = 0)$

and $W = mg.0 = \frac{1}{3}Mg.y_C + C = \frac{1}{3}Mga(\cos\theta + \cos\phi) + C$

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[\frac{1}{6}Ma^2(4\dot\theta + 2\dot\phi)] - 0 = \frac{1}{3}Mga(-\sin\theta) = -\frac{1}{3}Mga\theta$, $\because \theta$ is small

$\Rightarrow 2\ddot\theta + \ddot\phi = -c\theta$ (where $c = g/a$) —— (1)

And Lagrange's φ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$$

i.e. $\frac{d}{dt}[\frac{1}{6}Ma^2(2\dot\phi + 2\dot\theta)] - 0 = \frac{1}{3}Mga(-\cos\phi) = -\frac{1}{3}Mga\phi$ —— (2) $\because \phi$ is small

$\Rightarrow \ddot\theta + \ddot\phi = -c\phi$, where $c = g/a$

Equations (1) and (2) can be written as

$(2D^2 + c)\theta + D^2\phi = 0$ and $D^2\theta + (D^2 + c)\phi = 0$

— Eliminating φ between these two equations, we get

$[(D^2+c)(2D^2+c) - D^4]\theta = 0$.

$\Rightarrow (D^4 + 3cD^2 + c^2)\theta = 0$ —— (3)

Let the solution of (3) be given by $\theta = A\cos(pt+\delta)$

$\therefore D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$

Substituting in (2), we get

$(p^4 - 3cp^2 + c^2)\theta = 0 \Rightarrow p^4 - 3cp^2 + c^2 = 0 \quad \because \theta \neq 0$

$\therefore p^2 = \dfrac{3c \pm \sqrt{9c^2 - 4c^2}}{2} = \left(\dfrac{3 \pm \sqrt{5}}{2}\right)c = \left(\dfrac{3 \pm \sqrt{5}}{2}\right)g/a$

$\therefore$ one value of p^2 is $p_1^2 = \left(\dfrac{3-\sqrt{5}}{2}\right)g/a$

$\therefore$ one period of principal oscillation

$= \dfrac{2\pi}{p_1} = 2\pi\sqrt{\left[\dfrac{2}{3-\sqrt{5}}\cdot\dfrac{a}{g}\right]} = 2\pi\sqrt{\left[\dfrac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}\cdot\dfrac{a}{g}\right]}$

$= 2\pi\sqrt{\left[\left(\dfrac{6+2\sqrt{5}}{4}\right)\dfrac{a}{g}\right]} = 2\pi\sqrt{\left[\left(\dfrac{\sqrt{5}+1}{2}\right)^2\dfrac{a}{g}\right]}$

$= (\sqrt{5}+1)\pi\sqrt{a/g}.$

8(c). When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distance from its axis, show that path of each vortex is given by the equation

$(r^2\sin^2\theta - b^2)(r^2 - a^2)^2 = 4a^2b^2r^2\sin^2\theta$

θ being measured from the line through the centre $\perp$ar to the join of the vortices.

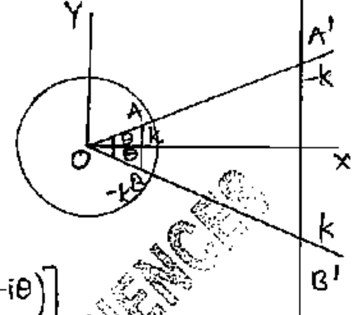
Sol'n: Let x-axis be the axis of the cylinder. Consider the vortices $+k$ at $A(r,\theta)$ and $-k$ at $B(-r,\theta)$ inside the cylinder s.t. distances of A and B from the axis are equal. Evidently, AB is $\perp$ar to x-axis.

The image of vortex $+k$ at A w.r.t the cylinder is a vortex $-k$ at A', the inverse point of A. Similarly the image of vortex $-k$ at B is a vortex $+k$ at B'.

$$OB \cdot OB' = a^2 = OA \cdot OA',$$

where a is the radius of the cylinder. Then

$$OB' = \frac{a^2}{r} = OA' \text{ as } OB = OA = r$$



The complex potential due to this system at $P(z)$ is.

$$w = \frac{ik}{2\pi}\Big[\log(z - re^{i\theta}) - \log\Big(z - \frac{a^2}{r}e^{i\theta}\Big)$$
$$- \log(z - re^{-i\theta}) + \frac{ik}{2\pi}\log\Big(z - \frac{a^2}{r}e^{-i\theta}\Big)\Big]$$

The motion of the vortex at A is due to other vortices. If W' be the complex potential for the motion of A, then

$$w' = w - \frac{ik}{2\pi}\log(z - re^{i\theta}) \text{ at } z = re^{i\theta}$$

$$= \frac{ik}{2\pi}\Big[-\log\Big(z - \frac{a^2 e^{i\theta}}{r}\Big) - \log(z - re^{-i\theta}) - \log\Big(z - \frac{a^2}{r}e^{-i\theta}\Big)\Big]$$

at $z = re^{i\theta}$

$$w' = \frac{-ik}{2\pi}\Big[\log\Big(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\Big) + \log(re^{i\theta} - re^{-i\theta}) - \log\Big(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\Big)\Big]$$

$$= \frac{-ik}{2\pi}\log(r^2 - a^2)e^{i\theta} - \log r + \log(2ir\sin\theta)$$
$$- \Big[\log\{(r^2-a^2)\cos\theta + i\sin\theta(r^2+a^2)\} + \log r\Big]$$

$$\psi = \frac{-k}{2\pi}\Big[\log|(r^2-a^2)e^{i\theta}| + \log|2ir\sin\theta| - \log|\{(r^2-a^2)\cos\theta + i\sin\theta(r^2+a^2)\}|$$

$$= \frac{-k}{2\pi}\Big[\log(r^2-a^2) + \log 2r\sin\theta\Big] - \frac{1}{2}\log\{(r^2-a^2)^2\cos^2\theta + \sin^2\theta(r^2+$$

streamlines are given by $\psi = \text{const}$. i.e.

$$\log\left\{\frac{(r^2-a^2)^2(2r\sin\theta)^2}{(r^2-a^2)^2\cos^2\theta + (r^2+a^2)^2\sin^2\theta}\right\} = \text{const} = \log 4b^2$$

$$\Rightarrow (r^2-a^2)^2 r^2\sin^2\theta = b^2\,[r^4 + a^4 - 2r^2a^2\cos 2\theta]$$

$$\Rightarrow (r^2-a^2)^2\,[r^2\sin^2\theta - b^2] = 4r^2a^2\sin^2\theta$$ This completes the

: Main Test Series–2016
Test–7 (Answer Key)
Full Length Test:

1.(a) (i) Let W be the vector space of 3×3 anti-symmetric matrices over K. Show that $\dim W = 3$ by exhibiting a basis of W.
(ii) Find a basis and the dimension of the subspace W of $\mathbb{R}^4$ spanned by $(1,-4,-2,1)$, $(1,-3,-1,2)$ and $(3,-8,-2,7)$.

Sol. (i) Let $W(K) = \left\{ \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \Big/ a,b,c \in K \right\}$

be the vector space of all 3×3 anti-symmetric matrices.

let $A = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \in W(K)$ then

$$= a\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\in L(S)$ where $S = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq W(K)$

$\therefore A \in W(K) \Rightarrow A \in L(S)$

$\therefore L(S) = W(K)$.

clearly S is Linearly independent subset of $W(K)$

$\therefore S$ is a basis of W and $\dim(W) = 3$.

(ii) Given that the subspace W of $\mathbb{R}^4$ spanned by the set $S = \{(1,-4,-2,1), (1,-3,-1,2), (3,-8,-2,7)\}$.

Let us construct a matrix A whose rows are given vectors of S and convert it into echelon form:

$$A = \begin{bmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

clearly it is in echelon form.

and the number of non-zero rows $= 2$.

$\therefore \dim(W) = 2$.

and basis set $B = \{(1,-4,-2,1), (0,1,1,1)\}$.

1(b) For what values of parameter λ will the following equations fail to have unique solution

$$3x - y + \lambda z = 1$$
$$2x + y + z = 2$$
$$x + 2y - \lambda z = -1 \ ?$$

Will the equations have any solution for these values of λ?

<u>Sol</u> We write a single matrix equation $AX = B$.

$$\begin{bmatrix} 3 & -1 & \lambda \\ 2 & 1 & 1 \\ 1 & 2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

We write augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & 21 & -12\lambda & -12 \end{array}\right] \quad R_3 \to -3R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & 0 & 2\lambda+7 & 16 \end{array}\right] \quad R_3 \to R_3 + 7R_2$$

Clearly which is in echelon form.

If $2\lambda+7 \neq 0$ then $\lambda \neq -7/2$

$\therefore \rho(A/B) = \rho(A) = 3 =$ The no. of unknown variables.

$\therefore$ The given system is consistent and has unique solution.

If $\lambda = -7/2$ then $\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$.

$\therefore$ The given system is inconsistent and has no solution.

1(c) Evaluate $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2x + b^2\cos^2x)^2}$.

Sol^n: put $a\sin x = b\cos x\tan\theta$.

$\Rightarrow a\tan x = b\tan\theta$.

$\Rightarrow a\sec^2x\,dx = b\sec^2\theta\,d\theta$

Hence $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2x + b^2\cos^2x)^2} = \frac{b}{a}\int_0^{\pi/2} \frac{\cos^2x\sec^2\theta\,d\theta}{b^4\cos^4x(\tan^2\theta+1)^2}$

$= \frac{1}{ab^3}\int_0^{\pi/2} \frac{\cos^2\theta}{\cos^2x}\,d\theta$

$= \frac{1}{ab^3}\int_0^{\pi/2} \cos^2\theta\left(1+\frac{b^2}{a^2}\tan^2\theta\right)d\theta$

$= \frac{1}{a^3b^3}\int_0^{\pi/2} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta$

$= \frac{1}{a^3b^3}\left[a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} + b^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$

$= \frac{\pi(a^2+b^2)}{4a^3b^3}$.

1(d) Let $f(x) = \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ \frac{-|x|}{2}+1 & \text{if } 2\le x \end{cases}$

what are the points of discontinuity of f, if any? what are the points where f is not differentiable, if any? Justify your answers.

Sol^n: Given that $f(x) = \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ \frac{-|x|}{2}+1 & \text{if } 2\le x \end{cases}$

i.e. $f(x) = \begin{cases} -\frac{x}{2}+1 & \text{if } x<0 \\ \frac{x}{2}+1 & \text{if } 0\le x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{x}{2}+1 & \text{if } 2\le x \end{cases}$

$\Rightarrow f(x) = \begin{cases} -\frac{x}{2}+1 & \text{if } x<0 \\ \frac{x}{2}+1 & \text{if } 0\le x<2 \\ -\frac{x}{2}+1 & \text{if } 2\le x \end{cases}$

f is linear function over the various subintervals.
$\Rightarrow$ f is continuous and differentiable over each subintervals.
The only doubtful points are the breaking points $x=0$ and $x=2$

At $x=0$, $f(x) = \frac{0}{2}+1 = 1$
i.e. $f(0)=1$

Now LHL : $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} -\frac{x}{2}+1 = 1$

RHL : $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x}{2}+1 = 1$

$\Rightarrow \lim_{x\to 0-} f(x) = 1 = \lim_{x\to 0+} f(x)$

$\therefore$ f is continuous at $x=0$.

Also LHD: $Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0-} \frac{-\frac{x}{2}+1-1}{x} = \lim_{x\to 0-} \frac{-x}{2x} = -\frac{1}{2}$

RHD $Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0+} \frac{\frac{x}{2}+1-1}{x}$

$= \lim_{x\to 0+} \frac{x/2}{x} = \frac{1}{2}$

$\Rightarrow Lf'(0) \neq Rf'(0)$

$\therefore$ f is not differentiable at $x=0$

At $x=2$, $f(x) = \frac{-2}{2}+1 = 0$; i.e. $f(2)=0$

Now **LHL**: $\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \frac{x}{2}+1 = 1+1 = 2$

RHL: $\lim_{x\to 2+} f(x) = \lim_{x\to 2+} -\frac{x}{2}+1 = -1+1 = 0$

$\Rightarrow \lim_{x\to 2-} f(x) \neq \lim_{x\to 2+} f(x)$

$\therefore$ f is not continuous at $x=2$

$\therefore$ f is not differentiable at $x=2$

Hence f is continuous for all values of x except at $x=2$

and also f is differentiable for all values of x except at $x=0$ & $x=2$

1(e) P is a point on the plane $lx+my+nz=p$. A point Q is taken on the line OP such that $OP.OQ=p^2$, Prove that the locus of Q is.

$$p(lx+my+nz)=x^2+y^2+z^2$$

Sol^n: Let Q be the point (α,β,γ) and $OQ=R$. Then the direction ratios of the line OQ are $\alpha-0,\beta-0,\gamma-0$, i.e. α,β,γ.

$\therefore$ The direction Cosines of OQ are α/R, β/R, γ/R, where

$$R=OQ=\sqrt{(\alpha^2+\beta^2+\gamma^2)} \quad \text{———} (1)$$

where r is the distance of any point from $(0,0,0)$

Let $OP=r$, then the Co-ordinates of P are $\left(\frac{\alpha r}{R},\frac{\beta r}{R},\frac{\gamma r}{R}\right)$

But it is given that P is a point on the plane $lx+my+nz=p$

$$\therefore l\frac{\alpha r}{R}+m\frac{\beta r}{R}+n\frac{\gamma r}{R}=p \Rightarrow \frac{r}{R}(l\alpha+m\beta+n\gamma)=p \quad \text{—} (2)$$

Again we are given that $OP.OQ=p^2$

$$\Rightarrow r.R=p^2, \quad \because OP=r, OQ=R$$

$$\Rightarrow r=p^2/R$$

$\therefore$ from (2) we get $(p^2/R^2)(l\alpha+m\beta+n\gamma)=p$

$\Rightarrow p(l\alpha+m\beta+n\gamma)=R^2=\alpha^2+\beta^2+\gamma^2$, from (1)

$\therefore$ The locus of $Q(\alpha,\beta,\gamma)$ is $p(lx+my+nz)=x^2+y^2+z^2$

2(a) Determine whether the following matrices are dependent or independent.

$$A=\begin{pmatrix}1 & 2 & -3\\ 4 & 0 & 1\end{pmatrix} \quad B=\begin{pmatrix}1 & 3 & -4\\ 6 & 5 & 4\end{pmatrix} \quad C=\begin{pmatrix}3 & 8 & -11\\ 16 & 10 & 9\end{pmatrix}$$

Sol^n: The coordinate vectors of matrix A relative to the usual basis as follows.

$$\begin{pmatrix}1 & 2 & -3\\ 4 & 0 & 1\end{pmatrix}=p\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}+q\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix}+r\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}+s\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}+t\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}+u\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}.$$

$\therefore$ $p=1$, $q=2$, $r=-3$, $s=4$, $t=0$, $u=1$.

Hence $[A]=[1,2,-3,4,0,1]$; whose components are the elements of A written row by row.

Similarly, the co-ordinate vectors of B and C relative to the usual basis are

$[B]=[1,3,-4,6,5,4]$, $[C]=[3,8,-11,16,10,9]$

Form the matrix M whose rows are the above co-ordinate vectors

$$M=\begin{bmatrix}1 & 2 & -3 & 4 & 0 & 1\\ 1 & 3 & -4 & 6 & 5 & 4\\ 3 & 8 & -11 & 16 & 10 & 9\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -3 & 4 & 0 & 1\\ 0 & 1 & -1 & 2 & 5 & 3\\ 0 & 2 & -2 & 4 & 10 & 6\end{bmatrix}\quad \begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3-3R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 2 & -3 & 4 & 0 & 1\\ 0 & 1 & -1 & 2 & 5 & 3\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\quad R_3\to R_3-2R_2$$

$\therefore$ Clearly which is in echelon form and the no. of non-zero rows are 2.

Since the echelon matrix has only two non-zero rows, the co-ordinate vectors $[A]$, $[B]$, and $[C]$ generate a space of dimension 2 and so are dependent. Accordingly, the original matrices A, B and C are dependent.

2(5)(i) Show that the volume common to the surface $y^2+z^2=4ax$ and $x^2+y^2=2ax$ is $\frac{2}{3}(3\pi+8)a^3$.

(ii) If $v = At^{-1/2}e^{-x^2/4a^2t}$, prove that $\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}$.

Sol: Clearly the limits of z are from $-\sqrt{4ax-y^2}$ to $\sqrt{4ax-y^2}$.

The limits of y are from $-\sqrt{2ax-x^2}$ to $\sqrt{2ax-x^2}$ and those of x from 0 to $2a$.

∴ The required volume

$$= 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\sqrt{4ax-y^2}} dx\,dy\,dz$$

$$= 4\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\sqrt{4ax-y^2}\,dx\,dy$$

$$= 4\int_0^{2a}\left[\frac{y}{2}\left(\sqrt{4ax-y^2}\right)+\frac{4ax}{2}\sin^{-1}\frac{y}{\sqrt{4ax}}\right]_0^{\sqrt{2ax-x^2}}dx$$

$$= 4\int_0^{2a}\frac{1}{2}\sqrt{2ax-x^2}\sqrt{2ax+x^2}+2ax\sin^{-1}\sqrt{\left(\frac{2a-x}{4a}\right)}\Bigg]\,dx$$

put $x = 2a\cos t \Rightarrow dx = -2a\sin t\,dt$

∴ The required volume

$$= 8a\int_{\pi/2}^{0} 4a^2\cos t\sin t\left[\frac{1}{2}\sin t^2+\frac{1}{2}t\right](-dt)$$

$$= 16a^3 \left\{ \int_0^{\pi/2} \cos t \sin^2 t \, dt + \int_0^{\pi/2} t \sin t \cos t \, dt \right\}$$

$$= 16a^3 \left[\frac{\sin^3 t}{3} \right]_0^{\pi/2} + 16a^3 \left(\frac{1}{2} t \sin^2 t \right)_0^{\pi/2}$$

$$= \frac{16a^3}{3} + 16a^3 \left[\frac{\pi}{4} - \frac{1}{2} \cdot \frac{\pi}{4} \right] = \frac{16a^3}{3} + 2\pi a^3$$

$$= \frac{2}{3}(3\pi + 8)a^3$$

(iv) we have $v = At^{-1/2} e^{-x^2/4a^2t}$.

$$\Rightarrow \frac{\partial v}{\partial x} = At^{-1/2} e^{-x^2/4a^2t} \left(\frac{-2x}{4a^2t} \right) = \frac{-xv}{2a^2t}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{-1}{2a^2t} \left[v + x \frac{\partial v}{\partial x} \right]$$

$$= \frac{-1}{2a^2t} \left[v + x \left(\frac{-xv}{2a^2t} \right) \right]$$

$$= \frac{v}{4a^4t^2} \left(x^2 - 2a^2t \right)$$

Again,

$$\frac{\partial v}{\partial t} = At^{-1/2} e^{-x^2/4a^2t} \left(\frac{x^2}{4a^2t^2} \right) - A \cdot \frac{1}{2} t^{-3/2} e^{-x^2/4a^2t}$$

$$= At^{-1/2} e^{-x^2/4a^2t} \left[\frac{x^2}{4a^2t^2} - \frac{1}{2t} \right]$$

$$= \frac{v}{4a^2t^2} \left(x^2 - 2a^2t \right)$$

Clearly, $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$

2(c) A Sphere whose centre lies in the positive octant passes through the origin and cuts the planes $x=0, y=0, z=0$ in circles of radii $a\sqrt{2}, b\sqrt{2}, c\sqrt{2}$ respectively. Find the equation of this Sphere.

Sol: The sphere passes through origin, so its equation can be taken as

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \quad \text{——} (1)$$

As this sphere meets the plane $z=0$; so putting $z=0$ in (1) we get $x^2+y^2+2ux+2vy=0$; which is evidently a circle on xy-plane and its radius $=\sqrt{(u^2+v^2)}$.

But we are given that the sphere meets the plane $z=0$ in a circle of radius $c\sqrt{2}$, so we have $\sqrt{(u^2+v^2)} = c\sqrt{2}$

$$\Rightarrow u^2+v^2 = 2c^2 \quad \text{——} (2)$$

Similarly as the sphere (1) meets the planes $x=0, y=0$ in circle of radii $a\sqrt{2}, b\sqrt{2}$, so we have $v^2+w^2=2a^2$ —— (3)

and $w^2+u^2=2b^2$ —— (4)

Adding (2), (3) and (4) we get $2(u^2+v^2+w^2)=2a^2+2b^2+2c^2$

$$\Rightarrow u^2+v^2+w^2 = a^2+b^2+c^2 \quad \text{——} (5)$$

Subtracting (3), (4) and (2) from (5) by turn we get

$$u^2 = b^2+c^2-a^2, \quad v^2 = c^2+a^2-b^2 \text{ and } w^2 = a^2+b^2-c^2$$

$$\Rightarrow u = \pm\sqrt{(b^2+c^2-a^2)}, \quad v = \pm\sqrt{(c^2+a^2-b^2)}, \quad w = \pm\sqrt{a^2+b^2-c^2}$$

Also it is given that the centre of the sphere (1) i.e. $(-u,-v,-w)$ lies in the +ve octant, so the values of u, v, w must be −ve.

$$\therefore u = -\sqrt{(b^2+c^2-a^2)}, \quad v = -\sqrt{(c^2+a^2-b^2)}, \quad w = -\sqrt{(a^2+b^2-c^2)}$$

Substituting these values of u, v, w in (1), the required equation is

$$x^2+y^2+z^2-2x\sqrt{(b^2+c^2-a^2)}-2y\sqrt{(c^2+a^2-b^2)} -2z\sqrt{(a^2+b^2-c^2)} = 0.$$

3(a) Find the range, rank, kernel and nullity of the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^4$ be defined by
$T(x,y,z) = (x+y+z,\ x+2y-3z,\ 2x+3y-2z,\ 3x+4y-z)$

Sol^n: Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$
be the standard basis set of $\mathbb{R}^3$

$\therefore$ the transformation T on S will be

$T(1,0,0) = (1,1,2,3)$

$T(0,1,0) = (1,2,3,4)$

$T(0,0,1) = (1,-3,-2,-1)$

form the matrix whose rows are the image vectors and row reduce to echelon form.

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & -3 & -2 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -4 & -4 & -4 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ which is in echelon form.

Clearly
$\therefore$ the non-zero rows of vectors $\{(1,1,2,3), (0,1,1,1)\}$ constitute the L.I. set by forming the basis of $R(T)$.

$\therefore \dim R(T) = 2.$

$\therefore$ Rank of $T = 2.$

Basis for null space of T.

set $T(v) = 0$ where $v = (x,y,z)$ and solve the homogeneous system

$T(x,y,z) = (x+y+z,\ x+2y-3z,\ 2x+3y-2z,\ 3x+4y-z) = (0,0,0,0)$

Thus
$x+y+z=0$
$x+2y-3z=0$
$2x+3y-2z=0$ (or) $x+y+z=0$
$3x+4y-z=0$ $\quad y-4z=0.$

The only free variable is z,
so $\dim(\ker T)=1$.
Set $z=1$ and get $y=4$ and $x=-5$.
$\therefore \{(-5,4,1)\}$ is a basis of $\ker T$.

3(b) (i) Show that if A is a non-singular matrix, then $\det A^{-1} = (\det A)^{-1}$.

(ii) If B is non-singular, prove that the matrices A and $B^{-1}AB$ have the same determinant, A and B being both square matrices of order n.

Sol^n (i) Since A is a non singular matrix
$\therefore \det A \neq 0$ and A^{-1} exists.

Now $AA^{-1}=I \Rightarrow \det(AA^{-1}) = \det I$
$\Rightarrow (\det A)(\det A^{-1}) = 1.$
$\Rightarrow (\det A^{-1}) = \frac{1}{\det A}$
$\Rightarrow (\det A^{-1}) = (\det A)^{-1}$

(ii) we have $\det(B^{-1}AB) = (\det B^{-1})(\det A)(\det B)$ ($\because B$ is non-singular $\Rightarrow B^{-1}$ exists)
$= (\det B^{-1})(\det B)(\det A)$
$= (\det B^{-1}B)\det A$
$= (\det I)(\det A) = \det A$

$\therefore A$ and $B^{-1}AB$ have the same determinant.

3(c) Let $f(x,y)$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2) & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

prove that f_{xy} and f_{yx} are not continuous at $(0,0)$ but $f_{xy}(0,0) = f_{yx}(0,0)$.

Sol^n: for $x \neq 0, y \neq 0$, we have

$$f_x = 2x\log(x^2+y^2) + \frac{(x^2+y^2)}{(x^2+y^2)}(2x)$$

$$= 2x[1+\log(x^2+y^2)]$$

Similarly $f_y = 2y[1+\log(x^2+y^2)]$

Also $f_{xy} = \frac{4xy}{x^2+y^2} \quad \forall (x,y) \neq (0,0)$

Now $f_x(0,0) = \lim_{h\to 0} \frac{[f(0+h,0) - f(0,0)]}{h} = \lim_{h\to 0} \frac{h^2 \log h^2}{h}$

$$= \lim_{h\to 0} 2h\log h = 2\lim_{h\to 0} \frac{\log h}{1/h} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= 2\lim\left(\frac{-h^2}{h}\right) = 0$$

Similarly $f_y(0,0) = 0$

Now $f_x(0,k) = \lim_{h\to 0} \frac{[f(0+h,k) - f(0,k)]}{h}$

$$= \lim_{h\to 0} \frac{(h^2+k^2)\log(h^2+k^2) - k^2\log k^2}{h} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h\to 0} 2h\log(h^2+k^2) + 2h \quad \text{(By L'-Hospital's rule)}$$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{(f_x(0,0+k) - f_x(0,0))}{k}$$

$= \lim_{k \to 0} \frac{0-0}{k} = 0$

Similarly we can show that $f_{xy}(0,0) = 0$.

Thus $f_{xy}(0,0) = f_{yx}(0,0)$.

Now $\lim_{(x,y)\to(0,0)} f_{xy} = \lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$

Along the paths $y = mx$

$= \lim_{x \to 0} \frac{4mx^2}{x^2(1+m^2)}$

$\therefore \lim_{(x,y)\to(0,0)} f_{xy} = \frac{4m}{1+m^2}$

and so $\lim_{(x,y)\to(0,0)} f_{xy}$ doesnot exist.

Similarly

$\lim_{(x,y)\to(0,0)} f_{yx}$ doesnot exist

Hence f_{xy} and f_{yx} are not continuous at $(0,0)$.

3(d) Show that the locus of the line of intersection of perpendicular tangent planes to the cone $ax^2+by^2+cz^2=0$ is the cone $a(b+c)x^2+b(c+a)y^2+c(a+b)^2z^2=0$.

Sol^n: Let $ux+vy+wz=0$ be any tangent plane to the cone $ax^2+by^2+cz^2=0$

Then the normal to this plane through the vertex of the cone viz. $x/u=y/v=z/w$ will be a generator of the reciprocal cone

$$x^2/a+y^2/b+z^2/c=0$$

$\therefore$ we have $(u^2/a)+(v^2/b)+(w^2/c)=0$

$$\Rightarrow bcu^2+cav^2+abw^2=0 \quad \text{——— (1)}$$

This equation being quadratic in u,v,w shows that there will be two tangent planes like

$$ux+vy+wz=0$$

Let the line of the intersection of these two tangent planes be. $x/l=y/m=z/n=0$ ——— (2)

Since this line lies on the plane $ux+vy+wz=0$

So we have $ul+vm+wn=0$ ——— (3)

– Now the direction of the normal to the plane viz u,v,w are given by the relations (1) and (3). Again if the two planes of the form $ux+vy+wz=0$ be perpendicular then we have $u_1u_2+v_1v_2+w_1w_2=0$ —— (4).

Now eliminating w between (1) & (3) we get

$$bcu^2+cav^2+ab\{-(ul+vm)/n\}^2=0$$

$$\Rightarrow bcn^2u^2+can^2v^2+ab(ul+vm)^2=0$$

$$\Rightarrow (bcn^2+abl^2)u^2+2ablmuv+(can^2+abm^2)v^2=0$$

$$\Rightarrow (bcn^2+abl^2)(u/v)^2+2ablm(u/v)+(can^2+abm^2)=0$$

which is a quadratic in u/v and if its roots are

u_1/v_1 and u_2/v_2 then we get

$$\frac{u_1}{v_1}\cdot\frac{u_2}{v_2} = \frac{abm^2+can^2}{ban^2+abl^2}$$

$$\Rightarrow \frac{u_1u_2}{abm^2+can^2} = \frac{v_1v_2}{bcn^2+abl^2} = \frac{w_1w_2}{cal^2+bcm^2}, \text{ by symmetry}$$

∴ from (4) we have.

$$(abm^2+can^2)+(bcn^2+abl^2)+(cal^2+bcm^2)=0$$

$$\Rightarrow a(b+c)l^2+b(c+a)m^2+c(a+b)n^2=0 \quad \text{--- (5)}$$

∴ The locus of the line of intersection (2), obtained by eliminating l, m, n between (2) and (5) is

$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0$, which evidently is a cone with vertex at origin, being homogeneous equation of second degree in x, y, z.

4(a) Let $A=\begin{pmatrix}4 & 1 & -1\\ 2 & 5 & -2\\ 1 & 1 & 2\end{pmatrix}$. Is A diagonalizable? If yes find P such that $P^{-1}AP$ is diagonal.

Solⁿ: The characteristic equation of A is

$$\begin{vmatrix}4-\lambda & 1 & -1\\ 2 & 5-\lambda & -2\\ 1 & 1 & 2-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[\lambda^2-7\lambda+12\right]-(6-2\lambda)-(\lambda-3)=0$$

$$\Rightarrow \lambda^3-11\lambda^2+39\lambda-45=0$$

$$\Rightarrow (\lambda-3)^2(\lambda-5)=0$$

$$\Rightarrow \lambda = 3, 3, 5.$$

∴ The eigen values of the matrix A are $3, 3, 5$

The eigen vectors X of A corresponding to the eigen value 3 are given by the equation $(A-3I)X=0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix of coefficients of these equations has rank 1. Therefore these equations have two linearly independent solutions. These equations reduce to the single equation

$$x_1 + x_2 - x_3 = 0$$

$\therefore X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are two linearly independent solutions of this equation.

Therefore X_1 and X_2 are two linearly independent eigen vectors of A corresponding to the eigen value 3. Thus the geometric multiplicity of the eigenvalue 3 is equal to its algebraic multiplicity.

Now the eigen vectors of A corresponding to the eigen value 5 are given by $(A-5I)X=0$

$$\Rightarrow \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix of coefficients of these equations has rank 2. Therefore these equations have $3-2=1$ linearly independent solution. These equations can be written as $-x_1+x_2-x_3=0$, $x_2-2x_3=0$.

From these, we get $x_1=1$, $x_2=2$, $x_3=1$.

$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 5.

The geometric multiplicity of the eigen value 5 is 1 and its algebraic multiplicity is also 1.

Since the geometric multiplicity of each eigen value of A is equal to its algebraic multiplicity, therefore A is similar to a diagonal matrix.

Let $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

The columns of P are linearly independent eigen vectors of A corresponding to the eigen values 3, 3, 5 respectively. The matrix P will transform A to diagonal form D which is given by the relation

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D.$$

4(c) Prove that the tangent planes to the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ which are parallel to tangent planes to the cone $\frac{b^2c^2x^2}{c^2-b^2}+\frac{c^2a^2y^2}{c^2-a^2}+\frac{a^2b^2z^2}{a^2+b^2}=0$ cut the surface in perpendicular generators.

Sol'n: We know that the equation of the cone reciprocal to the cone $ax^2+by^2+cz^2=0$ is $x^2/a+y^2/b+z^2/c=0$

$$\frac{c^2-b^2}{b^2c^2}x^2+\frac{c^2-a^2}{c^2a^2}y^2+\frac{a^2+b^2}{a^2b^2}z^2=0 \quad \text{--- (1)}$$

Let $lx+my+nz=0$ be a tangent plane to the given cone so that by definition its normal with direction ratio's l,m,n is a generator of its reciprocal cone (1).

$\therefore$ we know $\frac{c^2-b^2}{b^2c^2}l^2+\frac{c^2-a^2}{c^2a^2}m^2+\frac{a^2+b^2}{a^2b^2}n^2=0$ --- (2)

Let any plane parallel to the tangent plane to the given cone be $lx+my+nz=p$ --- (3)

If it is a tangent plane to the given hyperboloid, then

$$p^2=a^2l^2+b^2m^2-c^2n^2 \quad \text{--- (4)}$$

Again if it is a tangent plane at the point (x_1,y_1,z_1) then its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1 \quad \text{--- (5)}$$

Comparing (3) and (5), we get $\frac{x_1/a^2}{l}=\frac{y_1/b^2}{m}=\frac{z_1/c^2}{-n}=\frac{1}{p}$

$$\Rightarrow \frac{x_1}{la^2}+\frac{y_1}{mb^2}=\frac{z_1}{-nc^2}=\frac{1}{p} \quad \text{--- (6)}$$

Also the plane (3) cuts the given hyperboloid in ⊥lar generators if (x_1,y_1,z_1) lies on the director sphere $x^2+y^2+z^2=a^2+b^2-c^2$

$\therefore x_1^2 + y_1^2 + z_1^2 = a^2 + b^2 - c^2$

$\Rightarrow \left(\frac{a^2 l}{p}\right)^2 + \left(\frac{b^2 m}{p}\right)^2 + \left(\frac{-c^2 n}{p}\right)^2 = a^2 + b^2 - c^2$ from ⑥

$\Rightarrow a^4 l^2 + b^4 m^2 + c^4 n^2 = (a^2 + b^2 - c^2)p^2$

$= (a^2 + b^2 - c^2)(a^2 l^2 + b^2 m^2 - c^2 n^2)$ from ④

$\Rightarrow a^2 l^2 (b^2 - c^2) + b^2 m^2 (a^2 - c^2) - c^2 n^2 (a^2 + b^2) = 0$

$\Rightarrow \frac{l^2(c^2 - b^2)}{b^2 c^2} + \frac{m^2(c^2 - a^2)}{c^2 a^2} + \frac{n^2(a^2 + b^2)}{a^2 b^2}$

dividing each term by $-a^2 b^2 c^2$
which is true by virtue of ② Hence proved.

5(a) Solve $\sqrt{(1+x^2+y^2+x^2y^2)}+xy\left(\frac{dy}{dx}\right)=0$

Sol^n: Re writing the given differential equation, we have

$\sqrt{(1+x^2)(1+y^2)}+xy\frac{dy}{dx}=0$

$\Rightarrow \sqrt{(1+x^2)}\sqrt{(1+y^2)}+xy\frac{dy}{dx}=0$

$\Rightarrow \frac{\sqrt{1+x^2}\,dx}{x}+\frac{y\,dy}{\sqrt{1+y^2}}=0$

$\Rightarrow \frac{(1+x^2)}{x\sqrt{1+x^2}}dx+\frac{y\,dy}{\sqrt{1+y^2}}=0$

Integrating,

$\therefore \int\frac{dx}{x\sqrt{1+x^2}}+\int\frac{x\,dx}{\sqrt{1+x^2}}+\int\frac{y}{\sqrt{1+y^2}}dy=c$ ——(1)

Now, let $\int\frac{dx}{x\sqrt{1+x^2}}=\int\frac{\left(-\frac{1}{t^2}\right)dt}{\left(\frac{1}{t}\right)\sqrt{1+\left(\frac{1}{t}\right)^2}}$, by putting $x=\frac{1}{t}$, $dx=\frac{-1}{t^2}$.

$=-\int\frac{dt}{\sqrt{t^2+1}}=-\log\left(t+\sqrt{t^2+1}\right)$

$=-\log\left\{\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right\}$

$=-\log\left\{\frac{1+\sqrt{x^2+1}}{x}\right\}$

$=\log x-\log\left(1+\sqrt{1+x^2}\right)$ ——(2)

$\int\frac{x\,dx}{\sqrt{1+x^2}}=\int\frac{dt}{2\sqrt{t}}$, by putting $1+x^2=t$, $2x\,dx=dt$

$=\frac{1}{2}\int t^{-1/2}dt=t^{1/2}=(1+x^2)^{1/2}$ ——(3)

similarly $\int\frac{y\,dy}{\sqrt{1+y^2}}=(1+y^2)^{1/2}$ ——(4)

Using (2) (3) and (4), (1) gives the required solution as

$$\log x - \log(1+\sqrt{1+x^2}) + (1+x^2)^{1/2} + (1+y^2)^{1/2} = c$$

5(b) → Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^2+x$

Sol^n: Let $x = e^z$ so that $z = \log x$ —— (1)

and let $D_1 = \frac{d}{dz}$ —— (2)

Then $xD = D_1$, $x^2D^2 = D_1(D_1-1)$, $x^3D^3 = D_1(D_1-1)(D_1-2)$ —— (3)

Using (2) and (3), (1) reduces to

$$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 3D_1 - 3]y = e^{2z} + e^z$$

$$(D_1^3 - D_1^2 + 3D_1 - 3)y = e^{2z} + e^z \quad \text{—— (4)}$$

A.E of (4) is $D_1^3 - D_1^2 + 3D_1 - 3 = 0$

$$(D_1 - 1)(D_1^2 + 3) = 0$$

$$D_1 = 1, \pm\sqrt{3}\,i$$

$$\therefore C.F = c_1e^z + c_2\cos\sqrt{3}z + c_3\sin\sqrt{3}z$$

$$= c_1x + c_2\cos\sqrt{3}\log x + c_3\sin\sqrt{3}\log x$$

$$P.I = \frac{1}{D_1^3 - D_1^2 + 3D_1 - 3}(e^{2z} + e^z)$$

$$= \frac{1}{(D_1^2+3)(D_1-1)}e^{2z} + \frac{1}{(D_1^2+3)(D_1-1)}e^z$$

$$= \frac{1}{7}e^{2z} + \frac{1}{4}\frac{1}{D_1-1}e^z$$

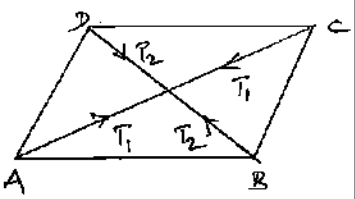
$$= \frac{1}{7}e^{2z} + \frac{1}{4}ze^z = \frac{1}{7}x^2 + \frac{1}{4}x\log x.$$

Hence the required solution is

$$y = c_1x + c_2\cos\sqrt{3}\log x + \sin\sqrt{3}\log x + \frac{1}{7}x^2 + \frac{1}{4}x\log x$$

5(c) Four rods are jointed together to form a parallelogram the opposite joints are joined by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.

Sol: A frame work ABCD in the form of a parallelogram and is placed on a smooth horizontal table. Let T_1 & T_2 be the tensions in the strings AC and BD respectively. Give the system a small displacement in the plane of the table in which AC changes to $AC+\delta(AC)$ and BD changes to $BD+\delta(BD)$. The lengths of the rods AB, BC, CD, DA do not change. During this displacement the weights of the rods do no work because the displacement of their points of application in the vertical direction is zero. The equation of virtual work is



$$-T_1\delta(AC)-T_2\delta(BD)=0$$

$$\Rightarrow \frac{\delta(AC)}{\delta(BD)}=-\frac{T_2}{T_1} \quad \text{(1)}$$

Now let us find a relation b/w the parameters AC and BD from the fig. Since in a parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of its sides, therefore

$$AC^2+BD^2=AB^2+BC^2+CD^2+DA^2=\text{constant} \quad \text{(2)}$$

Differentiating (2), we get

$$2AC\,\delta(AC)+2BD\,\delta(BD)=0$$

$$\Rightarrow \frac{\delta(AC)}{\delta(BD)}=-\frac{BD}{AC} \quad \text{(3)}$$

from (1) & (3) we get $-\frac{T_2}{T_1}=-\frac{BD}{AC} \Rightarrow \frac{T_1}{AC}=\frac{T_2}{BD}$, i.e. tensions are in the ratio of the lengths of the strings.

5(d) If $A = 5t^2 i + tj - t^3 k$ and $B = \sin t\, i - \cos t\, j$, find

(i) $\frac{d}{dt}(A\cdot B)$ (ii) $\frac{d}{dt}(A\times B)$ (iii) $\frac{d}{dt}(A\cdot A)$.

Sol: we have

$$\frac{dA}{dt} = 10t\, i + j - 3t^2 k \quad \text{and} \quad \frac{dB}{dt} = \cos t\, i + \sin t\, j.$$

(i) $\frac{d}{dt}(A\cdot B) = A\cdot\frac{dB}{dt} + \frac{dA}{dt}\cdot B$

$$= (5t^2 i + tj - t^3 k)\cdot(\cos t\, i + \sin t\, j) + (10t\, i + j - 3t^2 k)\cdot(\sin t\, i - \cos t\, j)$$

$$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t$$

$$= (5t^2 - 1)\cos t + 11t\sin t$$

(ii) $A\times B = (5t^2 i + tj - t^3 k)\times(\sin t\, i - \cos t\, j)$

$$= \begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= -t^3\cos t\, i - (0 + t^3\sin t)\, j + (-5t^2\cos t - t\sin t)\, k$$

$$= -t^3\cos t - t^3\sin t\, j - (5t^2\cos t + t\sin t)\, k$$

$$\therefore \frac{d}{dt}(A\times B) = (t^3\sin t - 3t^2\cos t)\, i - (t^3\cos t + 3t^2\sin t)\, j - (10t\cos t - 5t^2\sin t + \sin t + t\cos t)\, k$$

$$= t^2(t\sin t - 3\cos t)\, i - t^2(t\cos t + 3\sin t)\, j$$

(iii) $\frac{d}{dt}(A\cdot A) = \frac{dA}{dt}\cdot A + A\cdot\frac{dA}{dt} = 2A\cdot\frac{dA}{dt}$

$$= 2(5t^2 i + tj - t^3 k)\cdot(10t\, i + j - 3t^2 k)$$

$$= 2[50t^3 + t + 3t^5] = 100t^3 + 2t + 6t^5.$$

5(e) If $F = \cos y\,\hat{i} - x\sin y\,\hat{j}$, evaluate $\int_C F\cdot dr$, where C is the curve $y = \sqrt{1-x^2}$ in the x-y plane from $(1,0)$ to $(0,1)$.

Sol^n: We have $\int_C F\cdot dr = \int_C (\cos y\,dx - x\sin y\,dy)$

$$= \int_1^0 \cos\sqrt{1-x^2}\,dx - \int_0^1 \sqrt{1-y^2}\,\sin y\,dy$$

It is difficult to evaluate the integrals directly. However we observe that

$$\text{curl}\,F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos y & -x\sin y & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + (-\sin y + \sin y)k = 0$$

$\therefore$ The given line integral is independent of path.

$\cos y\, i - x\sin y\, j = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k.$

Then $\frac{\partial\phi}{\partial x} = \cos y \Rightarrow \phi = x\cos y + f_1(y,z)$ —(1)

$\frac{\partial\phi}{\partial y} = -x\sin y \Rightarrow \phi = x\cos y + f_2(x,z)$ —(2)

$\frac{\partial\phi}{\partial z} = 0 \Rightarrow \phi = f_3(x,y)$ —(3)

From (1), (2)(3), we see that $\phi = x\cos y$

The given line integral is equal to

$$\int_{(1,0)}^{(0,1)} d(x\cos y) = \left[x\cos y\right]_{(1,0)}^{(0,1)} = [0 - 1\cos 0] = -1$$

6(a) Solve $(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$

Sol'n: Given that

$(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$

put $x+y=u$, $x^2+y^2=v$.

$\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx}$; $2x+2y\frac{dy}{dx}=\frac{dv}{dx}$

$\Rightarrow 1+p=\frac{du}{dx}$; $2x+2yp=\frac{dv}{dx}$

$\therefore \frac{dv}{dx}=\frac{2(x+yp)}{1+p}$

$\Rightarrow P=\frac{2(x+yp)}{1+p}$; where $P=\frac{dv}{dx}$ & $p=\frac{dy}{dx}$

$\Rightarrow P(1+p)=2(x+yp)$

$\Rightarrow P+Pp=2x+2yp \Rightarrow P-2x=p(2y-P)$

$\Rightarrow p=\frac{P-2x}{2y-P}$ ——— (1)

using (1) the given equation becomes

$(x^2+y^2)\left[1+\frac{P-2x}{2y-P}\right]^2-2(x+y)\left(1+\frac{P-2x}{2y-P}\right)\left(x+y\frac{P-2x}{2y-P}\right)$
$+\left(x+y\left(\frac{P-2x}{2y-P}\right)\right)^2=0$

$\Rightarrow (x^2+y^2)\left[\frac{2y-2x}{2y-P}\right]^2-2(x+y)\left[\frac{2y-2x}{2y-P}\right]P\frac{(y-x)}{2y-P}+P^2\left(\frac{y-x}{2y-P}\right)^2=0$

$\Rightarrow (x^2+y^2)4(y-x)^2-4(x+y)(y-x)^2P+P^2(y-x)^2=0$

$\Rightarrow 4(x^2+y^2)-4P(x+y)+P^2=0$

$\Rightarrow 4v-4Pu+P^2=0$

$\Rightarrow v=Pu-\frac{P^2}{4}$

which ~~is~~ of Clairaut's form and its solution is

$v=uc-c^2/4$

i.e., $x^2+y^2=(x+y)c-c^2/4$

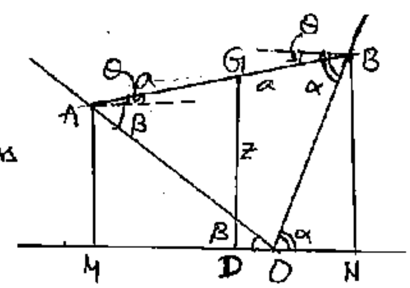
6(b). A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β ($\alpha > \beta$), show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position.

Sol^n: Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam takes an angle θ with the horizontal. We have

$\angle AOM = \beta$ and $\angle BON = \alpha$. The centre of gravity of the beam AB is its middle point G. Let z be the height of G above the fixed horizontal line MN. We shall express z as a function of θ.



We have $z = GD = \frac{1}{2}(AM + BN)$

$$= \frac{1}{2}(OA\sin\beta + OB\sin\alpha)$$

Now in the triangle OAB, $\angle OAB = \beta+\theta$, $\angle OBA = \alpha - \theta$ and $\angle AOB = \pi - (\alpha+\beta)$. Applying the Sine theorem for the ΔOAB, we have

$$\frac{OA}{\sin(\alpha-\theta)} = \frac{OB}{\sin(\beta+\theta)} = \frac{AB}{\sin[\pi-(\alpha+\beta)]} = \frac{2a}{\sin(\alpha+\beta)}$$

$$\therefore\ OA = \frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)},\quad OB = \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}$$

Substituting for OA and OB in (1) we have

$$z = \frac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta + \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha-\theta)\sin\beta + \sin(\beta+\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\cos\theta\sin\alpha\sin\beta\right]$$

$$\therefore \frac{dz}{d\theta} = \frac{a}{\sin(\alpha+\beta)}\left[\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta\right] \quad \text{--- (2)}$$

For equilibrium of the beam, we have $\frac{dz}{d\theta} = 0$

i.e., $\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta = 0$.

i.e. $2\sin\theta\sin\alpha\sin\beta = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1}{2}\left(\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)$$

$$\Rightarrow \tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha) \quad \text{--- (3)}$$

This gives the required position of equilibrium of the beam.

Differentiating (2), we have

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha+\beta)}\left[-\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\cos\theta\sin\alpha\sin\beta\right]$$

$$= \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[\frac{1}{2}\tan\theta(\cot\beta - \cot\alpha) + 1\right]$$

$$= \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[\tan^2\theta + 1\right] \quad [\text{by } (3)]$$

$=$ a negative quantity because θ, α and β are all acute angles and $\alpha+\beta<\pi$.

Thus in the position of equilibrium $\frac{d^2z}{d\theta^2}$ is negative i.e. z is maximum. Hence the equilibrium is unstable.

6(c). (i) Given that $\rho F = \nabla p$, where ρ, p, F are point functions, Prove that $F\cdot \operatorname{Curl} F = 0$.

(ii) Prove that $b\cdot\nabla\left(a\cdot\nabla\frac{1}{r}\right) = \frac{3(a\cdot r)(b\cdot r)}{r^5} - \frac{a\cdot b}{r^3}$ where a and b are constant vectors

Sol^n: we have

$$\operatorname{grad}\frac{1}{r} = -\frac{1}{r^2}\operatorname{grad} r = -\frac{1}{r^2}\frac{1}{r}\vec{r} = -\frac{1}{r^3}\vec{r}$$

$$\therefore a\cdot\left(\nabla\frac{1}{r}\right) = a\cdot\left(-\frac{1}{r^3}\vec{r}\right) = -\frac{a\cdot\vec{r}}{r^3}$$

$$\therefore b\cdot\nabla\left(a\cdot\nabla\frac{1}{r}\right) = b\cdot\nabla\left(-\frac{a\cdot\vec{r}}{r^3}\right) = b\cdot\Sigma i\frac{\partial}{\partial x}\left(-\frac{a\cdot\vec{r}}{r^3}\right)$$

$$= b\cdot\Sigma i\left\{-\frac{1}{r^3}\frac{\partial}{\partial x}(a\cdot\vec{r}) + (a\cdot\vec{r})\frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$

$$= b\cdot\Sigma i\left\{-\frac{1}{r^3}\left(a\cdot\frac{\partial\vec{r}}{\partial x}\right) + 3(a\cdot\vec{r})r^{-4}\frac{\partial r}{\partial x}\right\}$$

($\because$ a is a constant vector)

$$= b\cdot\Sigma i\left\{-\frac{a\cdot i}{r^3} + \frac{3x}{r^5}(a\cdot\vec{r})\right\} \quad \left[\because \frac{\partial\vec{r}}{\partial x} = i \;\&\; \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= b\cdot\Sigma\left\{-\frac{1}{r^3}(a\cdot i)i + \frac{3}{r^5}(a\cdot\vec{r})xi\right\}$$

$$= b\cdot\left\{\frac{-1}{r^3}a + \frac{3}{r^5}(a\cdot\vec{r})\vec{r}\right\} \quad \left[\because \Sigma(a\cdot i)i = a \text{ and } \Sigma xi = \vec{r}\right]$$

$$= -\frac{a\cdot b}{r^3} + \frac{3(a\cdot\vec{r})(b\cdot\vec{r})}{r^5}.$$

(i) Given that $\rho F = \nabla p$

$\Rightarrow F = \frac{1}{\rho}\nabla p$, where ρ and p are scalar functions

$$\text{curl}\,F = \text{curl}\left(\frac{1}{\rho}\nabla p\right)$$
$$= \nabla\left(\frac{1}{\rho}\right)\times\nabla p + \frac{1}{\rho}\,\nabla\times(\nabla p)$$
$$= \nabla\left(\frac{1}{\rho}\right)\times\nabla p \qquad (\because \nabla\times(\nabla p)=0)$$

Now
$$F\cdot\text{curl}\,F = \left(\frac{1}{\rho}\nabla p\right)\cdot\left(\nabla\left(\frac{1}{\rho}\right)\times\nabla p\right)$$
$$= \frac{1}{\rho}\nabla p,\ \nabla\left(\frac{1}{\rho}\right),\ \nabla p$$
$$= \frac{1}{\rho}\left[\nabla p,\ \nabla\frac{1}{\rho},\ \nabla p\right]$$
$= 0$, since the value of a scalar triple product is zero if two vectors are equal.

$\therefore F\cdot\text{curl}\,F = 0$

7(a) → Solve $xy_1 - y = (x-1)(y_2 - x + 1)$

Sol^n: Dividing by $(x-1)$, the given equation in standard form is

$$\left(\frac{x}{x-1}\right)y_1 - \frac{1}{x-1}\,y = y_2 - (x-1) \quad \text{(or)}$$
$$\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x-1}\,y = x-1 \quad \text{—— (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -x/(x-1),\quad Q = 1/(x-1),\quad R = x-1 \quad \text{—— (2)}$$

Here $P + Qx = 0$, showing that a part of C.F of (1) is

$$y = u = x \quad \text{—— (3)}$$

Let the required general solution be $y = uv$ —— (4)

Then v given by $\dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$

$$\Rightarrow \frac{d^2v}{dx^2}\left[-\frac{x}{x-1}+\frac{2}{x}\;\frac{dx}{dx}\right]\frac{dv}{dx} = \frac{x-1}{x} \quad \text{---} \; (5)$$

Let $\frac{dv}{dx} = q$ so that $d^2v/dx^2 = \frac{dq}{dx}$ ——— (6)

then (5) becomes $\frac{dq}{dx} + \left(\frac{2}{x} - \frac{x}{x-1}\right)q = \frac{x-1}{x}$ — (7)

Now $E = \int\left(\frac{2}{x} - \frac{x}{x-1}\right)dx = \int\left(\frac{2}{x} - \frac{x-1+1}{x-1}\right)dx = \int\left(\frac{2}{x} - 1 - \frac{1}{x-1}\right)dx$

$= 2\log x - x - \log(x-1) = \log x^2 - \log(x-1) - x$

$\therefore$ I.F. of (7) $= e^E = e^{\log x^2 - \log(x-1) - x} = e^{\log[x^2/(x-1)]}.e^{-x} = \left[\frac{x^2}{x-1}\right]e^{-x}$

and its solution is $q.\frac{x^2}{x-1}e^{-x} = \int\left(\frac{x-1}{x}.\frac{x^2}{x-1}\right)e^{-x}dx + c_1$

$= \int x e^{-x}dx + c_1 = x(-e^{-x}) - \int 1.(-e^{-x})dx + c_1$

$= -xe^{-x} - e^{-x} + c_1 = c_1 - e^{-x}(x+1)$

$\therefore q = \frac{dv}{dx} = \frac{x-1}{x^2}e^{x}\left[c_1 - e^{-x}(x+1)\right] = c_1\frac{x-1}{x^2}e^{x} - \frac{x^2-1}{x^2}$

$\Rightarrow \int dv = c_1\int\frac{1}{x}e^{x}dx - c_1\int\frac{1}{x^2}e^{x}dx - \int(1-x^{-2})dx + c_2$

$\Rightarrow v = c_1\left[\frac{1}{x}e^{x} - \int\left(\frac{-1}{x^2}\right)e^{x}dx\right] - c_1\int\frac{e^{x}}{x^2}dx - (x+x^{-1}) + c_2$

$\Rightarrow v = (c_1/x)e^{x} - x - \frac{1}{x} + c_2$ —— (8)

from (3), (4) and (8), the required general solution is

$y = xv = x\left[(c_1/x)e^{x} - x - \frac{1}{x} + c_2\right] = c_1e^{x} + c_2x - (x^2+1)$

7(b). A particle is moving with central acceleration $\mu(r^5 - c^4 r)$ being projected from an apse at a distance c with velocity $c^3\sqrt{(2\mu/3)}$, show that its path is the curve $x^4 + y^4 = c^4$.

Sol'n: Here the central acceleration

$$P = \mu(r^5 - c^4 r) = \mu\left(\frac{1}{u^5} - \frac{c^4}{u}\right)$$

$\therefore$ the differential equation of path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u^5} - \frac{c^4}{u}\right) = \mu\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right)$$

Multiplying bothsides by $2(du/d\theta)$ and then integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right) + A \quad \text{—— (1)}$$

where A is constant.

But initially, when $r = c$ i.e, $u = 1/c$, $du/d\theta = 0$ (at an apse) and $v = c^3\sqrt{(2\mu/3)}$.

$\therefore$ from (1), we have $\frac{2\mu c^6}{3} = \frac{h^2}{c^2} = \mu\left(-\frac{c^6}{3} + c^6\right) + A$

$$\therefore h^2 = \frac{2}{3}\mu c^8, \; A = 0$$

Substituting the values of h^2 and A in (1), we have

$$\frac{2}{3}\mu c^8\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right)$$

$$\Rightarrow c^8\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2u^6} + \frac{3c^4}{2u^2} - c^8u^2 = \frac{1}{u^6}\left[-\frac{1}{2} + \frac{3}{2}c^4u^4 - c^8u^8\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^8u^8 - \frac{3}{2}c^4u^4\right)\right] = \frac{1}{u^6}\left[-\frac{1}{2} - \left(c^4u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]$$

$$\therefore c^4u^3\frac{du}{d\theta} = \sqrt{\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]}$$

$$\Rightarrow d\theta = \frac{c^4u^3\,du}{\sqrt{\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]}}$$

Putting $c^4u^4 - \frac{3}{4} = z$, so that $4c^4u^3du = dz$, we have

$$4\,d\theta = \frac{dz}{\sqrt{\left[\left(\frac{1}{4}\right)^2 - z^2\right]}}$$

Integrating, $4\theta + B = \sin^{-1}\left(\frac{z}{\frac{1}{4}}\right) = \sin^{-1}(4z)$ where B is a constant.

$\Rightarrow 4\theta + B = \sin^{-1}(4c^4u^4 - 3)$.

But initially when $u = \frac{1}{c}$, $\theta = 0$, $\therefore B = \sin^{-1} 1 = \frac{\pi}{2}$.

$\therefore 4\theta + \frac{1}{2}\pi = \sin^{-1}(4c^4u^4 - 3)$

$\Rightarrow \sin\left(\frac{1}{2}\pi + 4\theta\right) = 4c^4u^4 - 3$

$\Rightarrow \cos 4\theta = 4c^4u^4 - 3$

$\Rightarrow 4c^4u^4 = 3 + \cos 4\theta$

$\Rightarrow 4c^4/r^4 = (3 + \cos 4\theta)$

$\Rightarrow 4c^4 = r^4\left[3 + (2\cos^2 2\theta - 1)\right] = 2r^4\left[1 + \cos^2 2\theta\right]$

$= 2r^4\left[(\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2\right]$

$= 4r^4(\cos^4\theta + \sin^4\theta)$

$\therefore c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$

$\Rightarrow c^4 = x^4 + y^4$ $[\because x = r\cos\theta \;\&\; y = r\sin\theta]$

which is the required equation of the path.

7(c) Evaluate $\iint_S \bar{F}\cdot n\,ds$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\bar{F} = 4xz\,i + xyz^2\,j + 3z\,k$.

Sol'n: By Divergence theorem, we have

$$\iint_S \bar{F}\cdot n\,ds = \iiint_V \operatorname{div}\bar{F}\,dv.$$

where V is the volume enclosed by S.

Here $\operatorname{div} F = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z) = 4z + xz^2 + 3$

Also V is the region bounded by the surfaces.

$z=0,\ z=4,\ z^2 = x^2+y^2$

$$\therefore \iiint_V \operatorname{div} F\, dv = \iiint_V (4z + xz^2 + 3)\, dx\,dy\,dz$$

$$= \int_{z=0}^{4} \int_{y=-z}^{z} \int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{z^2-y^2}} (4z + xz^2 + 3)\, dx\,dy\,dz$$

$$= 2\int_{z=0}^{4} \int_{y=-z}^{z} \int_{x=0}^{\sqrt{(z^2-y^2)}} (4z+3)\, dx\,dy\,dz$$

since $\displaystyle\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{z^2-y^2}} x\,dx = 0$

$$= 2\int_{z=0}^{4} \int_{y=-z}^{z} (4z+3)\sqrt{(z^2-y^2)}\, dy\,dz$$

on integrating w.r.to x

$$= 4\int_{z=0}^{4} (4z+3)\left[\frac{y}{2}\sqrt{(z^2-y^2)} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$= 4\int_0^4 (4z+3)\left[\frac{z^2}{2}\sin^{-1} 1\right] dz = \pi\int_0^4 (4z^3+3z^2)\,dz$$

$$= \pi\left[z^4 + z^3\right]_0^4 = \pi(256+64) = 320\pi$$

8(a) Apply the method of variation of parameters to solve $x^2 y_2 + 3xy_1 + y = 1/(1-x)^2$

Sol'n: Rewriting $y_2 + (3/x)y_1 + (1/x^2)y = x^{-2}(1-x)^{-2}$ —— (1)

Consider $y_2 + (3/x)y_1 + (1/x^2)y = 0 \Rightarrow (x^2D^2 + 3xD + 1)y = 0$ —— (2)

Let $x = e^z$, $\log x = z$ and $D_1 \equiv d/dz$ —— (3)

then $xD = D_1$ and $x^2D^2 = D_1(D_1-1)$ and so (2) becomes

$[D_1(D_1-1)+3D_1+1]y = 0 \Rightarrow (D_1+1)^2 y = 0$ so $D_1 = -1, -1$

$\therefore$ C.F of (1) $= (C_1 + C_2 z)e^{-z} = (C_1 + C_2 \log x)x^{-1}$

$\therefore$ C.F $= C_1 x^{-1} + C_2 x^{-1}\log x$ —— (4)

Let $u = x^{-1}$, $v = x^{-1}\log x$ and $R = x^{-2}(1-x)^{-2}$ —— (5)

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x^{-1} & x^{-1}\log x \\ -x^{-2} & x^{-2} - x^{-2}\log x \end{vmatrix} = x^{-3} \neq 0$

$\therefore$ P.I of (1) $= uf(x) + vg(x)$, where

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{x^{-1}\log x \cdot x^{-2}(1-x)^{-2}}{x^{-3}}dx = -\int (1-x)^{-2}\log x\, dx$$

$$= -\left[\frac{1}{1-x}\log x - \int \frac{1}{x(1-x)}dx\right], \text{ integrating by parts}$$

$$= -\frac{\log x}{1-x} + \int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx = -(1-x)^{-1}\log x + \log x - \log(1-x)$$

and $g(x) = \int \frac{uR}{W}dx = \int \frac{x^{-1}\cdot x^{-2}(1-x)^{-2}}{x^{-3}}dx = (1-x)^{-1}$

$\therefore$ P.I of (1) $= x^{-1}\{-(1-x)^{-1}\log x + \log x - \log(1-x)\} + x^{-1}\log x\cdot(1-x)^{-1}$

$$= x^{-1}\{\log x - \log(1-x)\} = x^{-1}\log\{x/(1-x)\}$$

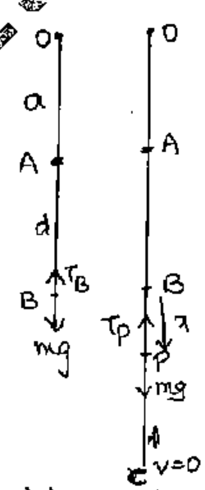
Hence the general solution of (1) is $y =$ C.F + P.I

i.e. $y = C_1 x^{-1} + C_2 x^{-1}\log x + x^{-1}\log\{x/(1-x)\}$

$$= x^{-1}\{C_1 + C_2\log x + \log(x/(1-x))\}$$

8(b). A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\frac{a}{g}}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, where a is the natural length of the string.

Sol'n: Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law, $T_B = \lambda\frac{OB-OA}{OA}$ where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B we have



$mg = T_B = \lambda\frac{d}{a} = mg\frac{d}{a}$ $[\because \lambda = mg, \text{ as given}]$

$\therefore d = a$

Now the particle is pulled down to a point C such that $OC = 4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP = x$

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P = \lambda\frac{a+x}{a} = \frac{mg}{a}(a+x)$ in the string OP acting in the direction PO, i.e. in the direction of x decreasing

(ii) The weight mg of the particle acting vertically downwards i.e. in the direction of x increasing

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = mg - \frac{mg}{a}(a+x) = -\frac{mgx}{a}$$

Thus $\frac{d^2x}{dt^2} = -\frac{g}{a}x$ ———— (1)

which is the equation of S.H.M with centre at the origin B and the amplitude $BC=2a$ which is greater than $AB=a$. Multiplying bothsides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + k, \quad \text{where } k \text{ is a constant.}$$

At the point C, $x = BC = 2a$, and the velocity $dx/dt = 0$

$$\therefore k = \frac{g}{a}4a^2$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}(4a^2 - x^2) \quad \text{—— (2)}$$

Taking square root of (2), we have

$$dx/dt = -\sqrt{(g/a)}\sqrt{4a^2-x^2}$$

the $-$ve sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\sqrt{a/g}\,\frac{dx}{\sqrt{(4a^2-x^2)}} \quad \text{—— (3)}$$

Let t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1} dt = -\sqrt{(a/g)}\int_{2a}^{-a}\frac{dx}{\sqrt{4a^2-x^2}}$$

$$\Rightarrow t_1 = \sqrt{a/g}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a} = \sqrt{\frac{a}{g}}\left[\cos^{-1}(-\tfrac{1}{2}) - \cos^{-1}(1)\right] = \sqrt{\frac{a}{g}}\cdot\frac{2\pi}{3}$$

Let v_1 be the velocity of the particle at A, Then at A

$x = -a$ and $(dx/dt)^2 = v_1^2$

So from (2), we have $v_1^2 = (g/a)(4a^2 - a^2)$

$\Rightarrow v_1 = \sqrt{3ag}$, the direction of v_1 being vertically upwards. Thus the velocity at A is $\sqrt{3ag}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{3ag}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by $0 = \sqrt{3ag} - g t_2$. So that $t_2 = \sqrt{\frac{3a}{g}}$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle return to

$C = 2(t_1 + t_2)$

$$= 2\left[\sqrt{\frac{a}{g}}\,\frac{2\pi}{3} + \sqrt{\frac{3a}{g}}\right] = \sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3} + 2\sqrt{3}\right].$$

8(c) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at (1,1,2) has maximum magnitude 4 in the direction parallel to y-axis.

Sol: we have $\text{grad}\,\phi = \left(\frac{\partial\phi}{\partial x}\right)\vec{i} + \left(\frac{\partial\phi}{\partial y}\right)\vec{j} + \left(\frac{\partial\phi}{\partial z}\right)\vec{k}$

$= 2ax\vec{i} + 2by\vec{j} + 2cz\vec{k}$

$= 2a\vec{i} + 2b\vec{j} + 2c\vec{k}$ at the point (1,1,2)

Now the directional derivative of ϕ at the point (1,1,2) is maximum in the direction vector grad ϕ at this point. According to the question this directional derivative is maximum in the direction parallel to y-axis. i.e, in the direction parallel to the vector $\vec{j}$. So if the direction of the vector $2a\vec{i}+2b\vec{j}+4c\vec{k}$. is parallel to the vector $\vec{j}$, we must have $2a=0$, $4c=0$, i.e, $a=0$ and $c=0$.

Then grad ϕ at $(1,1,2) = 2bj$

Also the maximum value of directional derivative

$$= |\text{grad}\,\phi|$$

$\therefore 4 = |2bj|$, Since according to the question the maximum value of directional derivative is 4.

$$\therefore 2b = 4 \Rightarrow b = 2$$

Hence $a=0$, $b=2$, $c=0$

86). Verify Stoke's theorem for the vector $A = 3y\vec{i} - xz\vec{j} + yz^2\vec{k}$ where S is the Surface of the paraboloid $2z = x^2+y^2$ bounded by $z=2$ and C is its boundary.

Sol^n: The boundary C of the surface S is the circle in the plane $z=2$ whose equations are $x^2+y^2=4$, $z=2$. The radius of this circle is 2 and centre $(0,0,2)$. Suppose $x = 2\cos t$, $y = 2\sin t$, $z=2$, $0 \le t < 2\pi$ are parametric equations of C. By Stoke's theorem.

$\oint A\cdot dr = \iint_S (\text{Curl}\,A)\cdot n\,ds$, where n is a unit vector along outward drawn normal to the surface S.

we have $\oint_C A\cdot dr = \oint_C (3y\vec{i} - xz\vec{j} + yz^2\vec{k})\cdot(dx\vec{i}+dy\vec{j}+dz\vec{k})$

$$= \oint_C (3y\,dx - xz\,dy + yz^2\,dz)$$

$$= \oint_C (3y\,dx - 2x\,dy), \text{ Since on C, } z=2 \text{ and } dz=0.$$

$$= \int_{2\pi}^{0}\left(3y\frac{dx}{dt} - 2x\frac{dy}{dt}\right)dt$$

$$= -\int_{0}^{2\pi}\left[3.2\sin t.(-2\sin t) - 2.2\cos t.2\cos t\right]dt$$

$$= -\int_{0}^{2\pi}\left[-12\sin^2 t - 8\cos^2 t\right]dt$$

$$= 4\left[12\int_0^{\pi/2}\sin^2 t\,dt + 8\int_0^{\pi/2}\cos^2 t\,dt\right]$$

$$= 4\left[12.\frac{1}{2}.\frac{\pi}{2} + 8.\frac{1}{2}.\frac{\pi}{2}\right] = 4 \ldots = 20\pi \quad \text{---(1)}$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'. By Gauss divergence theorem, we have

$$\iint_{S'}(\text{Curl } A).n\,ds = \iiint_V \text{div Curl } A\,dv$$

$$= 0 \quad \text{since div curl } A = 0$$

$$\therefore \iint_S (\text{Curl } A).n\,ds + \iint_{S_1}(\text{Curl } A).n\,ds = 0$$

[$\because S'$ consists of S & S_1]

$$\Rightarrow \iint_S(\text{Curl } A).n\,ds = -\iint_{S_1}(\text{Curl } A).n\,ds$$

$$= -\iint_{S_1}(\text{Curl } A).k\,ds \quad [\because \text{on } S_1,\ n = k]$$

Now

$$\text{curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial z}(-xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial z}(3y)\right] - \vec{k}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(3y)\right]$$

$$= (z^2+x)\vec{i} - (z+3)\vec{k}$$

$$\therefore \iint_S (\text{Curl } A)\cdot n\, ds = -\iint_{S_1} \left[(z^2+x)\vec{i} - (z+3)\vec{k}\right]\cdot \vec{k}\, ds$$

$$= \iint_{S_1} (z+3)\, ds = \iint_{S_1} 5\, ds, \text{ since on } S_1;\ z=2$$

$= 5S_1$, where S_1 is the area of a circle of radius 2.

$$= 5.\pi.2^2 = 20\pi \quad \text{—— (2)}$$

from (1) & (2), we see that

$$\oint_C A.\,dr = \iint_S (\text{curl } A)\cdot n\, ds$$

This verifies stokes theorem.

Main Test Series – 2016
Test – 8 (Answer Key)
Full Length Test

1.(a) In S_{10}, let $\beta = (13)(17)(265)(289)$, find an element in S_{10} that commutes with β but is not a power of β.

Sol $\because \beta = (13)(17)(265)(289)$
$= (13)(17)(28965)$
$= (173)(28965)$

let $\alpha \in S_{10}$ s.t $\alpha = (4\ 10)$ then $\alpha\beta = \beta\alpha$ because the product of disjoint cycles is commutative.

since $o(\beta) = 15 \therefore \beta, \beta^2, \beta^3 \text{---} \beta^{15} = I$.
and None of $\beta, \beta^2 \text{---} \beta^{15}$ will contain a cycle $(4\ 10)$ as β^n $(1 \le n \le 15)$ are obtained from re-arranging elements of individual cycle.

$\therefore \beta^n \ne \alpha$; $1 \le n \le 15$.

$\therefore \alpha = (4, 10)$ commute with β but is not a power of β.

1(b) consider the mapping from $M_2(Z)$ into Z given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a$. prove that this is a ring homomorphism. or disprove

Sol. $M_2(Z) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ a,b,c,d \in Z \right\}$ be a ring w.r.t + n & x n over Z (the ring of integers)

given that the mapping $f : M_2(Z) \to Z$ s.t

$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a$ —— ①.

let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(Z)$

s.t $f(A) = a_1$ $f(B) = a_2$

since $A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \in M_2(Z)$

and $AB = \begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{bmatrix} \in M_2(Z)$

$\therefore f(A+B) = a_1+a_2$ (by definition)

$\Rightarrow f(A+B) = f(A)+f(B)$

and

$f(AB) = a_1a_2+b_1c_2$

$\neq f(A)\cdot f(B)$.

$\therefore f : M_2(Z) \to Z$ is not a ring homomorphism

1.(c) A function f is $[0,1]$ by

$$f(x) = \begin{cases} (-1)^{n-1} & \text{when } \frac{1}{n+1} < x \le \frac{1}{n}, \quad n=1,2,3\ldots \\ 1 ; & x = 0 \end{cases}$$

prove that (i) f is integrable on $[0,1]$

(ii) $\int_0^1 f = \log\left(\frac{4}{e}\right)$.

Sol $f(x) = \begin{cases} 1 & ; \frac{1}{2} < x \le 1 \\ -1 & ; \frac{1}{3} < x \le \frac{1}{2} \\ 1 & ; \frac{1}{4} < x \le \frac{1}{3} \\ \vdots & \\ (-1)^{n-1} & ; \frac{1}{n+1} < x \le \frac{1}{n} \\ \vdots & \\ 1 & ; x = 0 \end{cases}$

(i) clearly f is discontinuous at an infinite number of points $\frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots$

since the set $\{\frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots\}$ having only one limit point '0'.

$\therefore$ f is integrable on $[0,1]$.

(ii) $\int_0^1 f = \int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} (-1)\,dx + \int_{1/4}^{1/3} 1\,dx + \cdots + \int_{1/(n+1)}^{1/n} (-1)^{n-1} dx + \cdots$

$\Rightarrow \int_0^1 f = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{4} - \frac{1}{4} + \frac{1}{5} \cdots$

$= 1 - 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{5}\right) \cdots$

$= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right) - 1$.

$= 2\log_e 2 - 1 = \log_e 4 - \log_e e = \log\left(\frac{4}{e}\right)$

4.(d) prove that the function $f(z) = u+iv$, where $f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

is continuous and that cauchy-Riemann equations at the origin, but differentiable at the origin.

Sol let $f(z) = u+iv$.

where $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$, $v = \left(\frac{x^3+y^3}{x^2+y^2}\right)$

let $x = r\cos\theta$, $y = r\sin\theta$.

let $\epsilon > 0$ be given

then we have $|f(r,\theta) - f(0,0)| = |f(r,\theta) - 0|$

$= r\left|(\cos^3\theta - \sin^3\theta) + i(\cos^3\theta + \sin^3\theta)\right|$.

$< 4r < \epsilon$ whenever $r < \frac{\epsilon}{4}$

choosing $\delta = \frac{\epsilon}{4}$.

$\therefore |f(r,\theta) - f(0,0)| < \epsilon$ whenever $r < \delta$

$\therefore f$ is continuous at $(0,0)$.

$\therefore u_x(0,0) = \underset{h\to 0}{Lt} \frac{u(h,0) - u(0,0)}{h} = 1$

$u_y(0,0) = \underset{k\to 0}{Lt} \frac{u(0,k) - u(0,0)}{k} = -1$.

$v_x(0,0) = \underset{h\to 0}{Lt} \frac{v(h,0) - v(0,0)}{h} = 1$

$\underset{k\to 0}{Lt} \frac{v(0,k) - v(0,0)}{k} = 1 \quad \therefore u_x = v_y$ &

$u_y = -v_x$ at $(0,0)$.

$\therefore$ Cauchy Riemann equations are satisfied at $(0,0)$.

since $f'(0) = \underset{(h,k)\to(0,0)}{Lt} \frac{f(h,k) - f(0,0)}{h+ik}$

$= \underset{(h,k)\to(0,0)}{Lt} \frac{\frac{h^3-k^3}{h^2+k^2} + i\left(\frac{h^3+k^3}{h^2+k^2}\right) - 0}{h+ik}$

let us approach (0,0) along the path $h = mk$ then

$$f'(0,0) = \frac{\frac{m^3-1}{m^2+1} + i\left(\frac{m^3+1}{m^2+1}\right)}{m+i}$$

does not exist because $f'(0,0)$ depends on 'm'.

$\therefore f(z)$ is not differentiable at (0,0)

1.(e) Solve graphically the following LPP.

Max $Z = 3x_1 + 2x_2$

S.C. $x_1 + 2x_2 \le 6$

$2x_1 + x_2 \le 8$

$-x_1 + x_2 \le 1$

$2x_2 \le 2$; $x_1, x_2 \ge 0$.

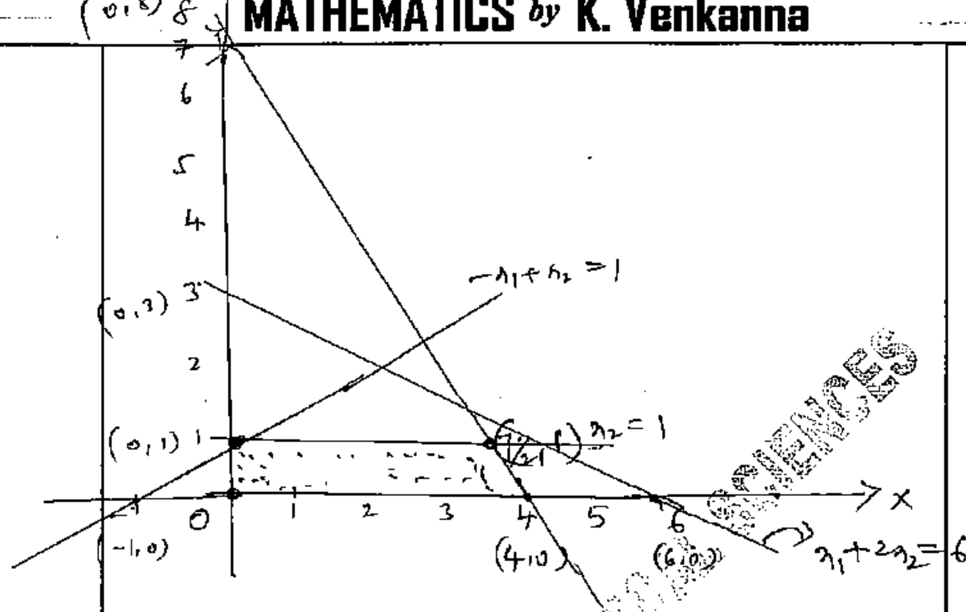
corresponding constraints:

$x_1 + 2x_2 = 6 \Rightarrow (0,3)$ & $(6,0)$;

$2x_1 + x_2 = 8 \Rightarrow (0,8)$ & $(4,0)$;

$-x_1 + x_2 = 1 \Rightarrow (0,1)$ & $(-1,0)$ and

$2x_2 = 2 \Rightarrow x_2 = 1 \therefore (0,1)$



Clearly the corner points of convex region (feasible) are $(0,0)$, $(0,1)$, $(4,0)$ & $\left(\frac{7}{2}, 1\right)$

At $(0,0)$ Max $Z = 0$.

At $(0,1)$, Max $Z = 2$.

At $(4,0)$, Max $Z = 12$.

and at $(7/2, 1)$: Max $Z = 3(7/2) + 2$.

$$= \frac{21}{2} + 2 = \frac{25}{2}.$$

∴ The required solution is

$x_1 = 7/2$, $x_2 = 1$

at which max $Z = \frac{25}{2}$.

2.(b) (i) $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \Big/ a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$?

(ii) construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field? Is it an integral domain?

Sol (i) let $GL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, a, b, c, d \in \mathbb{R} \right\}$.

let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in GL(2, \mathbb{R}) \Big/ a, b, d \in \mathbb{R}, ad \neq 0 \right\}$.

clearly $H \leq GL(2, \mathbb{R})$. and is not a normal subgroup of $GL(2, \mathbb{R})$.

because: let $g = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in GL(2, \mathbb{R})$

$$h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H$$

we have $gHg^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{(-1)}$

$$= \frac{1}{(-1)} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = (-1) \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \notin H.$$

(ii) let $Z_2[i] = \{ a + ib \mid a, b \in Z_2 = \{0, 1\} \}$ be the ring of gaussian integers where $(Z_2 = \{0, 1\}, +_2, \times_2)$ is a ring.

then $Z_2[i] = \{0, 1, i, 1+i\}$

let us construct multiplication table :-

$\times$	0	1	i	$1+i$
0	0	0	0	0
1	0	1	i	$1+i$
i	0	i	1	$1+i$
$1+i$	0	$1+i$	$1+i$	0

Clearly $Z_2[i]$ is not a field because from the table, the inverse the the non-zero element $1+i$ does not exist w.r.t. $\times^n$.

Also $Z_2[i]$ is not an integral domain because: it has zero divisors

i.e $(1+i)(1+i) = 0$ for which $1+i \neq 0+i0$.

2(b) (i) Give an example of family sets $\{I_n / n \in N\}$ of non-empty closed ~~intervals~~ such that $I_1 \supset I_2 \supset I_3 \supset \dots$ and $\bigcap_{n=1} I_n = \phi$.

(ii) Give an example of a family $\{I_n / n \in N\}$ of bounded open intervals such that $I_1 \supset I_2 \supset I_3 \supset \dots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$

Sol (i) Example: let $I_n = \{x \in R / x \geq n\}$
$= [n, \infty)$ then.

then $I_1 = [1, \infty)$, $I_2 = [2, \infty)$ ------
are all closed sets.
clearly $I_1 \supset I_2 \supset I_3 \supset$ ----
and $I_1 \cap I_2 \cap$ ---- $= \phi$.
ie $\bigcap_{n=1}^{\infty} I_n = \phi$.

(ii) let $I_n = (-1/n, 1/n) - \{0\}$ $\forall n \in N$.
then $I_1 = (-1, 1) - \{0\}$, $I_2 = (-\frac{1}{2}, \frac{1}{2}) - \{0\}$,
$I_3 = (-1/3, 1/3) - \{0\}$ - - $I_n = (-\frac{1}{n}, \frac{1}{n}) - \{0\}$ - - -
clearly $I_1, I_2, -- I_n --$ are bounded open intervals such that
$I_1 \supset I_2 \supset -- \supset I_n \supset$ ----
and $\bigcap_{n \in N} I_n = \phi$.

2.(c) Show that the series
$$1 - \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} - \frac{e^{-6x}}{6^2-1} + \cdots$$
converges uniformly for all $x \geq 0$.

Sol since $1 - \frac{e^{-2x}}{2^2-1} + \frac{e^{-4x}}{4^2-1} - \frac{e^{-6x}}{6^2-1} + \cdots$
$$= \sum_{n=1}^{\infty} (-1)^n \frac{e^{-2nx}}{(2n)^2-1} = \sum_{n=1}^{\infty} (-1)^n f_n(x)$$

where $f_n(x) = \dfrac{(-1)^n e^{-2nx}}{(2n)^2-1}$

$\because \left| f_n(x) \right| = \dfrac{e^{-2nx}}{4n^2-1} \quad \forall n \in \mathbb{N}, \; x \geq 0.$

$\leq \dfrac{1}{4n^2-1} \quad \left(\because e^{-2nx} \leq 1 \;\forall x \geq 0, \forall n \in \mathbb{N} \right)$

$< \dfrac{1}{n^2} \quad \forall n \in \mathbb{N}$

$\therefore \left| f_n(x) \right| < \dfrac{1}{n^2} \; (= M_n)$ say.

$\because \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is cgt.

$\therefore$ by Weierstrass's M-test the given series is convergent uniformly for all $x \geq 0$.

2(d). Use the method of contour integration to prove that $\displaystyle\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta = 2\pi a \left\{ 1 - \frac{a}{\sqrt{(a^2-1)}} \right\}$; where $a > 1$

Sol'n: $I = \int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta} d\theta = \int_0^{2\pi} \frac{2a\cos\theta}{2a+2\cos\theta}$.

$I = $ real part of $\int_0^{2\pi} \frac{2ae^{i\theta}}{2a+(e^{i\theta}+e^{-i\theta})} d\theta$

$=$ real Part of $\int_C \frac{2az}{2a+z+\frac{1}{z}} \frac{dz}{iz}$ writing $e^{i\theta}=z,\ d\theta = \frac{dz}{iz}$

where C is the unit circle $|z|=1$

$=$ real part of $\int_C \frac{-2iaz}{z^2+2az+1} dz$

$=$ real Part of $\int_C \frac{-2iaz}{(z-\alpha)(z-\beta)}$

where $\alpha = -a+\sqrt{a^2-1},\ \beta = -\sqrt{(a^2-1)}$

$=$ real part of $\int_C f(z)dz$ where $f(z) = \frac{-2iaz}{(z-\alpha)(z-\beta)}$

Poles of $f(z)$ are given by $(z-\alpha)(z-\beta)=0$

i.e. $z=\alpha$ and $z=\beta$ are the two simple poles.

The value of β is obviously greater than unity while that of α is less. Thus only the pole α lies within the Contour C.

Residue of $f(z)$ at the simple pole $z=\alpha$ is

$= \lim_{z\to\alpha} (z-a)f(z)$

$= \lim_{z\to\alpha} (z-a)\frac{-2iaz}{(z-\alpha)(z-\beta)} = \frac{-2aiz}{\alpha-\beta}$

$= \frac{-2ia\{-a+\sqrt{(a^2-1)}\}}{2\sqrt{(a^2-1)}} = ai\left[\frac{a}{\sqrt{a^2-1}} - 1\right]$

Hence by Cauchy's residue theorem we have

$$= 2\pi i \cdot ai\left\{\frac{a}{\sqrt{(a^2-1)}} - 1\right\}$$

$$= 2\pi a\left\{1 - \frac{a}{\sqrt{(a^2-1)}}\right\} \text{ which is purely real.}$$

Hence I = real part of $\int_C f(z)\,dz$

$$= 2\pi a\left\{1 - \frac{a}{\sqrt{(a^2-1)}}\right\}$$

3.(c) Suppose that a finite group is generated by two elements a and b (that is, every element of the group can be expressed as some product of a's and b's). Given that $a^3 = b^2 = e$ and $ba^2 = ab$, construct the cayley table for the group.

Sol: let G be a finite group generated by two elements a and b (i.e every element of the group can be expressed as some product of a's & b's)

then $G = \{a^n b^n / n \in I\} = \langle ab \rangle$

since $ba^2 = ab$ and $a^3 = b^2 = e$.

$\Rightarrow ba = a^2 b$.

$\therefore G = \{e, a, a^2, b, ab, a^2b\}$.

let us construct composition table:

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

2.(b) Let $H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle/ a, b, c \in Q \right\}$ under matrix multiplication.

(i) Find $Z(H)$

(ii) prove that $Z(H)$ is isomorphic to Q under addition.

(iii) prove that $\frac{Z}{Z(H)}$ is isomorphic to $Q \oplus Q$

(iv) Are your proofs for parts (a) and b valid when Q is replaced by R? Are they valid when Q is replaced by Z_p, where p is prime?

Sol (i) To determine the centre of H, suppose that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

is in the centre of H. Then for all choices of a', b' and c' we have

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a'+a & b'+ac'+b \\ 0 & 1 & c'+c \\ 0 & 0 & 1 \end{bmatrix}$$

and,

$$\begin{bmatrix}1&a'&b'\\0&1&c'\\0&0&1\end{bmatrix}\begin{bmatrix}1&a&b\\0&1&c\\0&0&1\end{bmatrix}=\begin{bmatrix}1&a+a'&b+a'c+b'\\0&1&c'+c\\0&0&1\end{bmatrix}$$

So, $b'+ac'+b = b+a'c+b'$

Thus, $a'c = ac' \quad \forall\ a', c'$

taking $a'=0,\ c'=1$ gives $a=0$.

taking $a'=1,\ c'=0$ gives $c=0$

Finally, note that $\begin{bmatrix}1&0&b\\0&1&0\\0&0&1\end{bmatrix}$ commute

with every element of H.

$\therefore\ Z(H) = \left\{\begin{bmatrix}1&0&b\\0&1&0\\0&0&1\end{bmatrix} \,\middle|\, b\in\mathbb{Q}\right\}$

(ii) The mapping

$$\phi\left(\begin{bmatrix}1&0&b\\0&1&0\\0&0&1\end{bmatrix}\right) = b$$

is an isomorphism. ϕ is 1-1 and onto by observation. To see that ϕ is operation preserving,

$$\phi\left(\begin{bmatrix}1&0&b\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&b'\\0&1&0\\0&0&1\end{bmatrix}\right) = \phi\left(\begin{bmatrix}1&0&bb'\\0&1&0\\0&0&1\end{bmatrix}\right) = bb'$$

$$= \phi\left(\begin{bmatrix}1&0&b\\0&1&0\\0&0&1\end{bmatrix}\right)\phi\left(\begin{bmatrix}1&0&b'\\0&1&0\\0&0&1\end{bmatrix}\right)$$

(iii) define $\phi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, c)$

by observation, ϕ is onto $Q \oplus Q$. To see that ϕ is operation preserving,

$$\phi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} 1 & a+a' & b+ac'+b' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= (a+a', c'+c) = (a, c)(a', c')$$

$$= \phi\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) \phi\left(\begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix}\right)$$

(iv) Yes, They are valid for both R and P.

3(d) Find the optimal solution of the following transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25.	

Ans 3d [Hints - Find the initial solⁿ by "Lowest Cost Entry Method." Although, "Vogel's Method" will also yield the same initial solⁿ, but will be longer one. One improvement is only required to reach optimal solⁿ.]

Ans - $X_{11} = 20$, $X_{13} = 10$, $X_{23} = 20$, $X_{24} = 10$,
$X_{25} = 10$ $X_{32} = 40$ $X_{35} = 10$ $X_{36} = 25$
$X_{45} = 20$
Min Cost = Rs. 430

4(b) Let F be the field modulo 5. Show that the polynomial x^2+2x+3 is irreducible over F. Use this to construct a field containing 25 elements.

9(a)) we have $F = Z_5 = \{0,1,2,3,4\}$

$$(x^2+2x+3) = (x+a)(x+b); \quad a,b \in F$$

$$2 = a+b \qquad \text{— (i)}$$
$$3 = ab \qquad \text{— (ii)}$$

(i) is satisfied for $(a,b) = (0,2), (1,1), (3,4), (2,0), (4,3)$.

for these values of a & b, $ab = 0, 1, 2, 0, 2$

i.e. (ii) is never satisfied.

consequently, x^2+2x+3 is irreducible over F.

Hence $\dfrac{F[x]}{\langle x^2+2x+3\rangle}$ is a field

Any element of the field is $f(x)+A$

where $f(x) \in F[x]$, $A = \langle x^2+2x+3\rangle$.

By division algorithm in $F[x]$, for $f(x) \in F[x]$, $x^2+2x+3 \in F[x]$, $\exists$ $t(x), r(x) \in F[x]$ such that

$$f(x) = (x^2+2x+3)\,t(x) + r(x) \qquad \text{—(iii)}$$
$$r(x) = 0 \text{ or } \deg r(x) < \deg(x^2+2x+3) = 2$$

we may take, $r(x) = \alpha x + \beta$ where $\alpha, \beta \in F$

$$f(x) + A = r(x) + (x^2+2x+3)\,t(x) + A$$

$$f(x) + A = r(x) + A$$
$$= \alpha x + \beta + A \qquad \text{—(iv)}$$

since $(x^2+2x+3)\,t(x) \in A = \langle x^2+2x+3 \rangle$

In (iv), we see that $\alpha, \beta \in F = Z_5$ and $O(Z_5) = 5$.

Consequently, each of α and β can be selected in 5 ways. Hence by (iv), no. of elements of field $\frac{F[x]}{\langle x^2+2x+3\rangle}$ is $5^2 = 25$.

48) prove that the function f defined by $f(x) = \begin{cases} \sin\frac{1}{x} & \forall x > 0 \end{cases}$

is continuous but not uniformly continuous on $\mathbb{R}$

4 - b let 'a' be any arbitary positive real number.

$\therefore f(a) = \frac{1}{\sin(a)} \sin\frac{1}{a} \quad \forall a \in \mathbb{R}^+$ --- (1)

Now,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \sin\frac{1}{x}$$

put $x = a-h$, $h>0$

$$\therefore \lim_{x\to a^-} f(x) = \lim_{h\to 0^+} \sin\frac{1}{a-h}$$

$$= \sin\frac{1}{a} \quad \text{--- (2)}$$

simillarly,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \sin\frac{1}{x}$$

put $x = a+h$, $h>0$

$$\therefore \lim_{x\to a^+} f(x) = \lim_{h\to 0^+} \sin\frac{1}{a+h}$$

$$= \sin\frac{1}{a} \quad \text{--- (3)}$$

Therefore, from (1), (2) and (3)

$$\lim_{x\to a^+} f(x) = f(a) = \lim_{x\to a^-} f(x)$$

$\therefore$ f is continuous at 'a'.

since 'a' is arbitary and $a \in \mathbb{R}^+$, f is continuous on $\mathbb{R}^+$.

Now,
let $\varepsilon > 0$. we shall show that for each $\delta > 0$
$\exists\ x_1, x_2 \in \mathbb{R}^+$ such that

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \varepsilon$$

consider for any $\delta > 0\ \exists\ m \in N$ such that

$$\frac{1}{2n\left(n\pi + \frac{\pi}{2}\right)} \leq \delta \quad \forall\ n \geq m$$

Take $x_1 = \dfrac{1}{m\pi + \frac{\pi}{2}}$ and $x_2 = \dfrac{1}{m\pi}$

$\therefore\ x_1, x_2 \in \mathbb{R}^+$ and

$$|x_1 - x_2| = \left|\frac{1}{m\pi + \frac{\pi}{2}} - \frac{1}{m\pi}\right|$$

$$= \left|\frac{-\frac{\pi}{2}}{m\pi\left(m\pi + \frac{\pi}{2}\right)}\right|$$

$$= \frac{1}{2m\left(m\pi + \frac{\pi}{2}\right)} < \delta$$

But,

$$|f(x_1) - f(x_2)| = \left|\sin\left(m\pi + \frac{\pi}{2}\right) - \sin m\pi\right|$$

$$= |\cos m\pi - 0|$$

$$= |(-1)^m|$$

$$= 1$$

which is not less than each $\varepsilon > 0$.

Hence, f is not uniformly continuous on $\mathbb{R}^+$.

4(c) Evaluate the integral $\displaystyle\int_\gamma \frac{z^2}{(z^2+1)(z-1)^2}\,dz$

where γ is the circle $|z| = 2$.

Sol. let $I = \displaystyle\int_\gamma \frac{z^2}{(z^2+1)(z-1)^2}\,dz = \int_\gamma f(z)\,dz$ say, where γ is the circle : $|z| = 2$.

Let $f(z) = \dfrac{z^2}{(z^2+1)(z-1)^2}$.

Then $f(z)$ has poles if $(z^2+1)(z-1)^2 = 0$.

$\Rightarrow z = \pm i,\ z = 1, 1.$

$\therefore f(z)$ has single poles at $z = \pm i$ and has double pole at $z = 1$.

$\therefore$ by residue theorem.

$$\int_{\gamma} f(z)\,dz = 2\pi i \begin{bmatrix} \text{residue at } 1 + \\ \text{residue at } i + \\ \text{residue at } -i \end{bmatrix} \quad \text{--- (1)}$$

$\therefore$ residue at $1 = \underset{z\to 1}{Lt} \frac{d}{dz}\left[\frac{\cancel{(z-1)^2}\cdot z^2}{(z^2+1)\cancel{(z-1)^2}}\right]$

$= \underset{z\to 1}{Lt} \frac{2z(z^2+1) - z^2(2z)}{(z^2+1)^2}$

$= \underset{z\to 1}{Lt} \frac{2z}{(z^2+1)^2} = \frac{1}{2}$

residue at i: $\underset{z\to i}{Lt}\left[\frac{\cancel{(z-i)}\,z^2}{\cancel{(z-i)}(z+i)(z-1)^2}\right]$

$= -\frac{1}{4}$

residue at $-i$: $\underset{z\to -i}{Lt} \frac{(z+i)\,z^2}{(z+i)(z-i)(z-1)^2} = -\frac{1}{4}$

$\therefore$ (1) $\equiv \int f(z)\,dz = 2\pi i\left[\frac{1}{2} - \frac{1}{4} - \frac{1}{4}\right]$

$= 0$

4(d) Show by solving the following LPP by simplex method that the problem has an unbounded solution Maximize $Z = 107x_1 + x_2 + 2x_3$

s.c. $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + x_2 - 6x_3 \le 5$

$3x_1 - x_2 - x_3 \le 0$

$x_1, x_2, x_3, x_4 \ge 0$.

Ans. 4(d)) Sol.

Standard form

Minimize $Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = 7/3$

$16x_1 + x_2 - 6x_3 + x_5 = 5$

$3x_1 - x_2 - x_3 + x_6 = 0$

$x_1, x_2, x_3, x_4 \geq 0$

$x_5, x_6 \geq 0$, slack variables

(T_1)

	c_j		107	1	2	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	MR
0	y_4	$\frac{7}{3}$	$\frac{14}{3}$	$\frac{1}{3}$	-2	1	0	0	$\frac{1}{2}$
0	y_5	5	16	1	-6	0	1	0	$5/16$
0	y_6	0	3	-1	-1	0	0	1	$0 \rightarrow$
$z_j - c_j$			$-107\uparrow$	-1	-2	0	0	0	

Introduce y_1 and drop y_6. 3 is the key element.

T_2

	c_j								
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
0	y_4	$\frac{7}{3}$	0	$17/9$	$-4/9$	1	0	$-14/9$	
0	y_5	5	0	$19/3$	$-2/3$	0	1	$-16/3$	$\leftarrow$
0	y_1	0	1	$-1/3$	$-1/3$	0	1	$1/3$	
$z_j - c_j$			0	$-\frac{110}{3}$	$-\frac{113}{3}$	0	0	$\frac{107}{3}$	

(T_2) shows that $z_3 - c_3 = -\frac{113}{3}$ being the most negative one. The column vector y_3 will not enters the basis since y_{i3} are non positive. This is a clear indication that the problem has an unbounded solⁿ.

5(a) Solve $(y+z)p + (z+x)q = x+y$.

Sol^n: Here the Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{——} \ (1)$$

Choosing 1, −1, 0 as multipliers, each fraction of

$$(1) = \frac{dx-dy}{(y+z)-(z+x)} = \frac{d(x-y)}{-(x-y)} \quad \text{——} \ (2)$$

Again, choosing 0, 1, −1 as multipliers, each fraction

$$\text{of } (1) = \frac{dy-dz}{(z+x)-(x+y)} = \frac{d(y-z)}{-(y-z)} \quad \text{——} \ (3)$$

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{——} \ (4)$$

$$(2), (3) \ \& \ (4) \Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{——} \ (5)$$

Taking the first two fractions of (5), $\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$

Integrating, $\log(x-y) = \log(y-z) + \log c_1$, c_1 being an arbitrary constant.

$\Rightarrow \log\{(x-y)/(y-z)\} = \log c_1 \Rightarrow (x-y)/(y-z) = c_1$ —— (6)

Taking the first and third fractions of (5)

$$\frac{2d(x-y)}{x-y} + \frac{d(x+y+z)}{x+y+z} = 0$$

Integrating $2\log(x-y) + \log(x+y+z) = \log c_2$

$$\Rightarrow (x-y)^2(x+y+z) = c_2 \quad \text{——} \ (7)$$

from (6) and (7), the required general solution is

$\phi\left[(x-y)^2(x+y+z), (x-y)/(y-z)\right] = 0$, ϕ being an arbitrary function.

5(b) Solve $(D+D'-1)(D+D'-3)(D+D')z = e^{x+y}\sin(2x+y)$.

Sol^n: C.F $= e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$P.I = \frac{1}{(D+D'-1)(D+D'-3)(D+D')} e^{x+y}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+1+D'+1-1)(D+1+D'+1-3)(D+1+D'+1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+1)(D+D'-1)(D+D'+2)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+2)(D^2+2DD'+D'^2-1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+2)}\cdot\frac{1}{-2^2-2(2\times1)-1^2-1}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{(D+D')^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{D^2+2DD'+D'^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{-2^2-2(2\times1)-1^2-4}\sin(2x+y)$$

$$= \frac{1}{130}\times e^{x+y}(D+D'-2)\sin(2x+y)$$

$$= \frac{1}{130}e^{x+y}\{2\cos(2x+y)+\cos(2x+y)-2\sin(2x+y)\}$$

$\therefore$ Solution is $z = e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)+\frac{1}{130}\cdot e^{x+y}\{3\cos(2x+y)-2\sin(2x+y)\}$

5(c) The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
velocity in km/hr	0	22	30	27	18	7	0

Apply simpson's rule to find distance covered by the car.

Sol'n: we know velocity $v = \frac{ds}{dt}$ —— (1) where s = distance, t = time

$\therefore$ (1) $\Rightarrow ds = v\,dt$

$\therefore$ so distance covered by car in 12 min. is

$$S = \int_0^S ds = \int_0^{12} v\,dt \quad \text{—— (2)}$$

Here given,

Time	$t_0=0$	$t_1=2$	$t_2=4$	$t_3=6$	$t_4=8$	$t_5=10$	$t_6=12$
velocity	$v_0=0$	$v_1=22$	$v_2=30$	$v_3=27$	$v_4=18$	$v_5=7$	$v_6=0$

(2) $\Rightarrow$ $S = \int_0^{12} v\,dt$

Using simpson's $\frac{1}{3}$ rule

we get $S = \frac{h}{3}\left[(v_0+v_6)+4(v_1+v_3+v_5)+2(v_2+v_4)\right]$

$= \frac{h}{3}\left[(0+0)+4(22+27+7)+2(30+18)\right]$ —— (3)

Since $\boxed{h = 2\text{ min} = \frac{2}{60} = \frac{1}{30}\text{ hour}}$

(3) $\Rightarrow$ $S = \frac{1/30}{3}\left[4(56)+2(48)\right] = 3.556\text{ km}$

Hence distance covered by car is 3.556 km

5(d) Sol^n: let the member one represents – M_1
similarly for 2nd — M_2
3rd — M_3

and if a person approve it then we'll represent it as – Y (yes)
if not – N (No)

Table;:

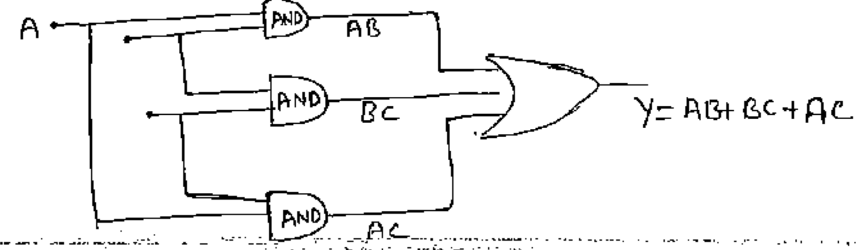
M_1	M_2	M_3	Result	
N	N	N	N	
N	N	Y	N	
N	Y	N	N	
Y	N	N	N	
N	Y	Y	Y	$\bar{A}BC$
Y	N	Y	Y	$A\bar{B}C$
Y	Y	N	Y –	$AB\bar{C}$
Y	Y	Y	Y –	$AB\bar{C}$

so solution is

$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

simplyfying we get $Y = AB + BC + AC$

Its circuit diagram



5(d) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved.

5(c). Use Hamilton's equations to find the equations of motion of a projectile in space.

Sol'n: Let (x, y, z) be the coordinates of a projectile in space at time t, if K and V are the kinetic and potential energies, then

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \text{ and } V = mgz$$

$$\therefore L = T - V = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Here x, y, z are the generalised coordinates,

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \text{--- (1)}$$

Since L doesnot contain t explicitly, therefore

$$H = T + V = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$$\Rightarrow H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz, \quad \text{(using relations (1))}$$

Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \text{ --- } (H_1) \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ --- } (H_2)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \text{ --- } (H_3) \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \text{ --- } (H_4)$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg \text{ --- } (H_5) \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \text{ --- } (H_6)$$

from (H_1) and (H_2), we have $\ddot{x} = \left(\frac{1}{m}\right)\dot{p}_x = 0$ --- (2)

from (H_3) and (H_4), we have $\ddot{y} = \left(\frac{1}{m}\right)\dot{p}_y = 0$ --- (3)

from (H_5) and (H_6) we have $\ddot{z} = \left(\frac{1}{m}\right)\dot{p}_z$ i.e. $\ddot{z} = -g$ --- (4)

Equations (2), (3), (4) are the equations of motion of projectile in space.

5(d) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved.

6(a). Form partial Differential equation by eliminating arbitrary functions f and g from $z = f(x^2-y)+g(x^2+y)$

Sol'n: Given $z = f(x^2-y)+g(x^2+y)$ —— ①

Differentiating ① partially w.r.to x and y, we get

$$\frac{\partial z}{\partial x} = 2xf'(x^2-y)+2xg'(x^2+y) = 2x\{f'(x^2-y)+g'(x^2+y)\} \text{ — ②}$$

and $\frac{\partial z}{\partial y} = -f'(x^2-y)+g'(x^2+y)$ — ③

Differentiating ② and ③ w.r.t x and y respectively, we get

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2-y)+g'(x^2+y)\} + 4x^2\{f''(x^2-y)+g''(x^2+y)\} \text{ — ④}$$

and $\frac{\partial^2 z}{\partial y^2} = f''(x^2-y)+g''(x^2+y)$ — ⑤

Again, ② $\Rightarrow f'(x^2-y)+g'(x^2+y) = \left(\frac{1}{2x}\right)\left(\frac{\partial z}{\partial x}\right)$ — ⑥

Substituting the values of $f''(x^2-y)+g''(x^2+y)$ and $f'(x^2-y)+g'(x^2+y)$ from ⑤ and ⑥ in ④, we have

$$\frac{\partial^2 z}{\partial x^2} = 2\times\left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}$$

which is the required partial differential equation.

6(b). Find a surface satisfying $r-2s+t=6$ and touching the hyperbolic paraboloid $z=xy$ along its section by the plane $y=x$.

Sol'n: Rewriting the given equation

$$\frac{\partial^2 z}{\partial x^2} - 2\left(\frac{\partial^2 z}{\partial x\partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 6$$

$$\Rightarrow (D^2-2DD'+D'^2)z = 6 \Rightarrow (D-D')^2 z = 6 \text{ — ①}$$

Its C.F $= \phi_1(y+x) + x\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $P.I = \frac{1}{(D-D')^2}\cdot 6 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2} 6$

$= \frac{1}{D^2}\left(1+\frac{2D'}{D}+\cdots\right)6 = \frac{1}{D^2}6 = 3x^2$

$\therefore$ General solution of (1) is

$z = C.F + P.I = \phi_1(y+x) + x\phi_2(y+x) + 3x^2$ —— (2)

Since the required surface (1) touches the given surface

$z = xy$ —— (3)

along the section $y=x$, the values of p and q for the surfaces must be equal for any point on the plane $y=x$ —— (4)

Now equating the values of p and q from (2) and (3)

$p = \phi_2(y+x) + x\phi_2'(y+x) + \phi_1'(y+x) + 6x = y$ —— (5)

and $q = x\phi_2'(y+x) + \phi_1'(y+x) = x$ —— (6)

Subtracting (6) from (5) and using (4), we get

$\phi_2(2x) = -6x = -3\times(2x)$

which gives $\phi_2(y+x) = -3(y+x)$ —— (7)

from (7) $\phi_2'(y+x) = -3$, Then (5) becomes

$-3x + \phi_1'(y+x) = x$ so that $\phi_1'(2x) = 2(2x)$ as $y=x$

Now, $\phi_1'(2x) = 2(2x) = \phi_1'(x) = 2x$ —— (8)

Integrating (8), $\phi_1(x) = x^2+c$ which gives $\phi_1(y+x) = (y+x)^2+c$ —— (9)

Putting the values of $\phi_2(y+x)$ and $\phi_1(y+x)$ given by (7) and (9) in (2), we get

$z = x\{-3(y+x)\} + (y+x)^2 + c + 3x^2 \Rightarrow z = x^2 - xy + y^2 + c$ —— (10)

Equating the values of z from (3) and (10), we get

$xy = x^2 - xy + y^2 + c \Rightarrow x^2 = x^2 - x^2 + x^2 + c$ using (4)

giving $c=0$. Hence required surface is $z = x^2 - xy + y^2$.

6(c). Reduce $y^2\left(\frac{\partial^2 z}{\partial y^2}\right)+\frac{\partial^2 z}{\partial x^2}=0$ to canonical form.

Sol^n: Given $y^2\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x^2}=0$ ——— (1)

comparing it with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

$\therefore R=1,\ S=0,\ T=x^2$

$\therefore S^2-4RT=-4x^2<0,\ x\neq 0.$

$\therefore$ equation (1) is elliptic.

The λ-quadratic $R\lambda^2+S\lambda+T=0$ reduces to

$\lambda^2+x^2=0 \Rightarrow \lambda=ix,\ -ix.$

The corresponding characteristic equations are given by

$$\frac{dy}{dx}+ix=0 \quad\text{and}\quad \frac{dy}{dx}-ix=0.$$

$$\therefore\quad y+i\left(\frac{x^2}{2}\right)=c_1 \quad\text{and}\quad y-i\left(\frac{x^2}{2}\right)=c_2$$

choose $u=y+i\left(\frac{x^2}{2}\right)=\alpha+i\beta$

and $v=y-i\frac{x^2}{2}=\alpha-i\beta$

where $\alpha=y$ and $\beta=\frac{x^2}{2}$

Now,

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}=x\frac{\partial z}{\partial \beta}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)=\frac{\partial z}{\partial \beta}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$=\frac{\partial z}{\partial \beta}+x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$$

$$= \frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2}$$

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$

$$\therefore \frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2} + x^2 \frac{\partial^2 z}{\partial \alpha^2} = 0$$

$$\therefore \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta} \frac{\partial z}{\partial \beta} \quad \text{as } \beta = \frac{x^2}{2}$$

6(d) ~~A~~ tightly stretched elastic ~~string~~ string of length l, with fixed end points $x=0$ and $x=l$ is initially in the position given by $y = c\sin^3\left(\frac{\pi x}{l}\right)$, c being constant.
It is released from the position of rest. Find the displacement $y(x,t)$

Ans.

6(d) Sol'n The P.D.E. of the tranverse vibrations of the given elastic string in given by.

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{a^2}\right)\left(\frac{\partial^2 y}{\partial t^2}\right) \quad \text{---} (1)$$

where $y(x,t)$ is the deflection of the string and a in a constant.

Given boundary and initial conditions are:

Boundary Condition (B.C.): $y(0,t) = y(l,t) = 0$ for all t.

Initial Conditions (I.C): $y(x,0) = C\sin^3(\pi x/l)$

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = y_t(x,0) = 0$. (Initial velocity).

let (1) be of the form $y(x,t) = X(x)\,T(t)$.

$$\therefore \quad X''T = \left(\frac{1}{a^2}\right) X\,T''$$

$$\therefore \quad \frac{X''}{X} = \frac{T''}{a^2 T} \quad \text{---} (2)$$

Since x and t are independent variables. eq^n(2) can only be true if each side is equal to the same constant, say μ.

$$X'' - \mu X = 0 \text{ and } T'' - \mu a^2 T = 0 \quad -(3)$$

$$X(0)\,T(t) = 0 \text{ and } X(l)\,T(t) = 0.$$

since $T(t)=0$ leads to $y=0$, hence we assume that $T(t) \neq 0$.

$\therefore\ X(0)=0$ and $X(l)=0$

Now, we solve $X''-\mu X=0$ under B.C.

Case - I

let $\mu=0$

$\therefore\ X(x)=Ax+B$

$\therefore\ X(0)=B$ and $X(l)=Al+B$

and $0=B$ and $0=Al+B$.

$\Rightarrow A=B=0$ so that $X(x)=0$

$\Rightarrow y\equiv 0$

but I.C. does not satisfy.

$\therefore\ \mu\neq 0$.

Case-II

let $\mu=\lambda^2$ where $\lambda\neq 0$.

$\therefore\ X(x)=Ae^{\lambda x}+Be^{-\lambda x}$

By using B.C., $A+B=0$ and $Ae^{l\lambda}+Be^{-l\lambda}=0$

$\Rightarrow A=B=0$ and hence $X(x)=0$.

$\therefore\ \mu\neq\lambda^2$

Case-III

$\mu=-\lambda^2$ where $\lambda\neq 0$.

$\therefore\ X(x)=A\cos\lambda x+B\sin\lambda x$

Using BC. $A=0$ and $A\cos\lambda l+B\sin\lambda l=0$

$\Rightarrow A=0$ and $\sin\lambda l=0$. where we have taken $B\neq 0$. since otherwise $X\equiv 0$ so that $y=0$

put $t=0$ and using I.C.,

$$C\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

$$\therefore \quad 0 = \sum_{n=1}^{\infty}\left(\frac{n\pi a F_n}{l}\right)\sin\frac{n\pi x}{l}$$

are fourier s.

$$E_n = \frac{2}{l}\int_0^l C\sin^3\left(\frac{\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

and $\dfrac{n\pi a F_n}{l} = \dfrac{2}{l}\displaystyle\int_0^l 0\cdot\sin\left(\frac{n\pi x}{l}\right)$

$\Rightarrow F_n = 0$.

Now, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\Rightarrow \quad \sin^3\theta = \frac{(3\sin\theta - \sin 3\theta)}{4}$$

$$\therefore \quad \sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$\therefore \quad E_n = \frac{2C}{l}\int_0^l \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{3C}{2l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx - \frac{C}{2l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

which does not satisfy I.C.

Now, $\sin \lambda l = 0 \Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$; $n = 1, 2, 3, \ldots$

from (3),

$$\frac{d^2T}{dt^2} + \frac{n^2\pi^2a^2}{l^2} T = 0$$

as $\mu = -\lambda^2 = -\frac{n^2\pi^2}{l^2}$

whose general solution is $T_n(t) = C_n \cos \frac{n\pi a t}{l} + D_n \sin \frac{n\pi a t}{l}$

$$\therefore \quad y_n(x,t) = X_n(t)\, T_n(t)$$

$$= \left(E_n \cos \frac{n\pi a t}{l} + F_n \sin \frac{n\pi a t}{l}\right) \sin \frac{n\pi x}{l}$$

are solutions of (1) satisfying (2) for $n = 1, 2, 3, \ldots$

Here $E_n (= C_n B_n)$ and $F_n (= B_n D_n)$ are new arbitary constants. In order to obtain a solⁿ also satisfying I.C, we consider most general solⁿ of the form.

$$y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi a t}{l} + F_n \sin \frac{n\pi a t}{l}\right) \sin \frac{n\pi x}{l}$$

Differentiating partially w.r.t. 't',

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{n\pi a E_n}{l} \sin \frac{n\pi a t}{l} + \frac{n\pi a F}{l} \cos \frac{n\pi a t}{l}\right) \sin \frac{n\pi x}{l}$$

$$\therefore\ I = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} 1/2 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

If $m = n$ then.

$$I = \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[x - \frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\right]_0^l$$

$$= \frac{l}{2}$$

if $m \neq n$.

$$I = \frac{1}{2}\int_0^l 2\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^l \left(\cos\frac{(n-m)\pi x}{l} - \cos\frac{(n+m)\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\frac{+1}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l} - \frac{1}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\right]_0^l$$

$$= 0$$

$$E_1 = \frac{3c}{2l}\times\frac{l}{2} = \frac{3c}{4}$$

and $E_3 = -\dfrac{c}{2l}\times\dfrac{l}{2} = -\dfrac{c}{4}$

Also, $E_n = 0$ for $n \neq 1$ and $n \neq 3$

$$y(x,t) = \frac{c}{4}\left[3\cos\frac{\pi a t}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi a t}{l}\sin\frac{3\pi x}{l}\right]$$

Since, $E_n = 0$.

$$y(x,t) = E_1 \cos\frac{\pi a t}{l}\sin\frac{\pi x}{l} + E_3\cos\frac{3\pi a t}{l}\sin\frac{3\pi x}{l}$$

hence,

$$y(x,t) = \frac{\epsilon}{4}\left[3\cos\frac{\pi a t}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi a t}{l}\sin\frac{3\pi x}{l}\right]$$

7(a) Solve the following system of linear equations correct to two decimal places by Gauss-Seidel method:

$$10x + y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22$$

Sol.

7(a) Sol'n For given system of linear equations,

$$x^{(k)} = \frac{1}{10}\left(9 - 2y^{(k-1)} - z^{k-1}\right) \quad \text{——} (1)$$

$$y^{k} = \frac{1}{20}\left(-44 - 2x^{k} + 2z^{k-1}\right) \quad \text{——} (2)$$

$$z^{k} = \frac{1}{10}\left(22 + 2x^{k} - 3y^{k}\right) \quad \text{——} (3)$$

For 1st iteration, taking $y^{(0)} = 0$, $z^{(0)} = 0$

$$x^{(1)} = \frac{1}{10}(9 - 0 - 0) = 0.9$$

$$\therefore \quad x^{(1)} = 0.9$$

$$y^{(1)} = \frac{1}{20}\left(-44 - 2\times 0.9 + 0\right) = -2.29$$

$$z^{(1)} = \frac{1}{10}\left(22 + 2\times 0.9 - 3\times(-2.29)\right) = 3.067$$

2nd iteration, $K = 2$

$$x^{(2)} = \frac{1}{10}\left(9 - 2\times(-2.29) - 3.067\right) = 1.0513$$

$$y^{(2)} = \frac{1}{20}\left(-44 - 2\times(1.0513) + 2\times 3.067\right) = -1.99843$$

$$z^{(2)} = \frac{1}{10}\left(22 + 2\times(1.0513) - 3\times -1.99843\right) = 3.009$$

3rd iteration

$$x^{(3)} = \frac{1}{10}\left(9 - 2\times(-1.998) - 3.009\right) = 0.9987$$

$$y^{(3)} = \frac{1}{20}\left(-44 - 2\times(0.9987) + 2\times 3.009\right) = -1.999$$

$$z^{(3)} = \frac{1}{10}\left[22 + 2\times(0.999) + 3\times(1.998)\right] = 2.9992$$

$\therefore$ Solution correct to two decimal places are $\boxed{x = 0.99, \ y = -1.99, \ z = 2.99}$

7(b) Given $\frac{dy}{dx} = 1+y^2$, where $y=0$ when $x=0$, find $y(0.2)$, $y(0.4)$ and $y(0.6)$.

Let $f(x,y) = \frac{dy}{dx} = 1+y^2$

Taking $h = 0.2$, $x_0=0$, $y_0=0$

Applying Runge-kutta method of order four;

$\because K_1 = hf(x_0, y_0) = 0.2(1+y_0^2) = 0.2(1+0) = 0.2$

$K_2 = hf(x_0+h/2, y_0+K_1/2) = (0.2)[1+(0.1)^2] = 0.202$

$K_3 = hf(x_0+h/2, y_0+K_2/2) = 0.2[1+0.101^2] = 0.20204$

$K_4 = h[f(x_0+h, y_0+K_3)] = 0.2[1+0.20204^2] = 0.20816$

$\therefore K = \frac{K_1+2K_2+2K_3+K_4}{6} = 0.2027$

$\therefore x_1 = x_0+h = 0+0.2 = 0.2$

$\therefore y_1(0.2) = y_0+K = 0+0.2027 = 0.2027$

For $y(0.4)$:

$x_1 = x_0+h = 0.2$, $y_1 = 0.2027$, $h = 0.2$

$K_1 = hf(x_1, y_1) = 0.2[1+(0.2027)^2] = 0.2082$

$K_2 = hf(x_1+h/2, y_1+K_1/2) = 0.2[1+(0.2027+0.1041)^2] = 0.2188$

$K_3 = hf(x_1+h/2, y_1+K_2/2) = 0.2[1+(0.2027+0.1094)^2] = 0.2195$

$K_4 = hf(x_1+h, y_1+K_3) = 0.2[1+(0.2+0.2195)^2] = 0.2357$

$\therefore K = \dfrac{k_1+2k_2+2k_3+k_4}{6}$

$= 0.2201$

$\therefore y_2(0.4) = y_1 + K = 0.2027 + 0.2201$

$\boxed{y_2(0.4) = 0.4228}$

For $y_3(0.6)$

$x_2 = x_1 + h = 0.4$, $y_2 = 0.4228$, $h = 0.2$

$K_1 = hf(x_2, y_2) = 0.2\left[1 + (0.4228)^2\right] = 0.2357$

$K_2 = hf(x_2 + h/2, y_2 + k_1/2) = 0.2\left[1 + (0.4228 + 0.1178)^2\right]$

$= 0.2584$

$K_3 = hf(x_2 + h/2, y_2 + k_2/2) = 0.2\left[1 + \left(0.4228 + \frac{0.2584}{2}\right)^2\right]$

$= \cancel{0.2605} = 0.2609$

$K_4 = hf(x_2 + h, y_2 + k_3) = 0.2\left[1 + (0.4228 + 0.2609)^2\right]$

$= 0.2935$

$\therefore K = \frac{1}{6}(k_1 + k_4 + 2k_2 + 2k_3) = 0.2613$

$\therefore y_3(0.6) = y_2(0.4) + K = 0.4228 + 0.2613$

$= 0.6841$

$\therefore$ $y(0.2) = 0.2027$

$y(0.4) = 0.4228$

$y(0.6) = 0.6841$

7(C) convert:

(i) 46655 given to be in the decimal system into one in base 6.

(ii) $(11110.01)_2$ into a number in the decimal system

7 (c) (10)

(i) Converting 46655 decimal system into system with base 6;

		Remainder
6	46655	
6	7775	5
6	1295	5
6	215	5
6	35	5
	5	

So, $(46655)_{10} = (55555)_6$

(ii) Converting $(11110.01)_2$ into decimal system

$$(11110\cdot01)_2 = (1\times2^4 + 1\times2^3 + 1\times2^2 + 1\times2^1 + 0\cdot2^0) + (0\times2^{-1} + 1\times2^{-2})$$

$$= 16+8+4+2+0+0+\frac{1}{4}$$

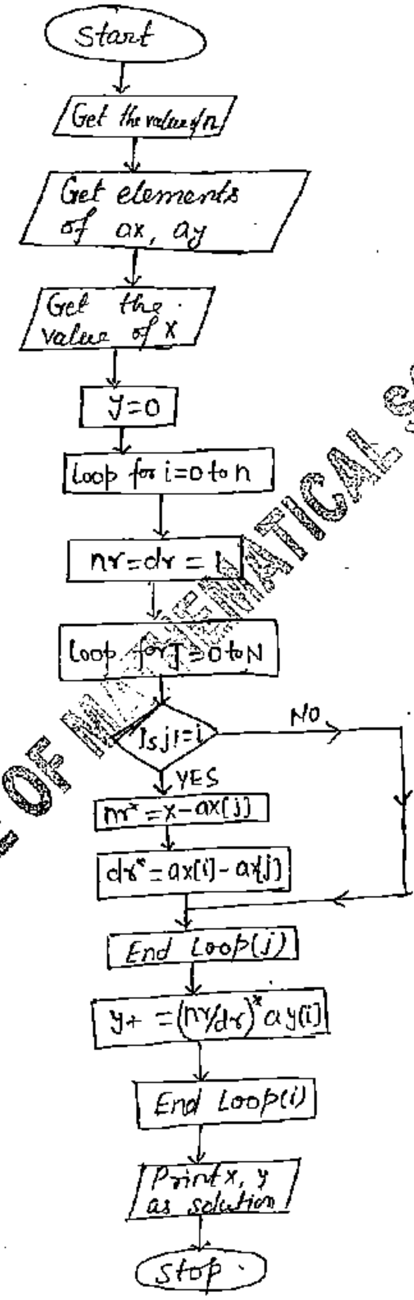
$$= 30 + 0.25$$

So $\boxed{(11110\cdot01)_2 = (30\cdot25)_{10}}$

7(d) Draw a flow chart for Lagrange's Interpolation formula:

(7) (d) Lagrange's Interpolation Formulae

Ans:-



Note:- MAX is the maximum value of n

ax is an array containing values of x $(x_0, x_1, \ldots x_n)$

ay is an array containing values of y $(y_0, y_1, \ldots y_n)$

x is the value of x at which value of y is wanted.

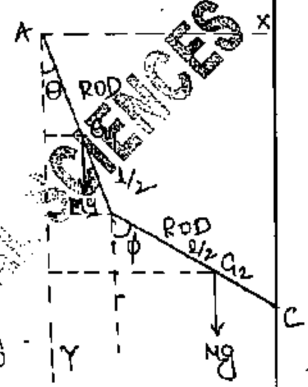
y is the calculated value of y.

nr - Numerator of the terms in expansion of y

dr - is denominator of the terms in expansion of y.

8(a) Two equal rods AB and BC, each of length l. Smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $2\pi/n$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

Sol'n: Let AB and BC be the rods of equal length l and mass M. At time t, let the two rods make angles θ and ϕ to the vertical respectively.



Referred to A as origin horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G. G_2 of rod BC are given by

$x_{G_1} = \frac{1}{2} l \sin\theta$, $y_{G_1} = \frac{1}{2} l \cos\theta$

$x_{G_2} = l \sin\theta + \frac{1}{2} l \sin\phi$, $y_{G_2} = l\cos\theta + \frac{1}{2} l \cos\phi$

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 and G_2

then $v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\frac{1}{2} l \cos\theta\dot\theta\right)^2 + \left(-\frac{1}{2} l \sin\theta\dot\theta\right)^2 = \frac{1}{4} l^2 \dot\theta^2$

$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta\dot\theta + \frac{1}{2} l \cos\phi\dot\phi\right)^2 + \left(-l\sin\theta\dot\theta - \frac{1}{2} l \sin\phi\dot\phi\right)^2$

$= l^2\left[\dot\theta^2 + \frac{1}{4}\dot\phi^2 + \dot\theta\dot\phi\cos(\theta-\phi)\right]$

$= l^2\left[\dot\theta^2 + \frac{1}{4}\dot\phi^2 + \dot\theta\dot\phi\right]$ ($\because \theta, \phi$ are small)

If T be the total K.E and W the work function of the system then.

$T =$ K.E of rod AB + K.E of rod BC

$= \left[\frac{1}{2} M \cdot \frac{1}{3}\left(\frac{1}{2}l\right)^2\dot\theta^2 + \frac{1}{2} M \cdot v_{G_1}^2\right] + \left[\frac{1}{2} M \cdot \frac{1}{3}\left(\frac{1}{2}l\right)^2\dot\phi^2 + \frac{1}{2} M \cdot v_{G_2}^2\right]$

$= \frac{1}{2} M\left[\frac{1}{12} l^2\dot\theta^2 + \frac{1}{4} l^2\dot\theta^2\right] + \frac{1}{2} M\left[\frac{1}{12} l^2\dot\phi^2 + l^2\left(\dot\theta^2 + \frac{1}{4}\dot\phi^2 + \dot\theta\dot\phi\right)\right]$

$= \frac{1}{2} M l^2\left(\frac{4}{3}\dot\theta^2 + \frac{1}{3}\dot\phi^2 + \dot\theta\dot\phi\right)$

and $W = Mg\,y_{G_1} + Mg\,y_{G_2} + C = Mg\left[\frac{1}{2} l \cos\theta + l\cos\theta + \frac{1}{2} l \cos\phi\right] + C$

$= \frac{1}{2}Mgl(3\cos\theta + \cos\phi)$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{2}Ml^2\left(\frac{8}{3}\dot\theta + \dot\phi\right)\right] - 0 = \frac{1}{2}Mgl(-3\sin\theta) = -\frac{3}{2}Mgl\theta$ ($\because \theta$ is small)

$\Rightarrow 8\ddot\theta + 3\ddot\phi = -9c\theta$, (where $c = g/l$) —— (1)

Equations (1) & (2) can be written as

$(8D^2+9c)\theta + 3D^2\phi = 0$ and $3D^2\theta + (2D^2+3c)\phi = 0$

Eliminating ϕ between these two equations we get

$\left[(2D^2+3c)(8D^2+9c) - 9D^4\right] = 0$

$\Rightarrow (7D^4 + 42cD^2 + 27c^2)\theta = 0$ —— (3)

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$\theta = A\cos(nt+B)$. $\therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta$.

Substituting in (3), we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

$\Rightarrow 7n^4 - 42cn^2 + 27c^2 = 0$ $\quad \because \theta \neq 0$

$\therefore n^2 = \frac{42c \pm \sqrt{(42c)^2 - 4.7.27c^2}}{2.7}$

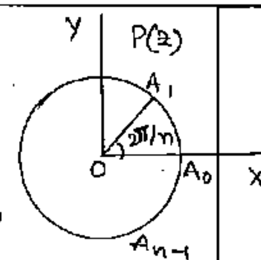
$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt7}\right)c = \left(3 \pm \frac{6}{\sqrt7}\right)\frac{g}{l}$ $\quad (\because c = g/l)$

8(b) → If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, Prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of the liquid.

Sol'n: The n vortices are at $A_0, A_1, A_2 \cdots A_{n-1}$

Such that

$\angle A_0OA_1 = \angle A_1OA_2 = \cdots = \angle A_{n-1}OA_1 = \frac{2\pi}{n}$



The coordinates of the points A_r are given by

$z = z_r = a e^{(2\pi/n) ir}$ where $r = 0, 1, 2, \cdots n-1$

these are n roots of the equation $z^n - a^n = 0$

[For $z^n - a^n = 0 \Rightarrow z^n = a^n e^{2\pi r i}$]

Hence $z^n - a^n = (z-z_0)(z-z_1)\cdots(z-z_{n-1})$

The complex potential due to n vortices at P is given by

$$w = \frac{ik}{2\pi}\left[\log(z-z_0) + \log(z-z_1) + \cdots + \log(z-z_{n-1})\right]$$

$$= \frac{ik}{2\pi}\log(z-z_0)(z-z_1)\cdots(z-z_{n-1}) = \frac{ik}{2\pi}\log(z^n - a^n) \quad \text{---(1)}$$

For the point A_0, $z = a$ so that $r = a$, $\theta = 0$

If w' is the complex potential at A_0, then

$$w' = w - \frac{ik}{2\pi}\log(z-a) = \frac{ik}{2\pi}\left[\log(z^n - a^n) - \log(z-a)\right]$$

$$\phi' + i\psi' = \frac{ik}{2\pi}\left[\log(r^n e^{in\theta} - a^n) - \log(re^{i\theta} - a)\right]$$

$$\therefore \psi' = \frac{k}{4\pi}\left[\log(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta) - \log(r^2 + a^2 - 2ra\cos\theta)\right]$$

$$\frac{\partial\psi'}{\partial r} = \frac{k}{4\pi}\left[\frac{2nr^{2n-1} - 2nr^{n-1}a^n\cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n\cos n\theta} - \frac{2r - 2a\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi}\left[\frac{2nr^n a^n \sin n\theta}{r^{2n} - 2r^n a^n\cos n\theta + a^{2n}} - \frac{2ra\sin\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a} = \frac{k}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1+\cos n\theta}\right) - \left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right] = \frac{k}{4\pi a}(n-1)$$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta} - \frac{\sin\theta}{1-\cos\theta}\right]$$

Since $\lim_{x\to 0}\frac{F(x)}{G(x)} = \lim_{x\to 0}\frac{F'(x)}{G'(x)} = \lim_{x\to 0}\frac{F''(x)}{G''(x)}$, $\left[\text{form } \frac{0}{0}\right]$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta} - \frac{\cos\theta}{\sin\theta}\right] \text{ as } \theta\to 0$$

$$= \frac{k}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta} - \frac{(-\sin\theta)}{\cos\theta}\right] \text{ as } \theta\to 0$$

$$= \frac{k}{4\pi}[0+0] = 0$$

finally, $\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi a}(n-1)$, $\frac{\partial\psi'}{\partial\theta} = 0$ as $r\to a$, $\theta\to 0$

Consequently, the velocity q_0 of the vortex A_0 is given by

$$q_0 = \left[\left(\frac{\partial\psi'}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi'}{\partial\theta}\right)^2\right]^{1/2} = \frac{k(n-1)}{4\pi a}$$

This proves that the whole of velocity is along the tangent and there is no velocity along the normal to the circle. Hence the vortices will move round the cylinder with uniform velocity $k(n-1)/4\pi a$. The time of one complete revolution

$$= \frac{\text{distance}}{\text{velocity}} = \frac{2\pi a}{k(n-1)/4\pi a} = \frac{8\pi^2 a^2}{(n-1)k}$$

8(c) An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r=c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}$.

Sol'n: Let v be the velocity, P the pressure at a distance x from the origin, then the equations of motion and continuity are respectively,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and $x^2 v = F(t)$ so that $v = \frac{F(t)}{x^2}$, $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$

$$\therefore \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v\right)^2 = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Integrating $\frac{F'(t)}{x} + \frac{1}{2}v^2 = \frac{2\mu}{\sqrt{x}} - \frac{p}{\rho} + c$ —— (1)

Boundary conditions are

→(2) when $x = \infty$, $v = 0$, $p = 0$

→(3) when $x = r$, (radius of Cavity), $p = 0$, $v = \dot{r}$

→(4) when $r = c$, $v = 0$ so that $F(t) = 0$

→(5) Let T be the required time of filling the cavity.

Subjecting (1) to the conditions (2) and (3)

$$0 + 0 = 0 - 0 + C \text{ and } \frac{-F'(t)}{r} + \frac{1}{2}(\dot{r}^2) = \frac{2\mu}{\sqrt{r}} - 0 + c$$

$$\Rightarrow \frac{-F'(t)}{r} + \frac{1}{2}\dot{r}^2 = \frac{2\mu}{\sqrt{r}}$$

Since $r^2(\dot{r}) = F(t) \Rightarrow r^2 dr = F(t)\,dt$

Multiplying by $2F(t)\,dt$ (or) $2r^2 dr$

$$\frac{-2F'(t)F(t)\,dt}{r} + \frac{F^2(t)}{r^4}\cdot r^2 dr = \frac{4\mu}{\sqrt{r}}\cdot r^2 dr$$

$$\Rightarrow d\left[\frac{-F^2(t)}{r}\right] = 4\mu r^{3/2}\,dr$$

Integrating, $-\frac{F^2(t)}{r} = 4\mu\cdot\frac{2}{5}r^{5/2} + A$ —— (6)

Subjecting (6) to (4) $\quad 0 = \frac{8\mu}{5}c^{5/2} + A$

Now (6) $\Rightarrow \frac{-(r^2\dot{r})^2}{r} = \frac{8\mu}{5}\left(r^{5/2} - c^{5/2}\right)$

$$\Rightarrow \frac{dr}{dt} = -\left[\frac{8\mu}{5r^3}\cdot\left(c^{5/2} - r^{5/2}\right)\right]^{1/2}$$

[−ve sign is taken as velocity increases when r decreases]

$$-\int_{c}^{0}\frac{r^{3/2}}{\left[c^{5/2}-r^{5/2}\right]^{1/2}}\,dr = \int_{0}^{T}\left(\frac{8\mu}{5}\right)^{1/2}dt$$

$$\Rightarrow \quad T = \left(\frac{5}{8\mu}\right)^{1/2}\int_{0}^{c}\frac{r^{3/2}}{\left(c^{5/2}-r^{5/2}\right)^{1/2}}\,dr \text{ —— } (7)$$

Put $r^{5/2} = c^{5/2}\sin^2\theta$, $\frac{5}{2}r^{3/2}\,dr = c^{5/2}\cdot 2\sin\theta\cos\theta\,d\theta$

$$T = \left(\frac{5}{8\mu}\right)^{1/2}\int_{0}^{\pi/2}\frac{4}{5}c^{5/2}\cdot\frac{\sin\theta\cos\theta\,d\theta}{c^{5/4}\cos\theta} = \left(\frac{5}{8\mu}\right)^{1/2}\cdot\frac{4}{5}c^{5/4}\left(-\cos\theta\right)_{0}^{\pi/2}$$

$$\Rightarrow T = \left(\frac{2}{5\mu}\right)^{1/2}\cdot c^{5/4}.$$

Main Test Series – 2016
Test – IX – Answer key
Paper I

1(a) (i) Show that the diagonal elements of the square of an anti-Hermitian matrix are either zero or negative.

(ii) Prove that the eigen values of a hermitian matrix are always real.

(iii) Use the above result to show that $\det(H - 3iI)$ cannot be zero, if H is a hermitian matrix and I is the unit matrix.

Soln: Let A be an anti-Hermitian matrix. then $A^{\theta} = -A$ —— (1).

Here the elements on the principal diagonal must be purely imaginary number (or) zero.

Ex:- $A = \begin{bmatrix} 4i & 2+i & 3 \\ -2+i & 0 & 4i \\ -3 & 4i & -3i \end{bmatrix}_{3\times3}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ be any anti-Hermitian matrix.

$a_{ii} = 0$ or Ki where K is any real number.

$\because (Ki)^2 = -K^2$ (Here K^2 is +ve)

$\therefore$ square of diagonal elements are either '0' (or) –ve.

(ii) let A be a Hermitian matrix then $A^\theta = A$.
Let λ be a characteristic root of A and X is a corresponding eigen vector then $AX = \lambda X$. ——— (1)

We have $X^\theta A X = \lambda X^\theta X$. ——— (2)

$\Rightarrow (X^\theta A X)^\theta = (\lambda X^\theta X)^\theta$

$\Rightarrow X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda} X^\theta (X^\theta)^\theta$

$\Rightarrow X^\theta A^\theta X = \bar{\lambda} X^\theta X$

$\Rightarrow X^\theta A X = \bar{\lambda} X^\theta X$ ——— (3)

from (2) & (3) we have

$\lambda X^\theta X = \bar{\lambda} X^\theta X$.

$\Rightarrow (\lambda - \bar{\lambda}) X^\theta X = 0$ ——— (4)

$\Rightarrow \lambda - \bar{\lambda} = 0 \quad (\because X^\theta X \neq 0)$.

$\lambda = \bar{\lambda}$

$\therefore \lambda$ is real. ($\because \bar{z} = z$ $\therefore z$ is real)

(iii) from (ii), the characteristic roots of Hermitian matrix are real.

$\therefore HX \neq (3i)X$.

$\Rightarrow (H - 3iI)X \neq 0$.

$\Rightarrow |H - 3iI| \neq 0$.

1(b)(i) If the vectors $(0,1,a)$, $(1,a,1)$, $(a,1,0)$ in $\mathbb{R}^3(\mathbb{R})$ are linearly dependent, find the value of a.

(ii) Show that the vector $(1+i, 2i)$, $(1, 1+i)$ are linearly dependent in $\mathbb{C}^2(\mathbb{C})$ and linearly independent in $\mathbb{C}^2(\mathbb{R})$.

Sol (i) Given that the vector $(0,1,a)$, $(1,a,1)$, $(a,1,0)$ are linearly dependent.

$$\therefore \begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} = 0 \Rightarrow +a + a(1-a^2) = 0$$

$$\Rightarrow a + a - a^3 = 0$$

$$\Rightarrow 2a - a^3 = 0$$

$$\Rightarrow a(2-a^2) = 0$$

$$\Rightarrow a = 0 \text{ (or) } a = \pm\sqrt{2}$$

(ii) let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{C})$.

Since one of the vector of 'S' is a scalar multiple of other.

i.e $(1+i, 2i) = (1+i)(1, 1+i)$.

S is L.D subset of $\mathbb{C}^2(\mathbb{C})$.

let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{R})$

Since 'no' vector of 'S' is a scalar multiple of ~~the~~ other over $\mathbb{R}$.

$\therefore$ S is L.I. subset of $\mathbb{C}^2(\mathbb{R})$.

1(c) Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

Sol^n: Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx$ ——— (1)

then $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x+\sin x}\,dx$ ——— (2)

Adding (1) and (2), we get

$2I = \int_0^{\pi/2} \frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}\,dx = \int_0^{\pi/2} \frac{dx}{\sin x+\cos x}$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{dx}{\sin\left(\frac{\pi}{4}+x\right)}$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \operatorname{cosec}\left(\frac{\pi}{4}+x\right) dx$

$\Rightarrow I = \frac{1}{\sqrt{2}}\left[\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right]_0^{\pi/4}$

$= \frac{1}{\sqrt{2}}\left[\log\left\{\operatorname{cosec}\left(x+\frac{\pi}{4}\right) - \cot\left(x+\frac{\pi}{4}\right)\right\}\right]_0^{\pi/4}$

$= \frac{1}{\sqrt{2}}\log\left(\frac{\operatorname{cosec}\frac{\pi}{2} - \cot\frac{\pi}{2}}{\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}}\right)$

$= \frac{1}{\sqrt{2}}\log\left(\frac{1-0}{\sqrt{2}-1}\right)$

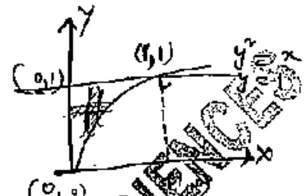
$I = \frac{1}{\sqrt{2}}\log(1+\sqrt{2})$

$\therefore \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\log(1+\sqrt{2})$

1(d) Evaluate the integral $\int_0^1\int_{\sqrt{x}}^1 e^{x/y}\,dx\,dy$, by changing the order of integration.

Sol: Given curves are $y=\sqrt{x}$; $y=1$
$\Rightarrow y^2=x$; $y=1$

Taking the limits of
y from 0 to 1.
x from 0 to y^2



$$\therefore \int_{y=0}^{1}\int_{x=0}^{y^2} e^{x/y}\,dx\,dy = \int_{y=0}^{1}\left[y e^{x/y}\right]_{x=0}^{y^2} dy$$

$$= \int_{y=0}^{1} \cancel{y}e^{y}\,dy$$

$$= \int_0^1 (ye^y - y)\,dy$$

$$= \left[ye^y - e^y - \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}.\ \underline{\text{Ans}}$$

1(e) Spheres are described to contain the circle $z=0$, $x^2+y^2=a^2$. Prove that the locus of the extremities of their diameters which are parallel to the x-axis is the rectangular hyperbola $x^2-z^2=a^2$, $y=0$.

Sol: The equation of the sphere through the given circle $x^2+y^2=a^2$, $z=0$ is

$$(x^2+y^2+z^2-a^2)+\lambda z=0 \quad \text{--- (1)}$$

Its centre is $(0,0,-\lambda/2)$ and radius $= \sqrt{\left(\frac{-\lambda}{2}\right)^2+a^2}$

$= \frac{\sqrt{\lambda^2+4a^2}}{2}$

Now the equations of the diameter of the sphere (1) and parallel to x-axis i.e., the line through the centre $(0,0,-\lambda/2)$ and parallel to the line with d.c's 1, 0, 0 are $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z+(\lambda/2)}{0}$.

The co-ordinates of any point on it at a distance r from the centre $(0,0,-\lambda/2)$ of the sphere (1) are $(r,0,-\lambda/2)$

If we take $r = \pm\frac{1}{2}\sqrt{\lambda^2+4a^2} = \pm$ radius of the sphere then we find that the co-ordinates of the extremities of the diameter parallel to x-axis are given by

$x = \pm\frac{1}{2}\sqrt{\lambda^2+4a^2}$, $y=0$, $z=-\lambda/2$ — (2)

Required locus is obtained by eliminating λ from (2).

From (2) we have

$4x^2 = \lambda^2+4a^2$, $y=0$, $2z = -\lambda$.

$4x^2 = (-2z)^2+4a^2$, $y=0$ on eliminating λ.

$x^2-z^2=a^2$, $y=0$ which is the required locus and is a rectangular hyperbola on the plane $y=0$

2(a) If $A=\begin{bmatrix}1&2\\-2&1\end{bmatrix}$, obtain A^2. find scalars a and b such that $I+aA+bA^2=O$.
Where I is the unit matrix and O is the null matrix both of order two.

Sol: Given $A=\begin{pmatrix}1&2\\-2&1\end{pmatrix}$

$$A^2=\begin{pmatrix}1&2\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\-2&1\end{pmatrix}=\begin{pmatrix}-3&4\\-4&-3\end{pmatrix}$$

If I and O are unit and null matrices respectively each of order 2.
we have $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $O=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$I+aA+bA^2=O$

$$\Rightarrow\begin{pmatrix}1&0\\0&1\end{pmatrix}+a\begin{pmatrix}1&2\\-2&1\end{pmatrix}+b\begin{pmatrix}-3&4\\-4&-3\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}a&2a\\-2a&a\end{pmatrix}+\begin{pmatrix}-3b&4b\\-4b&-3b\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1+a-3b&0+2a+4b\\0-2a-4b&1+a-3b\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$\Rightarrow$ $1+a-3b=0$; $2a+4b=0$
$-2a-4b=0$ $\quad 1+a-3b=0$

$\Rightarrow$ $1+a-3b=0$
$a+2b=0$

Solving we get $\boxed{a=-2/5,\ b=1/5}$

2(b) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $A=\begin{bmatrix}2&1&0\\0&1&-1\\0&2&4\end{bmatrix}$. Find the minimal polynomial for T.

Sol: The characteristic equation of it is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)[(1-\lambda)(4-\lambda)+2]=0$

$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-3)=0$

$\Rightarrow \lambda = 2, 2, 3.$

Hence the characteristic values of A are 2, 2, 3

The characteristic vector corresponding to $\lambda=2$

is given by $(A-2I)X=0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_2 \to R_2 + R_1$$

$\Rightarrow x_2 = 0, x_3 = 0$ and x_1 can be given any value

we take $x_1 = 1, x_2 = 0, x_3 = 0.$

Clearly there is only one L.I. vector corresponding to the characteristic value 2.

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the geometric multiplicity of the eigen value 2 is one while its algebraic multiplicity is 2. Since the geometric multiplicity of this eigen value is not equal to its algebraic multiplicity therefore A is not similar to a diagonal matrix.

i.e, T is not diagonalizable.

We know that the minimal polynomial for T divides its characteristic polynomial.

Thus the possible minimal polynomials for T can be either

$$p(\lambda)=(3-\lambda)(\lambda-2) \quad (\text{or}) \quad (\lambda-2)^2(3-\lambda).$$

Let us take $p(\lambda)=(3-\lambda)(\lambda-2)$.

we have

$$p(A)=(3I-A)(A-2I)=\begin{bmatrix}1 & -1 & 0\\ 0 & 2 & 1\\ 0 & -2 & \end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & -1\\ 0 & 2 & 2\end{bmatrix}$$

$$p(A)=\begin{bmatrix}0 & 2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\neq 0.$$

This shows that $p(\lambda)=(3-\lambda)(\lambda-2)$ is not the minimal polynomial for T.

Hence the minimal polynomial for T is

$p(\lambda)=(3-\lambda)(\lambda-2)$ which is same as the characteristic polynomial of T.

2(c)(i) Find $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$

(ii) If $v=At^{-1/2}e^{-x^2/4a^2t}$, prove that $\frac{\partial v}{\partial t}=a^2\frac{\partial^2 v}{\partial x^2}$.

Sol: (i) we have

$$y=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$$

$$\Rightarrow \log_e y=\lim_{x\to 0}\frac{1}{x}\log\left(\frac{\tan x}{x}\right)$$

$$=\lim_{x\to 0}\frac{\log\tan x-\log x}{x}$$

$$=\lim_{x\to 0}\left(\frac{\sec^2 x}{\tan x}-\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x-\sin x\cos x}{x\sin x\cos x}\right) = \lim_{x\to 0}\left(\frac{2x-\sin 2x}{x\sin 2x}\right) \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left(\frac{2-2\cos 2x}{\sin 2x+2x\cos 2x}\right) \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{4\sin 2x}{4\cos 2x-4x\sin 2x} = 0$$

$$\Rightarrow y = e^0 = 1$$

(ii) we have $v = At^{-1/2}e^{-x^2/4a^2t}$

$$\Rightarrow \frac{\partial v}{\partial x} = At^{-1/2}e^{-x^2/4a^2t}\left(\frac{-2x}{4a^2t}\right) = \frac{-x}{2a^2t}v$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{1}{2a^2t}\left[v + x\frac{\partial v}{\partial x}\right]$$

$$= -\frac{1}{2a^2t}\left[v + x\left(\frac{-xv}{2a^2t}\right)\right]$$

$$= \frac{v}{4a^4t^2}(-2a^2t + x^2)$$

Again, $\frac{\partial v}{\partial t} = At^{-1/2}e^{-x^2/4a^2t}\left(\frac{x^2}{4a^2t^2}\right) - A\frac{1}{2}t^{-3/2}e^{-x^2/4a^2t}$

$$= At^{-1/2}e^{-x^2/4a^2t}\left[\frac{x^2}{4a^2t^2} - \frac{1}{2t}\right]$$

$$= \frac{v}{4a^2t^2}(x^2 - 2a^2t)$$

$$\therefore \frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}$$

2(d) The Section of a cone with vertex at P and guiding curve $x^2/a^2 + y^2/b^2 = 1$, $z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $(x^2/a^2) + \{(y^2+z^2)/b^2\} = 1$.

Sol^n: Let the vertex P of the Cone by (α, β, γ)

Any line through $P(\alpha,\beta,\gamma)$ is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ — (1)

This line meets the plane $z=0$ is $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

and if this point lies on the given curve $x^2/a^2 + y^2/b^2 = 1$, we have

$$\frac{1}{a^2}\left(\alpha - \frac{l\gamma}{n}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m\gamma}{n}\right)^2 = 1 \quad \text{—— (2)}$$

Eliminating l, m, n between (1) & (2), the equation of the cone is

$$\frac{1}{a^2}\left[\alpha - \left(\frac{x-a}{z-\gamma}\right)\gamma\right]^2 + \frac{1}{b^2}\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 1$$

$$\Rightarrow b^2(\alpha z - x\gamma)^2 + a^2(\beta z - y\gamma)^2 = a^2b^2(z-\gamma)^2 \quad \text{—— (3)}$$

The Section of this cone by the plane $x=0$ gives the conic on yz-plane as

$$b^2\alpha^2z^2 + a^2(\beta z - y\gamma)^2 = a^2b^2(z-\gamma)^2$$

$$\Rightarrow a^2y^2\gamma^2 + (b^2\alpha^2 + a^2\beta^2 - a^2b^2)z^2 - 2a^2\beta\gamma yz + 2a^2b^2\gamma z - a^2b^2\gamma^2 = 0$$

It is represents a rectangular hyperbola on the yz-plane, then the sum of the coefficients of y^2 and z^2 must be zero.

i.e. $a^2\gamma^2 + (b^2\alpha^2 + a^2\beta^2 - a^2b^2) = 0$

$\Rightarrow \alpha^2/a^2 + [(\beta^2+\gamma^2)/b^2] = 1$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is $x^2/a^2 + [(y^2+z^2)/b^2] = 1$. Hence proved.

3(a)(i) Evaluate A^{50} for the matrix $A=\begin{bmatrix} 4/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 \end{bmatrix}$

(ii) prove that it is impossible to find a matrix P such that
$P^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}P=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ for any $\alpha,\beta \in R$.

Sol'n: Given that $A=\begin{pmatrix} 4/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 \end{pmatrix}$

The characteristic equation is

$$|A-\lambda I| = \begin{vmatrix} 4/3-\lambda & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \left(\tfrac{4}{3}-\lambda\right)\left(\tfrac{5}{3}-\lambda\right)-\tfrac{2}{9}=0$$

$$\Rightarrow \lambda^2-3\lambda+2=0$$

$$\Rightarrow (\lambda-1)(\lambda-2)=0$$

$\therefore$ The characteristic roots λ's are 1, 2

The eigen vector corresponding to $\lambda=1$, is

$$\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & 2/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{3} & 2/\sqrt{3} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1=\sqrt{2},\ x_2=-1$$

$\therefore X_1=\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to $\lambda=1$.

The eigen vector corresponding to $\lambda=2$ is

$$\begin{bmatrix} -\frac{2}{3} & \frac{\sqrt{2}}{3} \\ 2/\sqrt{3} & -1/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} -2/3 & \sqrt{2}/3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1=1,\ x_2=\sqrt{2}.$$

$\therefore P=\begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$

$P^{-1}AP=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}=D$

Obviously $|A|\neq 0$ i.e. matrix A is non-singular.

Hence $A^{50}=PD^{50}P^{-1}$.

$$= \frac{1}{3}\begin{pmatrix} 2^{50}+2 & (2^{50}-1)\sqrt{2} \\ (2^{50}-1)\sqrt{2} & 2^{51}+1 \end{pmatrix}$$

(ii) The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0$$

Hence the only distinct eigen value of A is $\lambda=1$.
The eigen vector corresponding to the eigen value $\lambda=1$ is $(A-I)X=0$ —— (1)

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_2 = 0$$

Hence the linear system (1) has only one L.I. solution namely $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
i.e. the matrix A has only one L.I. eigen vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding to the eigen value 1.
Hence A is not diagonalizable and so the conclusion of the problem follows.

3(b) (i) By using the transformation $x = u(1+v)$, $y = v(1+u)$ prove that $\int_0^1\int_0^x \left[(x+y+1)^2 - 4xy\right]^{-1/2} dx\,dy = \log 2 - \frac{1}{2}$.

(ii) Show that the function f defined by setting $f(x,y) = \frac{x^3y}{x^6+y^2}$, when $(x,y) \neq (0,0)$ and $f(0,0)=0$ is not continuous at the origin.

Sol: (i) Given that $x = u(1+v) = u+uv$
$y = v(1+u) = v+uv$

Now $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$
$$= 1+u+v+uv-uv = 1+u+v.$$

But $dxdy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = (1+u+v)^2\,du\,dv$

and $(x+y+1)^2-4xy = [u(1+v)+v(1+u)+1]^2-4uv(1+u)(1+v)$

$= u^2(1+v)^2+v^2(1+u)^2+1+2uv(1+u)(1+v)$
$\quad +2v(1+u)+2u(1+v)-4uv(1+u)(1+v)$

$= u^2(1+v)^2+v^2(1+u)^2+1-2uv(1+u)(1+v)$
$\quad +2u(1+v)+2v(1+u)$

$= [u(1+v)-v(1+u)]^2+2u(1+v)+2v(1+u)+1$

$= (u-v)^2+2u(1+v)+2v(1+u)+1$

$= u^2+v^2-2uv+2u+2uv+2v+2uv+1$

$= u^2+v^2+2uv+2u+2v+1$

$= (u+v+1)^2$

$[(x+y+1)^2-4xy]^{-1/2} = [(u+v+1)^2]^{-1/2} = (u+v+1)^{-1}$.

Now the given integration becomes

$I = \int_0^1\int_v^{2/(1+v)} (1+u+v)^{-1}(1+u+v)\,du\,dv$

Limits: $u: v \to \frac{2}{1+v}$
$v: 0$ to 1

$= \int_0^1\int_v^{2/1+v} du\,dv$

$= \int_0^1\left(\frac{2}{1+v}-v\right)dv$

$= \left[2\log v-\frac{v^2}{2}\right]_0^1$

$= 2\log 2-\frac{1}{2}$

(ii) suppose $(x,y)\to(0,0)$ along the path $y=mx$.

$$\lim_{x\to0} f(x,mx) = \lim_{x\to0}\frac{mx^4}{x^6+m^2x^2} = \lim_{x\to0}\frac{mx^4}{x^2(x^4+m^2)} = 0$$

for all values of m.

suppose now $(x,y)\to(0,0)$ along the curve $y=x^3$.

Then $\lim_{x\to0} f(x,x^3) = \lim_{x\to0}\frac{x^6}{x^6+x^6} = \lim_{x\to0}\frac{x^6}{2x^6} = \frac{1}{2}$.

Thus $f(x,y)$ tends to different limits as $(x,y)\to(0,0)$ along different paths and so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence the given function is not continuous at the origin.

3(c) A variable generator meets two generators of the same system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid in P and P' prove that $BP \cdot B'P' = a^2+c^2$

Sol: we know that the points of intersection of a generator of λ-system with a generator of μ-system for the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are given by

$$x=\frac{a(1+\lambda\mu)}{\lambda+\mu},\quad y=\frac{b(\lambda-\mu)}{\lambda+\mu},\quad z=\frac{c(1+\lambda\mu)}{\lambda+\mu} \quad (1)$$

The extremities of the minor axis of the principal elliptic section are $B(0,b,0)$ & $B'(0,-b,0)$

please continue in this way

please continue in this way

4(a) Let T be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$T(x_1, x_2, x_3) = (x_1 + x_2,\ 2x_3 - x_1)$.

If $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$, $\beta' = \{(0,1), (1,0)\}$ be ordered bases of $\mathbb{R}^3, \mathbb{R}^2$ respectively; then find the matrix of T relative to β, β'. Also find rank (T) and nullity (T).

Sol^n: Let T be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2,\ 2x_3 - x_1)$ ——— (1)

$\therefore$ from (1) $T(1,0,-1) = (1,-3) = -3(0,1) + 1(1,0)$

$T(1,1,1) = (2,1) = 1(0,1) + 2(1,0)$

$T(1,0,0) = (1,-1) = -1(0,1) + 1(1,0)$.

Hence the matrix of T relative to β, β' is

$$[T]_{\beta,\beta'} = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Let $(x_1, x_2, x_3) \in \operatorname{Ker} T$ be arbitrary.

Then $T(x_1, x_2, x_3) = (0, 0)$

$\Rightarrow (x_1 + x_2,\ 2x_3 - x_1) = (0,0)$

$\Rightarrow x_1 + x_2 = 0\ ;\ 2x_3 - x_1 = 0$

$\Rightarrow x_1 = 2x_3,\ x_2 = -x_1 = -2x_3$.

$\therefore (x_1, x_2, x_3) = (2x_3, -2x_3, x_3) = x_3(2, -2, 1)$.

Hence $\operatorname{Ker} T = \{x_3(2,-2,1) \mid x_3 \in \mathbb{R}\}$.

This shows that $\operatorname{Ker} T$ is spanned by $(2,-2,1)$ and so $(2,-2,1)$ is a basis of $\operatorname{Ker} T$.

$\therefore \dim \operatorname{Ker} T = N(T) = 1$. and $R(T) + N(T) = \dim \mathbb{R}^3 = 3$

$\Rightarrow R(T) = 3 - 1 = 2$

4(b) Find the points on the sphere $x^2+y^2+z^2=25$ where $f(x,y,z)=x+2y+3z$ has its maximum and minimum values.

Sol^n: Given that $f(x,y,z)=x+2y+3z$ —— (1)

Subject to the condition

$\phi(x,y,z)=x^2+y^2+z^2-25=0$ —— (2)

Let us write Lagranges auxiliary function

$F(x,y,z,\lambda)=f(x,y,z)+\lambda\phi(x,y,z)$. —— (3)

$\Rightarrow F(x,y,z,\lambda)=(x+2y+3z)+\lambda(x^2+y^2+z^2-25)$ —— (3)

we have

$F_x(x,y,z,\lambda)=1+2x\lambda$

$F_y(x,y,z,\lambda)=2+2\lambda y$

$F_z(x,y,z,\lambda)=3+3\lambda z$

$F_\lambda(x,y,z,\lambda)=x^2+y^2+z^2-25$

For extreme values

we have

$F_x=F_y=F_z=F_\lambda=0$

$\Rightarrow 1+2\lambda x=0$; $1+\lambda y=0$; $3+2\lambda z=0$

and $x^2+y^2+z^2-25=0$ —— (4)

$\Rightarrow \boxed{x=-\frac{1}{2\lambda}\ ;\ y=-\frac{1}{\lambda}\ ;\ z=-\frac{3}{2\lambda}}$

(4) $\equiv \frac{1}{4\lambda^2}+\frac{1}{\lambda^2}+\frac{9}{4\lambda^2}=25$

$\Rightarrow \boxed{\lambda=\pm\sqrt{14}/10}$

if $\lambda=-\sqrt{14}/10$ then $(x,y,z)=\left(\frac{10}{2\sqrt{14}},\frac{20}{2\sqrt{14}},\frac{30}{2\sqrt{14}}\right)$

if $\lambda=\sqrt{14}/10$ then $(x,y,z)=\left(\frac{-10}{2\sqrt{14}},\frac{-20}{2\sqrt{14}},\frac{-30}{2\sqrt{14}}\right)$

Now we have

$F_{xx} = 2\lambda$; $F_{xy} = F_{yx} = 0$

$F_{yy} = 2\lambda$; $F_{xz} = F_{zx} = 0$

$F_{zz} = 2\lambda$; $F_{zy} = F_{yz} = 0$

Now $d^2F(x,y,z,\lambda) = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2$

$= 2\lambda\left[(dx)^2 + (dy)^2 + (dz)^2\right]$

If $\lambda = -\frac{\sqrt{14}}{10}$ then $d^2F < 0$.

$\therefore$ At $\left(\frac{10}{2\sqrt{14}}, \frac{20}{2\sqrt{14}}, \frac{30}{2\sqrt{14}}\right)$

$f(x,y,z)$ has max.

and $f\left(\frac{10}{2\sqrt{14}}, \frac{20}{2\sqrt{14}}, \frac{30}{2\sqrt{14}}\right) = \frac{1}{2\sqrt{14}}[10+40+90]$

$= \frac{140}{2\sqrt{14}}$ is max. value.

If $\lambda = \frac{\sqrt{14}}{10}$ then $d^2F > 0$

$\therefore$ At $\left(\frac{-10}{2\sqrt{14}}, \frac{-20}{2\sqrt{14}}, \frac{-30}{2\sqrt{14}}\right)$

$f(x,y,z)$ has the minimum.

and $f\left(\frac{-10}{2\sqrt{14}}, \frac{-20}{2\sqrt{14}}, \frac{-30}{2\sqrt{14}}\right) = \frac{1}{2\sqrt{14}}(-10-40-90)$

$= \frac{-140}{2\sqrt{14}}$ is minimum value.

4(c) Prove that the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation to the plane in which they lie.

Sol^n: Given lines are coplanar, if

$$\begin{vmatrix} (a-d)-(b-c) & (a-b) & (a+d)-(b+c) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

Adding third column to first we get

$$\begin{vmatrix} 2(a-b) & a-b & (a+d)-(b+c) \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

The first column being twice the second column, the determinant on the left vanishes, hence the given lines are coplanar.

Also the equation of the plane in which the two given lines lie is

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+z-2a & y-a & z-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$ adding 3rd column to the first.

$$\Rightarrow \begin{vmatrix} (x+z-2a)-2(y-a) & y-a & z-a-d \\ 2\alpha-2(\alpha) & \alpha & \alpha+\delta \\ 2\beta-2(\beta) & \beta & \beta+\gamma \end{vmatrix} = 0$$, subtracting twice second column from first.

$$\Rightarrow \begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix} = 0$$

$\Rightarrow (x+z-2y)\left[\alpha(\beta+\gamma)-\beta(\alpha+\delta)\right]=0$

$\Rightarrow \quad x+z-2y=0.$

4(d) Show that the plane $8x-6y-z=5$ touches the paraboloid $\left(\frac{x^2}{2}\right)-\left(\frac{y^2}{3}\right)=z$, and find the point of contact.

Sol^n: Let the plane $8x-6y-z=5$ —— ①

touch the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z \Rightarrow 3x^2-2y^2=6z$ —— ②

at the point (α,β,γ)

The equation of the tangent plane to ② at (α,β,γ) is

$$3\alpha x-2\beta y=3(z+\gamma) \Rightarrow 3\alpha x-2\beta y-3z=3\gamma \text{ —— ③}$$

If the plane ① touches ② at (α,β,γ) then ① and ③ represent the same plane, and so comparing ① and ③, we get

$$\frac{3\alpha}{8}=\frac{-2\beta}{-6}=\frac{-3}{-1}=\frac{3\gamma}{5}$$

which gives $\alpha=8,\ \beta=9,\ \gamma=5$ —— ④

Also as (α,β,γ) lies on ②, so we find $3\alpha^2-2\beta^2=6\gamma$ —— ⑤

$\because$ Values of α,β,γ given by ④ satisfy ⑤, so the plane ① touches the paraboloid ② at (α,β,γ)

Also from ④, the coordinates of the point of contact are (α,β,γ) i.e. $(8,9,5)$.

5(a) Solve $(x^3D^3+2xD-2)y = x^2\ln x+3x$

Sol^n: Given that $(x^3D^3-2xD-2)y = x^2\ln x+3x$ —— (1)

put $x=e^z$ and $D_1=\frac{d}{dz}$

$\Rightarrow \log x = z$

Then from (1), we have

$(D_1(D_1-2)(D_1-2)+2D_1-2)y = e^{2z}z+3e^z$

$\Rightarrow [D_1(D_1^2-3D_1+2)+2D_1-2]y = ze^{2z}+3e^z$

$\Rightarrow [D_1^3-3D_1^2+2D_1+2D_1-2]y = ze^{2z}+3e^z$ —— (2)

Auxiliary equation of (2) is

$D_1^3-3D_1^2+4D_1-2=0$

$\Rightarrow D_1(D_1-1)(D_1-2)+2D_1-2=0$

$\Rightarrow (D_1-1)(D_1^2-2D_1+2)=0$

$\Rightarrow D_1=1,\quad D_1=\frac{2\pm\sqrt{4-8}}{2} = 1\pm i$

$\therefore$ The complementary function of (2) is

$C.F = c_1e^z+e^z(c_2\cos z+c_3\sin z)$

Now, the particular Integral (P.I)

$$= \frac{1}{D_1^3-3D_1^2+4D_1-2}\left[ze^{2z}+3e^z\right]$$

$$= \frac{1}{D_1^3-3D_1^2+4D_1-2}ze^{2z} + \frac{1}{D_1^3-3D_1^2+4D_1-2}(3e^z) \quad \text{—— (3)}$$

Let $\frac{1}{D_1^3-3D_1^2+4D_1-2}ze^{2z} = e^{2z}\frac{1}{(D_1+2)^3-3(D_1+2)^2+4(D_1+2)-2}z$

$$= e^{2z}\frac{1}{D_1^3+6D_1^2+12D_1+8-3D_1^2-12D_1-12+4D_1+8-2}z$$

$$= e^{2z}\frac{1}{D_1^3+3D_1^2+4D_1+2}z$$

$$= \frac{e^{2z}}{2}\cdot\frac{1}{\left[1+\left(\frac{D_1^3+3D_1^2+4D_1}{2}\right)\right]^{-1}}z$$

$$= \frac{e^{2z}}{2}\left[1-\left(\frac{D_1^3+3D_1^2+4D_1}{2}\right)+\cdots\right]z$$

$$= \frac{e^{2z}}{2}\left[z-2\right]$$

$$\frac{1}{D_1^3-3D_1^2+4D_1+2}3e^z = 3\frac{1}{(D_1-1)(D_1^2-2D_1+2)}e^z$$

$$= 3\frac{z}{(1)}e^z = 3ze^z$$

$\therefore$ from (3)

$$P.I = \frac{e^{2z}}{2}[z-2]+3ze^z$$

$\therefore$ The general solution of (2) is

$$y = C.F + P.I$$

$$= c_1e^z+e^z(c_2\cos z+c_3\sin z)+\frac{e^{2z}}{2}(z-2)+3ze^z$$

i.e. $y = c_1x+x(c_2\cos\log x+c_3\sin\log x)+\frac{x^2}{2}(\log x-2)+3x\log x$

which is the required solution of (1).

5(b) → Find the orthogonal trajectories of cardioids $r=a(1-\cos\theta)$, a being parameter.

Sol^n: The given family of cardioids is $r=a(1-\cos\theta)$ —(1)

Taking logarithm of bothsides of (1), we get

$$\log r = \log a + \log(1-\cos\theta) \quad \text{---} (2)$$

Differentiating (2) w.r.t θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \quad \text{---} (3)$$

Since (3) is free from parameter 'a', hence (3) is the differential equation of the given family (1)

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\theta/_2\cos\theta/_2}{2\sin^2\theta/_2}$$

$$\Rightarrow -r\frac{d\theta}{dr} = \cot\theta/_2$$

$$\Rightarrow \frac{dr}{r} = -\tan\theta/_2\, d\theta$$

Integrating, we get

$$\log r = 2\log\cos\theta/_2 + \log c$$

$$\Rightarrow \log r = \log(c\cos^2\theta/_2)$$

$$\Rightarrow r = \frac{c(1+\cos\theta)}{2}$$

$\Rightarrow r = b(1+\cos\theta)$ where $b = c/_2$ is arbitrary constant

$\therefore$ which gives another family of Cardioids.

5(c) A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight $W/_2$, the weight of the rod being W. this end is fastened a weight $W/_2$, the weight of the rod being W. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is lW/c.

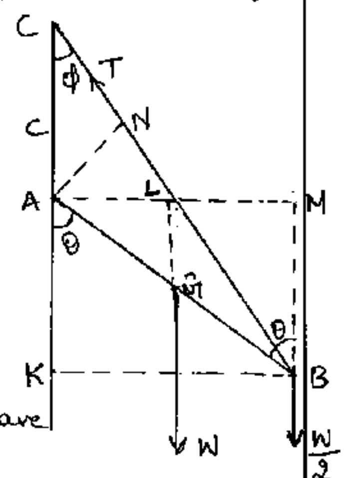
Sol^n: Let a rod AB of length $2a$ (say) be movable in a vertical plane about a smooth hinge at the end A. A weight $W/2$ is attached at the other end B of the rod and this end is fastened by a string BC of length l to a point C at a height $AC = c$ vertically over the hinge at A. The rod is in equilibrium under the action of the following forces:

(i) W, weight of the rod is at its mid point G, acting vertically downwards.

(ii) $W/2$, weight attached at the end B, acting vertically downwards.

(iii) T tension in the string along BC and

(iv) the reaction at the hinge at A.



Let θ and ϕ be the angles of inclination of the rod and the string respectively to the vertical.

To avoid reaction at A, taking moments about the point A, we have

$$T.AN = W AL + \tfrac{1}{2}W\,AM$$

$$\Rightarrow T.AC\sin\phi = W.AG\sin\theta + \tfrac{1}{2}W\cdot AB\sin\theta$$

$$\Rightarrow T.\,c\sin\phi = W.a\sin\theta + \tfrac{1}{2}W\cdot 2a\sin\theta \quad [\because AB = 2a]$$

$$\Rightarrow T = W\frac{2a\sin\theta}{c\sin\phi} \quad \text{--- (1)}$$

Now from the ΔCBK, $BK = BC\sin\phi = l\sin\phi$

and from the ΔABK, $BK = AB\sin\theta = 2a\sin\theta$

$$\therefore\ l\sin\phi = 2a\sin\theta \quad \text{——} \quad (2)$$

$\therefore$ from (1) and (2), we get

$$\boxed{T = Wl/c}$$

5(d) A particle whose mass is m is acted upon by a force $m\mu\left[x + \frac{a^4}{x^3}\right]$ towards origin; if it starts from rest at a distance a. Show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$.

Sol^n: Given $\frac{d^2x}{dt^2} = -\mu\left[x + \frac{a^4}{x^3}\right]$ —— (1)

the $-$ive sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2\left(\frac{dx}{dt}\right)$, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2 + \frac{a^4}{x^2}\right] + c$$

when $x = a$, $dx/dt = 0$, so that $c = 0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4 - x^4}{x^2}\right]$$

$$\Rightarrow \frac{dx}{dt} = \frac{\sqrt{\mu}\sqrt{a^4 - x^4}}{x} \quad \text{——} \quad (2)$$

the -ive sign is taken because the particle is moving in the direction x decreasing

If t_1 be the time taken to reach the origin, then integrating (3), we get

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{a^4 - x^4}}\,dx = \frac{1}{\sqrt{\mu}}\int_a^0 \frac{x\,dx}{\sqrt{a^4 - x^4}}$$

put $x^2 = a^2\sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$, when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$

$$\therefore t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta} = \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta = \frac{1}{2\sqrt{\mu}}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2} = \frac{\pi}{4\sqrt{\mu}}.$$

5(e) Using Green's theorem, evaluate $\int_C (x^2y\,dx + x^2dy)$ where C is boundary described counter clockwise of the triangle with vertices (0,0), (1,0), (1,1).

Sol^n: Green's Theorem in a plane is

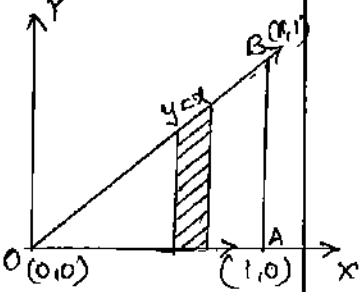
$$\oint_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy \quad \text{——} \; (1)$$

Given $M\,dx + N\,dy = x^2y\,dx + x^2dy$

Here $M = x^2y$ and $N = x^2$

$\Rightarrow \frac{\partial N}{\partial x} = 2x$ and $\frac{\partial M}{\partial y} = x^2$

$\therefore$ Equation (1) gives



$$\oint_C (x^2y\,dx + x^2dy) = \iint_S (2x - x^2)\,dx\,dy$$

$$= \int_0^1 (2x-x^2)dx\int_0^x dy$$

$$= \int_0^1 (2x-x^2)dx\left[y\right]_0^x$$

$$= \int_0^1 (2x-x^2)\,dx\cdot x$$

$$= \int_0^1 (2x^2-x^3)dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left[\frac{2}{3}-\frac{1}{4}\right] = \frac{5}{12}.$$

6(a). Solve the equation $xy''-2(x+1)y'+(x+2)y=(x-2)e^{x}$; $(x>0)$ by changing into normal form.

Soln: Given equation is.

$$xy''-2(x+1)y'+(x+2)y=(x-2)e^{x}.$$

$$y''-\frac{2}{x}(x+1)y'+\frac{1}{x}(x+2)y=\left(\frac{x-2}{x}\right)e^{x} \quad \text{---} (1)$$

It is given that e^{x} is a solution to its corresponding homogeneous differential equation

i.e. $y=u=e^{x}$ is the part of C.F of (1)

Let the general solution of (1) is $y=uv$

then v is given by $\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$

where $P=-\frac{2}{x}(x+1)$, $Q=\frac{x+2}{x}$, $R=\frac{(x-2)}{x}e^{x}$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-\frac{2}{x}(1+x)+\frac{2}{e^{x}}(e^{x})\right]\frac{dv}{dx}=\frac{(x-2)}{x}\frac{e^{x}}{e^{x}}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-\frac{2}{x}-\cancel{2}+\cancel{2}\right]\frac{dv}{dx}=\left(\frac{x-2}{x}\right) \quad \text{---} (2)$$

Let $\frac{dv}{dx}=q \Rightarrow \frac{d^2v}{dx^2}=\frac{dq}{dx}$

$\therefore$ from (2), we have

$$\frac{dq}{dx}+\left(-\frac{2}{x}\right)q=\frac{x-2}{x}$$

which is linear in q.

$$I.F=e^{-\int\frac{2}{x}dx}=e^{-(2\log x)}=e^{\log x^{-2}}=\frac{1}{x^2}.$$

$$\therefore q\,(IF)=\int\left(\frac{x-2}{x}\right)\cdot IF+c_1$$

$$q\left(\frac{1}{x^2}\right)=\int\left(\frac{x-2}{x}\right)\cdot\frac{1}{x^2}dx+c_1$$

$$=\int\frac{1}{x^2}dx-\int\frac{2}{x^3}dx+c_1$$

$= \int x^{-2}dx - 2\int x^{-3}dx + c_1$

$= -\frac{1}{x} - 2\cdot\left(\frac{1}{-2x^2}\right) + c_1 = -\frac{1}{x} + \frac{1}{x^2} + c_1$

$q = -x + 1 + c_1 x^2$

$dv = (-x + 1 + c_1 x^2)\,dx$

$v = -\frac{x^2}{2} + x + c_1\frac{x^3}{3} + c_2$

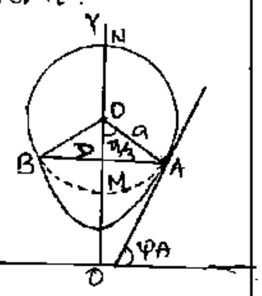
$y = uv$

$= e^x\left(-\frac{x^2}{2} + x + c_1\frac{x^3}{3} + c_2\right)$

which is the required solution.

6(b) Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two thirds of the circumference of the pulley is $a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}$.

Sol^n: Let ANBMA be the circular pulley of radius a and ANBCA the endless chain hanging over it.



Since the chain is in contact with the $\frac{2}{3}$ of the circumference of the pulley, hence the length of this portion ANB of the chain

$= \frac{2}{3}$ (Circumference of the pulley)

$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a$

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e. the vertex, CO'N the axis and OX the directrix of this Catenary.

Let $OC = c$ = the parameter of the catenary.

The tangent at A will be ⊥lar to the radius O'A

∴ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left(\frac{1}{3}\cdot 2\pi\right) = \frac{1}{3}\pi$$

from the triangle AO'D, we have

$$DA = O'A \sin\frac{1}{3}\pi = a\sqrt{3}/2$$

∴ from $x = c\log(\tan\psi + \sec\psi)$, for the point A, we have

$$x = DA = c\log(\tan\psi_A + \sec\psi_A)$$

$$\Rightarrow \frac{a\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right) = c\log(\sqrt{3}+2)$$

$$\therefore c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

from $s = c\tan\psi$ applied for the point A, we have

$$\text{arc } CA = c\tan\psi_A = c\tan\frac{1}{3}\pi = c\sqrt{3} = \frac{3a}{2\log(2+\sqrt{3})}$$

Hence the total length of the chain

= arc ABC + length of the chain in contact with the pulley

$$= 2\cdot(\text{arc } CA) + \frac{4}{3}\pi a$$

$$= 2\frac{3a}{2\log(2+\sqrt{3})} + \frac{4}{3}\pi a = a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}$$

6(c) (i) Show that $r^n\vec{r}$ is an irrotational vector for any value of n, but is solenoidal only if $n = -3$ (r is position vector of a point), (ii) Find the value of a, b and c such that

$F = (3x-4y+az)\hat{i} + (cx+5y-2z)\hat{j} + (x-by+7z)\hat{k}$ is irrotational.

Sol'n: (i) Let $F = r^n \vec{r}$

The vector F is irrotational if Curl F = 0 putting $\phi = r^n$ and $A = \vec{r}$

and we know that $\text{curl}(\phi A) = \nabla\phi \times A + \phi\, \text{curl}\, A$.

$$\therefore \text{Curl}(r^n \vec{r}) = \nabla r^n \times \vec{r} + r^n \,\text{curl}\,\vec{r}$$

$$= (n r^{n-1} \nabla r) \times \vec{r} + r^n(0) \quad (\because \nabla f(r) = f'(r)\nabla r)$$

$$= \left(n r^{n-1} \frac{1}{r}\vec{r}\right) \times \vec{r} \quad \left(\because \nabla r = \frac{1}{r}\vec{r}\right)$$

$$= n r^{n-2}(\vec{r} \times \vec{r}) = 0$$

The vector F is solenoidal if div F = 0

we know that $\text{div}(\phi A) = \phi(\text{div} A) + A\cdot(\text{grad}\,\phi)$

$$\Rightarrow \text{div}(r^n \vec{r}) = r^n \,\text{div}\,\vec{r} + \vec{r}\cdot \text{grad}\, r^n$$

$$= 3r^n + \vec{r}\cdot(n r^{n-1}\,\text{grad}\, r)$$

$$= 3r^n + \vec{r}\cdot\left(n r^{n-1}\cdot \frac{1}{r}\vec{r}\right)$$

$$(\because \text{div}\,\vec{r} = 3 \;\&\; \nabla f(r) = f'(r)\nabla r)$$

$$= 3r^n + \vec{r}\cdot\left(n r^{n-1}\cdot \frac{1}{r}\vec{r}\right)$$

$$= 3r^n + n r^{n-2}(\vec{r}\cdot\vec{r})$$

$$= 3r^n + n r^n$$

$$\text{div}(r^n \vec{r}) = r^n(n+3)$$

$\therefore$ The vector $r^n\vec{r}$ is solenoidal if $(n+3)r^n = 0$

i.e. $n+3 = 0$

$\Rightarrow n = -3$

(ii) For an irrotational vector F, Curl F = 0

$\therefore$ Curl F $= \nabla \times F$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x-4y+az) & (cx+5y-2z) & (x-by+7z) \end{vmatrix} = 0.$$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(x-by+7z) - \frac{\partial}{\partial z}(cx+5y-2z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(3x-4y+az) - \frac{\partial}{\partial x}(x-by+7z)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}(cx+5y-2z) - \frac{\partial}{\partial y}(3x-4y+az)\right] = 0$$

$$\Rightarrow \hat{i}(-b+2) + \hat{j}(a-1) + \hat{k}(c+4) = 0$$

As $\hat{i}, \hat{j}$ & $\hat{k}$ orthogonal and independent vectors, the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$ should be zero separately, therefore

$$(-b+2)=0, \ (a-1)=0 \text{ and } (c+4)=0$$

i.e. $b=2, a=1, c=-4$

there for the given vector F to be irrotational

$$a=1, \ b=2 \text{ and } c=-4.$$

7(a) By using Laplace transform method, solve

$(D^2+m^2)x = a\cos nt$, $t>0$ if $x = Dx = 0$ when $t=0$.

Sol'n: Given $y'' + m^2 y = a\cos nt$

Applying Laplace transform on both sides

$$s^2 L(y) - s\,y(0) - y'(0) + m^2 L(y) = aL(\cos nt)$$

$$(s^2+m^2)L(y) = a \cdot \frac{s}{s^2+n^2}$$

$$L(y) = a \cdot \frac{s}{(s^2+m^2)(s^2+n^2)}$$

Taking partial fractions.

$$\frac{s}{(s^2+m^2)(s^2+n^2)} = \frac{As+B}{s^2+n^2} + \frac{Cs+D}{s^2+m^2}$$

$$s = (As+B)(s^2+m^2) + (Cs+D)(s^2+n^2)$$

$$s = (A+C)s^3 + (B+D)s^2 + (Am^2+Cn^2)s + Bm^2+Dn^2$$

$$A+C=0, \quad B+D=0, \quad Am^2+Cn^2=1, \quad Bm^2+Cn^2=0$$

$$A = \frac{1}{m^2-n^2}, \quad C = \frac{1}{n^2-m^2}$$

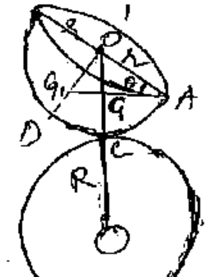
$$\therefore L(y) = a\left[\frac{1}{(m^2-n^2)}\left[\frac{s}{s^2+n^2}\right] + \frac{1}{(n^2-m^2)}\left[\frac{s}{s^2+m^2}\right]\right]$$

Taking inverse Laplace Transform

$$y = \frac{a}{m^2-n^2}\left[\cos nt - \sin mt\right]$$ Required solution.

7(b) A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle

Sol:



Let O' be the centre of the base of the hemispherical shell of radius r. Let weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the spherical shell is on its symmetrical radius $O'D$ and $O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$

Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre at O at the highest point C. for equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$. Also here $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$ i.e., $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

i.e., $\frac{1}{h} > \frac{R+r}{Rr}$

i.e., $h < \frac{rR}{R+r}$ —— (1)

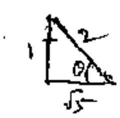
The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is maximum i.e., if $O'G$ is perpendicular to AG_1 or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$.

Then from right angled $\Delta O'G_1$

$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$.



$\therefore \sin\theta = \frac{1}{\sqrt5}$

$\therefore$ the minimum value of $O'G$

$= O'A\sin\theta = r\left(\frac{1}{\sqrt5}\right) = r/\sqrt5$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$= r - \frac{r}{\sqrt5} = \frac{r(\sqrt5-1)}{\sqrt5}$

Hence the equilibrium will be stable, whatever be the weight of the particle at A, if $\frac{r(\sqrt5-1)}{\sqrt5} < \frac{rR}{r+R}$ i.e., if $\frac{\sqrt5-1}{\sqrt5} < \frac{R}{R+r}$

i.e., if $(\sqrt5-1)R - (\sqrt5-1)r < R\sqrt5$

i.e., if $R/r > \sqrt5-1$

7(c) (i) Find the value of a if $A = a\hat{i}+\hat{j}+\sqrt{5}\hat{k}$ subtends and angle of $60°$ with $4\hat{i}-5\hat{j}+\sqrt{5}\hat{k}$.

(ii) Find the directional derivative of the scalar function $\phi = 4e^{(2x-y+z)}$ at the point $(1,1,-1)$ in a direction towards the point $(-3,5,6)$.

Sol^n: Given $A = a\hat{i}+\hat{j}+\sqrt{5}\hat{k}$, $B = 4\hat{i}-5\hat{j}+5\hat{k}$

Angle between A and B, $\theta = 60°$

$$A = |A| = \sqrt{a^2+1^2+(\sqrt{5})^2} = \sqrt{a^2+6} \quad \text{---}$$

$$B = |B| = \sqrt{4^2+(-5)^2+(\sqrt{5})^2} = \sqrt{46} \quad \text{---}$$

Also $A\cdot B = (a\hat{i}+\hat{j}+\sqrt{5}\hat{k})\cdot(4\hat{i}-5\hat{j}+\sqrt{5}\hat{k})$

$$= 4a-5+5 = 4a \quad \text{---} (3)$$

we have $A\cdot B = AB\cos\theta$

Substituting values from (1) (2) and (3)

$$4a = \sqrt{(a^2+6)}\,\sqrt{46}\,\cos 60°$$

$$\Rightarrow 4a = \sqrt{(a^2+6)}\,\sqrt{46}\times\frac{1}{2}$$

$$\Rightarrow 8a = \sqrt{46(a^2+6)}$$

Squaring on bothsides.

$$\Rightarrow 64a^2 = 46a^2+276$$

$$\Rightarrow 18a^2 = 276 \Rightarrow a^2 = \frac{46}{3}$$

$$\Rightarrow a = \sqrt{\frac{46}{3}}$$

(ii) $\text{grad}\,\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left\{4e^{(2x-y+z)}\right\} = 4e^{(2x-y+z)}(2\hat{i}-\hat{j}+\hat{k})$$

grad ϕ at the point $(1,1,-1)$ is

$$(\nabla\phi)_{1,1,-1} = 4e^{(2-1-1)}(2\hat{i}-\hat{j}+\hat{k}) = 4(2\hat{i}-\hat{j}+\hat{k})$$

If $A = (1,1,-1)$ and $B = (-3,5,6)$, then direction
$\vec{AB} = B - A$
$= (-3\hat{i}+5\hat{j}+6\hat{k}) - (\hat{i}+\hat{j}-\hat{k}) = -4\hat{i}+4\hat{j}+7\hat{k}$

Directional derivative of ϕ at point $(1,1,-1)$ in the direction of line $\vec{AB}$ is

$$= 4(2\hat{i}-\hat{j}+\hat{k}) \cdot \frac{(-4\hat{i}+4\hat{j}+7\hat{k})}{\sqrt{(-4)^2+4^2+7^2}}$$

$$= \frac{4(-8-4+7)}{\sqrt{81}} = \frac{-20}{9}$$

8(a). Reduce the equation $xyp^2-(x^2+y^2-1)p+xy=0$ to Clairaut's form. Hence show that the equation represents a family of conics touching the four sides of a square.

Sol^n. Let $x^2=u$, $y^2=v$

$$\frac{dv}{du} = \frac{2y}{2x}\frac{dy}{dx} = \frac{y}{x}\frac{dy}{dx}$$

$\therefore p = \frac{xP}{y}$ where $p = \frac{dy}{dx}$ and $P = \frac{dv}{du}$

putting it in the given equation, we get

$$xy\left(\frac{x^2P^2}{y^2}\right) - (x^2+y^2-1)\left(\frac{x}{y}P\right) + xy = 0$$

$$x^2P^2 - (x^2+y^2-1)P + y^2 = 0$$

$$\Rightarrow uP^2 - (u+v-1)P + v = 0$$

$$\Rightarrow v(1-P) = uP(1-P) - P$$

$$\Rightarrow v = uP - \frac{P}{1-P}$$

which is of Clairaut's form and hence (Replacing P by c)

its solution is $v = uc - \frac{c}{1-c}$

i.e. $y^2 = x^2c - \frac{c}{1-c}$, or $y^2 = cx^2 + \frac{c}{c-1}$

i.e. $c^2x^2 - c(x^2+y^2-1)+y^2=0$ —— ①
which represents a family of conics.

The given equation is
$xyp^2-(x^2+y^2-1)p+xy=0$ —— ②

From ①, the c-disc. relation is
$(x^2+y^2-1)^2-4x^2y^2=0$ —— ③

From ②, p-disc. relation is
$(x^2+y^2-1)^2-4x^2y^2=0$ —— ④

from ③ and ④, we notice that
$(x^2+y^2-1)^2-4x^2y^2=0$ —— ⑤
must be a singular solution since it is present once in both the discriminants.

Again $(x^2+y^2-1)^2-4x^2y^2=(x^2+y^2-1)^2-(2xy)^2$
$=[(x^2+y^2-1)+2xy][x^2+y^2-1-2xy]$
$=[(x+y)^2-1][(x-y)^2-1]$
$=(x+y-1)(x+y+1)(x-y-1)(x-y+1)$.

so by ⑤ $x+y+1=0$, $x+y-1=0$, $x-y+1=0$ and $x-y-1=0$ are four singular solutions.

Thus the given differential equation represents a family of conics given by ① which are touched by the four lines mentioned above. Furthermore it can be easily verified that the four lines form the four sides of a square.

8(b) Solve $\left(2+2x^2y^{1/2}\right)y\,dx+\left(x^2y^{1/2}+2\right)x\,dy=0$

Sol: Let $x^2y^{1/2}=v \Rightarrow y=\frac{v^2}{x^4}$.

$$dy=\frac{2v\,dv}{x^4}-\frac{4v^2}{x^5}dx.$$

$\therefore$ The given equation becomes

$$(2+2v)\frac{v^2}{x^4}dx+x(v+2)\left[\frac{2v}{x^4}dv-\frac{4v^2}{x^5}dx\right]=0$$

$$\frac{v^2}{x^4}(2+2v-4v-8)dx+x(v+2)\frac{2v}{x^4}dv=0$$

$$\Rightarrow \frac{v^2}{x^4}(-2v-6)dx+x(v+2)\frac{2v}{x^4}dv=0$$

$$\Rightarrow -2\frac{v^2}{x^4}(v+3)dx+x(v+2)\frac{2v}{x^4}dv=0$$

$$\Rightarrow v(v+3)dx-x(v+2)dv=0$$

$$\Rightarrow \frac{dx}{x}-\frac{2}{3}\frac{dv}{v}-\frac{1}{3}\frac{dv}{v+3}=0$$

Integrating, we get

$$\Rightarrow 3\log x-2\log v-\log(v+3)=\log c$$

$$\Rightarrow x^3=c\,v^2(v+3)$$

$$x^3=c\cdot x^4y\left(x^2y^{1/2}+3\right)$$

$$\Rightarrow 1=cxy\left(x^2y^{1/2}+3\right).$$

(or)

$$\boxed{xy\left(x^2y^{1/2}+3\right)=c_1}$$ where $c_1=\frac{1}{c}$

8(c) A particle moves with a central acceleration $\mu(r+a^4/r^3)$ being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$. Prove that it describes the curve $r^2(2+\cos\sqrt{3}\theta)=3a^2$.

Sol: Here, the central acceleration,

$P=\mu(r+a^4/r^3)=\mu\{(1/u)+a^4u^3\}$, where $u=\frac{1}{r}$.

∴ The differential equation of the path is

$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left(\frac{1}{u}+a^4u^3\right)$

$\Rightarrow h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\left(\frac{1}{u^3}+a^4u\right)$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and integrating w.r.t θ we have

$h^2\left[2\cdot\frac{u^2}{2}+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left(-\frac{1}{2u^2}+\frac{a^4u^2}{2}\right)+A$

$\Rightarrow v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{u^2}+a^4u^2\right)+A$ —(1)

where A is a constant.

Now initially the particle has been projected from an apse (say, the point A) at a distance a with velocity $2\sqrt{\mu}a$.

Therefore when $r=a$ i.e., $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ (at an apse)

and $v=2\sqrt{\mu}a$

∴ from (1), we have

$$\underset{(i)}{4\mu a^2}=\underset{(ii)}{h^2\left[\frac{1}{a^2}\right]}=\mu\left(-a^2+\underset{(iii)}{a^4\frac{1}{a^2}}\right)+A$$

From (i) & (ii), we have $h^2=4\mu a^4$ and from (i) & (iii) we have $4\mu a^2=0+A$ i.e, $A=4\mu a^2$.

Substituting the values of h^2 and A in (1).

we have $4\mu a^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{u^2}+a^4u^2\right)+4\mu a^2$.

$\Rightarrow 4a^4\left(\frac{du}{d\theta}\right)^2 = -4a^4u^2 - \frac{1}{u^2} + a^4u^2 + 4a^2$

$\Rightarrow 4a^4u^2\left(\frac{du}{d\theta}\right)^2 = (-1 - 3a^4u^4 + 4a^2u^2)$ —— ②

$\Rightarrow 2a^2u\frac{du}{d\theta} = \sqrt{(-1 - 3a^4u^4 + 4a^2u^2)}$

$\Rightarrow d\theta = \frac{2a^2u\,du}{\sqrt{(-1 - 3a^4u^4 + 4a^2u^2)}}$

$= \frac{2a^2u\,du}{\sqrt{3}\sqrt{\left[-\frac{1}{3} - \left(a^4u^4 - \frac{4}{3}a^2u^2\right)\right]}}$

$d\theta = \frac{2a^2u\,du}{\sqrt{3}\sqrt{\left[-\frac{1}{3} - \left(a^2u^2 - \frac{2}{3}\right)^2 + \frac{4}{9}\right]}}$

$\Rightarrow \sqrt{3}\,d\theta = \frac{2a^2u\,du}{\sqrt{\left[\left(\frac{1}{3}\right)^2 - \left(a^2u^2 - \frac{2}{3}\right)^2\right]}}$

$\Rightarrow \sqrt{3}\,d\theta = \frac{dz}{\sqrt{\left(\frac{1}{3}\right)^2 - z^2}}$ Substituting $a^2u^2 - \frac{2}{3} = z$ $\Rightarrow 2a^2u\,du = dz$.

Integrating

$\sqrt{3}\theta + B = \sin^{-1}(3z)$ —— ③ where B is a constant & $z = a^2u^2 - \frac{2}{3}$

Now take the apse line OA as the initial line.

Then initially $r = a$, $u = 1/a$ and $\theta = 0$

∴ from ③, $0 + B = \sin^{-1} 1 \Rightarrow B = \frac{\pi}{2}$

putting $B = \frac{\pi}{2}$ in ③, we have

$\sqrt{3}\,\theta + \frac{\pi}{2} = \sin^{-1}(3a^2u^2 - 2)$

$\Rightarrow 3a^2u^2 - 2 = \sin\left(\frac{\pi}{2} + \sqrt{3}\theta\right) = \cos\sqrt{3}\theta$.

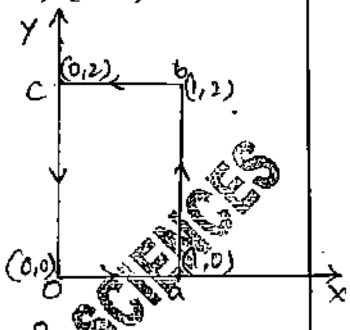
$\Rightarrow \frac{3a^2}{r^2} - 2 = \cos\sqrt{3}\,\theta$

$\Rightarrow 3a^2 - 2r^2 = r^2\cos\sqrt{3}\,\theta$

$\Rightarrow 3a^2 = r^2(2 + \cos\sqrt{3}\,\theta)$ which is the equation of the required curve.

8(d). Verify Stoke's theorem for the vector field $A = (3x-2y)\hat{i} + x^2 z\hat{j} + y^2(z+1)\hat{k}$ for a plane rectangular area with vertices at $(0,0)$, $(1,0)$, $(1,2)$, $(0,2)$ in the xy-plane.

Sol'n: Stoke's theorem is



$$\int_C A\cdot dr = \iint_S \text{Curl}\,A\cdot ds \quad \text{--- (1)}$$

Line integral along the boundary C of rectangular area is

$$\int_C A\cdot dr = \int_{oa} A\cdot dr + \int_{ab} A\cdot dr + \int_{bc} A\cdot dr + \int_{co} A\cdot dr$$

Here $A = (3x-2y)\hat{i} + x^2 z\hat{j} + y^2(z+1)\hat{k}$

In xy-plane $z=0$.

$\therefore\ A = (3x-2y)\hat{i} + y^2\hat{k}$

$$\therefore \int_o^a A\cdot dr = \int_{(0,0)}^{(1,0)} \left[(3x-2y)\hat{i} + y^2\hat{k}\right]\cdot(dx\,\hat{i}) = \int_{(0,0)}^{(1,0)} (3x-2y)\,dx$$

$$= \int_0^1 3x\,dx = \left[\frac{3x^2}{2}\right]_0^1 = \frac{3}{2}$$

$$\int_a^b A\cdot dr = \int_{(1,0)}^{(1,2)} \left[(3x-2y)\hat{i} + y^2\hat{k}\right]\cdot(dy\,\hat{j}) = 0$$

$$\int_b^c A\cdot dr = \int_{(1,2)}^{(0,2)} \left[(3x-2y)\hat{i} + y^2\hat{k}\right]\cdot(\hat{i}\,dx)$$

$$= \int_{(1,2)}^{(0,2)} (3x-2y)\,dx = \int_{x=1}^{0} (3x-4)\,dx = -\int_0^1 (3x-4)\,dx$$

$$= -\left[\frac{3x^2}{2} - 4x\right]_{x=0}^{1} = -\left(\frac{3}{2} - 4\right) = \frac{5}{2}$$

$$\int_C^0 A\cdot dr = \int_{(0,2)}^{(0,0)} [(3x-2y)\hat{i} + y^2\hat{k}]\cdot(\hat{j}\,dy) = 0$$

$$\therefore \int_C A\cdot dr = \frac{3}{2} + 0 + \frac{5}{2} + 0 = 4 \quad \text{——— (2)}$$

Also; $\text{Curl } A = \nabla\times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x-2y) & 0 & y^2 \end{vmatrix}$

$$= \hat{i}(2y) + \hat{j}(0) + \hat{k}(2) = 2y\hat{i} + 2\hat{k}$$

and ds in xy plane $= dx\,dy\,\hat{k}$

$$\therefore \iint_S \text{Curl } A\cdot ds = \iint_S (2y\hat{i} + 2\hat{k})\cdot(dx\,dy\,\hat{k}) = \iint_S 2\,dx\,dy$$

$$= 2\times \text{Area of rectangle} = 2\times 2 = 4 \quad \text{— (3)}$$

A comparison of (2) and (3) shows that equation (1) and hence Stoke's theorem is satisfied.

T10 — 2016 (KEY) (Paper-II)

T10-2016

Q.1 (a)

1(a) Let $g \in G$, $h \in H$

Consider ghg^{-1} and note that

$$ghg^{-1} = ghghh^{-1}g^{-2}$$
$$= (gh)^2 h^{-1} g^{-2}$$

Now, $h^{-1} \in H$ and by hypothesis $(gh)^2, g^{-2} \in H$.

This implies that $ghg^{-1} \in H$ which in turn shows that $gHg^{-1} \subseteq H$. Hence H is a normal subgroup of G. To show G/H is commutative, let $xH, yH \in G/H$.

We show that $xHyH = yHxH$ or $xyH = yxH$ or $(yx)^{-1}(xy) \in H$.

Now, $(yx)^{-1}xy = (x^{-1}y^{-1})(xy)$
$$= (x^{-1}y^{-1})^2 (yxy^{-1})^2 y^2$$

Since $a^2 \in H$ $\forall a \in G$. It follows that $(x^{-1}y^{-1})(yxy^{-1})^2 y^2 \in H$ and so $(yx)^{-1}(xy) \in H$. Hence, $\frac{G}{H}$ is commutative.

1(a) Let H be a subgroup of a group G. If $x^2 \in H$ then prove that H is normal subgroup of G and G/H is commutative.

Q.1 (b). Prove that order of a finite field F is p^n, for some prime p and some positive integer n.

Sol'n: Given that: F is a field (finite)

Now, we prove that $ch\,F \neq 0$.

If possible, let $ch.\,F = 0$

by defn, $\exists$ no +ve integer n such that

$$na = 0 \quad \forall a \neq 0 \in F$$

i.e. $na \neq 0 \quad \forall a \neq 0 \in F \;\&\; \forall n \in N$

It follows that $a, 2a, 3a, ----$ belong to F.

Since F is finite.

we must have $ia = ja$ for some +ve integer $i \& j$, $i > j$

$\Rightarrow (i-j)a = 0$

$\Rightarrow a = 0$

which is a contradiction. $\therefore ch\,F \neq 0$.

we know that $ch.\,F$ is either 0 or prime.

since $ch.\,F \neq 0 \Rightarrow ch.\,F = p$ (prime)

here p is the smallest number such that $pa = 0 \quad \forall a \in F$

$\Rightarrow o(a) = p$; treating $(F, +)$ as a group. Since $(F, +)$ is a finite group.

$\therefore$ By Lagrange's theorem $o(a)$ divides $o(F)$.

i.e. p divides $o(F)$, where p is prime.

$\therefore o(F) = p^n$ for some $n \in N$.

1.(c) prove that between any two real roots of the equation $e^x \sin x + 1 = 0$, there is at least one real root of the equation $\tan x + 1 = 0$.

Sol. Let $f(x) = e^{x} \sin x + 1$ — ①

let a & b be two roots of ①

then $f(a) = f(b) = 0$.

since f is continuous and differentiable for all $x \in \mathbb{R}$.

$\therefore$ by Rolle's theorem

$\exists$ at least one point $x \in (a, b)$

s.t $f'(x) = 0$.

$\Rightarrow e^x(\sin x + \cos x) = 0$.

$\Rightarrow \sin x + \cos x = 0$ ($\because e^x \neq 0 \; \forall x$)

$\Rightarrow \frac{\sin x}{\cos x} + 1 = 0$

$\Rightarrow \tan x + 1 = 0$.

Hence between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$

1(d). Locate and name the singularities in the finite z-plane $\frac{\ln(z-2)}{(z^2+2z+2)^4}$

Ans: please try yourself.

1(e) Does the following L.P.P. has a feasible solution $\max. Z = x_1 + x_2$

s.c. $x_1 - x_2 \geq 0$,

$3x_1 - x_2 \leq -3$.

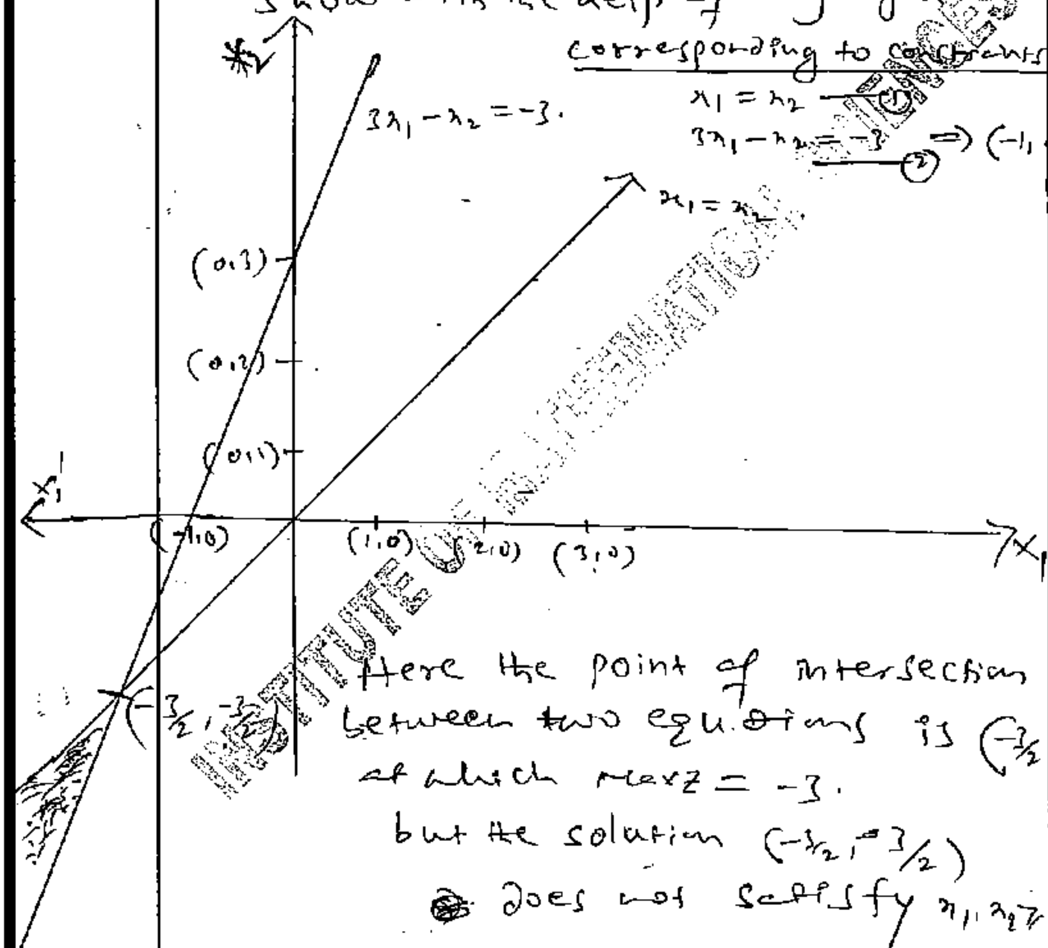
$x_1, x_2 \geq 0$.

Show with the help of a graph.

corresponding to constraints:

$x_1 = x_2$ — ①

$3x_1 - x_2 = -3$ — ② $\Rightarrow (-1, 0)$ & $(0, 3)$



Here the point of intersection between two equations is $(-\frac{3}{2}, -\frac{3}{2})$ at which $\max Z = -3$.

but the solution $(-\frac{3}{2}, -\frac{3}{2})$ does not satisfy $x_1, x_2 \geq 0$

$\therefore$ common convex region is not feasible region

2(a) Find all homomorphisms from $(Z_8, +)$ to $(Z_6, +)$.

Sol. $Z_8 = \langle [1] \rangle$.

Let $f : Z_8 \to Z_6$ be a homomorphism.

For any $[a] \in Z_8$, $f([a]) = a f([1])$ shows that f is completely known if $f([1])$ is known.

Now, $o(f[1])$ divides $o([1])$ and $|Z_6|$

i.e. $o(f[1])$ divides 8 and 6

hence $o(f[1]) = 1$ or 2

Thus, $f([1]) = [0]$ or $[3]$. If $f([1]) = [0]$ then f is trivial homomorphism.

If $(f[1]) = [3]$, implies that $f([a]) = [3a]$ $\forall [a] \in Z_8$.

Thus $f([a]+[b]) = f([a+b]) = [3(a+b)]$

$= [3a+3b] = [3a]+[3b] = f([a])+f([b])$.

$\therefore f : Z_8 \to Z_6$ is defined by

$f([a]) = [3a]$

Hence there are two homomorphisms from Z_8 into Z_6.

Q2(b) Factorise x^2+x+5 in $F[x]$ where F is a field of integers modulo 11.

Sol $F = Z_{11} = \{0, 1, \ldots 10\}$

$x^2+x+5 = (x+a)(x+b)$

here, $a+b = 1$ — (i)

$ab = 5$ — (ii)

(i) is satisfied by $(1,0), (2,10), (3,9)$
$(4,8), (5,7), (6,6)$

consequently,

$ab = 0, 9, 5, 10, 2$

We see, both satisfied when $a=3, b=9$

hence $x^2+x+5 = (x+3)(x+9)$ in Z_{11}.

Test-10

2(c) Test for convergence the series $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}+\ldots$

Sol^n:

Here, $u_n = \dfrac{1^2.3^2.5^2\ldots(2n-1)^2}{2^2.4^2.6^2\ldots(2n)^2}$

$$\therefore \quad u_{n+1} = \frac{1^2.3^2.5^2\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots(2n)^2(2n+2)^2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2(n+2)^2}{2(n+1)^2} = \frac{4n^2+8n+4}{4n^2+4n+1}$$

$\therefore \lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{4}{4} = 1$ and hence the ratio test fails.

Now, we shall apply Rabbe's test,

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{4n^2+8n+4}{4n^2+4n+1}-1\right]$$

$$= \frac{4n^2+3n}{4n^2+4n+1}$$

$\therefore \lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}}-1\right) = \dfrac{4}{4} = 1$ and hence the Rabbe's test also fails.

Further, we shall apply De Morgan's and Bertrand's test.

$$\therefore \left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]\log n = \left[\frac{4n^2+3n}{4n^2+4n+1}-1\right]\log n$$

$$= \frac{-n-1}{4n^2+4n+1}\log n$$

$$= \frac{\log n}{n}\cdot\frac{-1-\frac{1}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}$$

$$\therefore \lim_{n\to\infty}\left[(\log n)\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\log n}{n}\cdot\frac{-1-\frac{1}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}\right]$$

$$= 0.\left(\frac{-1}{4}\right)$$

$= 0$ which is < 1

Therefore, By De Morgan's and Bertrand's test the given series is divergent.

2(d) Apply calculus of residues to prove that.

$$\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right];\ a>b>0.$$

Sol'n:

2.(d) Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$.

Sol: put $z = e^{i\theta}$ $\therefore$ $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

& $\frac{dz}{iz} = d\theta$ $\quad\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

We get,

$$I = \int_C \frac{\left[(z-\frac{1}{z})/2i\right]^2}{\left[a + \frac{b}{2}(z+\frac{1}{z})\right]} \frac{dz}{iz}$$ where c is a unit circle $|z|=1$

$$= -\frac{1}{2ib}\int_C \frac{(z^2-1)^2}{z^2\left[z^2 + \frac{2a}{b}z + 1\right]}dz$$

$$= \frac{i}{2b}\int_C \frac{(z^2-1)^2}{z^2\left[z^2 + \frac{2a}{b}z + 1\right]}dz$$

where $f(z) = \frac{i}{2b}\frac{(z^2-1)^2}{z^2\left[z^2+\frac{2a}{b}z+1\right]} = \frac{i}{2b}\frac{(z^2-1)^2}{(z-\alpha)(z-\beta)}$

here $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ & $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

The singularities of $f(z)$ are given by

$z^2\left[z^2 + \frac{2a}{b}z + 1\right] = 0$ i.e. $z^2(z-\alpha)(z-\beta) = 0$

this yields $z = 0$ (a pole of order 2)

$z = \alpha$ (a pole of order 1)

& $z = \beta$ (a pole of order 1)

$\alpha + \beta = -\frac{2a}{b}$; $\alpha\beta = 1$; $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$

As $a > b > 0$ $\therefore |\beta| > 1$; $|\alpha||\beta| = 1$ $\therefore |\alpha| < 1$

i.e. α lies inside where β lies outside the contour C. Thus the poles lying inside the contour C are (i) a simple pole at $z = \alpha$ (ii) a double pole at $z = 0$

Res. of $f(z)$ at $z=\alpha$ is

$$\underset{z=\alpha}{\text{Res}}\, f(z) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{i}{2b}\cdot\frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}$$

$$= \frac{i}{2b}\frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)} = \frac{i}{2b}\frac{(\alpha-\frac{1}{\alpha})^2}{(\alpha-\beta)} = \frac{i}{2b}\frac{(\alpha-\beta)^2}{(\alpha-\beta)} \quad \{\because \alpha\beta=1$$

$$= \frac{i}{2b}(\alpha-\beta) = \frac{i}{2b}\times\frac{2\sqrt{a^2-b^2}}{b} = \frac{i\sqrt{a^2-b^2}}{b^2}$$

Res of $f(z)$ at $z=0$ is

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-0)^2 f(z)\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[z^2\cdot\frac{i}{2b}\frac{(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)}\right]$$

$$= \lim_{z\to 0}\frac{i}{2b}\left[\frac{2(z^2-1)2z\cdot\left(z^2+\frac{2a}{b}z+1\right)-\left(2z+\frac{2a}{b}\right)(z^2-1)^2}{\left(z^2+\frac{2a}{b}z+1\right)^2}\right]$$

$$= \frac{i}{2b}\left(-\frac{2a}{b}\right) = -\frac{ia}{b^2}$$

Hence by cauchy residue theorem

$$I = \int_C f(z)\,dz = 2\pi i\times(\text{sum of residues at poles within } C)$$

$$= 2\pi i\left[\frac{i\sqrt{a^2-b^2}}{b^2}-\frac{ia}{b^2}\right] = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

$$\int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

Q3(a) Prove that $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in \mathbb{Z}\}$ is a Euclidean domain.

8(5)

Sol. Define $\delta(a+b\sqrt{2}) = |a^2-2b^2|$
$\forall\ a+b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]\setminus\{0\}$.

clearly, $\delta(u) \geq 0\ \forall\ u \neq 0$ in $\mathbb{Z}[\sqrt{2}]$.

Also, verify that $a^2-2b^2=0$ iff $a=b=0$.
as $a,b \in \mathbb{Z}$.

Thus, $\delta(u) \geq 1\ \forall\ u \in \mathbb{Z}[\sqrt{2}]-\{0\}$.

Further, for any $u = a+b\sqrt{2}$, $v = c+d\sqrt{2}$,
$u, v \in \mathbb{Z}[\sqrt{2}]-\{0\}$

$$\delta(uv) = \delta\left((ac+2bd)+(bc+ad)\sqrt{2}\right)$$
$$= |(ac+2bd)^2 - 2(bc+ad)^2|$$
$$= |a^2c^2+4abcd+4b^2d^2 - 2(b^2c^2+2abcd+a^2d^2)|$$
$$= |a^2c^2+4b^2d^2-2b^2c^2-2a^2d^2|$$
$$= |(a^2-2b^2)(c^2-2d^2)|$$
$$= \delta(u)\,\delta(v) \geq \delta(u) \quad \text{as } \delta(v) \geq 1.$$

Next, let $u, v \in \mathbb{Z}[\sqrt{2}]$ with $v \neq 0$ then
$u = a+b\sqrt{2}$, $v = c+d\sqrt{2}$ for some $a,b,c,d \in \mathbb{Z}$
such that $(c,d) \neq (0,0)$.

Now, $\dfrac{u}{v} = \dfrac{(a+b\sqrt{2})(c-d\sqrt{2})}{c^2-2d^2} = \alpha+\beta\sqrt{2}$ (say)

where α, β are rational numbers.

Then $\exists$ integers m & n such that

$|m-\alpha| \le \frac{1}{2}$ & $|n-\beta| \le \frac{1}{2}$. So,

$u = (\alpha+\beta\sqrt{2})v = (m+n\sqrt{2})v + [(\alpha-m)+(\beta-n)\sqrt{2}]$

Now, $[(\alpha-m)+(\beta-n)\sqrt{2}]v = u-(m+n\sqrt{2})v \in Z[\sqrt{2}]$

as $Z[\sqrt{2}]$ is a ring. and $m+n\sqrt{2} \in Z[\sqrt{2}]$

Let $r = [(\alpha-m)+(\beta-n)\sqrt{2}]v$. Then

$\delta(r) = |(\alpha-m)^2 - 2(\beta-n)^2|\,|c^2-2d^2|$

$\le \left(\frac{1}{4}+\frac{2}{4}\right)|c^2-2d^2|$

$= \frac{3}{4}|c^2-2d^2| < |c^2-2d^2| = \delta(v)$.

Thus taking $q = m+n\sqrt{2}$, we have

$u = vq + r$ where $r=0$ or

$\delta(r) < \delta(v)$.

Hence, $Z[\sqrt{2}]$ is a Euclidean Domain.

3(b) A function f is defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{2^n}, & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}; (n=0,1,2,\ldots) \\ 0; & x=0. \end{cases}$$

prove that (i) f is integrable on $[0,1]$ (ii) $\int_0^1 f = 2/3$

Sol Let $f(x) = \begin{cases} \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}; & n=0,1,2\ldots \\ 0; \text{ when } x=0. \end{cases}$

then $f(x) = \begin{cases} \frac{1}{2^0} = 1 \text{ when } \frac{1}{2^1} < x \le \frac{1}{2^0} = 1 \\ \frac{1}{2} \text{ when } \frac{1}{2^2} < x \le \frac{1}{2^1} \\ \frac{1}{2^2}; \text{ when } \frac{1}{2^3} < x \le \frac{1}{2^2} \\ \vdots \\ \frac{1}{2^{n-1}} \text{ when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ \vdots \\ 0 \text{ when } x=0. \end{cases}$

Since f is bounded and continuous on $[0,1]$ except at the points $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

$\therefore$ The set of points of discontinuity of f on $[0,1]$ is $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$

which has only one limit point 0.

Since the set of points of discontinuity of 'f' on [0,1] has a finite number of limit points.

$\therefore$ f is integrable on [0,1].

Now $\int_{\frac{1}{2^n}}^{1} f(x)dx = \int_{\frac{1}{2}}^{1} f(x) + \int_{\frac{1}{2^2}}^{\frac{1}{2}} f(x)dx$

$- - - + \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx$

$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2}-\frac{1}{2^3}\right) +$

$- - - + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)$.

$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) + - -$

$- - - + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right)$

$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + - - - + \left(\frac{1}{2^2}\right)^{n-1}\right]$

$\therefore \int_{\frac{1}{2^n}}^{1} f(x)dx = \frac{1}{2}\left[\frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}\right] = \frac{2}{3}\left(1-\frac{1}{4^n}\right)$.

$\Rightarrow \lim_{n\to\infty} \int_{\frac{1}{2^n}}^{1} f(x)dx = \lim_{n\to\infty} \frac{2}{3}\left(1-\frac{1}{4^n}\right) \Rightarrow \int_0^1 f(x)dx = \frac{2}{3}$.

3(C) Solve the LPP.

$\max Z = 6x_1 + 4x_2$

S.C. $2x_1 + 3x_2 \le 30$

$3x_1 + 2x_2 \le 24$.

$x_1 + x_2 \ge 3$

$x_1, x_2 \ge 0$.

Is your answer unique? If not, give 3 different solutions.

Sol let us write the given LPP in standard form:

$\max Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - MA$

S.C.

$2x_1 + 3x_2 + s_1 + 0s_2 + 0A = 30$.

$3x_1 + 2x_2 + 0s_1 + s_2 + 0A = 24$

$x_1 + x_2 + 0s_1 + 0s_2 - s_3 + A = 3$.

$x_1, x_2, s_1, s_2, s_3, A \ge 0$.

Where s_1, s_2 are slack variables, s_3 is the surplus variable and A is the artificial variable.

Now IBFS is: $x_1 = x_2 = s_3 = 0$ (non-basic)

$s_1 = 30$, $s_2 = 24$, $A = 3$ (basic)

for which $Z = -3M$.

Now we put the above information in the Simplex tableau.

C_B	Basis	c_j: 6 x_1	4 x_2	0 s_1	0 s_2	0 s_3	$-M$ A	b	θ
0	s_1	2	3	1	0	0	0	30	30/2
0	s_2	3	2	0	1	0	0	24	24/3
$-M$	A	1	1	0	0	-1	1	3	3/1 →
$z_j = \Sigma C_B a_{ij}$		$-M$	$-M$	0	0	M	$-M$	$-3M$	
$C_j = c_j - z_j$		$6+M$ ↑	$M+4$	0	0	$-M$	0		

from the above table, x_1 is the entering variable, A is the outgoing variable and omit its column in the next simplex and (1) is the key element and all other elements in its column equal to zero.

Then the revised simplex table is

C_B	Basis	c_j: 6 x_1	4 x_2	0 s_1	0 s_2	0 s_3	b	θ
0	s_1	0	1	1	0	2	24	12
0	s_2	0	-1	0	1	3	15	5 →
6	x_1	1	1	0	0	-1	3	(-ve)
$z_j = \Sigma C_B a_{ij}$		6	6	0	0	-6		
$C_j = c_j - z_j$		0	-2	0	0	$+6$ ↑		

from the above table s_3 is entering variable and s_2 is the outgoing variable and 3 is the key element and make it unity and all other elements in its column equal to zero

Then the revised simplex table is.

	c_j	6	4	0	0	0	
C_B	Basic	x_1	x_2	s_1	s_2	s_3	b
0	s_1	0	$5/3$	1	$-2/3$	0	14
0	s_3	0	$-1/3$	0	$1/3$	1	5
6	x_1	1	$2/3$	0	$1/3$	0	8
$Z_j = \Sigma c_B a_{ij}$		6	4	0	2	0	
$C_j = c_j - z_j$		0	0	0	-2	0	

Since all $C_j \leq 0$, an optimum solution has been reached. The optimum basic feasible solution is

$$\boxed{x_1 = 8,\ x_2 = 0,\ z_{Max} = 48.}$$

from the optimum tableau. we observe that the net evaluation corresponding to non-basic variable x_2 is zero. This is an indication for the existence of an alternate basic feasible solution.

Thus we can bring x_2 into basis in place of s_1 or s_3. The resulting new basic solutions will also be an optimum solution.

Introducing x_2 and dropping s_1, the alternative optimum table is :

	c_j	6	4	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b
4	x_2	0	1	$3/5$	$-2/5$	0	$42/5$
0	s_3	0	0	$1/5$	$1/5$	1	$39/5$
6	x_1	1	0	$-2/5$	$3/5$	0	$12/5$
$Z_j = \sum c_B a_{ij}$		6	4	0	2	0	48
$C_j = c_j - z_j$		0	0	0	-2	0	

$\therefore$ An alternative optimum basic feasible solution is $\boxed{x_1 = 12/5,\ x_2 = 42/5,\ Z_{max} = 48}$

Similarly third optimal solution is $x_1 = \frac{26}{5}$, $x_2 = \frac{21}{5}$ and $\max Z = 48$

4(a) (i) $\beta = (123)(145)$, write β^{99} in cyclic notation.

Sol. $\beta = (123)(145)$

$\beta = (14523)$

$\beta^{99} = \beta^{4} = \beta^{-1}$

because order of β is 5.

$\therefore (\beta)^{99} = (\beta)^{5\times 19}(\beta)^{4} = (\beta)^{-1}$

$\therefore \beta^{-1} = (13254)$

$\therefore \boxed{\beta^{99} = (13254)}$

(ii) $\beta = (1357986)(2\,4\,10)$ in S_{10}. What is the smallest +ve integer n for which $\beta^n = \beta^{-5}$.

Sol. $|\beta| = \text{LCM}(7,3)$

$= 21$

$\therefore \beta^{21} = I$

$\beta^{16}\beta^{5} = I$

$\beta^{16} = \beta^{-5} \Rightarrow \boxed{n = 16}$

4(b). Show that sequence $<f_n>$ where $f_n(x) = nx(1-x)^n$ does not converge uniformly on $[0,1]$.

Sol^n: Given $f_n(x) = nx(1-x)^n$, Hence.

for $x=0$, $f_n(x)=0$

and for $0<x<1$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}}$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$= \lim_{n\to\infty} -\frac{x(1-x)^n}{\log(1-x)}$$

$$= 0 \qquad (\because (1-x)^n \to 0 \text{ as } n\to\infty)$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x\in[0,1]$$

Here,

$$M_n = \sup\{|f_n(x) - f(x)| / x\in[0,1]\}$$

$$= \sup\{nx(1-x)^n / x\in[0,1]\}$$

$$\geq n\cdot\frac{1}{n}\left(1-\frac{1}{n}\right)^n \quad \text{---- (consider } x=\tfrac{1}{n},\ x\in[0,1])$$

$$= \left(1-\frac{1}{n}\right)^n.$$

i.e. $M_n \longrightarrow \frac{1}{e}$ as $n\longrightarrow\infty$

since M_n cannot tend to zero as $n \to \infty$

Therefore, By M_n-test the sequence $<f_n>$ does not converge uniformly on $[0,1]$.

(Here, 0 is a point of non-uniform convergence $\because x \to 0$ as $n \to \infty$)

4(d). Solve the assignment problem represented by the following matrix.

	I	II	III	IV	V	VI
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	[illegible]
E	36	11	57	22	[illegible]	[illegible]
F	3	56	53	[illegible]	[illegible]	28

Sol^n^: Thus the optimal assignment is

$A \to IV,\ B \to I,\ C \to VI,\ D \to III,\ E \to II,\ F \to V,$

and minimum cost; $z = 11+43+33+27+11+17 = 142$

and Another optimal assignment is

$A \to IV,\ B \to VI,\ C \to II,\ D \to III,\ E \to V,\ F \to I$

and minimum cost

$$z = 11+48+28+27+25+3 = 142.$$

4(c) The real part of a complex analytic function is $u = x^6 - 3xy^4$. What is its imaginary part?
Express the complex function as a function as a function of $z = x+iy$.

Sol Let $u = x^6 - 3xy^4$.

then $\frac{\partial u}{\partial x} = 6x^5 - 3y^4$.

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 30x^4$.

$\therefore \frac{\partial u}{\partial y} = -12xy^3$.

$\therefore \frac{\partial^2 u}{\partial y^2} = -36xy^2$.

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$.

$\therefore$ u is not a ~~not~~ harmonic function.

(Here the problem has been presented wrongly)

5(a). Given $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$

The Lagrange's auxillary equations for given equation are;

$$\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{z(x-y)} \quad \text{---(1)}$$

Choosing 1, -1, 0 as multipliers each fraction of (1),

$$=\frac{dx-dy}{(x^2-y^2-yz)-(x^2-y^2-zx)}=\frac{dx-dy}{z(x-y)} \quad \text{---(2)}$$

Choosing $x, -y, 0$ as multipliers each fraction of (1),

$$\left.\begin{aligned}&=\frac{x\,dx-y\,dy}{x(x^2-y^2-yz)-y(x^2-y^2-zx)}\\&=\frac{x\,dx-y\,dy}{(x-y)(x^2-y^2)}\end{aligned}\right] \quad \text{---(3)}$$

from (1), (2) and (3)

$$\left.\begin{aligned}&\frac{dz}{z(x-y)}=\frac{dx-dy}{z(x-y)}=\frac{x\,dx-y\,dy}{(x-y)(x^2-y^2)}\\&\text{(OR)}\quad \frac{dz}{z}=\frac{dx-dy}{z}=\left(\frac{2x\,dx-2y\,dy}{2(x^2-y^2)}\right)\end{aligned}\right\} \quad \text{---(4)}$$

Taking the first two fractions of (4),

$dz = dx - dy \Rightarrow z - x + y = C_1$ --- (5)

Taking the first and third fractions of (4),

$\log(x^2 - y^2) - 2\log z = C_2$ — By integrating

or $\dfrac{(x^2-y^2)}{z^2} = C_2$ --- (6)

from (5) and (6), the solution is

$\phi\left(z - x + y, \dfrac{(x^2+y^2)}{z^2}\right) = 0$.

where ϕ is an arbitary function.

5(b) Given $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$

$\therefore \quad (D+3D')(D-2D')z = x^2\sin(x+y)$

let ϕ_1, ϕ_2 being arbitary functions.

$\therefore$ C.F. $= \phi_1(y-3x) + \phi(y+2x)$

$$P.I = \frac{1}{(D+3D')(D-2D')} x^2\sin(x+y)$$

$$= \frac{1}{D+3D'}\left(\frac{1}{(D-2D')}\cdot x^2\sin(x+y)\right)$$

$$= \frac{1}{D+3D'}\int x^2\sin(x+c-2x)\,dx$$

$$= \frac{1}{D+3D'}\int x^2\sin(c-x)\,dx$$

Here, $c = y+2x$

$$\therefore \quad P.I. = \frac{1}{D+3D'}\left[x^2\cos(c-x) - 2\int x\cos(c-x)dx\right]$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \left\{-2x\sin(c-x) + \int 2\sin(c-x)\,dx\right\}\right]$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)\right]$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y) + 2x\sin(x+y)\right]$$

$$\text{---} \; (\because c = y+2x)$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y) + 2x\sin(x+y)\right]$$

$$= \int \left[(x^2-2)\cos(x+c'+3x) + 2x\sin(x+c'+3x)\right] dx$$

Here, $c' = y-3x$.

$$\therefore \text{P.I.} = \int (x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')\,dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\,\frac{\sin(4x+c')}{4}\,dx +$$

$$2\int x\sin(4x+c')\,dx$$

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx$$

$$= \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[\frac{x}{4}\cos(4x+c') + \int \frac{\cos(4x+c')}{4}\,dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x)$$

$$+\frac{3}{32}\sin(4x+y-3x) \quad \ldots (\because c' = y-3x)$$

$$= \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y)$$

Hence, The solution is

$$z = \phi_1(y-3x) + \phi_2(y-2x) + \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y)$$

$$- \frac{3x}{8}\cos(x+y)$$

5(c). Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four places of decimal.

5.(c) Using Newton's Raphson Method,

Sol'n Let $f(x) = e^x \sin x - 1$ ——— (1)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Diff. (1) we get, $f'(x_n) = e^{x_n}[\sin x_n + \cos x_n]$

$$\therefore x_{n+1} = x_n - \left[\frac{e^{x_n}\sin x_n - 1}{e^{x_n}(\sin x_n + \cos x_n)}\right]$$

Let $x_0 = 0$, $n = 0$

$$x_1 = 0 - \left[\frac{-1}{1}\right] = 1$$

$$x_2 = x_1 - \left[\frac{e^{x_1}\sin x_1 - 1}{e^{x_1}(\sin x_1 + \cos x_1)}\right] = 1 - [0.34273] = 0.65725$$

$$x_3 = 0.65725 - \left[\frac{e^{0.65725}\sin(0.65725) - 1}{e^{0.65725}(\sin(0.65725) + \cos(0.65725))}\right]$$

$= 0.591183$

$$x_4 = 0.591183 - \left[\frac{e^{0.5912}\sin(0.5912) - 1}{e^{0.5912}(\sin 0.5912 + \cos 0.5912)}\right]$$

$= 0.5885$

$\therefore$ smallest root of $f(x) = 0.5885$

5(d). Draw a switching circuit that realizes the following switch function. If possible, draw a simpler switching circuit.

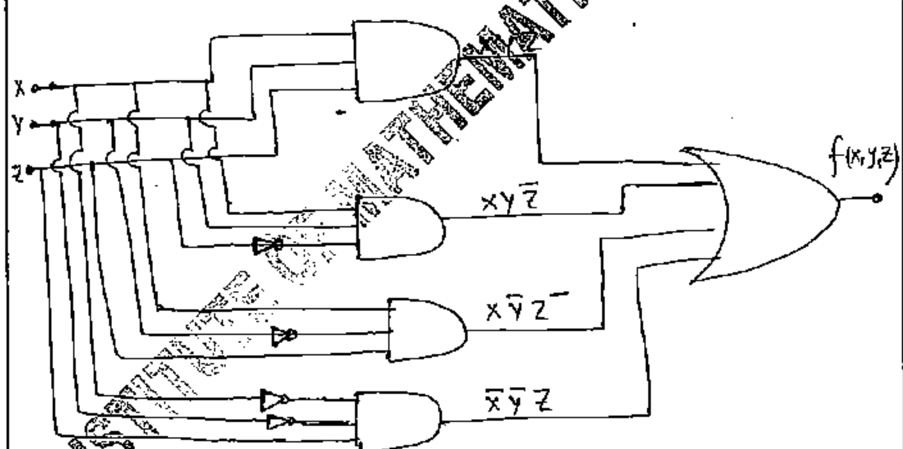
5(d) Sol^n:

x	y	z	$f(x,y,z)$	
1	1	1	1	xyz
1	1	0	1	xyz'
1	0	1	1	$xy'z$
1	0	0	0	
0	1	1	0	
0	1	0	0	
0	0	1	1	$x'y'z$
0	0	0	0	

Since Minterms are $xyz, xyz', xy'z, x'y'z$

so, $f(x,y,z) = xyz + xyz' + xy'z + x'y'z$

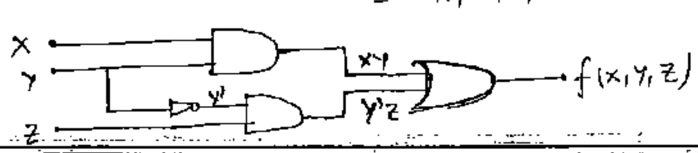
The same could be drawn as switching circuit given below:—



Reducing $f(x,y,z) = xyz + xy\bar{z} + x\bar{y}z + \bar{x}\bar{y}z$

$= xy(z+\bar{z}) + \bar{y}z(x+\bar{x})$

$= xy + \bar{y}z$



5(e) Given $u=-\omega y$, $v=\omega x$, $w=0$; show that the surfaces intersecting the streamlines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist.

Sol^n: Step I: To show that liquid motion is possible, we have to show that the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ is satisfied.}$$

Here $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0+0+0=0$ Hence the result I.

Step II: To show that the surfaces orthogonal to stream lines are planes through z-axis.

The required surfaces are solutions of

$$u\,dx+v\,dy+w\,dz=0 \text{ i.e.}$$

$$-\omega y\,dx+\omega x\,dy+0\,dz=0$$

$$\Rightarrow \frac{dx}{x}-\frac{dy}{y}=0$$

Integrating $\log\frac{x}{y}=\log a \Rightarrow \frac{x}{y}=a \Rightarrow x=ay$, which is a plane through z-axis.

Step III: To show that velocity potential ϕ does not exist

By def. $d\phi=-(u\,dx+v\,dy+w\,dz)$

$$=-\left[-\omega y\,dx+\omega x\,dy+0\,dz\right]$$

$$\Rightarrow d\phi=\omega y\,dx-\omega x\,dy=M\,dx+N\,dy, \text{ say}$$

Here $\frac{\partial M}{\partial y}=\omega$, $\frac{\partial N}{\partial x}=-\omega$. Hence $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

$\therefore$ the equation is not exact so that $d\phi=\omega y\,dx-\omega x\,dy$ cannot be integrated so that ϕ does not exist.

6(a) Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

6(a) soln → Given $\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 0$

$\therefore \quad r + 2s + t = 0 \qquad \text{---(1)}$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$\therefore R = 1,\ S = 2,\ T = 1$

$\therefore S^2 - 4RT = 0 \Rightarrow$ (1) is parabolic.

The λ-quadratic equation reduces to

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1, -1$.

The corresponding characteristic equation is,

$\frac{dy}{dx} - 1 = 0$ or $dx - dy = 0$

$\therefore \quad x - y = c$ — by integrating.

where c is an arbitrary constant.

let $u = x - y$ and $v = x + y$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= 1 \cdot 1 + 1 \cdot 1$$

$$= 2$$

$$\neq 0$$

from (2),

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{--- (3)}$$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$

$$= -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{--- (4)}$$

from (3) and (4),

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial v} - \frac{\partial}{\partial u}$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 v}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

also, $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 z}{\partial v^2} + \frac{\partial^2 z}{\partial v^2}$$

from (1),

$$\frac{\partial^2 z}{\partial v^2} = 0 \quad \text{or} \quad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0 .$$

By integrating partially w.r.t. v

$$z = \int \phi(u)\, dv + \psi(u)$$

$$= v\,\phi(u) + \psi(u)$$

where $\frac{\partial z}{\partial v} = \phi(u)$, ϕ is an arbitary function.

$$z = (x+y)\,\phi(x-y) + \psi(x-y)$$

6(b) Find a surface passing through the two lines $z=x=0$, $z-1=x-y=0$ satisfying $r-4s+4t=0$.

6(b) Sol: The given equation may be written as

$$\frac{\partial^2 z}{\partial x^2} - 4\left(\frac{\partial^2 z}{\partial x \partial y}\right) + 4\left(\frac{\partial^2 z}{\partial y^2}\right) = 0.$$

i.e. $(D - 2D')^2 z = 0$.

let ϕ_1, ϕ_2 are any arbitary functions.

$$z = \phi_1(y+2x) + x\,\phi_2(y+2x). \quad \text{---(1)}$$

from (1), the surface passes through $z=x=0$,

we have $0 = \phi_1(y)$.

$$\therefore \phi_1(y+2x) = 0$$

$$\therefore \quad z = x\,\phi_2(y+2x) \quad \text{---(2)}$$

equation (2) passes through $z-1 = x-y = 0$.

i.e. $z=1$ and $y=x$

$$\therefore \quad 1 = x\,\phi_2(3x) \quad \text{or} \quad \phi_2(3x) = \frac{3}{3x}$$

$$\therefore \quad \phi_2(y+2x) = \frac{3}{(y+2x)}$$

$3x = z(y+2x)$, is the required surface.

Q. 6(C) Using Newton's forward formula, estimate the number of persons earning wages b/w Rs. 60 and Rs. 70 from the following data:

Wages (Rs.)	Below 40	40–60	60–80	80–100	100–120
No. of persons (in thousands)	250	120	100	70	50

6(C) Sol'n: Newton's forward table :-

Wage x	No. of Persons y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		$+10$	
100	540		-20		
		50			
120	590				

Newton forward Interpolation formulae

$$y(x_0+nh) = y(x_0) + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 y_0 \quad \text{——— (1)}$$

$x_0 = 40$, $h = 20$

$70 = 40 + n \times 20 \Rightarrow n = 3/2$

$$\therefore y(70) = 250 + 3/2 \times 120 + \frac{3/2\,(3/2-1)}{2} \times (-20) + \frac{3/2 \times (3/2-1)(3/2-2)}{6} \times (-10) + \frac{3/2(3/2-1)(3/2-2)(3/2-3)}{24} \times 20$$

$$= 250 + 180 + (-7.5) + 0.625 + 0.46875$$

$y(70) = 423.59375$

$\because y(60) = 370$

∴ No. of persons with wages b/w Rs. 60 & 70 are:-

$$(423.59375 - 370) \times 1000$$

$$= 53593.75 \simeq 53593.75$$

$$\simeq \underline{53593}$$

6(d) The velocity v as a particle at a distance s from a point on its path is given by the table

s(ft)	0	10	20	30	40	50	60
v ft/sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60ft by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule.

Sol^n

we know $v = \frac{ds}{dt}$

$$dt = \int \frac{ds}{v}$$

$$\Rightarrow t = \int dt = \int \frac{ds}{v} \quad \text{—— (I)}$$

So,

	s_0	s_1	s_2	s_3	s_4	s_5	s_6
s	0	10	20	30	40	50	60
$y = 1/v$	0.02127	0.01724	0.015625	0.015385	0.016393	0.01923	0.02630
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using Simpson's 1/3 rule

$$(I) = \frac{h}{3}\left[y_0 + y_6 + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5) \right]$$

$$\because h = \frac{b-a}{n} = \frac{60-0}{6} = 10$$

$$\therefore T = \frac{10}{3}\left[0.02127 + 0.02631 + 4(0.017241 + 0.01538 + 0.01923) + 2(0.015625 + 0.01639)\right]$$

$t = 1.06538$ sec.

Simpson's 3/8 Rule

$$t = \int y\,ds = \frac{3h}{8}\left[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3 \right]$$

$$= \frac{30}{8}\Big[0.02127 + 0.02631 + (2 \times 0.01538) + 3(0.017241 + 0.015625 + 0.01639 + 0.01923)\Big]$$

$$= 1.032615 \text{ sec.}$$

$\therefore$ time 't' by simpson's 1/3 rd rule $= 1.06538$ sec

time 't' by Simpson's 3/8 Rule $= 1.032615$ sec.

$\therefore$ Difference $= \underline{0.032765}$

Q(a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2)=0$. What is the order of this partial differential equation?

Sol'n

Given $\phi(x+y+z, x^2+y^2-z^2)=0$

let $u = x+y+z$ and $v = x^2+y^2-z^2$

$\therefore \quad \phi(u,v)=0$

differentiating w.r.t. 'x',

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$\therefore \quad \frac{\partial u}{\partial x}=1, \ \frac{\partial u}{\partial z}=1, \ \frac{\partial v}{\partial x}=2x, \ \frac{\partial v}{\partial z}=-2z, \ \frac{\partial u}{\partial y}=1$

and $\frac{\partial v}{\partial y}=2y$.

$$\therefore \quad \frac{\partial \phi}{\partial u}(1+p)+2\left(\frac{\partial \phi}{\partial v}\right)(x-pz)=0$$

$$\therefore \quad \left(\frac{\partial \phi}{\partial u}\right)\Big/\left(\partial \phi/\partial v\right)=\frac{-2(x-pz)}{1+p} \quad \text{---(1)}$$

differentiating w.r.t. 'y',

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial u}{\partial y}\right)=0$$

$$\therefore \quad \frac{\partial \phi}{\partial u}(1+q)+2\left(\frac{\partial \phi}{\partial v}\right)(y-zq)=0$$

$$\therefore \quad \left(\frac{\partial \phi}{\partial u}\right)\Big/\frac{\partial \phi}{\partial v}=\frac{-2(y-qz)}{1+q} \quad \text{---(2)}$$

from (1) and (2),

$$\frac{(x-pz)}{1+p}=\frac{y-qz}{1+q} \Rightarrow (y+z)p-(x+z)q=x-y.$$

which is the required P.D.E. of first order.

7(b) A square plate is bounded by the lines $x=0$, $y=0$, $x=10$ & $y=10$. Its faces are insulated, the temp. along the upper horizontal edge is given by $u(x,10) = x(10-x)$ while the other three faces are kept at 0°C. Find the steady state temperature in the plane.

7(b) Sol'n The steady state temperature $u(x,y)$ is the solution of Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

subject to boundary conditions,

$u(0,y) = u(10,y) = 0 \quad , \quad 0 \le y \le 10.$

$u(x,0) = 0 \; , \; 0 \le x \le 10$

and $u(x,10) = 10x - x^2 \; , \; 0 \le x \le 10$

Here, $a=b=10$ and $u(x,b) = u(x,10) = f(x) = x(10-x)$

$$E_n = \frac{2}{10 \sinh n\pi} \int_0^{10} (10x - x^2) \sin \frac{n\pi x}{10} \, dx$$

By using chain rule,

$$E_n = \frac{1}{5 \sinh n\pi} \left[(10x - x^2)\left(\frac{-10}{n\pi}\right) \cos \frac{n\pi x}{10} - (10 - 2x) \left(-\frac{100}{n^2\pi^2}\right) \sin \frac{n\pi x}{10} + (-2) \frac{1000}{n^3\pi^3} \cos \frac{n\pi x}{10} \right]_0^{10}$$

$$= \frac{1}{5 \sinh n\pi} \left[-\frac{2000(-1)^n}{n^3\pi^3} + \frac{2000}{n^3\pi^3} \right]$$

$$= \frac{400\{1 - (-1)^n\}}{n^3\pi^3 \sinh n\pi}$$

$$\therefore E_n = \begin{cases} 0 & \text{if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ \dfrac{800 \operatorname{cosech} (2m-1)\pi}{(2m-1)^3 \pi^3} & \text{if } n = 2m-1, \; m = 1, 2, \ldots \end{cases}$$

$$\therefore \quad u(x,y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi x}{10} \sinh \frac{n\pi y}{10}$$

the required temperature is given by

$$u(x,y) = \frac{800}{\pi^3} \sum \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{10}$$

$$\sinh \frac{(2m-1)\pi y}{10} \operatorname{cosech} (2m-1)\pi$$

7(c) solve the following equations by Gauss seidel method $10x+y+2z=13$; $3x+10y+z=14$; $2x+3y+10z=15$

7(c) Sol^n: $\because$ For the given equation.

We get

$$x^{k+1} = \frac{1}{10}\left(13-y^{k}-2z^{k}\right)$$

$$y^{k+1} = \frac{1}{10}\left[14-3x^{k+1}-z^{k}\right)$$

$$z^{k+1} = \frac{1}{10}\left[15-2x^{k+1}-3y^{k+1}\right)$$

Let $\left(x^{(0)}, y^{(0)}, z^{(0)}\right) = (0,0,0)$

First iteration : $k=0$

$$x^{(1)} = \frac{1}{10}\left[13-y^{(0)}-2z^{(0)}\right] = 1.3$$

$$y^{(1)} = \frac{1}{10}\left[14-3x^{(1)}-z^{(0)}\right] = 1.01$$

$$z^{(1)} = \frac{1}{10}\left[15-2x^{(1)}-3y^{(1)}\right) = 0.937$$

For 2nd iteration : $k=1$

$$x^{(2)} = \frac{1}{10}\left(13-y^{1}-2z^{1}\right] = 1.0116$$

$$y^{(2)} = \frac{1}{10}\left(14-3x^{2}-z^{1}\right] = 1.00282$$

$$z^{(2)} = \frac{1}{10}\left(15-2x^{(2)}-3y^{(2)}\right] = 0.996834$$

For 3rd iteration

$$x^{(3)} = \frac{1}{10}\left[13-y^{(2)}-2z^{(2)}\right] = 1.00035$$

$$y^{(3)} = \frac{1}{10}\left(14-3x^{(3)}-z^{(2)}\right] = 1.0002116$$

$$z^{(3)} = \frac{1}{10}\left[15-2x^{(3)}-3y^{(3)}\right) = 0.999866$$

$\therefore$ By Gauss Seidel Method,

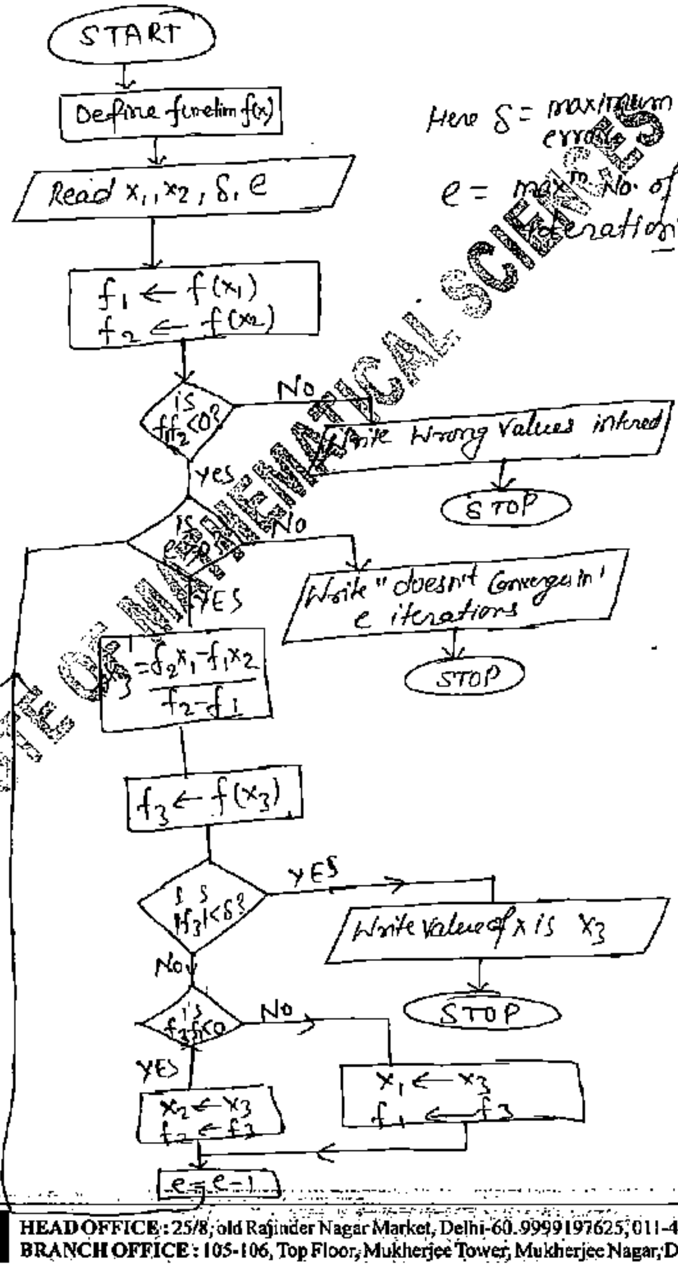
Sol^n are $x = 1.00035 \approx 1$

$y = 1.0002116 \approx 1$

$z = 0.999866 \approx 1$

7(d) Draw a flowchart for Regula falsi method.

7(d) Sol^n:



8(a) Write Hamilton's eqns in polar co-ordinates for a particle mass m moving in three dimensions in a force field of potential V.

Sol^n:- At time t, let (r, θ, ϕ) be the polar co-ordinates for a particle m at P. If (x, y, z) are the cartesian co-ordinates of P, then

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$\therefore$ K.E $= T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$$= \frac{1}{2}m\Big[(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi)^2 + (\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi)^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2\Big]$$

$$= \frac{1}{2}m\Big[\dot{r}^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta) + r^2\dot{\theta}^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta) + r^2\dot{\phi}^2\sin^2\theta(\sin^2\phi + \cos^2\phi)\Big]$$

$\therefore L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V$

Here r, θ and ϕ are the generalised co-ordinates.

$\therefore p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$; $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ and

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta$ —— (1)

Since L doesn't contain t explicitly,

$\therefore H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) + V$

eliminating $\dot{r}, \dot{\theta}, \dot{\phi}$, with the help of relations (1)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V$$

Hence the six Hamilton's eqns are (note that V is function of r, θ, ϕ and t)

$\dot{p}_r = -\frac{\partial H}{\partial r}$ i.e $\dot{p}_r = -\frac{1}{2m}\left(-\frac{2p_\theta^2}{r^3} - \frac{2p_\phi^2}{r^3\sin^2\theta}\right) - \frac{\partial V}{\partial r}$

or $\dot{p}_r = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{\partial V}{\partial r}$ — — (H1)

$\dot{r} = \frac{\partial H}{\partial p_r}$ ie $\dot{r} = p_r/m$ —— (H2)

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$ i.e $\dot{p}_\theta = -\frac{1}{2m}\left(-\frac{2\cos\theta}{r^2\sin^3\theta}p_\phi^2\right) - \frac{\partial V}{\partial \theta}$

or $\dot{p}_\theta = \frac{\cos\theta}{mr^2\sin^3\theta}p_\phi^2 - \frac{\partial V}{\partial \theta}$ —— (H3)

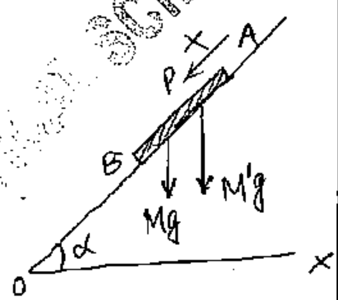
$\dot{\theta} = \frac{\partial H}{\partial p_\theta}$ i.e $\dot{\theta} = \frac{p_\theta}{mr^2}$ —— (H4)

$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi}$ —— (H5)

$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$ —— (H6)

8(b) A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end, walks down the plank so that it doesn't move, show that he gets to the other end in time $\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$, where a is the length of the plane.

Sol^n:- Let the plank AB of mass M and length a rest along the line of greatest slope of a smooth plane inclined at an angle α to the horizon. A man mass M' starts moving down the plank from the upper end A. Let the man move down the plank through a distance AP = x in time t. Since the plank does not move, therefore if $\bar{x}$ is the distance of the C.G of the plank and the man from A in this position, then



$$\bar{x} = \frac{M.AG + M'.AP}{M+M'} = \frac{M.(a/2) + M'x}{M+M'}$$

Differentiating twice w.r.t. t, we get

$$\ddot{\bar{x}} = \frac{M'}{M+M'}\ddot{x} \quad \text{——— (1)}$$

Now the total weight $(M+M')g$

will act vertically downwards at the c.G of the system. ∴ The eqn of motion of the c.G of the system is given by

$(M+M')\ddot{\bar{x}} = (M+M')g\sin\alpha$. —— (2)

∴ from (1) & (2), we get

$M'\ddot{x} = (M+M')g\sin\alpha$

Integrating, we get $M'\dot{x} = (M+M')g\sin\alpha\, t + c_1$

But initially when $t=0$; $x=0$ ∴ ...

∴ $M'x = (M+M')g\sin\alpha \cdot \frac{1}{2}t^2$

or, $t = \sqrt{\left\{\frac{2M'x}{(M+M')g\sin\alpha}\right\}}$

putting $x = AB = a$, the time to reach the other end B of the plank is given by

$$t = \sqrt{\left\{\frac{2M'x}{(M+M')g\sin\alpha}\right\}}$$

Main Test Series – 2016
Test – XI – Paper-I - Answer-key.

1(a) Let W be the subspace of $\mathbb{R}^5$ spanned by $u_1 = (1,2,-1,3,4)$, $u_2 = (2,4,-2,6,8)$, $u_3 = (1,3,2,2,6)$, $u_4 = (1,4,5,1,8)$, $u_5 = (2,7,3,3,9)$.
find a subset of the vectors which form a basis of W.

Sol^n: form the matrix whose rows are the given vectors and reduce the matrix to an echelon form but without interchanging any zero rows.

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$
$R_5 \to R_5 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$
$R_5 \to R_5 - 3R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

The non-zero rows are the first, third and fifth rows.

Hence u_1, u_3, u_5 form a basis of W.

1(c) (i) If A is a real skew-symmetric matrix and $A^2+I=0$, then show that A is orthogonal.

(ii) If H is a Hermitian matrix, what kind of matrix is e^{iH}?

Sol: If H be the Hermitian matrix. Then

$$A^T = -A. \quad \text{—— (1)}$$

Also we have $A^2+I=0$ (given)

$$\Rightarrow A^2 = -I$$

$$\Rightarrow A.A = -I$$

$$\Rightarrow A(-A^T) = -I \quad \text{(from (1))}$$

$$\Rightarrow A A^T = I$$

which is the condition for the matrix A to be orthogonal. Hence the matrix A is orthogonal matrix.

(ii) Let H be the Hermitian matrix.

Then $H^\theta = H$

For any matrix M,

we have $(e^M)^\theta = e^{M^\theta}$

Let $e^{iH} = A$. Then

$$A^\theta A = (e^{iH})^\theta e^{iH} \qquad \left(\because (iH)^\theta = \bar{i} H^\theta = -iH\right)$$

$$= e^{-iH} \cdot e^{iH}$$

$$= I \text{ (say)}$$

$\therefore A^\theta A = I$.

Which is the condition for the matrix $A = e^{iH}$ to be unitary.

Hence the matrix e^{iH} is unitary matrix.

1(c) Let $f(x, y) = \begin{cases} \dfrac{\sin(x-y)}{|x|+|y|}, & |x|+|y| \neq 0 \\ 0 & (x,y) = (0,0) \end{cases}$

Is f continuous at the origin? Justify

Sol: Let us approach $(0,0)$ along the x-axis, $y=0$.

$$\therefore \underset{(x,y)\to(0,0)}{Lt} \frac{\sin(x-y)}{|x+y|} = \underset{x\to 0}{Lt} \frac{\sin x}{|x|}$$

$$= \begin{cases} 1 ; & x>0 \\ -1 ; & x<0 \end{cases}$$

$\therefore$ the limit fails to exist

$\therefore$ f is not continuous at $(0,0)$.

1.(d) Given $w = (x, y)$ with $x = u+v$, $y = u-v$

Show that $\dfrac{\partial^2 w}{\partial u \partial v} = \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^2 w}{\partial y^2}$.

Sol Since $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial u}$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \quad \& \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}. \quad (1)$$

Again partially differentiating (1) w.r.t. v & u respectively, we get

$$\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial y}\right) \quad \&$$

$$\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 w}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial v} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\cdot\frac{\partial y}{\partial v} \quad \&$$

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial u} - \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{\partial^2 w}{\partial v \partial u} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

1(e) Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z+2=0$ in a circle of radius unity and find the equations of the sphere which has this circle for one of its great circles.

Sol'n: The centre of the given sphere is $(\frac{1}{2},0,-\frac{1}{2})$ and its radius

$$= \sqrt{\left[\left(\tfrac{1}{2}\right)^2 + 0^2 + \left(-\tfrac{1}{2}\right)^2 - (-2)\right]} = \sqrt{\tfrac{5}{2}} = R \text{ (say)}$$

Also length of $\perp$ lar from $(\frac{1}{2},0,-\frac{1}{2})$ to $x+2y-z-4=0$ is

$$\frac{\frac{1}{2}+2(0)-(-\frac{1}{2})-4}{\sqrt{[2^2+2^2+(-1)^2]}} = \frac{3}{\sqrt{6}} = \frac{1}{2}\sqrt{6} = p \text{ (say)}$$

Then radius of the circle $= \sqrt{R^2-p^2}$

$$= \sqrt{\left(\tfrac{5}{2}-\tfrac{6}{4}\right)} = 1$$

the equations of the circle are

$$x^2+y^2+z^2-x+z-2=0$$
$$x+2y-z-4=0$$

$\therefore$ The equations of a sphere through this circle is

$$(x^2+y^2+z^2-x+z-2) + \lambda(x+2y-z-4)=0$$

$$\Rightarrow x^2+y^2+z^2+(\lambda-1)x+2\lambda y+(1-\lambda)z-(2+4\lambda)=0 \quad \text{(i)}$$

Its centre is $[-\frac{1}{2}(\lambda-1),-\lambda,-\frac{1}{2}(1-\lambda)]$. If this circle is a great circle of the sphere (i), then the centre of (i) should lie on the plane of the circle i.e. the plane $x+2y-z-4=0$.

$$\therefore \quad -\tfrac{1}{2}(\lambda-1)+2(-\lambda)+\tfrac{1}{2}(1-\lambda)-4=0$$
$$-3\lambda-3=0 \Rightarrow \lambda=-1$$

$\therefore$ from (i), the equation of the required sphere is

$$x^2+y^2+z^2-2x-2y+2z=0.$$

2(a) (i) Let T be a linear operator on C^3 defined by
$T(1,0,0) = (1,0,i)$, $T(0,1,0) = (0,1,1)$, $T(0,0,1) = (i,1,0)$
Is T invertible? Justify your answer.

(ii) Let $T: R^3 \to R^3$ be a linear operator the matrix A of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$ Find a basis for the range of T and a basis for the null space of T.

Sol^n: (i) we know that

$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is a basis of C^3 over C.

However, $T(e_1)$, $T(e_2)$, $T(e_3)$ are not linearly independent over C,

Since $1(1,0,i) - i(0,1,1) + 1(i,1,0) = (0,0,0)$

Thus $\{e_1, e_2, e_3\}$ is a basis of C^3 but $\{T(e_1), T(e_2), T(e_3)\}$ is not a basis of C^3. Hence T is not invertible.

(ii) The set $\{e_1 = (1,0,0), e_2 = (0,1,0), (0,0,1)\}$ is the standard ordered basis of R^3, we have

$T(e_1) = (1,0,-1)$, $T(e_2) = (2,1,3)$, $T(e_3) = (1,1,4)$

Let $(x_1, x_2, x_3) \in \ker T$ be arbitrary.

Then $T(x_1, x_2, x_3) = (0,0,0)$

$\Rightarrow T(x_1 e_1 + x_2 e_2 + x_3 e_3) = (0,0,0)$

$\Rightarrow x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) = (0,0,0)$ as T is a L.T.

$\Rightarrow x_1(1,0,-1) + x_2(2,1,3) + x_3(1,1,4) = (0,0,0)$

$\Rightarrow x_1(1,0,-1)+x_2(2,1,3)+x_3(1,1,4)=(0,0,0)$

$\Rightarrow (x_1+2x_2+x_3,\ x_2+x_3,\ -x_1+3x_2+4x_3)=(0,0,0)$

$\Rightarrow x_1+2x_2+x_3=0$ —— (1)

$-x_1+3x_2+4x_3=0$ —— (2)

$x_2+x_3=0 \Rightarrow x_2=-x_3$

from (1) and (2), $x_1=x_3$

i.e. $(x_1,x_2,x_3)=(x_3,-x_3,x_3)=x_3(1,-1,1)$

Hence $\ker T=\{x_3(1,-1,1): x_3\in R\}$

Since $(1,-1,1)\neq(0,0,0)$, so $\{(1,-1,1)\}$ is a basis of $\ker T$ and $\dim \ker T=1$

we know that $\dim \text{Range } T+\dim \ker T=\dim R^3$

i.e. $\dim \text{Range } T=3-1=2$

we now show that $T(e_1), T(e_2)\in \text{Range } T$ are linearly independent

Let $\alpha T(e_1)+\beta T(e_2)=(0,0,0)$; $\alpha,\beta\in R$

$\alpha(1,0,-1)+\beta(2,1,3)=(0,0,0)$

$\Rightarrow (\alpha+2\beta,\ \beta,\ -\alpha+3\beta)=(0,0,0)\Rightarrow \alpha=0, \beta=0$

Thus $\{T(e_1), T(e_2)\}$ is a linearly independent subset of Range T. Since $\dim$ Range $T=2$,

$\{T(e_1)=(1,0,-1),\ T(e_2)=(2,1,3)\}$ is a basis of Range T.

2(b) A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x,y,z) on the probe's surface is

$T(x,y,z)=8x^2+4yz-16z+600$. Find the hottest point on the probe's surface.

Sol^n: $T(x,y,z)=8x^2+4yz-16z+600$

$\therefore g(x,y,z) = 4x^2+y^2+4z^2-16 = 0$

$\nabla T = 16x\,i + 4z\,j + (4y-16)k$ and

$\nabla g = 8x\,i + 2y\,j + 8z\,k.$

so that $\nabla T = \lambda \nabla g.$

$\Rightarrow 16x\,i + 4z\,j + (4y-16)k = \lambda(8x\,i + 2y\,j + 8z\,k)$

$\Rightarrow 16x = 8x\lambda,\ 4y-16 = 8z\lambda,\ 4z = 2y\lambda.$

$\Rightarrow \boxed{\lambda = 2}$ or $\boxed{x = 0}$

<u>Case(1):</u> $\lambda = 2 \Rightarrow 4z = 2y(2) \Rightarrow z = y.$

Then $4z - 16 = 16z \Rightarrow z = -\frac{4}{3}$

$\Rightarrow y = -\frac{4}{3}.$

$\therefore 4x^2+y^2+4z^2 = 4x^2 + \left(\frac{-4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16$

$\Rightarrow x = \pm\frac{4}{3}$

<u>Case(2):</u> $x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right)$

$\Rightarrow y^2 - 4y = 4z^2$

$\Rightarrow 4(0)^2 + y^2 + (y^2-4y) - 16 = 0$

$\Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y-4)(y+2) = 0$

$\Rightarrow y = 4$ or $y = -2.$

now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4).$

$\Rightarrow z = 0$

and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) = 4 + 8 = 12$

$z = \pm\sqrt{3}$

The temperatures are $T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

$= 642\frac{2}{3}^\circ.$

$T(0,4,0) = 600^\circ,\ T(0,-2,\sqrt{3}) = (600 - 24\sqrt{3})^\circ.$

and $T(0,-2,-\sqrt{3}) = (600 + 24\sqrt{3})^\circ \approx 641.6^\circ.$

$\therefore T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

2(c) (i) If the plane $2x-y+cz=0$ cuts the cone $yz+zx+xy=0$ in $\perp$ar lines, find the value of c.

(ii) Find the angle between the lines given $x+y+z=0$ and $\frac{yz}{q-r}+\frac{zx}{r-p}+\frac{xy}{p-q}=0$

Sol'n: (i) Let the plane $2x-y+cz=0$ cut the cone $yz+zx+xy=0$ in a line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Then $2l-m+cn=0$ and $mn+nl+lm=0$ ——(1)

Eliminating m between these relations we get

$(2l+cn)n+nl+l(2l+cn)=0$

$2l^2+(c+3)ln+cn^2=0 \Rightarrow 2\left(\frac{l}{n}\right)^2+(c+3)\frac{l}{n}+c=0$ ——(2)

If the roots of this equation are l_1/n_1 and l_2/n_2, then $\frac{l_1}{n_1}\cdot\frac{l_2}{n_2}=$ Product of the roots $=\frac{c}{2}\Rightarrow\frac{l_1l_2}{c}=\frac{n_1n_2}{2}$ ——(3)

Eliminating l b/w the relations (1) we get

$2nm+n(m-cn)+m(m-cn)=0$

$\Rightarrow m^2+(3-c)mn-cn^2=0\Rightarrow c\left(\frac{n}{m}\right)^2+(c-3)\frac{n}{m}-1=0$

$\therefore$ If the roots of this equation are n_1/m_1 and n_2/m_2

then $\frac{n_1}{m_1}\cdot\frac{n_2}{m_2}=$ Product of the roots $=-\frac{1}{c}$

$\Rightarrow\frac{n_1n_2}{1}=\frac{m_1m_2}{-c}\Rightarrow\frac{n_1n_2}{2}=\frac{m_1m_2}{-2c}$ ——(4)

from (3) and (4) we get $\frac{l_1l_2}{1}=\frac{m_1m_2}{-2c}=\frac{n_1n_2}{2}=k$ (say)

If the angle b/w the lines is a right angle, then we have

$l_1l_2+m_1m_2+n_1n_2=0\Rightarrow c+(-2c)+2=0\Rightarrow c=2$

2(c) (ii) Sol^n: The given plane is $x+y+z=0$ — (1)

and the cone $\frac{yz}{q-r}+\frac{zx}{r-p}+\frac{xy}{p-q}=0$ — (2)

put $\frac{1}{q-r}=a$, $\frac{1}{r-p}=b$, $\frac{1}{p-q}=c$. So that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=q-r+r-p+p-q=0$$

(or) $ab+bc+ca=0$.

Thus the cone (2) reduces to $ayz+bzx+cxy=0$ — (3)

Subject to the condition $ab+bc+ca=0$ — (4)

Now we have to find the angle b/w the lines of section of (1) and (3) subject to (4).

$$\tan\theta=\frac{\sqrt{3(a^2+b^2+c^2-2ab-2ca-2bc)}}{a+b+c}$$

$$=\frac{\sqrt{3(a^2+b^2+c^2+2ab+2bc+2ca)}}{a+b+c}\quad (\because ab+bc+ca=0 \text{ by } (4))$$

$$=\frac{\sqrt{3(a+b+c)^2}}{a+b+c}=\sqrt{3}$$

$\therefore \theta=\tan^{-1}\sqrt{3}=\pi/3$

Hence the plane cuts the cone in the lines which are inclined at an angle of 60°.

T_{13}

7(a) Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the diagonal form and diagonalizing matrix P

Sol: The characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\Rightarrow (1+\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)(1+\lambda)(3-\lambda) = 0.$$

$$\Rightarrow \lambda = -1, -1, 3$$

The characteristic roots of A are $-1, -1, 3$.

The characteristic roots corresponding to the eigen vectors X of A corresponding to the characteristic root -1 are given by

$$[A - (-1)I]X = 0$$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \to R_1 - R_3$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 4 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The rank of the coefficient matrix is 2

$\therefore$ The equations have $3-2=1$ L.I. solution.

$\therefore$ we have $4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$

$-8x_2 + 4x_3 = 0 \Rightarrow 2x_2 = x_3$

Taking $x_3 = 2$, then $x_2 = 1$ & $x_1 = 1$.

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is eigen vector of A corresponding to the eigen value 3.

$\therefore$ The geometric multiplicity of eigen value 3 is 1 and its algebraic multiplicity is also 1.

$\therefore$ A is similar to diagonal matrix.

$\therefore$ A is diagonalizable matrix.

Let $P = [X_1 \;\; X_2 \;\; X_3] = \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

The columns of P are LI eigen vectors of A corresponding to the eigen values $-1, -1, 3$ respectively. The matrix P will transform A to diagonal form D is given by the relation

$P^{-1}AP = D$

The rank of the coefficient matrix = 1

∴ The equations have $3-1=2$ linearly independent solutions.

∴ we have $-8x_1+4x_2+4x_3=0$

$\Rightarrow -2x_1+x_2+x_3=0$

Let $x_2=k_1$ and $x_3=k_2$; k_1, k_2 are arbitrary constants.

$\therefore x_1=\frac{k_1+k_2}{2}$.

$$\therefore X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{k_1+k_2}{2}\\k_1\\k_2\end{bmatrix}$$

$$=k_1\begin{bmatrix}1/2\\1\\0\end{bmatrix}+k_2\begin{bmatrix}1/2\\0\\1\end{bmatrix}$$

$$=k_1X_1+k_2X_2.$$

Here $X_1=\begin{bmatrix}1/2\\1\\0\end{bmatrix}$ & $X_2=\begin{bmatrix}1/2\\0\\1\end{bmatrix}$ are LI vectors of corresponding to characteristic root

∴ the geometric multiplicity of eigenvalue is equal to its algebraic multiplicity.

Now the eigen vectors X of A corresponding to the eigen value 3 are given by

$(A-3I)X=0$

$$\Rightarrow\begin{bmatrix}-12&4&4\\-8&0&4\\-16&8&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The transforming matrix $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

and diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

3(b) → Find the volume in the first octant bounded by the co-ordinate planes, the cylinder $x^2+y^2=4$ and the plane $z+y=3$

Ans: $\frac{(9\pi-8)}{3}$ please try yourself

3(c) ➤ Two perpendicular tangent planes to the paraboloid $x^2/a + y^2/b = 2z$ intersect in a line lying on the plane $x=0$. Prove that the line touches the parabola $x=0,\ y^2=(a+b).(2z+a)$.

Sol'n: Let the line of intersection of the two tangent planes be $my+nz=\lambda,\ x=0$ —— ①

Since this lies on the plane $x=0$ (given)

∴ Equation of the plane through the line ① is

$(my+nz-\lambda)+kx=0 \Rightarrow kx+my+nz=\lambda$ —— ②

If the plane ② touches the paraboloid, then

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{2pn}{c}=0$$

$\Longrightarrow ak^2+bm^2+2\lambda n=0$ —— ③

This being a quadratic in k, gives two values of k say k_1, k_2 such that

$k_1k_2=(bm^2+2\lambda n)/a$ —— ④

Also from ② the direction ratios of the normals to the two tangent planes whose line of intersection is ① are k_1, m, n and k_2, m, n.

Also as these two tangent planes are ⊥lar, so are their normals and consequently we have

$k_1k_2+m.m+n.n=0$

$\Rightarrow [(bm^2+2\lambda n)/a]+m^2+n^2=0$, from ④

$\Longrightarrow (a+b)m^2+an^2+2\lambda n=0$ —— ⑤

Now we are to prove that the line ① touches a parabola, so we are to find the envelope of ① which satisfies the condition ⑤

Eliminating λ between ① and ⑤, the equations of the line of intersection of two tangent planes is

$$(a+b)m^2 + an^2 + 2(my+nz)n = 0,\ x=0$$

$$\Rightarrow (a+b)(m/n)^2 + 2y(m/n) + (a+2z) = 0,\ x=0$$

It is quadratic in (m/n) so its envelope is given by

$$B^2 - 4AC = 0,\ x=0$$

$$\Rightarrow (2y)^2 - 4(a+b)(a+2z) = 0,\ x=0$$

$$\Rightarrow y^2 = (a+b)(a+2z),\ x=0$$ Hence proved.

4(a) Let $U = \text{span}\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$

$W = \text{span}\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$

be the subspaces of $\mathbb{R}^4$.

Find a basis and the dimension of $U+W$, U, W and $U \cap W$.

Sol. To find basis for $U+W$:

Since $U+W$ is the space spanned by all six vectors.

Hence form the matrix whose rows are the given six vectors, and then row reduce to echelon form.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1 \\ R_6 \to R_6 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix} \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \\ R_5 \to R_5 - R_2 \\ R_6 \to R_6 - 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_6 \to -\frac{1}{2} R_6 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \leftrightarrow R_6$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_4 \to R_4 + R_3 \\ R_5 \to R_5 + R_3. \end{array}$$

Clearly it is an echelon form.

The non-zero rows of the echelon matrix, $(1, 1, 0, -1)$, $(0, 1, 3, 1)$ and $(0, 0, -1, -2)$ form a basis of $U+W$. $\therefore \dim(U+W) = 3$.

To find a basis for U:

Reduce to echelon form the matrix whose rows span U:

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & +1 & 3 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Clearly it is an echelon form.

$\therefore$ The two non-zero rows of the echelon ~~form~~ matrix.

$\therefore$ $(1, 1, 0, -1)$ & $(0, 1, 3, 1)$ form a basis of U. and so $\dim(U) = 2$.

To find a basis of W:

Reduce to echelon form the matrix whose rows span W:

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2$$ clearly it is an echelon form.

$\therefore$ The two non-zero rows of the echelon matrix $(1, 2, 2, -2)$ and $(0, -1, -2, 1)$ form of W and so $\dim W = 2$.

To find $\dim(U \cap W)$:

Since $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$\Rightarrow 3 = 2 + 2 - \dim(U \cap W)$

$\Rightarrow \boxed{\dim(U \cap W) = 1}$.

4(b) (i) Evaluate $\underset{x \to 0}{Lt} \left(\frac{\sin x}{x}\right)^{1/x^2}$

(ii) verify Euler's theorem for $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.

Sol (i) The given limit is 1^∞-form.

Let $y=\left(\frac{\sin x}{x}\right)^{1/x^2}$ so that $\log y=\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$

$\lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log(\sin x/x)}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to 0}\frac{x}{\sin x}\left(\frac{x\cos x-\sin x}{x^2}\right)\Big/2x$

$=\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x}=\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3}$ $\left(\frac{0}{0}\text{ form}\right)$ as $\lim_{x\to 0}\frac{x}{\sin x}=1$

$=\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{6x^2}$

$=\lim_{x\to 0}\frac{-\sin x}{6x}=\frac{-1}{6}$ as $\lim_{x\to 0}\frac{\sin x}{x}=1$.

$\therefore \lim_{x\to 0}\log y=\log\lim_{x\to 0}y=-1/6$

Hence $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}=e^{-1/6}$

(ii) Write $z=z_1+z_2$ where $z_1=\sin^{-1}(x/y)$, $z_2=\tan^{-1}y/x$ are homogeneous functions of x and y of degree each zero.

we have $\frac{\partial z_1}{\partial x}=\frac{1}{\sqrt{1-x^2/y^2}}\left(\frac{1}{y}\right)=\frac{1}{\sqrt{y^2-x^2}}$

$\frac{\partial z_1}{\partial y}=\frac{1}{\sqrt{1-x^2/y^2}}\left(\frac{-x}{y^2}\right)=\frac{-x}{\sqrt{y^2-x^2}}$

$\therefore x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}=\frac{x}{\sqrt{y^2-x^2}}-\frac{x}{\sqrt{y^2-x^2}}=0$ —(1)

Now $\frac{\partial z_2}{\partial x}=\frac{-y}{x^2+y^2}$, $\frac{\partial z_2}{\partial y}=\frac{x}{x^2+y^2}$.

$\therefore x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}=\frac{-xy}{x^2+y^2}+\frac{xy}{x^2+y^2}=0$.

$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\frac{\partial}{\partial x}(z_1+z_2)+y\frac{\partial}{\partial y}(z_1+z_2)$

$=\left(x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}\right)+\left(x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}\right)=0+0=0$

Hence Euler's Theorem is verified for z.

4(c) Show that the ⊥lar from the origin on the generator of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ lie on the curve

$$\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2}$$

Sol'n: we know that the equations to a generator of the hyperboloid through any point of the principal elliptic section.

$x^2/a^2 + y^2/b^2 = 1,\ z=0$ are $\dfrac{x-a\cos\theta}{a\sin\theta} = \dfrac{y-b\sin\theta}{-b\cos\theta} = \dfrac{z}{c}$ —(1)

Equations of any line through the origin are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \quad \text{——(2)}$$

If the line (2) is ⊥lar to generator (1), then

$$al\sin\theta - bm\cos\theta + cn = 0 \quad \text{——(3)}$$

Also if (1) & (2) are coplanar they

$$\begin{vmatrix} a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & c \\ l & m & n \end{vmatrix} = 0$$

$\Rightarrow a\cos\theta(-nb\cos\theta - mc) - b\sin\theta(an\sin\theta - lc) = 0$

$\Rightarrow -anb(\cos^2\theta + \sin^2\theta) - amc\cos\theta + lbc\sin\theta = 0$

$\Rightarrow bcl\sin\theta - acm\cos\theta - abn = 0$ ——(4)

Solving (3) and (4) simultaneously for $\sin\theta$ & $\cos\theta$, we get

$$\frac{\sin\theta}{ab^2mn + ac^2mn} = \frac{\cos\theta}{bc^2nl + a^2bln} = \frac{1}{-a^2clm + b^2clm}$$

$$\Rightarrow \frac{\sin\theta}{amn(b^2+c^2)} = \frac{\cos\theta}{bnl(c^2+a^2)} = \frac{1}{-clm(a^2-b^2)}$$

$$\Rightarrow \sin\theta = \frac{an(b^2+c^2)}{-cl(a^2-b^2)}, \quad \cos\theta = \frac{bn(c^2+a^2)}{-cm(a^2-b^2)}$$

$$\Rightarrow \left[\frac{an(b^2+c^2)}{-cl(a^2-b^2)}\right]^2 + \left[\frac{bn(c^2+a^2)}{-cm(a^2-b^2)}\right]^2 = 1 \qquad \because \cos^2\theta+\sin^2\theta=1$$

$$\Rightarrow \frac{a^2(b^2+c^2)^2}{l^2} + \frac{b^2(c^2+a^2)^2}{m^2} = \frac{c^2(a^2-b^2)^2}{n^2}$$

This shows that the line ⑤ lies on the curve

$$\frac{a^2(b^2+c^2)^2}{x^2} + \frac{b^2(c^2+a^2)^2}{y^2} = \frac{c^2(a^2-b^2)^2}{z^2} \quad \text{Proved.}$$

5(a) Solve $16(x+1)^4 y_4 + 96(x+1)^3 y_3 + 104(x+1)^2 y_2 + 8(x+1)y_1 + y = x^2+4x+3$.

Sol'n: Given $16(x+1)^4 \frac{d^4y}{dx^4} + 96(x+1)^3 \frac{d^3y}{dx^3} + 104(x+1)^2 \frac{d^2y}{dx^2}$
$$+ 8(x+1)\frac{dy}{dx} + y = x^2+4x+3 \quad \text{---} (1)$$

Let $1+x = v$ so that $dy/dx = dy/dv$, d^2y/dv^2 ...

Then (1) reduces to

$$16v^4 \frac{d^4y}{dv^4} + 96v^3\frac{d^3y}{dv^3} + 104v^2\frac{d^2y}{dv^2} + 8v\frac{dy}{dv} + y = (v-1)^2 + 4(v-1)+3$$

$$\left(16v^4\frac{d^4}{dv^4} + 96v^3\frac{d^3}{dv^3} + 104\frac{d^2}{dv^2} + 8v\frac{d}{dv} + 1\right)y = v^2+2v \quad \text{---} (3)$$

Now, put $v = e^z$ i.e. $z = \log v$ and let $D_1 \equiv d/dz$ --- (4)

Then (3) reduces to

$$16D_1(D_1-1)(D_1-2)(D_1-3) + 96D_1(D_1-1)(D_1-2) + 104D_1(D_1-1) + 8D_1 + 1]y = e^{2z} + 2e^z$$

$$\Rightarrow (16D_1^4 - 8D_1^2 + 1)y = e^{2z} + 2e^z \quad \text{———} (5)$$

A.E. of (5) is $16D_1^4 - 8D_1^2 - 1 = 0 \Rightarrow (4D_1^2-1)^2 = 0$

Giving $D_1 = \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$

$$\therefore C.F = (C_1 + C_2 z)e^{z/2} + (C_3 + C_4 z)e^{-z/2}$$
$$= (C_1 + C_2\log v)v^{1/2} + (C_3 + C_4 \log v)v^{-1/2}, \text{ by } (4)$$
$$= [C_1 + C_2\log(1+x)](1+x)^{1/2} + [C_3 + C_4\log(1+x)](1+x)^{-1/2}$$

P.I corresponding to e^{2z}

$$= \frac{1}{(4D_1^2-1)^2}e^{2z} = \frac{1}{(16-1)^2}e^{2z} = \frac{1}{225}v^2, \text{ by } (4)$$

$$= \frac{1}{225}(1+x)^2, \text{ by } ②$$

P.I corresponding to $2e^z$

$$= \frac{1}{(4D_1^2-1)^2} 2e^z = 2\frac{1}{(4-1)^2}e^z = \frac{2}{9}v, \text{ by } ④$$

$$= \frac{2}{9}(1+x), \text{ by } ②$$

$$\therefore y = [C_1 + C_2 \log(1+x)](1+x)^{1/2} + [C_3 + C_4 \log(1+x)](1+x)^{-1/2}$$

$$+ \frac{1}{225}(1+x)^2 + \frac{2}{9}(1+x)$$

5(b) Find the equation of the system of orthogonal trajectories of the parabolas $r = 2a/(1+\cos\theta)$, where a is the parameter.

Sol'n: Given that $r = \frac{2a}{1+\cos\theta}$ —— ①.

Taking logarithm on both sides

we get $\log r = \log 2a - \log(1+\cos\theta)$

Differentiating ① w.r.t θ, we get.

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2 \text{ —— } ②$$

which is the differential equation of the given family of curves ①

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in ②, so we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan\theta/2$$

$$\Rightarrow \frac{dr}{r} = -\cot\theta/2 \, d\theta$$

Integrating, $\log r = -2\log\sin\theta/2 + \log C$

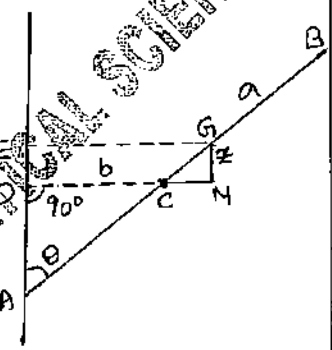
$$\Rightarrow r = C/\sin^2\theta/2$$

$$\Rightarrow \boxed{r = \frac{2c}{1-\cos\theta}}$$

5(c) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable.

Sol^n: Let AB be a uniform rod of length $2a$. The end A of the rod rests against smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b i.e. $CD = b$.

Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C. i.e. $GM = z$. We shall express z in terms of θ.



We have

$$z = GM = ED = AE - AD$$

$$= AG\cos\theta - CD\cot\theta = a\cos\theta - b\cot\theta$$

$$\therefore dz/d\theta = -a\sin\theta + b\,\text{cosec}^2\theta$$

and $\dfrac{d^2z}{d\theta^2} = -a\cos\theta - 2b\,\text{cosec}^2\theta\cot\theta$

For equilibrium of the rod, we have $dz/d\theta = 0$

i.e. $-a\sin\theta + b\,\text{cosec}^2\theta = 0$

$$\Rightarrow a\sin\theta = b\,\text{cosec}^2\theta$$

$$\Rightarrow \sin^3\theta = b/a \Rightarrow \sin\theta = (b/a)^{1/3} \Rightarrow \theta = \sin^{-1}(b/a)^{1/3}$$

This gives the position of equilibrium of rod

Again $\dfrac{d^2z}{d\theta^2} = -(a\cos\theta + 2b\,\text{cosec}^2\theta\cot\theta) = -$negative for all acute values of θ.

Thus $d^2z/d\theta^2$ is $-$ve in the position of equilibrium & so z is maximum. Hence the equilibrium is unstable.

5(d). A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. Show that the greatest height attained by the particle is $\frac{1}{2k}\log\frac{g+ku^2}{g}$.

Sol^n: Let a particle of mass m be projected vertically upwards from a point O with velocity u. If v is the velocity of the particle at time t at a distance x from the starting point O, then the resistance on the particle is mkv^2 in the downward direction i.e, in the direction of x decreasing. The weight mg of the particle also acts vertically downwards. So the equation of motion of the particle during its upward motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

$$v\frac{dv}{dx} = -(g+kv^2), \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

$$\frac{2kv\,dv}{g+kv^2} = -2k\,dx, \text{ separating the variables.}$$

Integrating, $\log(g+kv^2) = -2kx + A$, where A is a constant

But initially $x=0, v=u$; $\therefore A = \log(g+ku^2)$

$$\therefore \log(g+kv^2) = -2kx + \log(g+ku^2)$$

$$\Rightarrow 2kx = \log(g+ku^2) - \log(g+kv^2)$$

$$\Rightarrow x = \frac{1}{2k}\log\frac{g+ku^2}{g+kv^2} \quad \text{——— (1)}$$

which gives the velocity of the particle at a distance x.
If h is the greatest height attained by the particle then at $x=h$, $v=0$. Therefore from (1) we have $h = \frac{1}{2k}\log\frac{g+ku^2}{g}$.

5(e). Apply Green's theorem to evaluate the line integral $\oint_C (4x-2y)\,dx + (2x-4y)\,dy$. where C is the circle $(x-2)^2 + (y-2)^2 = 4$.

Sol: By Green's theorem in the plane we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$$

Here $M = 4x-2y$, $N = 2x-4y$.

and C is the circle $(x-2)^2+(y-2)^2=4$. $r=2$

$$\frac{\partial M}{\partial y} = -2, \quad \frac{\partial N}{\partial x} = 2$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint (2-(-2))\,dx\,dy = \iint 4\,dx\,dy$$

$$= 4\iint dx\,dy$$

$= 4$ (Area of the circle of radius 2 with its centre (2, 2))

$$= 4(\pi(2)^2)$$

$$= 16\pi.$$

$$\therefore \oint_C M\,dx + N\,dy = 16\pi \text{ sq. units}$$

6(a) Solve $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$

Sol^n: $\dfrac{dy}{dx} = -\dfrac{x^3+3xy^2}{y^3+3x^2y} = -\dfrac{1+3(y/x)^2}{(y/x)^3+3(y/x)}$ —— (1)

Take $y/x = v$ i.e. $y = vx$

$\therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— (2)

from (1) & (2) $v + x\dfrac{dv}{dx} = -\dfrac{1+3v^2}{v^3+3v}$

$\Rightarrow x\dfrac{dv}{dx} = -\dfrac{1+3v^2}{v^3+3v} - v = -\dfrac{v^4+6v^2+1}{v^3+3v}$

$4\dfrac{dx}{x} = -\dfrac{4v^3+12v}{v^4+6v^2+1}dv$

Integrating, $4\log x = -\log(v^4+6v^2+1)+\log c$

$\log x^4 = \log[c/(v^4+6v^2+1)]$

$\Rightarrow x^4(v^4+6v^2+1) = c$

$\Rightarrow y^4+6x^2y^2+x^4 = c$, as $y/x = v$

6(b) Transform the equation $(2x^2+1)p^2+(x^2+2xy+y^2+2)p+2y^2+1=0$ to Clairaut's form by the substitution $x+y=u$, $xy-1=v$ and interpret it. Find its singular solution also.

Ans: General solution is $c^2+(x+y)c+1-xy=0$. Singular solution is $(x+y)^2-4(1-xy)=0$

6(c) Solve $(1-x^2)y_2+xy_1-y = x(1-x^2)^{3/2}$.

Sol^n: Dividing by $(1-x^2)$, the given equation is standard form is $\dfrac{d^2y}{dx^2}+\dfrac{x}{1-x^2}\dfrac{dy}{dx}-\dfrac{1}{1-x^2}y = x(1-x^2)^{1/2}$ —— (1)

Comparing ① with $y'' + Py' + Qy = R$, we have

$P = x/(1-x^2)$, $Q = -1/(1-x^2)$, $R = x(1-x^2)^{1/2}$

Here $P + Qx = 0$, showing that a part of C.F of ① is

$y = u = x$ ——— ③

Let the required general solution be $y = uv$ ——— ④

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{x}{1-x^2} + \frac{2}{x}\frac{dx}{dx}\right)\frac{dv}{dx} = \frac{x(1-x^2)^{1/2}}{x}$ ⑤

Let $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$ ——— ⑥

Then ⑤ reduces to $\frac{dq}{dx} + \left(\frac{2}{x} + \frac{x}{1-x^2}\right)q = (1-x^2)^{1/2}$ ——— ⑦

Here $E = \int\left(\frac{2}{x} + \frac{x}{1-x^2}\right)dx = \int \frac{2}{x}dx - \frac{1}{2}\int\frac{(-2x)dx}{1-x^2}$

$= 2\log x - \frac{1}{2}\log(1-x^2)$

$= \log x^2 - \log(1-x^2)^{1/2} = \log\left(\frac{x^2}{\sqrt{1-x^2}}\right)$

$\therefore$ I.F. of ⑦ $= e^E = e^{\log[x^2/\sqrt{1-x^2}]} = x^2/(1-x^2)^{1/2}$

$\therefore q.\frac{x^2}{(1-x^2)^{1/2}} = \int (1-x^2)^{1/2}.\frac{x^2}{(1-x^2)^{1/2}}dx + c_1 = \frac{1}{3}x^3 + c_1$

$\Rightarrow q = dv/dx = \frac{1}{3}x(1-x^2)^{1/2} + (c_1/x^2)(1-x^2)^{1/2}$

$\Rightarrow \int dv = \int(\frac{1}{3})(-\frac{1}{2})(1-x^2)^{1/2}(-2x)dx + c_1\int x^{-2}(1-x^2)^{1/2}dx + c_2$

Integrating, $v = -\left(\frac{1}{6}\right)\frac{(1-x^2)^{3/2}}{3/2} + c_1\left[(1-x^2)^{1/2}(-x^{-1})\right.$

$\left.-\int\frac{1}{2}(1-x^2)^{-1/2}(-2x)(-x^{-1})dx\right] + c_2$ [Integrating by parts second integral]

$\Rightarrow v = -\frac{1}{9}(1-x^2)^{3/2} - \frac{c_1}{x}(1-x^2)^{1/2} - c_1\int\frac{dx}{\sqrt{1-x^2}} + c_2$

$\Rightarrow v = -\left(\frac{1}{9}\right)(1-x^2)^{3/2} - \left(\frac{c_1}{x}\right)(1-x^2)^{1/2} - c_1 \sin^{-1}x + c_2$ — (8)

from (3), (4) and (8), the required general solution is

$y = uv = x\left[-\left(\frac{1}{9}\right)(1-x^2)^{3/2} - \left(\frac{c_1}{x}\right)(1-x^2)^{1/2} - c_1 \sin^{-1}x + c_2\right]$

$\Rightarrow y = -c_1\left[(1-x^2)^{1/2} + x\sin^{-1}x\right] + c_2 x - \frac{x}{9}(1-x^2)^{3/2}$

6(d). By using Laplace transform method solve

$$\frac{d^2y}{dt^2} + y = t\cos 2t \text{ if } y=0,\ \frac{dy}{dt} = 0 \text{ when } t=0.$$

Sol'n: Given that $(D^2+1)y = t\cos 2t$

(or) $y'' + y = t\cos 2t$

Taking the Laplace transform of both sides of the given equation, we have

$$L\{y''\} + L\{y\} = L\{t\cos 2t\}$$

$$p^2 L\{y\} - py(0) - y'(0) + L\{y\} = -\frac{d}{dp}\left(L\{\cos 2t\}\right)$$

$$(p^2+1)L\{y\} = -\frac{d}{dp}\left(\frac{p}{p^2+4}\right)$$

$$= -\frac{1}{p^2+4} + \frac{2p^2}{(p^2+4)^2}$$

$$L\{y\} = \frac{p^2-4}{(p^2+4)^2(p^2+1)}$$

$$= -\frac{5}{9(p^2+1)} + \frac{5}{9(p^2+4)} + \frac{8}{3(p^2+4)^2}$$

$$y = -\frac{5}{9}L^{-1}\left\{\frac{1}{p^2+1}\right\} + \frac{5}{9}L^{-1}\left\{\frac{1}{p^2+4}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(p^2+4)^2}\right\}$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\int_0^t \frac{1}{2}\sin 2x \cdot \frac{1}{2}\sin 2(t-x)\,dx$$

by the Convolution theorem

$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\int_0^t \{\cos 2(t-2x) - \cos 2t\}dx$

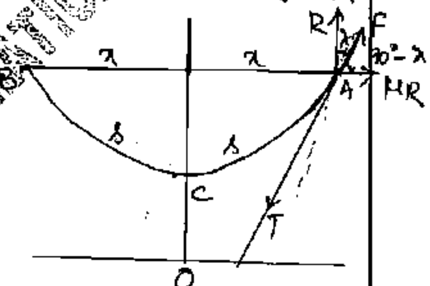
$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\left[-\frac{1}{4}\sin 2(t-2x) - x\cos 2t\right]_0^t$

$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{12}\sin 2t - \frac{t}{3}\cos 2t + \frac{1}{12}\sin 2t$

$= -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t$, which is the required sol'n

7(a) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is $\mu \log \left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$ where μ is the coefficient of friction.

Sol'n: Let the end links A and B of a uniform chain slide along fixed rough horizontal rod. If AB be the maximum span, then A and B are in the state of limiting equilibrium. Let R be the reaction of the rod at A acting $\perp$lar to the rod. Then the frictional force μR will act at A along the rod in the outward direction BA as shown in fig. The resultant F of the force R and μR at A will make an angle λ with the direction of R.



For the equilibrium of A the resultant F of R and μR at B A will be equal and opposite to the tension T at A.
Since the tension at A acts along the tangent to the chain at A, therefore the tangent to the catenary at A makes an angle $\varphi_A = \frac{1}{2}\pi - \lambda$ to the horizontal.
Thus for the point A of the catenary, we have
$\varphi = \varphi_A = \frac{1}{2}\pi - \lambda$.

$\therefore$ The length of the chain $= 2s = 2c\tan\varphi_A = 2c\tan(\frac{1}{2}\pi - \lambda)$

$= 2c\cot\lambda = \frac{2c}{\mu}$ $[\because \tan\lambda = \mu]$

If (x_A, y_A) are the coordinates of the point A, then the maximum span $AB = 2x_A$

$= 2c\log(\tan\varphi_A + \sec\varphi_A)$

$= 2c\log\left[\tan\varphi_A + \sqrt{(1+\tan^2\varphi_A)}\right]$

$$= 2c\log\left\{\cot\lambda + \sqrt{(1+\cot^2\lambda)}\right\} \quad \left[\because \psi_A = \tfrac{1}{2}\pi - \lambda\right]$$

$$= 2c\log\left\{\frac{1}{\mu} + \sqrt{\left(1+\frac{1}{\mu^2}\right)}\right\} = 2c\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}$$

Hence the required ratio

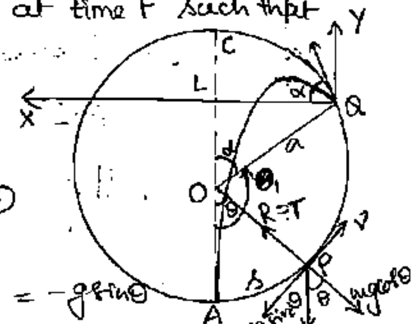
$$= \frac{2x}{2s} = \frac{2c\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}}{(2c/\mu)}$$

$$= \mu\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}.$$

7(b). A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$, Prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point. Also find the height of the highest point of the parabola subsequently described.

Sol^n: Take $R = T$ (i.e. the tension in the string)

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$. If P is the position of the particle at time t such that $\angle AOP = \theta$, then the equations



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{(1)}$$

and $$m\frac{v^2}{l} = T - mg\cos\theta \quad \text{(2)}$$

Also $$s = l\theta \quad \text{(3)}$$

from (1) and (3), we have $l\frac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying bothsides by $2l\frac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$$

But at the point A, $\theta = 0$ and $v = 2\sqrt{(gl)}$

$\therefore\ 4gl = 2lg + A$ so that $A = 2gl$

$\therefore\ v^2 = 2lg(\cos\theta + 1)$ —— (4)

from (2) and (4), we have

$T = \frac{m}{l}(v^2 + gl\cos\theta) = mg(3\cos\theta + 2)$ —— (5)

If the cord becomes slack at the point Q, where $\theta = \theta_1$, then from (5), we have

$T = 0 = mg(3\cos\theta_1 + 2)$

giving $\cos\theta_1 = -2/3$

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta_1$ and $\cos\alpha = 2/3$.

If v_1 is the velocity of the particle at Q, then $v = v_1$ where $\theta = \theta_1$. Therefore from (4), we have

$v_1^2 = 2lg(1 + \cos\theta_1) = 2lg(1 - 2/3) = 2lg/3$

Now $OL = l\cos\alpha = \frac{2}{3}l$

Thus the particle leaves the circular path at the point Q at a height $2l/3$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path. Max. height H of the particle above Q.

$$= \frac{v_1^2\sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1 - \cos^2\alpha) = \frac{\frac{2}{3}lg}{2g}\left(1 - \frac{4}{9}\right) = \frac{5l}{27}$$

$\therefore$ Height of the highest point of the parabolic path above the fixed point $O = OL + H = \frac{2}{3}l + \frac{5l}{27}$

$$= \frac{23l}{27}.$$

7(c). A particle moves under a force $m\mu\{3au^4 - 2(a^2-b^2)u^5\}$, $a>b$ and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu/(a+b)}$. Show that the equation of its path is $r = a + b\cos\theta$.

Sol'n: Here the Central acceleration

$$P = \mu\{3au^4 - 2(a^2-b^2)u^5\}$$

∴ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\{3au^4 - 2(a^2-b^2)u^5\}$$

$$\Rightarrow h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\{3au^2 - 2(a^2-b^2)u^3\}$$

Multiplying bothsides by $2\left(\frac{du}{d\theta}\right)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\{au^3 - 2(a^2-b^2)u^4 + A$$

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\{2au^3 - (a^2-b^2)u^4\} + A \quad \text{—(1)}$$

where A is a constant.

But initially at an apse $r = a+b$, $u = \frac{1}{(a+b)}$, $\frac{du}{d\theta} = 0$ and $v = \sqrt{\mu}/(a+b)$

∴ from (1) we have

$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right] + A$$

∴ $h^2 = \mu$ and $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2} + \frac{2a}{r^3} - \frac{(a^2-b^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}\left[-r^2 + 2ar - (a^2-b^2)\right]$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - a^2 + b^2 = b^2 - (r^2 - 2ar + a^2)$$
$$= b^2 - (r-a)^2$$

$$\therefore \frac{dr}{d\theta} = \sqrt{\{b^2 - (r-a)^2\}} \Rightarrow d\theta = \frac{dr}{\sqrt{\{b^2 - (r-a)^2\}}}$$

Integrating $\theta + B = \sin^{-1}\left(\frac{r-a}{b}\right)$ —— (3)

But initially when $r = a+b$, let us take $\theta = 0$.
Then from (3), $B = \sin^{-1}(1) = \pi/2$

Substituting in (3), we have

$$\theta + \tfrac{1}{2}\pi = \sin^{-1}\left(\frac{r-a}{b}\right) \Rightarrow r - a = b\sin\left(\tfrac{1}{2}\pi + \theta\right)$$

$r = a + b\cos\theta$, which is required equation of the path.

8(a) (i) Find the most general differentiable function $f(r)$ so that $f(r)\vec{r}$ is solenoidal.

(ii) Find $A\times(\nabla\times B)$ and $(A\times\nabla)\times B$ at the point $(1,-1,2)$, if $A = xz^2 i + 2y j - 3xz k$ and $B = 3xz i + 2yz j - z^2 k$.

Sol'n: (i) Given that $f(r)\vec{r}$ is solenoidal.

i.e. $\text{div}(f(r)\vec{r}) = 0$

$$\Rightarrow f(r)\,\text{div}\,\vec{r} + \vec{r}\cdot\text{grad}\,f(r) = 0$$

$$\Rightarrow f(r)(3) + \vec{r}\cdot\text{grad}\,f(r) = 0 \quad \text{---} (1)$$

Now $\text{grad}\,f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$

$$= \hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left(\hat{i}\cdot\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= \frac{f'(r)}{r}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{f'(r)}{r}\vec{r}$$

$\therefore$ from (1), we have

$$3f(r) + \vec{r}\cdot\left(\frac{f'(r)}{r}\vec{r}\right) = 0$$

$$\Rightarrow 3f(r) + \frac{f'(r)}{r}(\vec{r}\cdot\vec{r}) = 0$$

$$\Rightarrow 3f(r) + f'(r)\cdot\frac{r^2}{r} = 0$$

$$\Rightarrow 3f(r) + r f'(r) = 0$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

$$\Rightarrow \log f(r) = -3\log r + \log c$$

$$\Rightarrow \log f(r) + \log r^3 = \log c$$

$$\Rightarrow f(r)\cdot r^3 = c$$

$\Rightarrow f(r) = c/r^3$, where c is an arbitrary constant.

(ii) Ans: $A\times(\nabla\times B) = 18\hat{i} - 12\hat{j} + 16\hat{k}$, $(A\times\nabla)\times B = 4\hat{i} + 76\hat{k}$

8(b). If a is a constant vector, prove that

$$\text{Curl}\,\frac{\vec{a}\times\vec{r}}{r^3} = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r}).$$

Soln: we have $\text{curl}\,\dfrac{\vec{a}\times\vec{r}}{r^3} = \nabla\times\left(\dfrac{\vec{a}\times\vec{r}}{r^3}\right) = \Sigma\left\{\hat{i}\times\dfrac{\partial}{\partial x}\left(\dfrac{\vec{a}\times\vec{r}}{r^3}\right)\right\}$ — (1)

Now $\dfrac{\partial}{\partial x}\left(\dfrac{\vec{a}\times\vec{r}}{r^3}\right) = -\dfrac{3}{r^4}\dfrac{\partial r}{\partial x}(\vec{a}\times\vec{r}) + \dfrac{1}{r^3}\left(\vec{a}\times\dfrac{\partial\vec{r}}{\partial x}\right) + \dfrac{1}{r^3}\left(\dfrac{\partial\vec{a}}{\partial x}\times\vec{r}\right)$

Now $\dfrac{\partial\vec{a}}{\partial x} = 0$ because $\vec{a}$ is a constant vector.

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\therefore \dfrac{\partial\vec{r}}{\partial x} = \hat{i}$

further $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\therefore$ (1) becomes $\dfrac{\partial}{\partial x}\left(\dfrac{\vec{a}\times\vec{r}}{r^3}\right) = -\dfrac{3}{r^4}\dfrac{x}{r}(\vec{a}\times\vec{r}) + \dfrac{1}{r^3}(\vec{a}\times\hat{i})$

$$= -\frac{3x}{r^5}(\vec{a}\times\vec{r}) + \frac{1}{r^3}(\vec{a}\times\hat{i})$$

$\therefore \hat{i}\times\dfrac{\partial}{\partial x}\left(\dfrac{\vec{a}\times\vec{r}}{r^3}\right) = -\dfrac{3x}{r^5}\hat{i}\times(\vec{a}\times\vec{r}) + \dfrac{1}{r^3}\hat{i}\times(\vec{a}\times\hat{i})$

$$= -\frac{3x}{r^5}\left[(\hat{i}\cdot\vec{r})\,\vec{a} - (\hat{i}\cdot\vec{a})\,\vec{r}\right] + \frac{1}{r^3}\left[(\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\hat{i}\right]$$

$$= -\frac{3x}{r^5}x\vec{a} + \frac{3x}{r^5}a_1\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_1\hat{i}$$

$[\because \hat{i}\cdot\vec{r} = x$ and $\hat{i}\cdot\vec{a} = a_1$ if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}]$

$$= -\frac{3x^2}{r^5}\vec{a} + \frac{3}{r^5}a_1x\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_1\hat{i} \quad (2)$$

Similarly,

$$\left(\hat{j}\times\frac{\partial}{\partial y}\right)\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{-3y^2}{r^5}\vec{a} + \frac{3}{r^5}a_2y\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_2\hat{j} \quad (3)$$

and $\left(\hat{k}\times\dfrac{\partial}{\partial z}\right)\left(\dfrac{\vec{a}\times\vec{r}}{r^3}\right) = \dfrac{-3z^2}{r^5}\vec{a} + \dfrac{3}{r^5}a_3z\vec{r} + \dfrac{1}{r^3}\vec{a} - \dfrac{1}{r^3}a_3\hat{k}$ (4)

Substituting (3), (4) and (5) in (1), we get,

$$\text{Curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3x^2\vec{a}}{r^5} + \frac{3}{r^5}a_1 x\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_1\hat{i}$$
$$-\frac{3y^2}{r^5}\vec{a} + \frac{3}{r^5}a_2 y\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_2\hat{j}$$
$$-\frac{3z^2}{r^5}\vec{a} + \frac{3}{r^5}a_3 z\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_3\hat{k}$$
$$= -\frac{3}{r^5}(x^2+y^2+z^2)\vec{a} + \frac{3}{r^5}(a_1x+a_2y+a_3z)\vec{r} + \frac{3}{r^3}\vec{a} - \frac{1}{r^3}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$$
$$= -\frac{3}{r^5}(r^2)\vec{a} + \frac{3}{r^5}(\vec{r}\cdot\vec{a})\vec{r} + \frac{3}{r^3}\vec{a} - \frac{\vec{a}}{r^3}$$
$$= \cancel{-\frac{3}{r^3}\vec{a}} + \frac{3}{r^5}(\vec{r}\cdot\vec{a})\vec{r} + \cancel{\frac{3}{r^3}\vec{a}} - \frac{\vec{a}}{r^3} \quad \left(\because x^2+y^2+z^2=r^2,\ a_1\hat{i}+a_2\hat{j}+a_3\hat{k}=\vec{a},\ a_1x+a_2y+a_3z=\vec{a}\cdot\vec{r}\right)$$
$$= \frac{-\vec{a}}{r^3} + \frac{3}{r^5}(\vec{r}\cdot\vec{a})\vec{r}$$

8(c). If $F = (2y+3)\hat{i} + xz\hat{j} + (yz-x)k$, evaluate $\int_C F\cdot dr$ where C is the path consisting of the straight lines from (0,0,0) to (0,0,1) then to (0,1,1) and then to (2,1,1)

Sol: We have. $F\cdot dr = [(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$
$$= (2y+3)dx + xz\,dy + (yz-x)\,dz$$

Let C_1 denote the straight line joining (0,0,0) to (0,0,1), C_2 denote the straightline joining (0,0,1) to (0,1,1) and C_3 denote the straightline joining (0,1,1) to (2,1,1).

Along C_1, $x=0$, $y=0$ so that $dx=0$, $dy=0$

Also along C_1, z varies from 0 to 1.

Along C_2, $x=0$, $z=1$ so that $dx=0$, $dz=0$.
Also along C_2, y varies from 0 to 1.
along C_3, $y=1$, $z=1$ so that $dy=0$, $dz=0$
Also along C_3, x varies from 0 to 2.

$$\therefore \int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr + \int_{C_3} F\cdot dr$$

$$= \int_{z=0}^{1} (0.z-0)dz + \int_{y=0}^{1} (0\cdot 1)\,dy + \int_{0}^{2} (2+3)\,dx$$

$$= 0 + 0 + 5[x]_0^2 = 10$$

8(d). Evaluate $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot n\,ds$ where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane and bounded by this plane.

Sol^n: By divergence theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot n\,ds$$

$$= \iiint_V \text{div}\,(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,dV,$$

where V is the volume enclosed by S.

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\,dV = 2\iiint_V zy^2\,dV.$$

we shall use spherical polar coordinates (r,θ,ϕ) to evaluate this triple integral. In polars $dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$ $= r^2\sin\theta\,dr\,d\theta\,d\phi$. Also $z = r\cos\theta$, $y = r\sin\theta\sin\phi$. To cover V the limits of r will be 0 to 1, those of θ will be 0 to $\pi/2$ and those of ϕ will be 0 to 2π. The triple integral is

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^5 \sin^3\theta\cos\theta\sin^2\phi\, dr\,d\theta\,d\phi$$

$$= 2\cdot\tfrac{1}{6}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin^3\theta\cos\theta\sin^2\phi\, d\theta\,d\phi$$

on integrating w.r.t r.

$$= \frac{1}{3}\cdot\frac{2}{4.2}\int_{0}^{2\pi}\sin^2\phi\,d\phi$$

on integrating w.r.t θ.

$$= \frac{1}{12}\cdot 4\int_{0}^{\pi/2}\sin^2\phi\,d\phi = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi/12$$

T12–2016 (Paper II)

Q.1 (a) $|a| = 12$, $|b| = 22$

$\therefore a^{12} = e$, $b^{22} = e$

$\langle a \rangle = \{e, a, a^2, a^3, \ldots\ldots a^{11}\}$.

$\langle b \rangle = \{e, b, b^2, \ldots\ldots b^{21}\}$.

Now, $\langle a \rangle \cap \langle b \rangle$ is a subgroup of $\langle a \rangle$ and $\langle b \rangle$. It must divide $|\langle a \rangle|$ & $|\langle b \rangle|$.

Thus, possibilities of $|\langle a \rangle \cap \langle b \rangle| = 1$ or 2

But given $\langle a \rangle \cap \langle b \rangle \neq \{e\}$.

So, $|\langle a \rangle \cap \langle b \rangle| = 2$

Now, a^6 is only element of order 2 in $\langle a \rangle$ and a^{11} is only element of order 2 in $\langle b \rangle$.

$\Rightarrow a^6 = b^{11}$.

1(a) Q: Let a & b belong to a group. $|a| = 12$, $|b| = 22$, and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$ prove that $a^6 = b^{11}$.

1(b) Q: S.T $S = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \middle/ a, b \in I \right\}$ is a left ideal in the ring M_2 of 2×2 matrices over I. Further S.T 'S' is not right ideal in M_2

Q. 1(b).

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} ; a, b \in I \right\}$$

Now, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S \Rightarrow S \neq \phi$

let $x, y \in S$ such that

$$x = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \quad y = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}$$

$$\therefore x - y = \begin{pmatrix} a-c & 0 \\ b-d & 0 \end{pmatrix} \in S$$

Thus S is a subgroup of M_2.

let $A \in M_2$, so that $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $p, q, r, s \in Z$

then $Ax = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} pa+qb & 0 \\ ra+sb & 0 \end{pmatrix} \in S$.

Hence S is a left ideal of M_2.

again $xA = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

$$= \begin{pmatrix} ap & aq \\ bp & bq \end{pmatrix} \notin S$$

Hence, S is not a right ideal of M_2.

1(c) Show that the function f defined by
$$f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$$
is integrable on $[0, m]$, m being a positive integer.

Sol'n: $f(x) = \begin{cases} 0, & \text{if } x = 0, 1, 2 \ldots m \\ 1, & \text{if } r-1 < x < r, \; r = 1, 2 \ldots m \end{cases}$

$\Rightarrow f$ is bounded and has only $m+1$ points of finite discontinuity at $0, 1, 2 \ldots m$.

Since the points of discontinuity of f on $[0, m]$ are finite in number, therefore, f is integrable on $[0, m]$.

1(d) If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$, find $f(z)$ subject to the condition $f(\pi/2) = 0$.

Sol'n

1(d) Sol'n: we have $u+iv=f(z)$ $\therefore iu-v=if(z)$

on adding, we get

$$(u-v)+i(u+v)=(1+i)f(z)=F(z)\text{ say}$$

i.e $(u-v)+i(u+v)=F(z)$

Let $u-v=U$ and $u+v=V$ then

$U+iV=F(z)$ is an analytic function.

Now, $$U=\frac{\cos x+\sin x-e^{-y}}{2\cos x-2\cosh y}=\frac{1}{2}\left[1+\frac{\sin x+\sinh y}{\cos x-\cosh y}\right]$$

differentiating,

$$\frac{\partial U}{\partial x}=\frac{1}{2}\,\frac{1+\sin x\sinh y-\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_1(x,y)$$

$$\frac{\partial U}{\partial y}=\frac{1}{2}\,\frac{-1+\sin x\sinh y+\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_2(x,y)$$

By Milne's method, we have

$$F'(z)=\phi_1(z,0)-i\phi_2(z,0)$$

$$=\frac{1}{2}\cdot\frac{1}{1-\cos z}+\frac{i}{2}\cdot\frac{1}{1-\cos z}$$

$$=\frac{1}{4}(1+i)\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

Integrating it we get

$$F(z)=-\frac{1}{2}(1+i)\cot\left(\frac{z}{2}\right)+c$$

$$\Rightarrow (1+i)f(z)=-\frac{1}{2}(1+i)\cot\left(\frac{z}{2}\right)+c$$

$$\Rightarrow f(z)=-\frac{1}{2}\cot\left(\frac{z}{2}\right)+c_1\quad ;\ c_1=\frac{c}{1+i}$$

But given that when $z=\frac{\pi}{2}$; $f\left(\frac{\pi}{2}\right)=0$ $\therefore c_1=\frac{1}{2}$

so, $$f(z)=\frac{1}{2}\left(1-\cot\frac{z}{2}\right)$$

1.(e) Qn: Find the maximum and minimum value of $z = 5x_1 + 3x_2$

s.c. $x_1 + x_2 \le 6$,

$2x_1 + 3x_2 \geq 3$

$0 \le x_1 \geq 3$,

$0 \le x_2 \geq 3$.

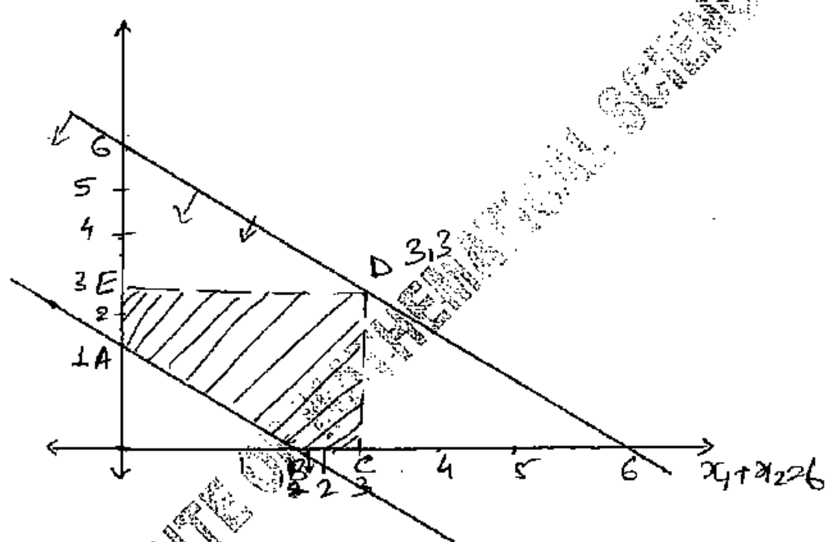
Ans-1(e)

Using Graphical method find max and Min value. Max/Min $z = 5x_1+3x_2$

Let $L_1: x_1+x_2 \leq 6$ and $L_2 = 2x_1+3x_2 \geq 3$

where $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 3$

Plotting graph for $x_1+x_2=6$, $2x_1+3x_2=3$



Shaded portion ~~stat~~ satisfies the Contraint Equ^n

By corner point method, Max/min occurs at corner pt.

Value at A = (0,1) = z = 3

Value at B = $(\frac{3}{2}, 0)$ = z = $15/2$

Value at C = (3,0) = z = 15

Value at D = (3,3) = z = 24

Value at E = (0,3) = ~~z = 9~~

Max at (3,3) = 24, Min at (0,1) = 3

1.(e) Qn: Find the maximum and minimum value of $z = 5x_1 + 3x_2$

s.c. $x_1 + x_2 \leq 6$,

$2x_1 + 3x_2 \geq 3$

$0 \leq x_1 \geq 3$,

$0 \leq x_2 \geq 3$.

Q.2. (a) (i) An element of order 5 in A_6 must be a 5 cycle.

No of ways to create a 5 cycle

$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

$= 720$ ways.

But $(abcde) = (bcdea) = (cdeab)$

$= (deabc) = (eabcd)$

These 5 cycles are equivalent.

Thus,

No of distinct 5 cycles $= \frac{720}{5} = 144$

Q.2 (a)(ii) $\alpha, \beta \in S_5$

2(c)(ii)

let $\alpha = (123)$

$\beta = (145)$

here $|\alpha| = 3$ & $|\beta| = 3$

$\alpha\beta = (14523)$

$o(\alpha\beta) = 5$

because, $\alpha\beta$ is a 5-cycle.

$\therefore |\alpha\beta| = 5$

2(c) Q: (i) How many elements of order 5 are there in A_6?

(ii) Find group elements α and β in S_5 such that $|\alpha| = 3$, $|\beta| = 3$ and $|\alpha\beta| = 5$.

3(c) Solve the LPP by Simplex method.

Max $Z = 5x_1 + 3x_2$

subject to $x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$3x_1 + 8x_2 \le 12$

$x_1, x_2 \geq 0$.

Ans: optimal solution is $x_1 = 2$, $x_2 = 0$ and Max $Z = 10$.

(please try yourself)

2(b) Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x>0$

Deduce that $\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2$,

n being a positive.

Sol^n: Let $f(t) = \log(1+t)$ $\forall\, t \in [0, x]$.

where $x > 0$.

and $f'(t) = \frac{1}{1+t}$ $\forall\, t \in (0, x)$

By Lagrange's Mean value theorem

$\exists\, c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{—— (1)}$$

Since $c \in (0, x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ — (by (1))

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$

i.e. $\frac{x}{1+x} < \log(1+x) < x$

Now, we have $\log(1+x) < x$.

Let $x = \frac{1}{n+1}$

Then $\log\left(1 + \frac{1}{n+1}\right) < \frac{1}{n+1}$ i.e. $\log\left(\frac{n+2}{n+1}\right) < \frac{1}{n+1}$

$\log\left(1+\frac{1}{n+2}\right) < \frac{1}{n+2}$ i.e, $\log\left(\frac{n+3}{n+2}\right) < \frac{1}{n+2}$

Similarly

$\log\left(\frac{n+4}{n+3}\right) < \frac{1}{n+3}$

$\log\left(\frac{n+5}{n+4}\right) < \frac{1}{n+4}$

$\vdots$

$\log\left(1+\frac{1}{n+n}\right) < \frac{1}{n+n}$ i.e, $\log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+n}$

$\therefore$ Adding all the above, we get

$$\log\left(\frac{n+2}{n+1}\right)+\log\left(\frac{n+3}{n+2}\right)+\log\left(\frac{n+4}{n+3}\right)+\cdots+\log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{n+n}$$

$$\Rightarrow \log\left(\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdot\frac{n+4}{n+3}\cdots\frac{2n}{2n-1}\cdot\frac{2n+1}{2n}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

$$\Rightarrow \log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n} \quad \text{(2)}$$

Also, we have $\frac{x}{1+x} < \log(1+x)$

Let $x=\frac{1}{n}$, then $\frac{1/n}{1+\frac{1}{n}} < \log\left(1+\frac{1}{n}\right) \Rightarrow \frac{1}{n+1} < \log\left(\frac{n+1}{n}\right)$

$x=\frac{1}{n+1}$ then $\frac{1/n+1}{1+\frac{1}{n+1}} < \log\left(1+\frac{1}{n+1}\right) \Rightarrow \frac{1}{n+2} < \log\left(\frac{n+2}{n+1}\right)$

$\vdots$

$x=\frac{1}{2n-1}$ then $\frac{\cancel{\frac{1}{2n-1}}}{1+\frac{1}{2n-1}} < \log\left(1+\frac{1}{2n-1}\right) \Rightarrow \frac{1}{2n} < \log\frac{2n}{2n-1}$

Adding, we get

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n} < \log\left(\frac{n+1}{n}\right)+\log\left(\frac{n+2}{n+1}\right)+\cdots\log\left(\frac{2n}{2n-1}\right)$$

$$= \log \frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdots\frac{2n}{2n-1}$$

$$= \log\left(\frac{2n}{n}\right) = \log 2$$

$\therefore \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} < \log 2$ ——— (3)

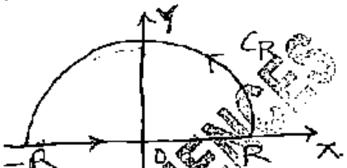
$\therefore$ from (2) & (3), $\log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} < \log 2$

2(c) Use a suitable contour integration to show that $\int_{-\infty}^{\infty} \frac{\cos x + x\sin x}{1+x^2}\,dx = \frac{2\pi}{e}$.

2(c) Sol^n To evaluate the given integral, consider the integral.

$$\int_C \frac{e^{iz}}{z-i}\,dx = \int_C f(z)\,dz \qquad \text{——} (1)$$

$f(z) = \frac{e^{iz}}{z-i}$ and c is the semicircular closed contour.



By cauchy residue theorem,

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum R^+ \qquad \text{——} (ii)$$

where $\sum R^+$ is the sum of residues of function $f(z)$ at its poles in the upper half-plane.

The poles of $f(z)$ are given by

$(z-i) = 0 \Rightarrow z = i$ is the only the simple pole in the upper half-plane.

The residue of $f(z)$ at $z = i$ is given by

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} (z-i)\frac{e^{iz}}{z-i} = e^{-1}$$

$$|f(z)| = \left|\frac{e^{iz}}{z-i}\right| \le \frac{|e^{iz}|}{|z|-|i|} \le \frac{1}{|z|-1} \to 0$$

As $|z| = R \to \infty$, so that by Jordan's Lemma

$$\lim_{R\to\infty} \int_C f(z)\,dz = 0$$

Hence eqn (ii) in the limit $R \to \infty$ gives

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \times e^{-1}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{e^{ix}}{x-i}\,dx = i\cdot\frac{2\pi}{e}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{(\cos x + i\sin x)(x+i)}{(x-i)(x+i)}\,dx = i\cdot\frac{2\pi}{e}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{(x\cos x - \sin x) + i(x\sin x + \cos x)}{x^2+1}\,dx = i\cdot\frac{2\pi}{e}$$

equating imaginary parts on both sides, we have

$$\int_{-\infty}^{\infty} \frac{x\sin x + \cos x}{x^2+1}\,dx = \frac{2\pi}{e}$$

3(c) (i) If R is a ring with unity 1 and f is a homomorphism of R into an integral domain R' with $\ker f \neq R$. prove that $f(1)$ is unity of R'.

(ii) prove that the characteristic of any integral domain is either zero (or) prime number.

Sol^n: If $\text{Char } R = 0$, there is nothing to prove.

If $\text{Char } R = n \neq 0$, then n is the least positive integer such that $na = 0\ \forall a \in R$. we shall pt n is prime. If n is not prime, then $n = lm$, for some integers l and m: $1 < l, m < n$.

Now $na = 0 \Rightarrow (lm)a = 0 \Rightarrow (lm)ab = 0 = 0b = 0,\ b \in R$

$\Rightarrow \underbrace{ab + ab + \cdots + ab}_{lm \text{ times}} = 0\ \forall a, b \in R.$

$\Rightarrow (\underbrace{a + a + \cdots + a}_{l \text{ times}})(\underbrace{b + b + \cdots + b}_{m \text{ times}}) = 0\ \forall a, b \in R.$

$\Rightarrow (la)(mb) = 0\ \forall a, b \in R$ ——— (1)

Since R is an integral domain, it follows from (1) that $la = 0\ \forall a \in R$, $mb = 0\ \forall b \in R$ when $1 < l < n$, $1 < m < n$.

The above two statements contradict the fact that n is the least ~~+ve integer such that $na = 0\ \forall a \in R$. Hence n must be a prime number~~

Q.3-(a) Let $a' \in R'$ be arbitrary.

we shall prove that

$f(1)a' = a'f(1) = a'$

Obviously, $f(1)a' - f(1)a' = 0'$

$\Rightarrow f(1.1)a' - f(1)a' = 0'$, since 1 is the unity of R.

$\Rightarrow f(1)f(1)a' - f(1)a' = 0'$ since f is a homomorphism

$\Rightarrow f(1)[f(1)a' - a'] = 0'$

$\Rightarrow f(1) = 0'$ or $f(1)a' - a' = 0'$

since R is an I.D.

If $f(1) = 0'$ then $1 \in \ker f$ where $\ker f$ is an ideal of R.

consequently, $x = 1 \cdot x \in \ker f \;\forall x \in R$

So $\ker f = R$

This is contrary to given hypothesis, so

$f(1)a' - a' = 0'$ or $f(1)a' = a' \;\forall a' \in R'$

similarly, we can prove $a'f(1) = a' \;\forall a' \in R'$

Hence, $f(1)$ is the unity of R'.

3(b). S.T the sequence of functions $\langle f_n \rangle$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on $[0,1]$.

Q.3 (b) If char $R = 0$, then there is nothing to prove.

Let char $R = n \neq 0$, then n is the least +ve integer such that $na = 0$ $\forall a \in R$.

We shall prove that n is prime.

let n is not prime $\therefore n = lm$, $1 < l, m < n$

now, $na = 0$.

$lm(a) = 0$

$lm(ab) = 0 \cdot b = 0$, $b \in R$

$\underbrace{ab + ab + ab - - - + ab}_{(lm \text{ times})} = 0$

$\underbrace{(a + a + - - - + a)}_{(l \text{ times})} \underbrace{(b + b + - - - b)}_{(m \text{ times})} = 0$

$(la)(mb) = 0$ $\forall a, b \in R$

Since, R is an ID, it follow that either $la = 0$ or $mb = 0$.

the two statements contradict the fact that n is the least +ve integer. Hence, n must be a prime.

4(a) Show that the centre of a division ring is a field.

Q.4 (a)

Let R be a division ring.

The centre of R is defined as

$$Z(R) = \{a \in R : xa = ax \ \forall x \in R\}.$$

We know $Z(R)$ is a subring of R.

Thus, $Z(R)$ itself is a ring. We have to show $Z(R)$ is a field.

Let $a, b \in Z(R)$

$\therefore ax = xa$ & $bx = xb \ \forall x \in R$

in particular, $ab = ba \ \forall a, b \in Z(R)$

$\Rightarrow Z(R)$ is a commutative ring.

Since, R is a division ring, $1 \in R$

and $1x = x1 \ \forall x \in R$

Thus $1 \in Z(R)$

Finally we show that each non-zero element of $Z(R)$ has its multiplicative inverse in $Z(R)$.

Let $a \neq 0 \in Z(R)$ be arbitrary

$\Rightarrow a \neq 0 \in R$

$\Rightarrow a^{-1} \in R$, $\because R$ is a division ring.

Let $x \neq 0 \in R$ thus $x^{-1} \in R$.

we have

$$a^{-1}x = (x^{-1}a)^{-1}$$
$$= (ax^{-1})^{-1}$$

since $a \in Z(R) \Rightarrow ax^{-1} = x^{-1}a$

$\therefore a^{-1}x = xa^{-1}, x \neq 0 \in R$

obviously, $a^{-1}0 = 0a^{-1}$.

Thus, $a^{-1}x = xa^{-1} \; \forall x \in R$

It means that $a^{-1} \in Z(R) \; \forall a \in Z(R)$

Hence $Z(R)$ is a field.

(c) Show that $\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$ exists iff $n < m+1$

Sol: Here $f(x) = \frac{\sin^m x}{x^n} = \left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}$.

$$\lim_{x\to 0^+} f(x) = \begin{cases} 0 & \text{if } n-m<0 \\ 1 & \text{if } n-m=0 \\ \infty & \text{if } n-m>0 \end{cases}$$

$\therefore$ the given integral is a proper integral if $n-m \leq 0$.

i.e, if $n \leq m$ and an improper integral if $n-m>0$; 0 being the only point of infinite discontinuity of f on $[0, \frac{\pi}{2}]$.

when $n-m>0$, take $g(x) = \frac{1}{x^{n-m}}$.

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^m = 1.$$ which is non-zero and finite

Also $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{dx}{x^{n-m}}$ is convergent iff $n-m<1$

i.e, $n < m+1$

$\therefore$ By comparison test, the given integral is convergent iff $n < m+1$

4(c) Sol^n We have, $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

(i) when $|z| < 2$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}[-1]^n[z/2]^n - \frac{8}{3}\sum_{n=0}^{\infty}[-1]^n[z/3]^n$ this is the req. expansion of $f(z)$ in Taylor's series within a circle $|z| = 2$

(ii) when $2 < |z| < 3$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$ be the req. expansion of $f(z)$ in a Laurent's series within the annulus $2 < |z| < 3$

(iii) when $|z| > 3$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$ be the req. expansion of $f(z)$ for $|z| > 3$.

4(c) Find the Taylor's and Laurent's series which represent the function

$\frac{z^2-1}{(z+2)(z+3)}$; (i) when $|z| < 2$ (ii) when $2 < |z| < 3$ (iii) when $|z| > 3$.

4(d) ~~A~~ company has 4 warehouses and 6 stores; the cost of shipping one unit from warehouse i to store j is C_{ij}.

If $$C = (C_{ij}) = \begin{bmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & [illegible] \end{bmatrix}$$

and the requirements of the six stores are 4, 4, 6, 2, [illegible], 2 and quantities at the warehouse are 5, 6, 2, 9. Find the minimum cost solution.

Ans 4(d) Using "lowest cost entry Method" an initial soln having the transportation cost Rs 70 is obtained as below:

		5(7)				5
	4(1)	Δ (5)			2(3)	6+Δ = 6
2(4)						2
2(4)		1(9)	2(0)	4(0)		9

Since the number of allocations (8) in the initial BFS is less than m+n-1(= 9), introduce a negligible quantity Δ in the independent empty cell (2,3)

Starting Iteration Table

						u_i
+ 7 2	+ (10) 3	5 (7)	+ (4) −2	+ (7) −2	+ (8) 5	−2
+ (5) 0	4−θ (1)	Δ+θ (5)	+ (5) −4	(3) −4	2 (3)	−4
2−θ (4)	−2 +θ (3) 5	−2 (9) 9	+ (9) 0	+ (1) 0	+ (9) 7	0
2+θ (4)	+ (6) 5	1−θ (9)	+ 2 (0) 5	4 (0)	+ (8) 7	0
v_j: 4	5	9	0	0	7	

Min. $[4-\theta, 2-\theta, 1-\theta] = 0 \Rightarrow \theta = 1$.

Since all the net-evaluations for the non-basic (empty) cells are not non negative, the initial BFS is not optimal. The empty cell (3,2) must be allocated the maximum possible amount $\theta = 1$ to this cell

Consequently, cell (4,3) becomes empty.

First Iteration Table:- Vacate the cell (4,3) and occupy the cell (3,2).

						u_i
+ (7) 4	+ (10) 3	5 (7)	+ (4) 0	+ (7) 0	+ (8) 5	2
+ (5) 2	3 (1)	1 (5)	+ (5) −2	+ (3) −2	2 (3)	0
1 (4)	1 (3)	0 (7) 7	+ (9) 0	+ (?) 0	+ (9) 5	2
3 (4)	+ (6) 3	+ (9) 7	2 (0)	4 (0)	+ (8) 5	2
v_j: 2	1	5	−2	−2	3	

Since all the net evaluations are non negative, the current soln is optimum. Hence the optimum soln is given by $x_{13}=5$, $x_{22}=3$, $x_{23}=1$, $x_{26}=2$, $x_{31}=1$, $x_{32}=1$, $x_{41}=3$, $x_{44}=2$, $x_{45}=4$

The optimum transportation cost is given by

$$Z = 5(7)+3(1)+1(5)+2(3)+1(4)+3(4)+1(3)+2(0)+4(0) = 68$$

5(a) solve $x(x^2+3y^2)p - y(3x^2+y^2)q = 2z(y^2-x^2)$

sol^n — Langrange's auxillary equation for given equation is

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{-y(3x^2+y^2)} = \frac{dz}{2z(y^2-x^2)} \quad \text{---(1)}$$

Choosing $\frac{1}{x}, \frac{1}{y}, -\frac{1}{z}$ as multipliers, each fraction of above equation,

$$= \frac{(\frac{1}{x})dx + (\frac{1}{y})dy - (\frac{1}{z})dz}{0}$$

so that

$$\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0$$

By integrating,

$$\log x + \log y - \log z = \log c_1$$

$$\therefore \quad \frac{xy}{z} = c_1$$

Taking two ratios of (1),

$$\frac{dy}{dx} = \frac{y(3x^2+y^2)}{x(x^2+3y^2)} = \left(\frac{-y}{x}\right)\frac{3+(\frac{y}{x})^2}{1+3(\frac{y}{x})^2}$$

put $y = xv$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore \quad v + x\dfrac{dv}{dx} = -v\,\dfrac{3+v^2}{1+3v^2}$

$\therefore \quad x\dfrac{dv}{dx} = -v\left[\dfrac{3+v^2}{1+3v^2} + 1\right]$

$\therefore \quad x\dfrac{dv}{dx} = \dfrac{4(1+v^2)v}{1+3v^2}$

$\therefore \quad 4\dfrac{dx}{x} + \dfrac{1+3v^2}{v(1+v^2)}\,dv = 0$

By integrating,

$\therefore \quad 4\log x + \log v + \log$ or $x^4 v(1+v^2) = C_2'$

$x^4\left(\dfrac{y}{x}\right)\left[1+\left(\dfrac{y}{x}\right)^2\right] = C_2'$ or $\Rightarrow xy(x^2+y^2) = C_2'$

$\therefore \quad C_1 z\,(x^2+y^2) = C_2'$

$\therefore \quad z\,(x^2+y^2) = \dfrac{C_2'}{C_1}$

let $\dfrac{C_2'}{C_1} = C_2$

$\therefore \quad z\,(x^2+y^2) = C_2$

Therefore, the solution is

$$\phi\left(z(x^2+y^2),\ \frac{xy}{z}\right) = 0$$

where ϕ is an arbitary function.

5(b) $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{1/2}$

Sol^n Here the auxillary equation is,

$$m^3 - 4m^2 + 5m - 2 = 0$$

$\therefore$ C.F. $= \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x)$ --- (1)

where ϕ_1, ϕ_2 and ϕ_3 are arbitary function.

$\therefore$ P.I. corresponding to e^{y+2x}

$$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} e^{y+2x}$$

$$= \frac{1}{D - 2D'}\left\{\frac{1}{(D-D')^2} e^{y+2x}\right\}$$

$$= \frac{1}{D-2D'} \frac{1}{(2-1)^2} \iint e^v \, dv \, dv$$

$$= \frac{1}{D-2D'} \int e^v dv \qquad \text{---} (v = y+x)$$

$$= \frac{1}{D-2D'} e^v$$

$$= \frac{1}{(1\times D - 2\times D')^1} e^{y+2x}$$

$$= \frac{x}{1\times 1!} e^{y+x}$$

$$= x e^{y+x} \qquad \text{---} (2)$$

Now, P.I. corresponding to $(y+x)^{1/2}$

$$= \frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}(y+x)^{1/2}$$

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{D-2D'}(y+x)^{1/2}\right\}$$

$$= \frac{1}{(D-D')^2}\times\frac{1}{1-(2\times1)}\int v^{1/2}dv$$

$$= \frac{1}{D-D'}\times\frac{2}{3}v^{3/2} \quad \text{--- } v=y+x$$

$$= -\frac{2}{3}\frac{1}{(D-D')^2}(y+x)^{3/2}$$

$$= -\frac{2}{3}\times\frac{x^2}{1^2\times 2!}(y+x)^{3/2}$$

$$= -\left(\frac{x^2}{3}\right)\times(y+x)^{3/2} \quad \text{--- (3)}$$

from (1), (2) and (3)

$$z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x) + x.e^{y+x} - \frac{x^2}{3}\times(y+x)^{3/2}$$

5(c) The velocity of a particle at distance S from a point on its path is given by the following table

S(meters)	0	10	20	30	40	50	60
V(m/sec)	47	58	64	65	~~61~~	~~52~~	38

Estimate the time taken to travel the first 60 meters using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule.

Sol'n: As we know, Velocity $v = \frac{ds}{dt}$

$\therefore dt = \frac{ds}{v}$.

$$\therefore \boxed{t = \int dt = \int_{s_0}^{s} \frac{ds}{v} = \int_{s_0}^{s} (y)\cdot ds} \quad \text{—(1)}$$

where $y = 1/v$

s	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38
$y = 1/v$	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using simpson's 1/3rd Rule

$$t = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right].$$

$\because h = \frac{60-0}{6} = 10.$

$$\therefore t = \frac{10}{3}\left[(0.0213 + 0.0263) + 2(0.0156 + 0.0164) + 4(0.0172 + 0.0154 + 0.0192)\right]$$

$= 1.0635$ sec.

Now Using Simpson's 3/8 Rule

$$t = \frac{3h}{8}\left[(y_0+y_6) + 2y_3 + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3\times 10}{8}\left[(0.0213 + 0.0263) + 2\times 0.0154 + 3\times(0.0172 + 0.0156 + 0.0164 + 0.0192)\right]$$

$\simeq 1.06437$ sec

$\therefore$ Difference b/w 1/3rd Rule & 3/8 Rule $= 0.00087$ sec

5(d) A majority function is a digital circuit whose output is '1' iff the majority of the inputs are 1. The output is '0' otherwise. Obtain the truth table of a 3-input majority function and show that the circuit of a majority function can be obtained with 4 NAND gates.

sol:- Let x, y, z be the three inputs of majority function $f(x,y,z)$

Its truth-table

x	y	z	output, $f(x,y,z)$	
1	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\bar{x}yz$
1	0	1	1	$x\bar{y}z$
1	1	0	1	$xy\bar{z}$
1	1	1	1	xyz
0	0	0	0	

$\therefore$ output f:n $f(x,y,z) = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz$

Now simplifying the output f:n

$f(x,y,z) = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz$

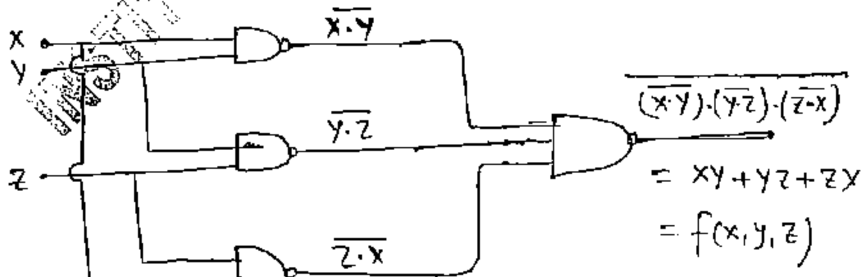
$= xy(\bar{z}+z) + \bar{x}yz + x\bar{y}z$

$$f(x,y,z) = xy + \bar{x}yz + x\bar{y}z$$
$$= xy + (\bar{x}y + x\bar{y})z$$
$$= (xy + \bar{x}y + x\bar{y})(xy + z)$$
$$[\because A + BC = (A+B)(A+C)]$$
$$= (y + x\bar{y})(xy + z)$$
$$= (x+y)(xy+z)$$
$$= x^2y + xz + xy^2 + yz$$
$$= xy + xz + xy + yz$$
$$= 2xy + xz + yz$$
$$= xy + yz + zx$$

(or) by k-map we can directly write

$$f(x,y,z) = xy + yz + zx$$

Now circuit of majority $f: n = f(x,y,z)$ with four NAND gets



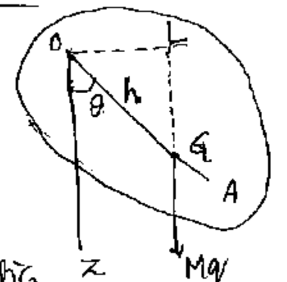
[Note :- $\overline{(\overline{x\cdot y})\cdot(\overline{y\cdot z})\cdot(\overline{z\cdot x})} = \overline{\overline{(xy)}+\overline{(yz)}+\overline{(zx)}}$

$\overline{(\overline{x\cdot y})\cdot(\overline{y\cdot z})\cdot(\overline{z\cdot x})} = xy+yz+zx$]

5(e) Q. Write the Hamiltonian function and equation of motion of a compound pendulum.

5(e) Sol^n:

At time t, let θ be the angle b/w the vertical plane through the fixed axis (plane fixed in the body). Let $OG = h$.



If T and V are the kinetic and potential energies of the pendulum then

$$T = \frac{1}{2} Mk^2\dot{\theta}^2 \quad \text{and} \quad V = -Mgh\cos\theta$$

(-ve sign is taken because G is below the fixed axis)

$$\therefore L = T - V = \frac{1}{2}Mk^2\dot{\theta}^2 + Mgh\cos\theta$$

Here θ is the only generalised co-ordinate,

$$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}Mk^2\dot{\theta}$$

since L does not t explicitly,

$$\therefore H = T + V = \frac{1}{2}Mk^2\dot{\theta}^2 - Mgh\cos\theta = \frac{1}{2Mk^2}p_\theta^2 - Mgh\cos\theta,$$

Hence the two Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -Mgh\sin\theta \quad \text{---}(H_1)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{Mk^2}p_\theta \quad \text{---}(H_2)$$

Differentiating (H_2) and substituting from (H_1), we get

$$\ddot{\theta} = \frac{1}{Mk^2}\dot{p}_\theta = \frac{1}{Mk^2}(-Mgh\sin\theta)$$

or $\ddot{\theta} = -\frac{gh}{k^2}\sin\theta$ which is the eqn. of motion of a compound pendulum.

6(a) Find the integral surface of $x^2p+y^2q+z^2=0$, $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$ which passes through the hyperbola $xy=x+y$, $z=1$.

Sol'n Given $x^2p+y^2q+z^2=0$.

i.e. $x^2p+y^2q=-z^2$. ---(1)

Given curve is given by $xy=x+y$ and $z=1$.

The langrange's auxillary equations of (1) are,

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(-z)^2} \quad \text{---(2)}$$

Taking the first and third fractions of (1),

$$x^{-2}dx+z^{-2}dz=0.$$

$$\therefore \quad -\frac{1}{x}-\frac{1}{z}=-C_1 \quad \text{---- (Integrating)}$$

$$\therefore \quad \frac{1}{x}+\frac{1}{z}=C_1 \quad \text{--- (3)}$$

Taking the second and ~~fourth~~ third fractions of (1),

$$y^{-2}dy+z^{-2}dz=0.$$

$$\therefore \quad -\frac{1}{y}-\frac{1}{z}=-C_2 \quad \text{-- (Integrating)}$$

$$\therefore \quad \frac{1}{y}+\frac{1}{z}=C_2 \quad \text{--- (4)}$$

from (3) and (4),

$$\frac{1}{x}+\frac{1}{y}+\frac{2}{z}=C_1+C_2$$

$$\therefore \quad \frac{x+y}{xy}+\frac{2}{z}=C_1+C_2$$

$$\therefore \quad \frac{xy}{xy}+2=C_1+C_2 \quad -- (\because z=1 \ \& \ x+y=xy)$$

$$\therefore \quad C_1+C_2=3$$

from (3) and (4),

$$\frac{1}{x}+\frac{1}{z}+C_1\frac{1}{y}+\frac{1}{z}=3$$

$$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{2}{z}=3$$

$$\therefore \quad yz+2xy+xz=3xyz$$

6(b) Form a partial differential equation by eliminating the arbitary function ϕ from $\phi(x^2+y^2+z^2, z^2-2xy)=0$

Sol: let $u = x^2+y^2+z^2$ and $v = z^2-2xy$

$\therefore\ \phi(u,v)=0$

$\therefore\ \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)$

here $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$

Now, $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=2y$, $\frac{\partial u}{\partial z}=2z$ and

$\frac{\partial v}{\partial x}=-2y$, $\frac{\partial v}{\partial y}=-2x$, $\frac{\partial v}{\partial z}=2z$

$\therefore\ \frac{\partial\phi}{\partial u}\left(2x+2zp\right)+\frac{\partial\phi}{\partial v}\left(-2y+p2z\right)=0$

$\therefore\ \frac{\partial\phi}{\partial u}(x+pz) = (y-pz)\left(\frac{\partial\phi}{\partial v}\right)$ ---(1)

Now, differentiating $\phi(u,v)=0$ w.r.t. y.

$\therefore\ \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$

$\therefore\ \frac{\partial\phi}{\partial u}\left(2y+q2z\right)+\frac{\partial\phi}{\partial v}\left(-2x+q2z\right)=0$

$\therefore\ (y+qz)\left(\frac{\partial\phi}{\partial u}\right) = (x-qz)\frac{\partial\phi}{\partial v}$ ---(2)

Dividing (1) by (2),

$$\therefore \quad \frac{(x+pz)}{(y+qz)} = \frac{(y-pz)}{(x-qz)}$$

$$\therefore \quad pz(y+x) - qz(y+x) = y^2 - x^2$$

$$\therefore \quad z(p-q) = y-x$$

6(c) Reduce $\frac{\partial^2 z}{\partial x^2} = (1+y)^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

Sol'n: $r - (1+y^2)t = 0.$ --- (1)

comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$.

$\therefore$ $R=1$, $S=0$ and $T = -(1+y)^2$

$\therefore$ $S^2 - 4RT = (1+y^2) > 0$ for $y \neq -1$.

equation (1) is hyperbolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (1+y^2) = 0$.

$$\therefore \lambda = 1+y, -(1+y)$$

$\therefore$ the corresponding characteristic equations are,

$$\frac{dy}{dx} + (1+y) = 0 \text{ and } \frac{dy}{dx} - (1+y) = 0.$$

$$\therefore \log(1+y) + x = C_1 \text{ and } \log(1+y) - x = C_2$$

let $u = \log(1+y) + x$ and $v = \log(1+y) - x$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

from p, $\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$

$$\therefore r = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left\{\frac{1}{1+y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\right\}$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}\right.$$

$$\left.+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= \frac{1}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

from (1), putting the values of r and t,

$$\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right.$$

$$\left.-\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)=0$$

6(d) A string of length 'l' is initially at rest in its equilibrium position and each of its points is given the velocity

$$\left(\partial y/\partial t\right)_{t=0} = V_0 \sin^3\left(\frac{\pi x}{l}\right) \quad \text{Where } 0<x<l.$$

Find the displacement function.

Soln: —

~~Differentiat~~

The Wave eqn of string;

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——(1)}$$

Let $y = X(x).T(t)$ ——

substituting the value in eqn (1)

We get,

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = -\mu^2 \text{ (say)} \text{——}$$

$$\Rightarrow \left.\begin{array}{l} X''(x)+\mu^2 X(x) = 0 \\ T''(t)+\mu^2 c^2 T(t) = 0 \end{array}\right\} \text{——(4)}$$

Given, Boundary Conditions

$$y(0,t) = y(l,t) = 0.$$

$$\frac{\partial y}{\partial t}\Big/_{(0,t)} = \frac{\partial y}{\partial t}(l,t) = 0$$

Initial Conditions

$$\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3\left(\frac{\pi x}{l}\right); \qquad 0<x<l$$

$$y(x,0) = 0$$

~~$\Rightarrow X(x)=$~~

Case-(i) : if $\mu = 0$

$$X = a_1 x + b_1 \quad , \quad T = c_1 t + d_1 \quad \text{---} (5)$$

using boundary conditions.

$$X(0) = 0 = X(l) \Rightarrow X(x) = 0$$

$$\therefore y(x,t) = X(x)\cdot T(t) = 0\cdot T(t) = 0$$

$\therefore$ This is rejected

Case-II if $\mu = +ve = \lambda^2$

then $X = a_2 e^{\lambda x} + b_2 e^{-\lambda x}$ ——— (6)

$$T = c_2 e^{\lambda ct} + d_2 e^{-\lambda ct} \quad \text{---} (7)$$

$\because X(0) = 0 = a_2 + b_2 \Rightarrow a_2 = -b_2$

$X(l) = 0 = a_2(e^{\lambda l} - e^{-\lambda l}) = 0 \Rightarrow a_2 = 0$

$\Rightarrow b_2 = 0$

$\therefore X(x) = 0$

$\therefore y(x,t) = X(x)\cdot T(t) = 0 \times T(t) = 0$

$\therefore$ This solution is rejected

Case-III for $\mu = -\lambda^2 = -ve$

$$X = a_3 \cos\lambda x + b_3 \sin\lambda x \quad \text{---} (8)$$

$$T = c_3 \cos\lambda ct + d_3 \sin\lambda ct \quad \text{---} (9)$$

$X(0) = 0 \Rightarrow a_3 = 0$

$X(l) = 0 \Rightarrow b_3 \sin\lambda l = 0 \Rightarrow \lambda l = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$

$T(0) = 0 \Rightarrow c_3 = 0$

~~$\therefore y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi ct}{l} \cdot \sin\frac{n\pi}{l}x$~~

$\therefore y_n(x,t) = x_n(x) \cdot T_n(t)$

$= \phi_3 \sin\left(\frac{n\pi x}{l}\right) \times d_3 \sin\left(\frac{n\pi c}{l}\right)t$

$= E_n \sin\left(\frac{n\pi c}{l}\right)t \cdot \cos\left(\frac{n\pi}{l}\right)x$

$\therefore y(x,t) = \sum_{n=0}^{\infty} y_n(x,t)$

$$\boxed{y_1(x,t) = \sum_{n=0}^{\infty} E_n \sin\left(\frac{n\pi c}{l}\right)t \cdot \cos\left(\frac{n\pi}{l}\right)x} \quad \text{(10)}$$

Now differentiating the eqn (10)

we get,

$\frac{\partial y}{\partial t} = \left[\sum_{n=0}^{\infty} E_n \cos\left(\frac{n\pi c}{l}\right)t \cdot \cos\left(\frac{n\pi}{l}\right)x\right] \times \left(\frac{n\pi c}{l}\right)$

$= \left(\frac{\pi c}{l}\right) \sum_{n=0}^{\infty} E_n \, n \cos\left(\frac{n\pi c}{l}\right)t \cdot \cos\left(\frac{n\pi}{l}\right)x$ (11)

$\because \left.\frac{\partial y}{\partial t}\right|_{t=0} = V_0 \sin^3\frac{\pi x}{l}$

as we know, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

i.e. $V_0 \sin^3\left(\frac{\pi x}{l}\right) = V_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$ (12)

Comparing (11) and (12)

We get,

$$\sum E_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} = V_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\Rightarrow E_1\left(\frac{\pi c}{l}\right) = \frac{3V_0}{4}$$

$$E_3\left(\frac{3\pi c}{l}\right) = \frac{-V_0}{4}$$

$\because E_n = 0 \quad \forall n \neq 1,3$

$$\therefore E_1 = \frac{3V_0 l}{4\pi c} \quad \& \quad E_3 = -\frac{l V_0}{12\pi c}$$

$$\therefore y(x,t) = \frac{3V_0 l}{4\pi c}\sin\left(\frac{\pi c t}{l}\right)\sin\frac{\pi x}{l} - \frac{V_0 l}{12\pi c}\sin\frac{3\pi c t}{l}\sin\frac{3\pi x}{l}$$

7(a) The equation $x^2+ax+b=0$ has two real roots α & β. Show that the iteration method $x_{k+1} = -\frac{(ax_k+b)}{x_k}$ is convergent near $x=\alpha$ if $|\alpha|>|\beta|$ and that $x_{k+1} = \frac{-b}{x_k+a}$ is convergent near $x=\alpha$ if $|\alpha|<|\beta|$. Show also that iteration method $x_{k+1} = -\frac{(x_k^2+b)}{a}$ is convergent near $x=\alpha$ if $2|\alpha|<|\alpha+\beta|$.

Sol: The iterations are given by

$$x_{k+1} = -\frac{(ax_k+b)}{x_k} = g(x_k) \quad (\text{say})$$
$$k=0,1,2 \ldots$$

By the known theorem

If $g(x)$ and $g'(x)$ are continuous in an interval about a root α of the equation $x=g(x)$ and if $|g'(x)|<1$ for all x in the interval, then the successive approximations $x_1, x_2, \ldots$ given by

$$x_k = g(x_{k-1}),\ k = 1, 2, 3, \ldots$$

converges to the root α provided that the initial approximation x_0 is chosen in the interval.

$\therefore$ These iterations converge to α if

$|g'(x)| < 1$ near α.

i.e. $|g'(x)| = \left|-\frac{b}{x^2}\right| < 1$

Note that $g'(x)$ is continuous near α.

If the iterations converge to $x = \alpha$, then we require $|g'(\alpha)| = \left|\frac{-b}{\alpha^2}\right| < 1$

Thus $|b| < |\alpha|^2$

i.e. $|\alpha|^2 > |b|$ —— (1)

Given that α and β are roots of the equation $x^2 + ax + b = 0$

then $\alpha + \beta = -a$ and $\alpha\beta = b \Rightarrow |b| = |\alpha||\beta|$ —— (2)

Substituting (2) in (1), we get

$|\alpha|^2 > |b| = |\alpha||\beta|$

$\Rightarrow |\alpha|^2 > |\alpha||\beta|$

$\Rightarrow |\alpha| > |\beta|$.

Now, if $x = \frac{-b}{x+a}$

The iteration $x_{k+1} = \frac{-b}{x_k + a} = g(x_k)$ (say)

converges to α if $|g'(x)| = \left|\frac{b}{(x+a)^2}\right| < 1$ in an interval containing α.

In particular we require

$$|g'(\alpha)| = \left|\frac{b}{(\alpha+a)^2}\right| < 1$$

$$\Rightarrow (\alpha+a)^2 > |b|$$

But we have $\alpha+\beta=-a$ & $\alpha\beta=b$

$$\Rightarrow \beta^2 > |b| = |\alpha||\beta|$$

$$\Rightarrow |\beta|^2 > |\alpha||\beta|$$

$$\Rightarrow |\beta| > |\alpha|$$

$\therefore x_{k+1} = \frac{-b}{x_k+a}$ is convergent near α if $|\beta| > |\alpha|$

8(a) A particle of mass m moves in a conservative forces field. Find (i) the Lagrangian function and (ii) the eqⁿ of motion in cylindrical coordinates (ρ, ϕ, z)

8.(a) Let P be the position of the particle of mass m whose cylindrical co-ordinates referred to axes OX, OY, OZ are (ρ, ϕ, z)

$\therefore$ If (x, y, z) are its cartesian co-ordinates, then

$x = OA = \rho\cos\phi$

$y = OB = \rho\sin\phi, \; z = z$

If $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along OX, OY, OZ respectively, then $\vec{OP} = r = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

If $\hat{\rho}_1$ and $\hat{\phi}_1$ are the unit vectors in the directions of ρ and ϕ increasing respectively, then

$$\hat{\rho}_1 = \frac{\partial\vec{r}}{\partial\rho} \Big/ \left|\frac{\partial\vec{r}}{\partial\rho}\right| = \frac{\cos\phi\,\hat{i} + \sin\phi\,\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{\phi}_1 = \frac{\partial\vec{r}}{\partial\phi} \Big/ \left|\frac{\partial\vec{r}}{\partial\phi}\right| = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Now $\vec{v} = \dot{\vec{r}} = (\dot{\rho}\cos\phi - \rho\sin\phi\dot{\phi})\hat{i} + (\dot{\rho}\sin\phi + \rho\cos\phi\dot{\phi})\hat{j} + \dot{z}\hat{k}$

$$= \dot{\rho}(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + \rho\dot{\phi}(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) + \dot{z}\hat{k}$$

$$= (\dot{\rho})\hat{\rho}_1 + (\rho\dot{\phi})\hat{\phi}_1 + \dot{z}\hat{k}$$

$$\therefore v^2 = \dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2$$

Total K.E $= T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

Let $V = V(\rho, \phi, z)$ be the potential function.

(i) Lagrangian function, $L = T - V$

i.e $L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z)$

(ii) Lagrange's ρ equation is, $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\rho}}\right) - \frac{\partial L}{\partial\rho} = 0$

or $\frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial\rho}\right) = 0$

i.e $m\ddot{\rho} - m\rho\dot{\phi}^2 = -\frac{\partial V}{\partial\rho}$ —— (1)

Lagrange's ϕ equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0$

or $\frac{d}{dt}(m\rho^2\dot{\phi})-\left(-\frac{\partial V}{\partial \phi}\right)=0$ or $\frac{d}{dt}(m\rho^2\dot{\phi})=-\frac{\partial V}{\partial \phi}$ —— ②

and Lagrange's z equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=0$

or $\frac{d}{dt}(m\dot{z})-\left(-\frac{\partial V}{\partial z}\right)=0$

or $m\ddot{z}=-\frac{\partial V}{\partial z}$ —— ③

Mains · Test Series – 2016

Test – 13 (Paper – I)

1(c) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1,1,2)$, $\alpha_2 = (0,1,3)$, $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

Sol. Let $S_1 = \{\alpha_1, \alpha_2\}$ then we construct a matrix A whose rows are the elements of A and reduce it into row reduced echelon form.

$$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

Let $S_2 = \{\alpha_3, \alpha_4\}$ then $B = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 0 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 0 & -1 \\ 2 & 4 & 5 \end{bmatrix} R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 5 \end{bmatrix} R_1 \to (-1) R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 3 \end{bmatrix} R_2 \to R_2 - 2R_1 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/4 \end{bmatrix} R_2 \to \tfrac{1}{4} R_2$$

We have $L(S_1) = \{x(1,0,-1) + y(0,1,3) / x, y \in \mathbb{R}\}$

$= \{(x, y, -x+3y) / x, y \in \mathbb{R}\}$ — (1)

and $L(S_2) = \{a(1,0,1) + b(0,1,3/4) / a,b \in \mathbb{R}\}$

$= \{(a, b, a+\tfrac{3}{4}b) / a,b \in \mathbb{R}\}$. —— (2)

from (1) & (2), we have

$x=a,\ y=b\ ;\ -x+3y = a+\tfrac{3}{4}b$.

$\Rightarrow -a+3b = a+\tfrac{3}{4}b$

$\Rightarrow (3-\tfrac{3}{4})b = 2a$

$\Rightarrow \frac{9}{4}b = 2a$

$\Rightarrow \boxed{b = \frac{8}{9}a}$

$a+\frac{3}{4}\cdot\frac{8}{9}a$

$a+\frac{2}{3}a$

$\frac{5a}{3}$.

$\therefore L(S_1) \cap L(S_2) = \{(a, \tfrac{8}{9}a, \tfrac{5}{3}a) / a \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

clearly it is a subspace of $\mathbb{R}^3$.

and it contains only one free variable 'a'. its dimension is 1.

and its basis is $\{(1, 8/9, 5/3)\}$.

1(b) Obtain eigen values and eigen vectors of the differential operator $D: P_2 \to P_2$
$D(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$ for $a_0, a_1, a_2 \in \mathbb{R}$

Sol: Given that $D: P_2 \to P_2$

$D(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$ for $a_0, a_1, a_2 \in \mathbb{R}$

Let $\beta = \{1, x, x^2\}$ is a basis of P_2

Then $D(1) = 0(1) + 0(x) + 0(x^2)$

$D(x) = 1(1) + 0(x) + 0(x^2)$

$D(x^2) = 0(1) + 2(x) + 0(x^2)$

$\therefore [D]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The characteristic polynomial of D is

$$|D - \lambda I| = \begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 2 \\ 0 & 0 & 0-\lambda \end{vmatrix} = -\lambda^3$$

$\therefore$ The eigen values are given by $\lambda^3 = 0$
$\Rightarrow \lambda = 0$.

Now the eigen vector corresponding to eigen value $\lambda = 0$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

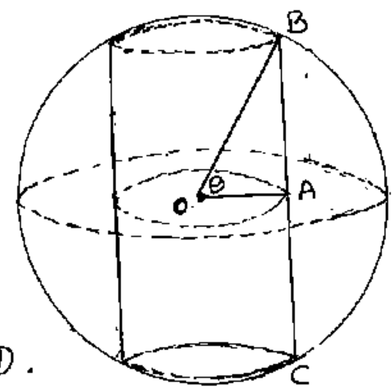
$\Rightarrow x_2 = 0,\ x_3 = 0,\ x_1 = k$ where k is an arbitrary constant.

Hence eigen vector is $k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

1(c). Find the Surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r.

Sol'n: we construct a cylinder, OA is the radius of the base and CB is the height of this cylinder.

Let $\angle AOB = \theta$, so that θ lies b/w 0 and π



we have $\frac{OA}{OB} = \cos\theta$

$\Rightarrow OA = OB\cos\theta = r\cos\theta$

Also $\frac{AB}{OB} = \sin\theta$

$\Rightarrow AB = OB\sin\theta = r\sin\theta$

If S be the surface, we have

$S = 2\pi \cdot OA^2 + 2\pi \cdot OA \cdot BC$

$= 2\pi r^2(\cos^2\theta + \sin 2\theta)$ —— (1).

$\Rightarrow \frac{dS}{d\theta} = 2\pi r^2(-2\cos\theta\sin\theta + 2\cos 2\theta)$

$= 2\pi r^2(2\cos 2\theta - \sin 2\theta) \Rightarrow \frac{dS}{d\theta} = 0$ for

$2\cos 2\theta - \sin 2\theta = 0 \Leftrightarrow \tan 2\theta = 2$ —— (2)

$\tan 2\theta = 2$ admits of only one value of $\theta \in [0, \pi/2]$ as its solution.

Let $\theta_1 \in [0, \pi/2]$ be the root of $\tan 2\theta = 2$.

Now $\tan 2\theta_1 = 2 \Rightarrow \sin 2\theta_1 = 2/\sqrt{5}$ and $\cos 2\theta_1 = 1/\sqrt{5}$

from (1) we see that

$\theta = 0 \Rightarrow S = 2\pi r^2$

$\theta = \pi/2 \Rightarrow S = 0$

$\theta = \theta_1 \Rightarrow S = 2\pi r^2\left(\frac{1+\cos 2\theta_1}{2} + \sin 2\theta_1\right)$

$= \frac{\pi r^2(5+5\sqrt{5})}{5}$

which is greater than $3\pi r^2$.

Hence $\frac{\pi r^2(5+5\sqrt{5})}{5}$ is the required greatest surface.

1(d) If $v = \log_e \sin\left\{\dfrac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$, find the value $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} + z\dfrac{\partial v}{\partial z}$ when

$x = 0,\ y = 1,\ z = 2.$

Sol'n : Given

$$v = \log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow e^v = \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow \sin^{-1}e^v = \frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}} = u \text{ say} \quad \text{--- (1)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \quad \text{--- (2)}$$

where $n = 1 - \frac{2}{3} = \frac{1}{3}$

But from (1)

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-e^{2v}}}\, e^v \frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-e^{2v}}}\, e^v \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{1}{\sqrt{1-e^{2v}}}\, e^v \frac{\partial v}{\partial z}$$

i.e from (2)

$$\frac{e^v}{\sqrt{1-e^{2v}}}\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z}\right] = \frac{1}{3}\left(\sin^{-1}e^v\right)$$

$$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = \frac{1}{3}\left(\sin^{-1}e^v\right)\frac{\sqrt{1-e^{2v}}}{e^v} \quad \text{--- (3)}$$

when $(x, y, z) = (0, 1, 2)$

$$v = \log_e \sin\left\{\frac{\pi (1)^{1/2}}{2(4+4)^{1/3}}\right\}$$

$$= \log_e \sin\left[\frac{\pi}{2(8)^{1/3}}\right] = \log_e \sin\left(\frac{\pi}{4}\right)$$

$$v = \log_e\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow e^v = \frac{1}{\sqrt{2}}$$

and $u = \sin^{-1} e^v = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\therefore$ from (3)

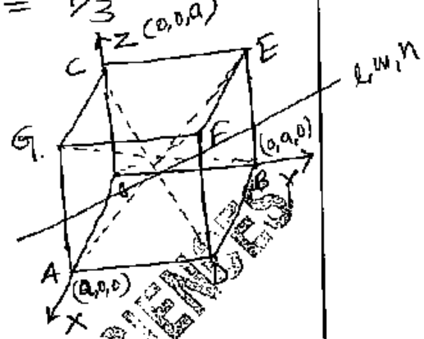
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\left(\frac{\pi}{4}\right)}{3} \cdot \frac{\sqrt{1-\frac{1}{2}}}{1/\sqrt{2}}$$

$$= \frac{\pi}{12}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right)$$

$$= \frac{\pi}{12}$$

1.(e) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube; prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$$



Sol^n: Let $A \equiv (a,0,0)$
$B \equiv (0,a,0)$
$C \equiv (0,0,a)$

$F \equiv (a,a,a)$; $D \equiv (a,a,0)$
$E \equiv (0,a,a)$; $G \equiv (a,0,a)$

So, d.c's of diagonals of are $\left(\frac{a}{\sqrt{3}a^2}, \frac{a}{\sqrt{3}a^2}, \frac{a}{\sqrt{3}a^2}\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$DC = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

$AE \equiv \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$BG \equiv \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Let l, m, n be the d.c's of the line which makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of the cube. Then

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{1}{\sqrt{3}}(l+m+n)$$

$$\text{lly}, \cos\beta = \frac{1}{\sqrt{3}}(l+m-n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(-l+m+n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(l-m+n)$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}\left[\begin{array}{l}(l+m+n)^2+(l+m-n)^2 \\ +(-l+m+n)^2+(l-m+n)^2\end{array}\right]$$

so, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= \frac{1}{3}\left[4(l^2+m^2+n^2)\right]$

$= \frac{4}{3} \quad \{\because l^2+m^2+n^2=1\}$.

2(a) (i) Show that the diagonal entries of a skew symmetric matrix are all zero, but the converse is not true.
(ii) let $Q' = \{ax^2+bx+c \,/\, c \neq 0,\ a, b, \ldots\}$
Is Q' a complex vectorspace? Justify your answer.

Sol let A be a skew-symmetric matrix.
Then $A^T = -A \Rightarrow A^T = -A$

$$\therefore \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & -a_{22} & \cdots & -a_{n2} \\ \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & & -a_{nn} \end{bmatrix}$$

$\therefore$ for any $i = 1, 2, 3 \ldots n$,
$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

but the converse is not true!
for example, the diagonal entries of $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ are zero but it is not a skew-symmetric matrix.

(ii) $Q' = \{ax^2+bx+c \,/\, a, b, c \in \mathbb{C}\}$ ($c \neq 0$)
let $Q = \{ax^2+bx+c \mid a, b, c \in \mathbb{C}\}$.
clearly $Q' \subseteq Q$.

addition is a binary operation on $\wp$, since $(ax^2+bx+c)+(dx^2+ex+f)$
$= (a+d)x^2+(b+e)x+(c+f)$
$\forall\ a,b,c,d,e,f \in \mathbb{C}$

Scalar multiplication from $\mathbb{C}\times\wp$ to $\wp$ is also well-defined
since $\alpha(ax^2+bx+c) = (\alpha a)x^2+(\alpha b)x+\alpha c$
$\forall\ \alpha, a, b, c \in \mathbb{C}$

clearly on the above lines, we can easily show that $\wp$ is a complex vector space.

Here clearly addition is closed on $\wp$ but not on $\wp'$.
because, for example $2x^2 \in \wp'$
and $(-2)x^2 \in \wp'$
but $2x^2+(-2)x^2 \notin \wp'$.

$\therefore$ $\wp'$ cannot be a vector space under the usual operations

2(c) Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \left(\frac{1}{2}\log 2 - \frac{5}{8}\right)$ throughout the volume bounded by the coordinate planes and the plane $x+y+z=1$.

Sol^n: clearly are as follows.

z from 0 to $1-x-y$

y from 0 to $1-x$

and x from 0 to 1

$$\therefore I = \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(1+x+y+z)^{-2}}{-2}\right]_0^{1-x-y} dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{(1+x+y)^2} - \frac{1}{4}\right] dx\,dy$$

$$= \frac{1}{2}\int_0^1 \left\{\frac{(1+x+y)^{-1}}{-1} - \frac{1}{4}y\right\}_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{x+1} - \frac{1}{2} - \frac{(1-x)}{4}\right] dx$$

$$= \frac{1}{2}\left[\log(x+1) - \frac{x}{2} + \frac{(1-x)^2}{8}\right]_0^1$$

$$= \frac{1}{2}\log 2 - \frac{1}{2} + 0 - \frac{1}{8}$$

$$= \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

2(d) Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1$, $x=0$; and parallel to line $\frac{x}{a}-\frac{z}{c}=1$, $y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if $2d$ is the S.D. prove that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

Sol^n: Any plane through the line $\frac{y}{b}+\frac{z}{c}-1=0$, $x=0$ —— (1)

is $\left(\frac{y}{b}+\frac{z}{c}-1\right)+\lambda x=0$.

i.e. $\lambda x+\frac{y}{b}+\frac{z}{c}-1=0$ —— (2)

This will be $\parallel$el to the line $\frac{x}{a}-\frac{z}{c}=1$, $y=0$

i.e, $\frac{x-a}{a}=\frac{y}{0}=\frac{z}{c}$ —— (3)

if $al+bm+cn=0$

i.e. if $\lambda\cdot a+\frac{1}{b}\cdot 0+\frac{1}{c}\cdot c=0$

$\therefore$ if $\lambda a+1=0 \Rightarrow \lambda=-\frac{1}{a}$

Putting this value in (2), the plane through (1) and parallel to (3) is $\left(\frac{y}{b}+\frac{z}{c}-1\right)-\frac{1}{a}\cdot x=0$.

$$\Rightarrow \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0 \quad \text{—— (4)}$$

Now, one point on the line (3) is $(a,0,0)$

$\therefore$ $2d$ = S.D = $\perp$ distance of $(a,0,0)$ from the plane (4)

$$=\frac{\frac{a}{a}-0-0+1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=\frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow \frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

3(a) ii) Let A be 3×3 upper triangular matrix with real entries. If $a_{11}=1$, $a_{22}=2$ and $a_{33}=3$, determine α, β and γ such that $A^{-1}=\alpha A^2+\beta A+\gamma I$.

Sol'n: Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$ where $a,b,c \in \mathbb{R}$

then characteristic equation of A is

$|A-\lambda I|=0$ —— (1) where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{vmatrix} 1-\lambda & a & b \\ 0 & 2-\lambda & c \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda)=0$

$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$ —— (2)

we know that by Caley's theorem every square matrix satisfies its own characteristic equation.

$\therefore$ (2) $\equiv A^3-6A^2+11A-6I=0$ —— (3)

$\Rightarrow A^2-6A+11I-6A^{-1}=0$ (by multiplying with A^{-1})

$\Rightarrow A^{-1}=\frac{1}{6}\left[A^2-6A+11I\right]$ —— (4)

Given that $A^{-1}=\alpha A^2+\beta A+\gamma I$

$\Rightarrow \frac{1}{6}\left[A^2-6A+11I\right]=\alpha A^2+\beta A+\gamma I$ (from (4))

$\Rightarrow A^2-6A+11I=(6\alpha)A^2+(6\beta)A+(6\gamma)I$

$\Rightarrow 6\alpha=1$, $6\beta=-6$, $6\gamma=11$

$\Rightarrow \alpha=\frac{1}{6}$, $\beta=-1$, $\gamma=\frac{11}{6}$.

3(b) (ii) Find a non-zero vector common to the space spanned by (1,2,3), (3,2,1) and the space spanned by (1,0,1) and (3,4,3)

Sol let $\mathbb{R}^3 = \{(a,b,c) / a,b,c \in \mathbb{R}^3\}$ be the given vector space.

let $S_1 = \{(1,2,3), (3,2,1)\} \subseteq \mathbb{R}^3$

then $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix} R_2 \to R_2 - 3R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} R_2 \to R_2(-\frac{1}{4}) \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} R_1 \to R_1 - 2R_2$

clearly it is in row reduced echelon form.

let $S_2 = \{(1,0,1), (3,4,3)\} \subseteq \mathbb{R}^3$ then

$B = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix} R_2 \to R_2 - 3R_1 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} R_2 \to \frac{1}{4}R_2$

clearly it is in row reduced echelon form.

we have

$L(S_1) = \{x(1,0,2) + y(0,1,\frac{1}{2}) / x,y \in \mathbb{R}\}$

$= \{(x, y, 2x + \frac{1}{2}y) / x,y \in \mathbb{R}\}$ —— (1) &

$L(S_2) = \{(x_1, y_1, x_1) / x_1, y_1 \in \mathbb{R}\}$. —— (2)

∴ from (1) & (2), we have

$x = x_1,\ y = y_1,\ 2x + \frac{1}{2}y = x_1 \Rightarrow 2x_1 + \frac{1}{2}y_1 = x_1$

$\Rightarrow \frac{1}{2} y_1 = -x_1$

$\Rightarrow \boxed{y_1 = -2x_1}$

$\therefore L(S_1) \cap L(S_2) = \{(x_1, -2x_1, x_1) / x_1 \in \mathbb{R}\} \subseteq \mathbb{R}^3$

Clearly it is also a subspace of $\mathbb{R}^3$.

let $\alpha = (1, -2, 1) \in \mathbb{R}^3$ s.t.

α is a non-zero vector common to the space spanned by $(1,2,3)$, $(3,1,1)$ and the space spanned by $(1,0,1)$ and $(3,4,3)$.

3(c) Find $\lim \frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$, $(x\to 0)$

Sol^n: We write

$$\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}=\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\left(\frac{x}{\tan x}\right)^2$$

so that $\lim\limits_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$

$$=\lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\lim_{x\to 0}\left(\frac{x}{\tan x}\right)^2$$

$$=\lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}\cdot 1$$

$$=\lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$$

To evaluate the limit on the R.H.S, we notice that the numerator and denominator both become 0 for $x=0$.

$$\therefore \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}=\lim_{x\to 0}\frac{\cos x-\sin x-\left[\frac{1}{1-x}\right]}{3x^2}$$

$$=\lim_{x\to 0}\frac{-\sin x+\cos x-\left[\frac{1}{(1-x)^2}\right]}{6x}$$

$$=\lim_{x\to 0}\frac{-\cos x-\sin x-\left[\frac{2}{(1-x)^3}\right]}{6}$$

$$=\frac{-3}{6}=\frac{1}{2}$$

3(d) Find the limiting points of coaxial systems defined by the spheres $x^2+y^2+z^2+2x+2y+4z+2=0$

Sol^n: The equation of the coaxial system of spheres is $(x^2+y^2+z^2+2x+2y+4z+2)+\lambda(x^2+y^2+z^2+x+y+3z+2)=0$

$$\Rightarrow x^2+y^2+z^2+\left(\frac{2+\lambda}{1+\lambda}\right)x+\left(\frac{2+\lambda}{1+\lambda}\right)y+\left(\frac{4+2\lambda}{1+\lambda}\right)z+2=0$$

Its centre is $\left(-\frac{2+\lambda}{1+\lambda}, -\frac{2+\lambda}{1+\lambda}, -\frac{4+2\lambda}{1+\lambda}\right)$ — (1)

and equating its radius to zero, we get

$$\frac{(2+\lambda)^2}{(1+\lambda)^2}+\frac{(2+\lambda)^2}{(1+\lambda)^2}+\frac{(4+2\lambda)^2}{(1+\lambda)^2}-2=0$$

$$\Rightarrow 6(2+\lambda)^2=2(1+\lambda)^2 \Rightarrow 3(\lambda^2+4\lambda+4)=\lambda^2+2\lambda+1$$

$$\Rightarrow 2\lambda^2+10\lambda+11=0 \Rightarrow \lambda=\frac{-10\pm\sqrt{100-88}}{4}=\frac{-5\pm\sqrt{3}}{2}$$

$$\Rightarrow \lambda=\frac{-5+\sqrt{3}}{2}, \frac{-5-\sqrt{3}}{2}$$

Substituting these values of λ in the coordinates of the above centre, the required limiting points are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \text{ and } \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$$

4(a) (i) If $P=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ then find P^{50}.

(ii) Find the dimension of the Subspace $W=\left\{(x,y,z,w)\in\mathbb{R}^4 \;\middle|\; \begin{array}{l} x+y+z+w=0,\\ x+y+2z=0,\\ x+3y=0\end{array}\right\}\subseteq\mathbb{R}^4$.

Sol (i) Let $P=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$ then

Characteristic equation is $(\lambda-1)^3=0$.

$$\lambda^3-1-3\lambda^2+3\lambda=0.$$

$$\Rightarrow \lambda^3-1=0.$$

We know that every square matrix satisfies its own characteristic equation.

$\therefore\; P^3-I=0 \Rightarrow P^3=I$

$$\Rightarrow P^3=P^6=\cdots P^{30}=P^{33}=P^{36}=P^{39}$$
$$=P^{42}=P^{45}=P^{48}=P^{51}=I.$$

$\because P^{51}=I \Rightarrow P^{50}=P^{-1}=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}$.

(ii) $\because\; x+y+z+w=0$

$x+y+2z=0$

$x+3y=0.$

$\Rightarrow AX=O$ —— (1)

where $A=\begin{bmatrix}1&1&1&1\\1&1&2&0\\1&3&0&0\end{bmatrix}$; $X=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$; $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_3.$$

we write a single matrix equation

$x+y+z+w=0$

$2y-z-w=0$

$z-w=0$

$\Rightarrow \boxed{z=w}$; $2y-w-w=0$

$2y=2w$

$\Rightarrow \boxed{y=w}$

$\therefore x-w+w+w=0 \Rightarrow \boxed{x=-w}$.

$\therefore W=\{(-w,w,w,w)\in\mathbb{R}^4 \mid w\in\mathbb{R}\}$.

clearly it contains only one free variable w.

$\therefore \boxed{\dim(W)=1}$

4(b) Examine for continuity at $x=a$, the function f where

$$f(x)=\begin{cases} \frac{x^2}{a}-a ; & 0<x<a \\ 0 ; & x=a \\ a-\frac{a^3}{x^2} ; & x>a. \end{cases}$$

Also examine if the function is derivable at

Sol continuity:-

LHL: $\underset{x \to a^-}{Lt} f(x) = 0$

RHL: $\underset{x \to a^+}{Lt} f(x) = 0$.

$\therefore$ LHL = RHL = $f(a)$.

$\therefore$ f is continuity at $x=a$

Differentiability!

LHD! $Lf'(a) = \underset{x \to a^-}{Lt} \frac{f(x) - f(a)}{x-a}$

$= \underset{x \to a^-}{Lt} \frac{\frac{x^2}{a} - 0}{x-a}$

$= \underset{x \to a^-}{Lt} \frac{x^2 - a^2}{a(x-a)} = 2a$.

RHD: $Rf'(a) = \underset{x \to a^+}{Lt} \frac{f(x) - f(a)}{x-a}$

$= \underset{x \to a^+}{Lt} \frac{a - \frac{a^3}{x^2} - 0}{x-a}$

$= \underset{x \to a^+}{Lt} \frac{ax^2 - a^3}{x^2(x-a)} = \underset{x \to a^+}{Lt} \frac{a(x+a)\cancel{(x-a)}}{x^2\cancel{(x-a)}}$

$= \underset{x \to a^+}{Lt} \frac{a(x+a)}{x^2} = 2$.

$\therefore$ LHD = RHD

$\therefore$ f is differentiability at $x=a$

4(c) Prove that the lines drawn from the origin parallel to the normal $ax^2+by^2+cz^2=1$ at its points of intersection with the plane $lx+my+nz=p$ generate the cone $p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$

Sol'n: Let (α,β,γ) be the point of intersection of the given conicoid and given plane, then we have $a\alpha^2+b\beta^2+c\gamma^2=1$ —— (1)

and $l\alpha+m\beta+n\gamma=p$ —— (2)

Also the equations of the normal to the given conicoid at (α,β,γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$$

∴ The equations of the line through the origin parallel to this line are

$$x/a\alpha=y/b\beta=z/c\gamma \quad —— (3)$$

from (1) and (3), we have

$$a\alpha^2+b\beta^2+c\gamma^2=\left(\frac{l\alpha+m\beta+n\gamma}{p}\right)^2$$

$$\Rightarrow p^2\left[\frac{(a\alpha)^2}{a}+\frac{(b\beta)^2}{b}+\frac{(c\gamma)^2}{c}\right]=\left[\frac{l(a\alpha)}{a}+\frac{m(b\beta)}{b}+\frac{n(c\gamma)}{c}\right]^2$$

$$\Rightarrow p^2\left[\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right]=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

from (3) eliminating α,β,γ.

Hence the line (3) generates the above cone.

5(a) Show that $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the real line, even though the Wronskian cannot be computed.

Sol: On the real line, let $K_1, K_2 \in \mathbb{R}$ s.t

$$K_1 y_1(x) + K_2 y_2(x) = 0(x) = 0 \quad \text{--- (1)}$$

$$\Rightarrow K_1 x + K_2 |x| = 0 \quad \text{--- (2)}$$

let us define,

$$K_1 x + K_2 |x| = \begin{cases} K_1 x - K_2 x \ ; \ x < 0 & \text{--- (3)} \\ 0 \ ; \ x = 0 & \text{--- (4)} \\ K_1 x + K_2 x \ ; \ x > 0 & \text{--- (5)} \end{cases}$$

from (2) & (3), we have

$$K_1 x - K_2 x = 0 \Rightarrow (K_1 - K_2) x = 0$$

$$\Rightarrow K_1 = K_2 \ (\because x \neq 0) \quad \text{--- (6)}$$

from (2) & (5), we have

$$K_1 x + K_2 x = 0 \Rightarrow (K_1 + K_2) x = 0$$

$$\Rightarrow K_1 + K_2 = 0 \ (\because x \neq 0) \quad \text{--- (7)}$$

from (4) we have

$x = 0$ is not possible bcz

both vectors $y_1(x)$ & $y_2(x)$ are zero.

In this case, clearly they are L.D.

from (6) & (7) we have $K_1 = K_2 = 0$

$\therefore y_1(x), y_2(x)$ are linearly independent.

Now we have, $W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} x & |x| \\ 1 & \text{depends on } x \end{vmatrix}$.

$\therefore$ The Wronskian can't be calculated as it varies different values of x

Here for $x > 0$; $W(y_1, y_2) = 0$ | but Wronskian can't be
for $x < 0$; $W(y_1, y_2) = 0$ | calculated at $x = 0$

$|x|$ does not differentiable

5(b) ⇒ Prove that $\frac{1}{(x+y+1)^4}$ is an integrating factor of $(2xy-y^2-y)\,dx + (2xy-x^2-x)\,dy = 0$ and find the solution of this equation.

Sol'n: Given that $(2xy-y^2-y)dx + (2xy-x^2-x)dy = 0$ —— (1)

Multiplying the equation (1) by $\frac{1}{(x+y+1)^4}$

∴ from (1):

$$\frac{2xy-y^2-y}{(x+y+1)^4}\,dx + \frac{2xy-x^2-x}{(x+y+1)^4}\,dy = 0 \quad \text{—— (2)}$$

equation (2) is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$.

where $M = \frac{2xy-y^2-y}{(x+y+1)^4}$ & $N = \frac{2xy-x^2-x}{(x+y+1)^4}$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2x-2y-1}{(x+y+1)^4} - \frac{4(2xy-y^2-y)}{(x+y+1)^5} - \frac{2y-2x-1}{(x+y+1)^4} + \frac{4(2xy-x^2-x)}{(x+y+1)^5}$$

$$= \frac{4(x-y)}{(x+y+1)^4} - \frac{4(y^2+y-x^2-x)}{(x+y+1)^5}$$

$$= \frac{4(x-y)(x+y+1) - 4(y^2+y-x^2-x)}{(x+y+1)^5}$$

$$= 0$$

i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. ∴ Equation (1) is exact.

∴ $\frac{1}{(x+y+1)^4}$ is Integrating factor of the given equation

∴ The solution of (1) is

$$\int_y M\,dx + \int N\,dy = C$$

$$\int \frac{2xy-y^2-y}{(x+y+1)^4}\,dx = C$$

$$\Rightarrow 2y\int \frac{x+(y+1)-(y+1)}{(x+y+1)^4}\,dx - \int \frac{y(y+1)}{(x+y+1)^4}\,dx = C$$

$$2y\int\frac{1}{(x+y+1)^3}\,dx-\int\frac{3y(y+1)}{(x+y+1)^4}\,dx=c$$

$$\Rightarrow 2y\left[\frac{-1}{2(x+y+1)^2}\right]-\frac{3y(y+1)}{-3(x+y+1)^3}=c$$

$$\Rightarrow \frac{-y}{(x+y+1)^2}+\frac{y(y+1)}{(x+y+1)^3}=c$$

$$\Rightarrow -y(x+y+1)+y(y+1)=c(x+y+1)^3$$

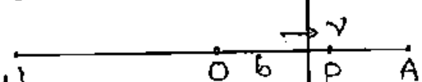
$$\Rightarrow \boxed{xy+c(x+y+1)^3=0}$$

which is the required solution.

5(d). A particle is performing a S.H.M. of period T about a centre O and it passes through a point P where OP = b with velocity v in the direction OP; P.T. that the time which elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$

Sol^n: Let the equation of the S.H.M with centre O as origin be $d^2x/dt^2=-\mu x$

The time period $T=2\pi/\sqrt{\mu}$



Let the amplitude be a. Then A'

$$\left(\frac{dx}{dt}\right)^2=\mu(a^2-x^2) \quad \text{---} (1).$$

when the particle passes through P its velocity is given to be v in the direction OP. Also OP = b.

So putting $x=b$ and $dx/dt=v$ in (1) we get

$$v^2=\mu(a^2-b^2) \quad \text{———} (2).$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P. In S.H.M the time from P to A is equal to the time from A to P.

$\therefore$ the required time $= 2$, time from A to P.

Now for the motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2-x^2} \Rightarrow dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}$$

Let t_1 be the time from A to P. Then at A, $t=0$, $x=a$ and at P, $t=t_1$ and $x=b$. Therefore integrating (3), we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{a^2-x^2}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$= \frac{1}{\sqrt{\mu}}\left[\cos^{-1} b/a - \cos^{-1} 1\right] = \frac{1}{\sqrt{\mu}}\cos^{-1} b/a.$$

Hence the required time $= 2t_1 = \frac{2}{\sqrt{\mu}}\cos^{-1} b/a$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{a^2-b^2}}{b}\right\}$$

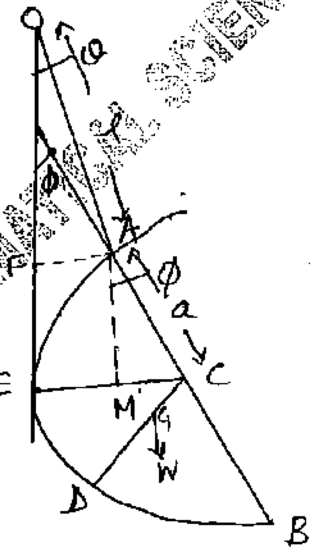
$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right) \qquad \left[\because \text{from (2)}, \sqrt{a^2-b^2} = \frac{v}{\sqrt{\mu}}\right]$$

$$= \frac{2}{2\pi/T}\tan^{-1}\left\{\frac{v}{b(2\pi/T)}\right\} \qquad \left[\because T = \frac{2\pi}{\sqrt{\mu}} \text{ so that } \sqrt{\mu} = 2\pi/T\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

Q.5(c) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to vertical, prove that:

Sol^n O is a fixed point in the wall to which one end of the string has been attached. Let l be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C. The weight W of the hemisphere acts at its centre of gravity G which lies on symmetrical radius CD and is such that



$$CG = \frac{3}{8}a$$

The hemisphere touches the wall at E. We have $\angle OEC = 90$ so that EC is horizontal. The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall

The depth of G below $O = OF + AM + NG$

$$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$$

[Note that $\angle NCG = 90^\circ - \angle ACM = 90^\circ - (90^\circ - \phi) = \phi$]

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point O remains fixed. The length of the string AO doesn't change so that the work done by its tension is zero and point G is slightly displaced. The $\angle OEC$ remains 90°. The only force that contributes to the equ^n of virtual work is the weight W of hemisphere acting at G whose depth below the fixed point O has been found above. The equation of virtual work is

$$W\delta(l\cos\theta + a\cos\phi + \tfrac{3}{8}a\sin\phi) = 0$$

or $-l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$

$$l\sin\theta\,\delta\theta = a\left(\frac{3}{8}\cos\phi - \sin\phi\right)\delta\phi \quad \text{--- (1)}$$

from the figure $EC = a$

Also $EC = EM + MC = FA + MC = l\sin\theta + a\sin\phi$

$\therefore\ a = l\sin\theta + a\sin\phi$

Differentiating $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$

or $-l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi$ --- (2)

Dividing (1) by (2), we get

$-\tan\theta = 3/8 - \tan\phi$ or $\tan\phi = 3/8 + \tan\theta$

5(e) If $f = \nabla(\vec{a}\cdot\nabla r^{-1})$, show that $\operatorname{div} f = 0$ and $f = \operatorname{curl} g$, where $g = -\vec{a} \times \nabla(r^{-1})$

Sol^n: ~~we~~ have $\nabla r^{-1} = \nabla\left(\frac{1}{r}\right)$

$$= -\frac{1}{r^2}\nabla r$$

$$= -\frac{1}{r^2}\cdot\frac{1}{r}\vec{r} = -\frac{1}{r^3}\vec{r}$$

$$\Rightarrow \vec{a}\cdot(\nabla r^{-1}) = \vec{a}\cdot\left(-\frac{1}{r^3}\vec{r}\right) = \frac{-\vec{a}\cdot\vec{r}}{r^3}$$

Now $f = \nabla(\vec{a}\cdot\nabla r^{-1})$

$$= \nabla\left(\frac{-\vec{a}\cdot\vec{r}}{r^3}\right)$$

$$= \sum i\frac{\partial}{\partial x}\left(\frac{-\vec{a}\cdot\vec{r}}{r^3}\right)$$

$$= \sum i\left\{-\frac{1}{r^3}\frac{\partial}{\partial x}(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$

$$= \sum i\left\{-\frac{1}{r^3}\left(\vec{a}\cdot\frac{\partial\vec{r}}{\partial x}\right) + 3(\vec{a}\cdot\vec{r})\, r^{-4}\frac{\partial r}{\partial x}\right.$$

$$= \sum i\left\{\frac{-\vec{a}\cdot\vec{i}}{r^3} + \frac{3x}{r^5}(\vec{a}\cdot\vec{r})\right\}$$

$$= \sum\left\{-\frac{1}{r^3}(\vec{a}\cdot\vec{i})\vec{i} + \frac{3}{r^5}(\vec{a}\cdot\vec{r})x\vec{i}\right\} \quad \left(\because \frac{\partial \vec{r}}{\partial x} = i \;\&\; \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$= -\frac{1}{r^3}\vec{a} + \frac{3}{r^5}(\vec{a}\cdot\vec{r})\vec{r} \quad \text{(1)} \quad \left(\because \sum(\vec{a}\cdot\vec{i})\vec{i} = \vec{a} \;\&\; \sum x\vec{i} = \vec{r}\right)$$

By using above we can easily prove that $\operatorname{div} f = 0$ and $f = \operatorname{curl} g$.

To show that $f = \operatorname{Curl} g$:

$$\operatorname{curl} g = \nabla\times g = \nabla\times(-\vec{a}\times\nabla r^{-1})$$

$$= -\nabla\times\left(\vec{a}\times\frac{1}{r^3}\vec{r}\right)$$

$= \nabla \times \left[\frac{1}{r^3}(\vec{a} \times \vec{r})\right]$ (It is in the form of $\nabla \times (\phi \vec{A})$ $= (\text{curl}\,\phi) \times A + \phi\,\text{curl}\,A$; $\phi = \frac{1}{r^3}$ & $A = \vec{a} \times \vec{r}$)

$= \nabla\left(\frac{1}{r^3}\right) \times (\vec{a} \times \vec{r}) + \frac{1}{r^3}\, \nabla \times (\vec{a} \times \vec{r})$

$= -\frac{3}{r^4}\left(\frac{\vec{r}}{r}\right) \times (\vec{a} \times \vec{r}) + \frac{1}{r^3}\left[\vec{a}(\nabla \cdot \vec{r}) - \vec{r}(\nabla \cdot \vec{a}) + (\vec{r} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{r}\right]$

$= -\frac{3}{r^5}\vec{r} \times (\vec{a} \times \vec{r}) + \frac{1}{r^3}(3\vec{a} - 0 + 0 - \vec{a})$

$= -\frac{3}{r^5}\left[(\vec{r} \cdot \vec{r})\vec{a} - (\vec{r} \cdot \vec{a})\vec{r}\right] + \frac{1}{r^3}(2\vec{a})$ ($\because \nabla \cdot \vec{a} = 0$, $(\vec{r} \cdot \nabla)\vec{a} = 0$ & $(\vec{a} \cdot \nabla)\vec{r} = a$)

$= -\frac{3}{r^5}(r^2 \vec{a}) + \frac{3}{r^5}(\vec{r} \cdot \vec{a})\vec{r} + \frac{2\vec{a}}{r^3}$

$= -\frac{3\vec{a}}{r^3} + \frac{2\vec{a}}{r^3} + \frac{3}{r^5}(\vec{r} \cdot \vec{a})\vec{r}$

$= -\frac{\vec{a}}{r^3} + \frac{3}{r^5}(\vec{r} \cdot \vec{a})\vec{r} = f$ (by ①)

$\therefore f = \text{curl}\, g$

6(a) Solve $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$

Sol: Given $(x^3D^3+2x^2D^2+2)y=10\left(x+\frac{1}{x}\right)$ —(1), $D\equiv\frac{d}{dx}$

Let $x=e^z \Rightarrow z=\log x$ and $D_1=\frac{d}{dz}$ —(2)

Then $xD=D_1$, $x^2D^2=D_1(D_1-1)$; $x^3D^3=D_1(D_1-1)(D_1-2)$ —(3)

Using (2) and (3), (1) reduces to

$$[D_1(D_1-1)(D_1-2)+2D_1(D_1-1)+2]y=10(e^z+e^{-z})$$

$$\Rightarrow (D_1^3-D_1^2+2)y=10e^z+10e^{-z} \quad —(4)$$

A.E. of (4) is $D_1^3-D_1^2+2=0$

$$\Rightarrow (D_1+1)(D_1^2-2D_1+2)=0$$

$$\Rightarrow D_1=-1,\ 1\pm i$$

$$\therefore C.F = c_1e^{-z}+e^{z}(c_2\cos z+c_3\sin z)$$

$$=c_1x^{-1}+x\left(c_2\cos(\log x)+c_3\sin(\log x)\right)$$

P.I. corresponding to $10e^z = 10\frac{1}{(D_1+1)(D_1^2-2D_1+2)}e^z$

$$=10\frac{1}{2(1-2+2)}e^z$$

$$=5x.$$

and P.I. corresponding to $10e^{-z}=10\frac{1}{(D_1+1)(D_1^2-2D_1+2)}e^{-z}$

$$=10\frac{1}{(D_1+1)}\frac{1}{1+2+2}e^{-z}$$

$$=2\frac{1}{D_1+1}e^{-z}\cdot 1$$

$$=2e^{-z}\frac{1}{D_1-1+1}(1)$$

$$=2e^{-z}\frac{1}{D_1}(1)=2ze^{-z}=2x^{-1}\log x$$

$$\therefore y=c_1x^{-1}+x[c_2\cos(\log x)+c_3\sin(\log x)]+5x+2x^{-1}\log x.$$

6(b) Find the solution of the differential equation $y = 2xp - yp^2$ where $p = \frac{dy}{dx}$. Also find the singular solution.

Sol:- Given $y = 2xp - yp^2$ —— (1)

Solving (1) for x,

(1) ⇒ $x = y/2p + yp/2$ —— (2)

Diff (2) w.r.t y and put $\frac{dx}{dy} = 1/p$, we get

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} + \frac{p}{2} + \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \frac{y}{2}\frac{dp}{dy}\left(1 - \frac{1}{p^2}\right) + \frac{p}{2}\left(1 - \frac{1}{p^2}\right) = 0$$

$$\Rightarrow \frac{1}{2}\left(1 - \frac{1}{p^2}\right)\left(y\frac{dp}{dy} + p\right) = 0$$

Omitting the first factor, for general solution we have

$$y\frac{dp}{dy} + p = 0 \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$$

$$\Rightarrow \log p + \log y = \log c$$

$$\Rightarrow py = c$$

$$\Rightarrow p = c/y \quad \text{—— (3)}$$

Eliminating p from (1) and (3), the general solution is $y = 2xc/y - \frac{yc^2}{y^2}$.

$$\Rightarrow y^2 = 2cx - c^2 \quad \text{—— (4)}$$

The p-disc. relation from (1) is $4x^2 - 4y^2 = 0$

$\Rightarrow x^2 - y^2 = 0$

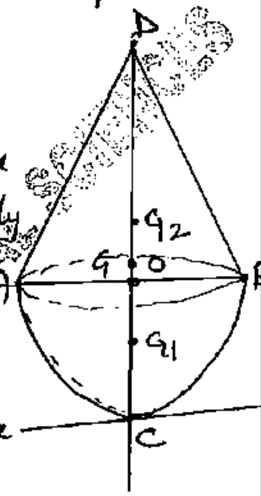
i.e. $(x-y)(x+y) = 0$.

The c-disc. relation from (4) is $4x^2 - 4y^2 = 0 \Rightarrow x^2 - y^2 = 0$
$\Rightarrow (x+y)(x-y) = 0$

Hence $\boxed{x - y = 0 \text{ and } x + y = 0}$ are singular solutions because these appear once in both the discriminants and clearly they satisfies the given diff equation.

Q.6C A body, consisting of cone and a hemisphere on the same base, rest on rough horizontal table the hemisphere being in contact with the table, show that the greatest height of the the cone so that the equilibrium may be stable is $\sqrt{3}$ times the radius of the hemisphere

Ans.6C



$OG_1 = 3r/8$, $OG_2 = H/4$

If h be the height of the centre of gravity (G) of the combined body composed of the hemisphere and cone above the contact point C then using the formula

$$x = \frac{w_1x_1 + w_2x_2}{w_1+w_2}$$ we have

$$h = \frac{\frac{1}{3}\pi r^2 H . CG_2 + \frac{2}{3}\pi r^3 . CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{\frac{1}{3}\pi r^2 H\left(r + \frac{H}{4}\right) + \frac{2}{3}\pi r^3 . \frac{5}{8}r}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{H\left(r + \frac{1}{4}H\right) + 5/4\, r^2}{H + 2r}$$

here ρ_1 = The radius of curvature at the point of contact C of the upper body which is spherical $= r$

ρ_2 = the radius of curvature of the lower body at the point of contact $= \infty$

$\therefore$ The equilibrium will be stable if

$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$ i.e, $\frac{1}{h} > \frac{1}{r} + \frac{1}{\infty}$ i.e, $\frac{1}{h} > \frac{1}{r}$

i.e $h < r$

i.e, $\dfrac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r} < r$ i.e

$Hr + \frac{1}{4}H^2 + \frac{5}{4}r^2 < Hr + 2r^2$

i.e $\frac{1}{4}H^2 < \frac{3}{4}r^2$ i.e $H^2 < 3r^2$

i.e $H < r\sqrt{3}$

hence the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere

6(d) In what direction from the point (1,3,2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum.

Sol: We know that the directional derivative of $\phi = 2xz - y^2$ at the point (x, y, z) is maximum in the direction of the normal to the surface ϕ = constant

i.e. in the direction of the vector grad ϕ.

Now $\text{grad}\,\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

$= 2z\,i + (-2y) j + 2x\,k$

$\text{grad}\,\phi \Big|_{(1,3,2)} = 2(2) i - 2(3) j + 2(1) k$

$= 4i - 6j + 2k$

Hence the directional derivative of ϕ at the point $(1,3,2)$ will be maximum in the direction of the vector $4i - 6j + 2k$.

Also the magnitude of this maximum directional derivative

$=$ modulus of grad ϕ at $(1,3,2)$

$= |4i - 6j + 2k|$

$= \sqrt{16+36+4}$

$= \sqrt{56}$

$= 2\sqrt{14}$

7(a). Solve $y_2 - 2y_1 + y = xe^x \log x$, $x > 0$ by the method of variation of parameters.

Soln: Given that

$y_2 - 2y_1 + y = xe^x \log x$, $x > 0$

This can be written as

$$(D^2 - 2D + 1)y = xe^x \log x \quad \text{--- (1)}$$

Consider $(D^2 - 2D + 1)y = 0$

$$\Rightarrow (D-1)^2 y = 0 \quad \text{--- (2)}$$

The auxiliary equation of (2) is

$$(D-1)^2 = 0 \Rightarrow D = 1, 1.$$

$\therefore$ Complementary function of (1) is

$$= (C_1 + C_2 x)e^x.$$

$$= C_1 e^x + C_2 xe^x \quad \text{--- (3)}$$

Now let $u = e^x$, $v = xe^x$ & $R = xe^x \log x$.

Let $y_p = Au + Bv$ be a particular integral of (1)

where A and B are functions of x.

and $u = e^x$, $v = xe^x$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = \begin{vmatrix} e^x & e^x \\ xe^x & (1+x)e^x \end{vmatrix}$

$$= (1+x)e^{2x} - xe^{2x}$$

$$= e^{2x} \neq 0.$$

$\therefore A = \int \frac{-vR}{\ldots} dx = -\int \frac{xe^x \cdot xe^x \log x}{2x} dx$

$$= -\int x^2 \log x \, dx$$

$$= -\left[\log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx\right]$$

$$= -\left[\frac{x^3}{3}\log x - \frac{x^3}{9}\right]$$

and $B = \int \frac{uR}{uv' - u'v} dx = \int \frac{e^x \cdot xe^x \log x}{e^{2x}} dx$

$$= \int x \log x \, dx$$

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}.$$

$$\therefore y_p = -\left[\frac{x^3}{3}\log x - \frac{x^3}{9}\right]e^x + \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]xe^x$$

$$= \left(-\frac{1}{3} + \frac{1}{2}\right)x^3 \log x \cdot e^x + x^3 e^x \left[\frac{1}{9} - \frac{1}{4}\right]$$

$$= \frac{x^3}{6}\log x \cdot e^x + x^3 e^x \left(-\frac{5}{36}\right)$$

$$= \frac{1}{6}x^3 e^x \log x - \frac{5}{36}x^3 e^x.$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 x e^x + \frac{1}{6}x^3 e^x \log x - \frac{5}{36}x^3 e^x$$

7(b) A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{a/g}\log(\sqrt{5}+\sqrt{6})$

Sol'n: Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.

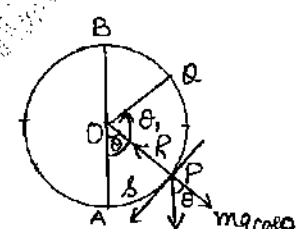
If P is the position of the particle at any time t.

Such that $\angle AOP=\theta$ & arc $AP=s$, then the equations of motion of the particle along the tangent & normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{—— (1)}$$

and $$m\frac{v^2}{a} = R-mg\cos\theta \quad \text{—— (2)}$$

Also $$s=a\theta \quad \text{—— (3)}$$



from (1) & (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying bothsides by $2a\left(d\theta/dt\right)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

But according to the question $v=0$ at the highest point B, where $\theta=\pi$.

$$\therefore\ 0 = 2ag\cos\pi + A \Rightarrow A = 2ag$$

$$\therefore\ v^2=\left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta+2ag \quad \text{—— (4)}$$

from (2) and (4), we have

$$R = \frac{m}{a}\left(v^2+ag\cos\theta\right) = \frac{m}{a}\left(2ag+3ag\cos\theta\right) \quad \text{—— (5)}$$

If the reaction $R=0$ at the point Q where $\theta=\theta_1$, then from (5), we have

$$0 = \frac{m}{a}\left(2ag+3ag\cos\theta_1\right)$$

$$\Rightarrow \cos\theta_1 = -2/3 \quad \text{——} \quad (6)$$

from (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag(\cos\theta+1) = 2ag.2\cos^2\tfrac{1}{2}\theta = 2ag\cos^2\tfrac{1}{2}\theta$$

$\therefore \frac{d\theta}{dt} = 2\sqrt{g/a}\cos\frac{1}{2}\theta$, the +ve sign being taken before the radical sign because θ increases as t increases.

$$\Rightarrow dt = \tfrac{1}{2}\sqrt{a/g}\sec\tfrac{1}{2}\theta\, d\theta$$

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t = \tfrac{1}{2}\sqrt{a/g}\int_{\theta=0}^{\theta_1}\sec\tfrac{1}{2}\theta\, d\theta$$

$$= \sqrt{a/g}\left[\log\left(\sec\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\theta\right)\right]_0^{\theta_1}$$

$$= \sqrt{a/g}\log\left(\sec\tfrac{1}{2}\theta_1+\tan\tfrac{1}{2}\theta_1\right) \quad \text{——} \quad (7)$$

from (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1 - 1 = -\tfrac{2}{3}$$

$$2\cos^2\tfrac{1}{2}\theta_1 = \tfrac{1}{3}$$

$$\cos^2\tfrac{1}{2}\theta_1 = \tfrac{1}{6} \Rightarrow \sec^2\tfrac{1}{2}\theta_1 = 6$$

$$\therefore \sec\tfrac{1}{2}\theta_1 = \sqrt{6}$$

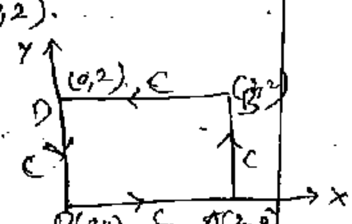
$$\&\ \tan\tfrac{1}{2}\theta_1 = \sqrt{(\sec^2\tfrac{1}{2}\theta_1 - 1)} = \sqrt{(6-1)} = \sqrt{5}$$

Substituting in (7), the required time is given by

$$t = \sqrt{a/g}\log\left(\sqrt{5}+\sqrt{6}\right)$$

7(c) Verify Stokes theorem for $A=(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}$, where S is the surface of the cube $x=0, y=0, z=0$, $x=2, y=2, z=2$ above the xy-plane.

Sol: The xy-plane cuts the surface of the cube in a square. Thus the curve C bounding the surface S is the square, say OABD, in the xy-plane whose vertices in the xy-plane are points $O(0,0)$, $A(2,0)$, $B(2,2)$, $D(0,2)$.



By Stokes theorem, we have

$$\iint_S (\nabla\times A)\cdot \hat{n}\, dS = \oint_C A\cdot dr$$

$$\oint_C A\cdot dr = \int_C \left[(y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}\right]\cdot\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)$$

$$= \int (y-z+2)dx + (yz+4)dy - xz\,dz$$

$$= \int (y+2)dx + 4dy \qquad (\because \text{on } C,\ z=0 \text{ and } dz=0)$$

$$= \int_{OA} (y+2)dx+4dy + \int_{AB} (y+2)dx+4dy + \int_{BD} (y+2)dx+4dy + \int_{DO} (y+2)dx+4dy$$

$$= \int_0^2 2dx + \int_0^2 4dy + \int_2^0 4dx + \int_2^0 4dy$$

$\left(\begin{array}{l}\because \text{on } OA,\ y=0,\ dy=0\ \&\ x \text{ varies from } 0 \text{ to } 2\\ \text{on } AB,\ x=2,\ dx=0\ \&\ y \text{ varies from } 0 \text{ to } 2\\ \text{on } BD,\ y=2,\ dy=0\ \&\ x \text{ varies from } 2 \text{ to } 0\\ \text{on } DO,\ x=0,\ dx=0\ \&\ y \text{ varies from } 2 \text{ to } 0\end{array}\right)$

$= 2[x]_0^2 + 4[y]_0^2 + 4[x]_2^0 + 4[y]_2^0$

$= 4+8-8-8 = -4.$

Now curl $A = \nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$= i(0-y) + j(-1+z) + k(0-1)$

$= -y\hat{i} + (z-1)\hat{j} - \hat{k}$

$\iint (\nabla \times A)\cdot \hat{n}\, dS = \iint (-y\hat{i} + (z-1)\hat{j} - \hat{k})\cdot \hat{n}\, ds$

please try yourself in this way.

8(a). By using Laplace transform method solve, the differential equation $(D^2-D-2)y = 20\sin 2t$, subject to initial conditions $y=-1, Dy=2$ when $t=0$.

Sol^n: Given that $(y''-y'-2y) = 20\sin 2t$ —(1)

Subject to initial conditions $y(0)=-1$ & $y'(0)=2$ —(2)

Taking Laplace transform on both sides of (1), we get

$L(y'') - L(y') - 2L(y) = 20L(\sin 2t)$.

$s^2 L\{y\} - s y(0) - y'(0) - s[L\{y\} - y(0)] - 2L\{y\} = \frac{40}{s^2+4}$

$\Rightarrow (s^2-s-2)L\{y\} + s-2-1 = \frac{40}{s^2+4}$ (using (2))

$\Rightarrow L\{y\} = \frac{3-s}{(s-2)(s+1)} + \frac{40}{(s-2)(s+1)(s^2+4)}$ —(3)

resolving into partial fractions, we get

$\frac{3-s}{(s-2)(s+1)} = \frac{1}{3(s-2)} - \frac{4}{3(s+1)}$ —(4)

and $\frac{40}{(s-2)(s+1)(s^2+4)} = \frac{5}{3(s-2)} - \frac{8}{3(s+1)} + \frac{s-6}{s^2+4}$ —(5)

using (4) & (5), (3) reduces to $\frac{s-1}{s^2+4} = \frac{2}{s-2} - \frac{4}{s+1} + \frac{s}{s^2+4} - \frac{6}{s^2+4}$

$L\{y\} = \frac{1}{3(s-2)} - \frac{4}{3(s+1)} + \frac{5}{3(s-2)} - \frac{8}{3(s+1)} + \frac{s-1}{s^2+4}$ —(6)

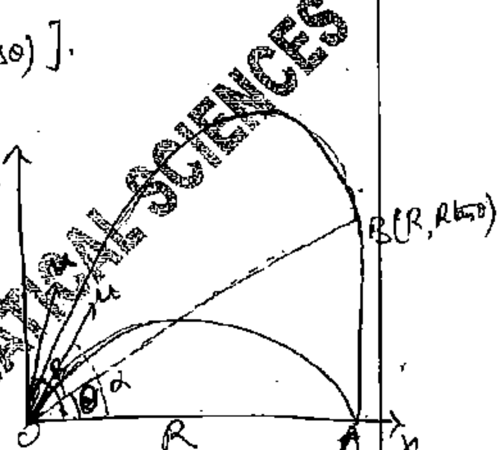
$y = L^{-1}\left\{\frac{2}{s-2}\right\} - L^{-1}\left\{\frac{4}{s+1}\right\} + L^{-1}\left\{\frac{s}{s^2+4}\right\} - L^{-1}\left\{\frac{6}{s^2+4}\right\} = 2e^{2t} - 4e^{-t} + \cos 2t - 3\sin 2t$.

8(b) A shot fired at an elevation is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is

$$\frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha\cos\theta)\right].$$

Sol: Let AB be the tower and O the point of projection. It is given that $\angle AOB = \theta$.

Let u be the velocity of projection of the shot. When the shot is fired at an elevation α from O, it strikes the foot A of the tower AB.



Let $OA = R$.

Then $R = \frac{u^2\sin 2\alpha}{g}$

Referred to the horizontal and vertical lines OX and OY lying in the plane of motion as the co-ordinate axes, the co-ordinates of the top B of the tower are $(R, R\tan\theta)$

If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is

$$y = x\tan\beta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\beta}$$

$$\therefore R\tan\theta = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$\tan\theta = \tan\beta - \frac{1}{2}g\frac{R}{u^2\cos^2\beta} \qquad (\because R \neq 0)$$

Substituting the value of R from ①, we get

$$\tan\theta = \tan\beta - \frac{1}{2}g\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

$$\tan\theta = \tan\beta - \frac{\sin 2\alpha}{2\cos^2\beta}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\beta}{\cos\beta} - \frac{\sin 2\alpha}{2\cos^2\beta}$$

Multiplying both sides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta = 2\sin\beta\cos\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow (1+\cos 2\beta)\sin\theta = \sin 2\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow \sin 2\beta\cos\theta - \cos 2\beta\sin\theta = \sin\theta + \cos\theta\sin 2\alpha$$

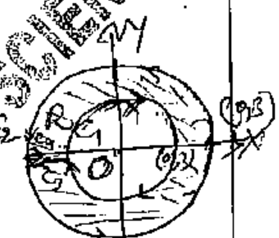
$$\Rightarrow \sin(2\beta - \theta) = \sin\theta + \cos\theta\sin 2\alpha$$

$$\Rightarrow 2\beta - \theta = \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow 2\beta = \theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow \beta = \frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)\right].$$

8(c) Verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.



Sol: By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

The boundary of the curve C is given by $C = C_1 + C_3 + C_2 + C_4$ and R is the region bounded by the circles $x^2+y^2=1$, and $x^2+y^2=3$

$\therefore \int M\,dx + N\,dy = \int_{C_1+C_2+C_3+C_4} M\,dx + N\,dy$.

Here note that along C_3 & C_4

i.e. $\int_{C_3+C_4} M\,dx + N\,dy = 0$

$\because$ C_3 & C_4 are in opposite directions

Along C_3 & C_4: $y=0 \Rightarrow dy=0$

$\therefore \int_{C_3} 2x\,dx + \int_{C_4} 2x\,dx$

$= \int_1^3 2x\,dx + \int_3^1 2x\,dx$

$= \int_1^3 2x\,dx - \int_1^3 2x\,dx$

$= 0$

Now

LHS $= \int_C M\,dx + N\,dy = \int_{C_1+C_2} M\,dx + N\,dy$

$= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy \quad \text{---(1)}$

Let $\int_{C_1} M\,dx + N\,dy = \int_{C_2} (2x-y^3)\,dx - xy\,dy$

putting $x = 3\cos\theta$, $y = 3\sin\theta$.
$dx = -3\sin\theta$, $dy = 3\cos\theta$.

$$\int_{C_1} Mdx + Ndy = \int_0^{2\pi} (6\cos\theta - 27\sin^3\theta)(-3\sin\theta)\,d\theta - 27\cos^3\theta\sin\theta\,d\theta$$

$$= \int_0^{2\pi} \left(-18\cos\theta\sin\theta + 81\sin^4\theta - 27\sin\theta\cos^3\theta\right) d\theta$$

$$= 0 + 81\cdot(4)\int_0^{\pi/2} \sin^4\theta\,d\theta + 27\left.\frac{\cos^4\theta}{3}\right]_0^{2\pi}$$

$$= 81\cdot 4\cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 0$$

$$= \frac{243\pi}{4}$$

$$\int_{C_2} Mdx + Ndy = \int_{2\pi}^{0} (2\cos\theta - \sin^3\theta)(-\sin\theta\,d\theta) - \cos^3\theta\sin\theta\,d\theta$$

by putting $x = \cos\theta$, $y = \sin\theta$, $\theta : 2\pi \to 0$.

$$= 0 + \int_{2\pi}^{0} \sin^4\theta\,d\theta - 0$$

$$= -4\int_0^{\pi/2} \sin^4\theta\,d\theta = -4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = -3\pi/4$$

$\therefore$ from ①

$$\int_C Mdx + Ndy = -\frac{3\pi}{4} + \frac{243\pi}{4} = \frac{240\pi}{4} = 60\pi$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (3y^2 - y)\,dx\,dy$$

using the polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{3} (3r^2\sin^2\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{3r^4}{4}\sin^2\theta - \frac{r^3}{3}\sin\theta\right)_1^3 d\theta = \int_0^{2\pi} \left(60\sin^2\theta - \frac{26}{3}\sin\theta\right) d\theta$$

$$= 60\left[\frac{\theta - \frac{\sin 2\theta}{2}}{2}\right]_0^{2\pi}$$

$$= 60\cdot\frac{2\pi}{2} = 60\pi.$$

$\therefore$ Green's theorem is verified

$\int_0^{2\pi} \sin\theta\,d\theta = 0$ and $\sin^2\theta = \left(\frac{1-\cos 2\theta}{2}\right)$, $\left.\frac{\theta}{2} - \frac{\sin 2\theta}{2\cdot 2}\right]_0^{2\pi}$

8(d) Find the workdone in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field F is given by
$F = (2x-y+2z)\hat{i} + (x+y-z)\hat{j} + (3x-2y-5z)k.$

Sol: In the xy-plane, we have $z=0$.

$\therefore F = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)K$

The given circle C is given by $x^2+y^2=4$

(or) $x = 2\cos t$, $y = 2\sin t$.

$\therefore \vec{r} = x\hat{i} + y\hat{j} = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$.

$\Rightarrow \frac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$.

Also $F = (4\cos t - 2\sin t)\hat{i} + (2\cos t + 2\sin t)\hat{j} + (6\cos t - 4\sin t)k$

In moving round the circle once t will vary from 0 to 2π.

The required work done is $= \int_C F\cdot d\vec{r}$

$$= \int_0^{2\pi} F\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^{2\pi} [-2\sin t\,(4\cos t - 2\sin t) + 2\cos t\,(2\cos t + 2\sin t)]\,dt$$

$$= \int_0^{2\pi} [4(\sin^2 t + \cos^2 t) - 4\sin t\cos t]\,dt$$

$$= \int_0^{2\pi} (4 - 4\sin t\cos t)\,dt$$

$$= \left[4t - 2\sin 2t\right]_0^{2\pi}$$

$$= 8\pi.$$

[redacted]

TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/21-8-16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

LINEAR ALGEBRA, CALCULUS AND 3D

Test- 01

Time: Three Hours | ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) Find the rank of the matrix **[10]**

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

1. (b) If
$\alpha_1 = (1, -1), \beta_1 = (1, 0)$
$\alpha_2 = (2, -1), \beta_2 = (0, 1)$
$\alpha_3 = (-3, 2), \beta_3 = (1, 1)$
is there a linear transformation T from R^2 into R^2 such that $T\alpha_i = \beta_i$ for $i = 1, 2$ and 3? **[10]**

1. (c) Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere. **[10]**

1. (d) In the closed interval (–1, 1), let f(x) defined as $x^2 \sin \frac{1}{x^2}$ for $x \neq 0$, and $f(0) = 0$. In the given interval,
(i) Is the function bounded ?
(ii) Is it continuous ?
(iii) Is it uniformly continuous ?
(iv) Is it absolutely continuous ? **[10]**

1. (e) Show that the two circles
$x^2 + y^2 + z^2 - y + 2z = 0,\ x - y + z - 2 = 0;$
$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0,\ 2x - y + 4z - 1 = 0;$
lie on the same sphere and find its equation. **[10]**

2. (a) Let V be a vector space and T a linear transformation from V into V. Prove that the following two statements about T are equivalent.
(i) The intersection of the range of T and the null space of T is the zero subspace of V.
(ii) If $T(T\alpha) = 0$, then $T\alpha = 0$. **[10]**

2. (b) Show that the minimal polynomial for the matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \text{ is } (x-1)(x-2)$$ **[10]**

2. (c) Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2y$ **[15]**

4. Generally we make lots of silly mistakes while solving a question. It is best to catch these errors early and not repeat them in exam hall. The best strategy for this is to maintain a notebook of errors that you usually commit and their mitigation measures. For example, I commit a lot of mistakes when doing Integration by parts and usually the error involves missing negative (-) sign etc. Therefore whenever I come across such type of question I try to devote extra 1 minute to re-check all my steps.

5. maths.stackexchange.com is the best online resource for preparation. You can create an account and get your maths questions answered within minutes.

Why did I score only 262?

Among all the students in the final list who had Maths as an optional, I have scored the least. My paper - 1 was a complete disaster and I only scored 92 marks in it. In fact I could only attempt 160 marks paper and had to leave 90 marks paper completely.

The reasons for the above situation in Paper - 1 are as follows:

1. **Lack of written practice:** In many topics (especially statics and dynamics) I used to just look at a question and its solution without solving it first. As a result I forgot the exact method in the exam hall!

2. **Left many topics:** I prepared only 25% 3-D, 80% Calculus and 25% Statics & Dynamics and had to pay a heavy price in the exam.

On the other hand my preparation for paper - 2 was excellent and therefore I scored an amazing 170 marks in it

Bhavesh Mishra

AIR-58 in CSE-2014

2. (d) Prove that the equation to the two planes inclined at an angle α to xy-plane and containing the line $y = 0$, $z \cos\beta = x \sin\beta$ is $(x^2 + y^2)\tan^2\beta + z^2 - 2zx\tan\beta = y^2\tan^2\alpha$. **[15]**

3. (a) Let T be the linear trnsformation from R^3 into R^2 defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

(i) If β is the standard ordered basis for R^3 and β' is the standard ordered basis for R^2, what is the matrix of T relative to the pair β, β' ?

(ii) If $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\beta' = \beta_1, \beta_2$ where $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 (1, 0, 0), \beta_1 = (0, 1), \beta_2 (1, 0)$ what is the matrix of T relative to the pair β, β'? **[17]**

3. (b) (i) Evaluate $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx$.

(ii) If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

so that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ and $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ **[17]**

3. (c) Prove that the area of the section of the cone $bcx^2 + cay^2 + abz^2 = 0$ by the plane $lx + my + nz = p$ is $\pi p^2\sqrt{(abc)}/(al^2 + bm^2 + cn^2)^{3/2}$ **[16]**

4. (a) Let V be the vector space over the complex numbers of all functions from R into C, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(i) Prove that f_1, f_2 and f_3 are linearly independent.

(ii) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that $g_i = \sum_{i=1}^{3} P_{ij}f_i$ **[16]**

4. (b) (i) Prove or disprove : $f_{xy}(0, 0) = f_{yx}(0, 0)$, where

$$f(x,y) = \frac{(x^2y + xy^2)\sin(x-y)}{x^2 + y^2}, (x,y) \neq (0,0)$$

$f(0, 0) = 0$.

(ii) Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ has a minima at the orgin. **[18]**

4. (c) Reduce the equation $2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$ to the standard form. What does it represent ? **[16]**

SECTION-B

5. (a) What is the smallest subspace of 3 by 3 matrices that contains all symmetric matericer and all lower triangular matrices ?What is the largest subspace that is contained in both of those subspaces ? **[10]**

5. (b) Is there a linear transformation T from R^3 into R^2 such that $T(1,-1,1)=(1,0)$ and $T(1,1,1)=(0,1)$? **[10]**

5. (c) Evaluate $\lim_{x\to+\infty}\dfrac{a^{1/x}-b^{1/x}}{\log\{x/(x-1)\}}$ **[10]**

5. (d) How far is the point (4, 1, 1) from the line of intersection of $x+y+z-4=0=x-2y-z-4$? **[10]**

5. (e) Find the equation of the sphere whose centre is the point(1, 2, 3) and which touches the plane $3x+2y+z+4=0$. Find also theradius of the circle in which the sphere is cut by the plane $x+y+z=0$. **[10]**

6. (a) Let V be the real vector space spanned by the rows of the matrix **[15]**

$$A=\begin{bmatrix}3 & 21 & 0 & 9 & 0\\ 1 & 7 & -1 & -2 & -1\\ 2 & 14 & 0 & 6 & 1\\ 6 & 42 & -1 & 13 & 0\end{bmatrix}$$

(i) Find a basis for V.

(ii) Tell which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.

(iii) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its condinates in the basis chosen in part (a) ?

6. (b) Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$T(x_1, x_2, x_3)=(x_1-x_2+2x_3, 2x_1+x_2, -x_1-2x_2+2x_3)$.

(i) Verify that T is a linear transformation.

(ii) If (a, b, c) is a vector in F^3, what are the conditions on a, b and c that the vector be in the range of T ? What is the rank of T ?

(iii) What are the conditions on a, b and c that (a, b, c) be in the null space of T ? What is the nullity of T ? **[15]**

6. (c) Find the characteristic values and bases of the corresponding responding characteristic spaces of the matrix $A=\begin{bmatrix}2 & 1 & 0\\ 0 & 1 & -1\\ 0 & 2 & 4\end{bmatrix}$

Myths around science subjects.

Coaching institutions have mastered the art of brainwashing students and creating an atmosphere of gloom and doom around science subjects. There are lots of myths circulating among students. Let's bust these myths.

1. **Maths optional is only for students from IITs: Definitely not.** Anyone willing to put in hard work can easily score very high marks. The best example being **Nitish K (Rank 8) who is not from any IIT**.

2. **There is heavy scaling:** Let the data speak for itself. I attempted 240 marks in Paper 2 and got 170 marks. Now would you call it a scaling?

3. **It plays no role in GS:** Yes it's true that science optional subjects don't overlap with GS but it's equally true that GS has never been a rank decider in UPSC IAS.

4. **There are 3 major things that decides your rank:** Essay, Optional and Interview. Even if one puts in 5 years of efforts in GS the advantage in terms of marks would be around 30 marks or so but 1 year of dedicated effort in maths would give you 50+ marks advantage straightaway.

Do's and Dont's:

1. Practice, Practice and Practice. The key to success in maths is filling up as many notebooks as you can, during the preparation stage. The more you sweat during preparation the less you will bleed in the battlefield!

2. Don't read Maths book / notes like GS. It is a recipe for disaster. Rather always study with pen, paper and calculator.

3. While solving examples don't jump to see solution first. Try giving your best shot and after making sure that you are not able to solve it using your present knowledge then only look at the answer. This will ensure that better retention.

PREPARATION STRATEGY

for IAS/IFoS Mathematics Optional

by

Successful Candidate

AIR-58 in IAS-2014 Examination

Bhavesh Mishra - Classroom Student

Why Maths ?

Simply because it is the best performing optional subject in UPSC/IAS.

Extremely high scoring: If you get your maths optional right then you will make it to the final list. This year one of my batch mate in IMS **Nitish K (Rank 8)** has got a mind boggling 346 marks.

Certainty: If you have attempted your paper well then you are sure that you will get good marks. For example this year just by attempting 400 marks paper you could get a decent 260+ marks. Even if you don't get good marks in first attempt but you can be sure that you will increase your marks in subsequent attempt(s).

Fun: Mathematics is a delightful subject and therefore doing maths takes you away from somewhat boring humanities.

Good Impression: The fact that you have taken Maths makes a good impression on interview board members

(it happened in my case !). They are very pleased to see that you have opted for a tough optional.

Easy paper: The difficulty level of paper is quite moderate and almost all questions are directly picked from the IMS Test Series / Standard Textbooks.

Who should take it?

Anyone who has done B.Tech/M.Tech/B.Sc/M.Sc and has an interest in Maths.

Is A similar to a diagonal matrix? Give reasons. **[20]**

7. (a) Let $f(x) = x\left\{1+\frac{1}{3}\sin\log\left(x^2\right)\right\}$ for $x \neq 0$, $f(x)=0$ for $x=0$. Show that f(x) is continuous and monotone. **[11]**

7. (b) Show that

$$\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}dy \neq \int_0^1 dy\int_0^1 \frac{x-y}{(x+y)^3}dx$$ **[11]**

Find the values of the two integrals.

7. (c) Find the minimum value of $x^4+y^4+z^4$, where $xyz = c^3$. **[13]**

7. (d) If $0<x<1$, show that

$$2x < \log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)$$

Deduce that

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e.e^{\frac{1}{12n(n+1)}}$$ **[15]**

8. (a) The plane x/a + y/b + z/c = 1 meets the axes OX, OY, OZ which are rectangular, in A, B, C. Prove that the planes through the axes and the internal bisector of the angles of the triangle ABC pass through the line

$$\frac{x}{a\sqrt{b^2+c^2}} = \frac{y}{b\sqrt{c^2+a^2}} = \frac{z}{c\sqrt{(a^2+b^2)}}$$ **[15]**

8. (b) Find the equation of the sphere inscribed in the tetrahedron formed by the planes whose equations are
$y+z=0, z+x=0, x+y=0, x+y+z=1.$ **[15]**

8. (c) The generators through P of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ meets the principal elliptic section of A and B. If the median of the triangle APB through P is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, show that P lies on the surface
$z(\alpha x+\beta y)+\gamma(c^2+z^2)=0$ **[20]**

PREPARATION STRATEGY

for IAS/IFoS Mathematics Optional

by Successful Candidate AIR-5 in IFoS-2014 Examination AIR-299 in IAS-2014 Examination

Parth Jaiswal - Classroom Student

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

Why I Chose Mathematics?

I chose **Mathematics** because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

- This subject is vast in syllabus and takes more time to study than other optionals.

award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

All the best!

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/ calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will

- It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.
- In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.
- So again it gives you the flexibility of giving IFoS Exam.

From where to study?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

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No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

Thank You

Parth Jaiswal

AIR-5 in IFoS-2014,

AIR-299 in CSE-2014

PREPARATION STRATEGY

for IAS/IFoS Mathematics Optional
by Successful Candidate of
AIR-8 in IAS-2015
AIR-13 in IFoS-2014 Examination & AIR-143 in IAS-2014 Examination

Kumbhejkar Yogesh Vijay - Classroom Student

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

TEST SERIES (MAIN)-2016

Test Code: PAPER-II: IAS (M)/28-8-16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

MODERN ALGEBRA, REAL ANALYSIS AND COMPLEX & LPP.

Test- 02

Time: Three Hours | ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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Ph: 09999197625, 011-45629987

SECTION – A

1. **(a)** Let $G=\{a\in\mathbb{R}:-1<a<1\}$ .Define * on G by $a*b=\dfrac{a+b}{1+ab}$ for all $a,b\in G$.Show that *is a binary operation on G .Hence prove that (G,*)is a group. **(10)**

1. (b) Show that x^2+1 is irreducible over the integers mod 7. **(10)**

1. (c) Show that the series **(10)**

$$\sum_{n=1}^{\alpha}\frac{3.6.9,\ldots,3n}{7.10.13...(3n+4)}x^n, x>0$$

Converges for $x\le 1$,and diverges for x>1

1. (d) Prove that the function

$$u=x^3-3xy^2+3x^2-3y^2+1$$

Satisfies Laplace's equation and determine the corresponding analytic function u+iv. **(10)**

(e)A farm is engaged in breeding pigs.The pigs are fed various products grown on the farm. in view of the need to ensure certain nutrient constituents $N_1,N_2,N_3,$ it is necessary to buy two additional products P_1 and P_2. One unit of P_1 contains 36 units of N_1,3 units of N_2 and 20 units of N_3 .One unit of products P_2 contains 6 units of N_1 ,12 units of N_2 and 10 units of N_3 .The minimum requirement of N_1,N_2 and N_3 is 108 units,36 units and 100units respectively.product P_1 cost Rs. 20per unit and P_2 costs Rs. 40per unit.

Formulate this Diet problem as an LP model and solve it graphically. **[10]**

2. (a) **(i)** Let R be the ring of all real valued ,continuous function on [0,1].Show that the set

$$S=\left\{f\in R:f\left(\frac{1}{2}\right)=0\right\}$$ is an ideal of R

(ii) Give an example of two ideals A and B of R such that $A\subseteq B\subseteq R$,where A is an ideal of B,B is an ideal of R,but A is not an ideal of R. **(17)**

(b) **(i)** Show that the function f defined by

5. maths.stackexchange.com is the best online resource for preparation. You can create an account and get your maths questions answered within minutes.

Why did I score only 262?

Among all the students in the final list who had Maths as an optional, I have scored the least. My paper - 1 was a complete disaster and I only scored 92 marks in it. In fact I could only attempt 160 marks paper and had to leave 90 marks paper completely.

The reasons for the above situation in Paper - 1 are as follows:

1. **Lack of written practice:** In many topics (especially statics and dynamics) I used to just look at a question and its solution without solving it first. As a result I forgot the exact method in the exam hall!

2. **Left many topics:** I prepared only 25% 3-D, 80% Calculus and 25% Statics & Dynamics and had to pay a heavy price in the exam.

On the other hand my preparation for paper - 2 was excellent and therefore I scored an amazing 170 marks in it

Bhavesh Mishra

AIR-58 in CSE-2014

$$f(x) = x\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, \text{ if } X \neq 0$$

$=0, \qquad \text{if } x=0$

is continuous at x=0.

(ii) Test for convergence $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ **(15)**

(c) Find different developments of $\frac{1}{(z-1)(z-3)}$ in powers of z; according to the position of the poin z in the plane. Expand the function in a Tyalor's series about z= 2. and indicate the circle of covergence. **(18)**

3. (a) **(i)** Find all the homomorphisms of the group $(\mathbb{Z},+)$ to the group $(\mathbb{Z},+)$

(ii) Let G be a group. Show that if $G/Z(G)$ is cyclic,then G is abelian. **(16)**

(b) **(i)** Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty]$.

(ii) Show that $\frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2} dx \le \frac{2}{\pi}$ **(16)**

(c) Solve the following by simplex method. **(18)**

Maximise $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$

$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

4.(a) Prove that $z\left[\sqrt{-5}\right]$ is not a U.F.D. **(13)**

(b) Give examples of each of the following (Justifying your answers): **(12)**

(i) a set having no limit point
(ii) every point of the set is its limit point.
(iii) an infinite number of limit points
(iv) exactly one limit point
(v) exactly two limit points
(vi) Finte number of limit points

(c) What kind of singularity have the following function: **(12)**

(i) $\dfrac{1}{1-e^{z}}$ at $z = 2\pi i$ **(ii)** $\dfrac{1}{\sin z - \cos z}$ at $z = \dfrac{\pi}{4}$

(iii) $\dfrac{\cot \pi z}{(z-a)^{2}}$ at z=0 and $z = \infty$.

(d) Find the optimal (maximization) solution of the following transportation problem apply VAM to find intial basic feasible solution **(13)**

	Sale Agencies					Capacity
	S_1	S_2	S_3	S_4	S_5	a_i
F_1	15	17	12	11	11	140
F_2	5	9	7	15	7	190
F_3	14	15	16	20	10	115
Demand b_j	74	94	69	39	119	

Section -B

5. (a) Let $\alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 6\,4\,7\,5\,2\,3\,1 \end{pmatrix}, \beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,4\,6\,7\,3\,5\,2 \end{pmatrix}$ be elements of S_7 . **(10)**

(i) write α as a product of disjoint cycles.

(ii) write β as a product of 2-cycles.

(iii) Is β an even permutation?

(iv) Is α^{-1} an even permutation?

(b) Show that a finte integral domain is a field. **(10)**

(c) Show that the series **(10)**

Myths around science subjects.

Coaching institutions have mastered the art of brainwashing students and creating an atmosphere of gloom and doom around science subjects. There are lots of myths circulating among students. Let's bust these myths.

1. **Maths optional is only for students from IITs: Definitely not.** Anyone willing to put in hard work can easily score very high marks. The best example being **Nitish K (Rank 8) who is not from any IIT**.

2. **There is heavy scaling:** Let the data speak for itself. I attempted 240 marks in Paper 2 and got 170 marks. Now would you call it a scaling?

3. **It plays no role in GS:** Yes it's true that science optional subjects don't overlap with GS but it's equally true that GS has never been a rank decider in UPSC IAS.

4. **There are 3 major things that decides your rank:** Essay, Optional and Interview. Even if one puts in 5 years of efforts in GS the advantage in terms of marks would be around 30 marks or so but 1 year of dedicated effort in maths would give you 50+ marks advantage straightaway.

Do's and Dont's:

1. Practice, Practice and Practice. The key to success in maths is filling up as many notebooks as you can, during the preparation stage. The more you sweat during preparation the less you will bleed in the battlefield!

2. Don't read Maths book / notes like GS. It is a recipe for disaster. Rather always study with pen, paper and calculator.

3. While solving examples don't jump to see solution first. Try giving your best shot and after making sure that you are not able to solve it using your present knowledge then only look at the answer. This will ensure that better retention.

4. Generally we make lots of silly mistakes while solving a question. It is best to catch these errors early and not repeat them in exam hall. The best strategy for this is to maintain a notebook of errors that you usually commit and their mitigation measures. For example, I commit a lot of mistakes when doing Integration by parts and usually the error involves missing negative (-) sign etc. Therefore whenever I come across such type of question I try to devote extra 1 minute to re-check all my steps.

PREPARATION STRATEGY

for IAS/IFoS Mathematics Optional

by

Successful Candidate

AIR-58 in IAS-2014 Examination

Bhavesh Mishra - Classroom Student

Why Maths ?

Simply because it is the best performing optional subject in UPSC/IAS.

Extremely high scoring: If you get your maths optional right then you will make it to the final list. This year one of my batch mate in IMS **Nitish K (Rank 8)** has got a mind boggling 346 marks.

Certainty: If you have attempted your paper well then you are sure that you will get good marks. For example this year just by attempting 400 marks paper you could get a decent 260+ marks. Even if you don't get good marks in first attempt but you can be sure that you will increase your marks in subsequent attempt(s).

Fun: Mathematics is a delightful subject and therefore doing maths takes you away from somewhat boring humanities.

Good Impression: The fact that you have taken Maths makes a good impression on interview board members

(it happened in my case !). They are very pleased to see that you have opted for a tough optional.

Easy paper: The difficulty level of paper is quite moderate and almost all questions are directly picked from the IMS Test Series / Standard Textbooks.

Who should take it?

Anyone who has done B.Tech/ M.Tech/ B.Sc/ M.Sc and has an interest in Maths.

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...$$

converges uniformly in $0 < a \le x \le b < 2\pi$.

(d) Show that the function **(10)**

$$f(z) = \sqrt{(|x\, y|)}$$

is not regular at the origin,although Cauchy-Riemann equations are satisfied at the point.

(e) If $x_1 = 2, x_2 = 3, x_3 = 1$ is a feasible solution of the LPP **(10)**

Maximise $Z = x_1 + 2x_2 + 4x_3$

subject to $2x_1 + x_2 + 4x_3 = 11$

$3x_1 + x_2 + 5x_3 = 14$

$x_1, x_2, x_3 \ge 0.$

find a basic feasible solution of the problem.

6. (a) Show that there does not exist any isomorphism form the group $(\mathbb{R}, +)$ to the group $(\mathbb{R}^*, \bullet)$. **(06)**

(b) Give an example that the field with nine elements and write composition tables **(15)**

(c) Let $R = M_2(\mathbb{C})$, the ring of all 2×2 complex matrices and let S be a subset of R consisting of the matrices of the form $B = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$. Show that S is a subring of R Find the center of S. **(14)**

(d) In the ring $\mathbb{Z}[i]$, show that $I = \{a + bi \in \mathbb{Z}[i] | a, b$ are both even$\}$ is an ideal of $\mathbb{Z}[i]$, but not a maximal ideal of $\mathbb{Z}[i]$. **(15)**

7 (a) Find the dervied sets of the following sets: **(08)**

(i) $\left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$ **(ii)** $\left\{\frac{1 + (-1)^n}{n} : n \in N\right\}$

(iii) $\left\{(-1)^n + \frac{1}{n} : n \in N\right\}$ **(iv)** $\left\{2^n + \frac{1}{2^n} : n \in N\right\}$

(b) Show that the series

$$2xe^{-x^2} = \sum_{1}^{\infty} 2x\left[\frac{1}{n^2}e^{-x^2/n^2} - \frac{1}{(n+1)^2}e^{-x^2/(n+1)^2}\right]$$

can be integrated term by term between any two finite limits.Can the function defined by the series be integrated between the limits 0 and ∞ ?If so,what is the value of this integral given by integrating the series term by term between these limits? **(15)**

(c) A sequence $< a_n >$ is defined as **(12)**

$a_1 = 1, a_{n+1} = (4+3a_n)/(3+2a_n), n \geq 1.$ Show that $< a_n >$ converges and find its limit.

(d) Show that $\lim\{I_n\}$, where **(15)**

$$I_n = \int_0^{nh} \frac{\sin nx}{x}dx, n \in 1$$

exists and that the limit is equal to $\pi/2$.

8. (a) Prove that all the roots $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2.$ **(10)**

(b) Use cauchys theorem and/or cauchy integral formula to evaluater the following integrals **(07)**

(i) $\displaystyle\int_{|z|=1} \frac{\cos z}{z(z-4)}dz$ **(ii)** $\displaystyle\int_{|z-a|=1} \frac{e^{2\pi z}}{(z-a)}dz$

(c) show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}.$ **(15)**

(d) From the optimal table of the following primal problem,find the optimal solution to the associated dual problem. **(18)**

Maximise $Z = 30x_1 + 23x_2 + 29x_3$

Subject to
$$6x_1 + 5x_2 + 3x_3 \leq 26$$
$$4x_1 + 2x_2 + 5x_3 \leq 7$$
$$x_1, x_2, x_3 \geq 0.$$

should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

All the best!

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/ calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

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PREPARATION STRATEGY

for IAS/IFoS Mathematics Optional

by Successful Candidate AIR-5 in IFoS-2014 Examination AIR-299 in IAS-2014 Examination

Parth Jaiswal - Classroom Student

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

Why I Chose Mathematics?

I chose **Mathematics** because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

- This subject is vast in syllabus and takes more time to study than other optionals.
- It also requires consistent practise. But the positive part is - If you are

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thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

- In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.
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I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

Thank You

Parth Jaiswal

AIR-5 in IFoS-2014,

AIR-299 in CSE-2014

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by Successful Candidate of
AIR-8 in IAS-2015
AIR-13 in IFoS-2014 Examination & AIR-143 in IAS-2014 Examination

Kumbhejkar Yogesh Vijay - Classroom Student

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/04-9-16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

ODE, STATICS & DYNAMICS & VA

Test- 03

Time: Three Hours | ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$. **(10)**

1. (b) Using Laplace transform, Solve $ty'' + 2y + ty = \cos t$ given that $y(0) = 1$. **(10)**

1. (c) The extremities of a heavy string of length $2l$ and weight $2lw$, are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to lw. Show that the distance apart of the rings is. $2\ell\log\left(1+\sqrt{2}\right)$ **(10)**

1. (d) A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, prove that $\tan\alpha = \tan A + \tan B$. **(10)**

1. (e) Verify the Green's theorem for $M = \dfrac{-y}{x^2+y^2}$, $N = \dfrac{x}{x^2+y^2}$. **(10)**

$R = \left\{(x,y)/h^2 \le x^2 + y^2 \le 1\right\}$, where $0 < h < 1$

2. (a) (i) Find the orthogonal trajectories of the family of $r^n = a^n \sin n\theta$. **(15)**

(ii) Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x$.

2. (b) Five equal uniform rods, freely jointed at their ends, form a regular pentagon ABCDE and BE is joined by a weightless bar. The system is suspended from A in a vertical plane. Prove that the thrust in BE is $W\cot\dfrac{1}{10}\pi$, where W is the weight of the rod. **(20)**

2. (c) Evaluate $\int_S F\cdot ds$ where $F = 4x\,\mathbf{I} - 2y^2\,\mathbf{J} + z^2\,\mathbf{K}$ and S is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. Using the divergence theorem **(15)**

3. (a) (i) Solve $(D^2 - 1)\,Y = x\sin 3x + \cos x$.

(ii) Solve $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$ **(15)**

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

3. (b) A particle move under a central force $m\lambda\ (3a^3u^4 + 8au^2)$. It is projected from an apse at a distance a from the centre of force with velcocity $\sqrt{(10\lambda)}$. Show that the second apsidal distance is half of the first and that the equation to the path is

$$2r = a\left[1 + \text{sech}(\theta/\sqrt{5})\right]$$ **(17)**

3. (c) If $F = (y^2+z^2-x^2)\,\mathbf{i} + (z^2+x^2-y^2)\,\mathbf{j} + (x^2+y^2-z^2)\,\mathbf{k}$, evaluate $\iint \mathbf{curl\,F \cdot n\,ds}$ taken over the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ above the plane $z = 0$, and verify stoke's theorem. **(18)**

4. (a) Apply the method of variation of parameters to solve,
$y_2 = 4y = \sec 2x$ **(15)**

4. (b) A heavy uniform chain AB hangs freely under gravity. with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angles 60° and 30° respectively with the horizontal. Prove that the ratio of these lengths is $(\sqrt{3}-1):1$. **(12)**

4. (c) A particle starts from rest at a distance a fromthe centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is a $\sqrt{(\pi/2\mu)}$. **(10)**

4. (d) **(i)** If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.
(ii) If $F = (x+y+az)\,\mathbf{I} + (bx+2y-z)\,\mathbf{J} + (x+cy+2z)\,\mathbf{K}$, find a, b, c such that curl $\mathbf{F} = 0$, then find ϕ such that $\mathbf{F} = \nabla\phi$. **(12)**

SECTION – B

5. (a) Solve the following differential equation,

$$x^4\frac{dy}{dx} + x^3y + \text{cosec}\,(xy) = 0.$$ **(10)**

5. (b) A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1}\frac{3}{8}$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable. **(10)**

5. (c) Show that in a simple harmonic motion of amplitude a and period 'T', the velocity v at a distance x from the centre is given by the relation $v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$.

Find the new amplitude if the velocity were doubled when the particle is at a distance $\frac{1}{2}$ a from the centre ; the period remaining unaltered. **(10)**

5. (d) If $a = \sin\theta\mathbf{i} + \cos\theta\mathbf{j} + \theta\mathbf{k},\ b = \cos\theta\mathbf{i} - \sin\theta\mathbf{j} - 3\mathbf{k}$, and $c = 2\mathbf{i} + 3\mathbf{j} - 3k$,

find $\frac{d}{d\theta}\left\{a \times \left(b \times c\right)\right\}$ at $\theta = \frac{\pi}{2}$. **(10)**

5. (e) Find the constant a so that V is a conservative vector field, where $V = (axy - z^3)\,\mathbf{I} + (a-2)x^2\,\mathbf{J} + (1-a)\,az^2\mathbf{K}$. **(10)**

6. (a) Find the inverse Lapalce transforms of the following : **(12)**

(i) $\log\frac{s^2+1}{s\left(s+1\right)}$ (ii) $\tan^{-1}\left(\frac{2}{s^2}\right)$

6. (b) Obtain the singular solution of the equaiton

$p^2 y^2 \cos^2\alpha - 2\,pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$.

Directly from the equation and also from its complete primitive, explaining the geometrical significance of the irreleveant factors that present themselves. **(12)**

6. (c) Solve $y'' + (1 - \cot x)\,y' - y\cot x = \sin^2 x$. **(12)**

6. (d) Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \log x\frac{\sin\left(\log x\right)+1}{x}$. **(13)**

7. (a) A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either $\frac{\pi}{4}$ or $\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right)$. **(18)**

7. (b) A particle is Projected with a velocity u from a point on an inclined plane whose inclination to the horizontal is β. and strikes it at right angles. Show that

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

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your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

(i) the time of flight is $\dfrac{2u}{g\sqrt{\left(1+3\sin^2\beta\right)}}$,

(ii) the range on the inclined plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$.

and (iii)the vertical height of the point struck, above the point of projection is $\dfrac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$. **(16)**

7. (c) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag)}/V\right]$. **(16)**

8. (a) (i) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.

(ii) If F $(3x^2 y - z)$ **i** + $(xz^2 + y^4)$ **j** $-2x^3z^2$ **k**, find $\nabla(\nabla\cdot F)$ at the point $(2, -1, 0)$. **(12)**

8. (b) Find the curvature and torsion of the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$. **(09)**

8. (c) A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\ \mathbf{i} + \sin\omega t\ \mathbf{j}$ where ω is a constant; show that

(i) the velocity of the particle is perpendicular to **r**

(ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin

(iii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$ is a constant vector. **(12)**

8. (d) Verify divergence theorem for the function $F = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ over the cylindrical region bounded by $x^2 + y^2 = a^2$, $z = 0$ and $z = h$. **(17)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

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THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN
AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-II: IAS (M)/11-9-16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

PDE, NA & CP AND MECHANICS & FLUID DYNAMICS

Test- 04

Time: Three Hours *Maximum Marks: 250*

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Each question is printed only in English.

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Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) Find a partial differential equation by eliminating a, b, c from $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. **(10)**

1. (b) $\left(D^2 - DD' - 2D'^2 + 2D + 2D'\right) z = xy + \sin(2x + y)$. **(10)**

1. (c) Apply Newton-Raphson method to determine a root of the equation $\cos x = x e^x$ correct to three decimal places. **(10)**

1. (d) (i) Draw a logic circuit for the Boolean equation $Y = AB + AC + BC + CD$.
(ii) Simplify the expression and draw logic circuit for the simplified expression. **(10)**

1. (e) Show that M.I. of a rectangle of mass M and sides 2a, 2b about a diagonal is $\dfrac{2M}{3}\dfrac{a^2b^2}{a^2+b^2}$. **(10)**

2. (a) (i) Form a partial differential equation by eliminating the arbitrary functions f and g from, $z = y f(x) + x g(y)$. **(16)**
(ii) Find the surface which is orthogonal to the one parameter system $z = cxy(x^2 + y^2)$ which passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$.

2. (b) Solve the following system of equations.

$$10x - 7y + 3z + 5w = 6$$
$$-6x + 8y - z - 4w = 5$$
$$3x + y + 4z + 11w = 2$$
$$5x - 9y - 2z + 4w = 7$$

by Gauss Seidel method. **(17)**

2. (c) A solid body of density ρ is in the shape of the solid formed by the revolution of the cardiod $r = a(1 + \cos\theta)$ about the initial line, show that its M.I. about a straight line through the pole and perpendicular to the initial line is $\dfrac{352}{105}\pi\rho a^5$. **(17)**

3. (a) Solve the boundary value problem $\partial^2u/\partial x^2 = (1/k)(\partial u/\partial t)$ satisfying the conditions $u(0, t) = u(\ell.t) = 0$ and $u(x, 0) = x$ when $0 \le x \le l/2$; $u(x, 0) = l - x$ when $l/2 \le x \le l$. **(15)**

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

3. (b) Using fourth order Runge-Kutta method, find the solution of $x(dy + dx) = y(dx - dy)$, $y(0) = 1$ at $x = 0.1$ and 0.2, by taking $h = 0.1$. **(17)**

3. (c) In two dimensional irrotational fluid motion, show that if the stream lines are confocal ellipses.

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

$$\psi = A \log\left[\sqrt{\left(a^2+\lambda\right)} + \sqrt{\left(b^2+\lambda\right)}\right] + B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point. **(18)**

4. (a) Prove that for the equation $z + px + qy - 1 - pq\,x^2 y^2 = 0$ the characteristic strips are given by $x = (B + C\,e^{-t})^{-1}$, $y = (A + D\,e^{-t})^{-1}$, $z = E - (AC + BD)\,e^{-t}$, $p = A(B + C\,e^{-t})^2$ $q = B\,(A + D\,e^{-t})^2$ where A, B, C, D and E are arbitray constants. Hence find the integral surface which passes through the line $z = 0$, $x = y$. **(18)**

4. (b) Draw a flow chart for Trapezoidal rule. **(16)**

4. (c) Test whether the motion specified by

$$q = \frac{k^2\left(x\mathbf{j} - y\mathbf{i}\right)}{x^2+y^2}\left(k = \text{const.}\right)$$

is a possible motion for an incompressible fluid. If so, determine the equation of stream lines. Also tell whether the motion is of the potential kind and if it determines the velocity potential. **(16)**

SECTION – B

5. (a) Solve $\cos(x+y)\,p + \sin(x+y)\,q = z$. **(10)**

5. (b) Solve $\left(D^2 - 6DD' + 9D'^2\right)z = \tan(y+3x)$ **(10)**

5. (c) Find Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$. Hence find $y(5)$. **(10)**

5. (d) For a simple pendulum (i) find the Lagrangian function and (ii) Obtain an equation describing its motion. **(10)**

5. (e) Find the stream function ψ for the given velocity potential $\phi = cx$, where c is constant. **(10)**

6. (a) Find a surface satisfying the equation $D^2z = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$. **(13)**

6. (b) Reduce $x^2\left(\partial^2 z/\partial x^2\right) - y^2\left(\partial^2 z/\partial y^2\right) = 0$ to canonical form and hence solve it. **(15)**

6. (c) Obtain temperature distribution $y(x, t)$ in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Further it is given that $y(x, 0) = 1 - x$, $0 < x < 1$. **(22)**

7. (a) A missile is launched from a ground station. The acceleration during its first 80 seconds of flight, as recorded, is given in the following table :

$t(s)$	0	10	20	30	40	50	60	70	80
$a(m/s^2)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

compute the velocity of the missile when t = 80 s, using Simpson's 1/3 rule. **(12)**

7. (b) Using modified Euler's method, obtain the solution of the differential equation

$$\frac{dy}{dt} = t + \sqrt{y} = f(t, y)$$

with the inital condition $y_0 = 1$ at $t_0 = 0$ for the range $0 \le t \le 0.6$ in steps of 0.2. **(13)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

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your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination
AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

7. (c) Obtain the principal disjunctive and conjunctive normal forms of

$$p \to \left[(p \to q) \wedge \sim(\sim q \vee \sim p)\right] \qquad \textbf{(13)}$$

7. (d) In Boolean algebra $[B, +, \cdot, 1]$, show that

$$(x \cdot y' + y \cdot z) \cdot (x \cdot z + y \cdot z') = x \cdot z \qquad \textbf{(12)}$$

8. (a) A uniform rod O A, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or $\cos^{-1}\left(3g/4a\omega^2\right)$. **(16)**

8. (b) A uniform rod, of length 2a, which has one end attached to a fixed point by a light inextensible string of length 5a/12, is performing small oscillations in a vertial plane about its position of equilibrium. Find its position at any time, and show that the period of its principal oscillations are

$$2\pi\sqrt{(5a/3g)} \text{ and } \pi\sqrt{(a/3g)} \qquad \textbf{(17)}$$

8. (c) If a vortex pair is situated within a cylinder show that it will remain at rest if the distance of either from the centre is given by a $(\sqrt{5}-2)^{1/2}$, where a is the radius of the cylinder. **(17)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN

AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

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diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/18-9-16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 05

Time: Three Hours | *Maximum Marks: 250*

INSTRUCTIONS

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SECTION – A

1. (a) Find a basis and dimension of the subspace W of V spanned by the polynomials $v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = 2t^3 - 3t^2 + 9t - 1$, $v_3 = t^3 + 6t - t$, $v_4 = 2t^3 - 5t^2 + 7t + 5$. **(10)**

1. (b) For what values of η the equations

$$x + y + z = 1,$$
$$x + 2y + 4z = \eta,$$
$$x + 4y + 10z = \eta^2$$

have a solution and solve them completely in each case. **(10)**

1. (c) An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least. **(10)**

1. (d) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. **(10)**

1. (e) If the edges of a rectangular parallepiped be a, b, c show that the angles between the four diagonals are given by

$$\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right]$$ **(10)**

2. (a) Find the range, rank, kernal and nullity of the linear transformation $T : R^4 \to R^3$ defined by
$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$. **(13)**

2. (b) (i) Is the vector (3, – 1, 0, – 1) in the subspace of R^4 spanned by the vectors $\alpha_1 = (2,-1,3,2)$, $\alpha_2 = (-1,1,1-3)$ and $\alpha_3 = (1,1,9-5)$? **(12)**

(ii) If α is a characteristic root of a non-singular matrix A, then prove that $\dfrac{|A|}{\alpha}$ is a characteristic root of Adj A.

2. (c) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomal in A. **(10)**

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For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

2. (d) Let $B = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}$, a symmetric matrix. Find a nonsingular matrix P such that $P^T BP$ is diagonal and find the diagonal matrix $P^T BP$. **(15)**

3. (a) Show that the function $f: R^2 \to R$ defined by setting

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{when } xy \neq 0 \\ x\sin(1/x), & \text{when } x \neq 0, y = 0 \\ y\sin(1/y), & \text{when } x = 0, y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$

is continuous but not differentiable at (0,0) **(15)**

3. (b) (i) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(ii) Determine

$$\lim\left(\frac{\pi}{2} - x\right)^{\tan x} \text{ as } x \to \left(\frac{\pi}{2} - 0\right)$$ **(12)**

3. (c) Evaluate the following integrals by changing to polar corrdiantes.

$$\int_0^a \int_y^a \frac{x\,dy\,dx}{yx^2 + y^2}$$ **(13)**

3. (d) Evaluate $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx$ **(10)**

4. (a) Find the incentre of the tetrahedron formed by the planes x = 0, y = 0, z = 0 and x + y + z = a. **(07)**

4. (b) Find the equation of the sphere that passes through the points (4, 1, 0), (2, – 3, 4), (2, – 3, 4), (1, 0, 0) and touches the plane 2x + 2y – z = 11. **(12)**

4. (c) Show that the feet of the normals from the point (α, β, γ) on the paraboloid $x^2 + y^2 = 2az$ lie on a sphere. **(15)**

4. (d) Prove that the projections of the generators of a hyperboloid on coordinate plane are tangnets to the section of the hyperboloid by that plane. **(16)**

SECTION – B

5. (a) Solve sin x (dy/dx) + 3y = cos x. **(10)**

5. (b) Solve $(x^2 - 4)p^2 - 2xyp - x^2 = 0$ and examine for singular solutions and extraneous loci. **(10)**

5. (c) A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$. **(10)**

5. (d) A particle of mass m, is falling under the influence of gravity through a medium whose resistance equal μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is

$$\frac{gm^2}{\mu^2}\left[e^{-(\mu/m)t} - 1 + \frac{\mu t}{m}\right]$$ **(10)**

5. (e) Show that for the curve $R = a(3t - t^3)\,\mathbf{I} + 3at^2\mathbf{J} + a(3t + t^2)\,\mathbf{K}$, the curvature equals torsion. **(10)**

6. (a) Find th orthogonal trajectories of the family of co-axial circles $x^2 + y^2 + 2gx + c = 0$, where g is the parameter. **(15)**

6. (b) A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is

$$\frac{2l}{\lambda}\log\left\{\lambda + \sqrt{(1+\lambda^2)}\right\}$$, where $\frac{1}{\lambda} = \mu(2n+1)$ and μ is the coefficient of friction. **(18)**

6. (c) (i) A person going east wards with a velocity of 4 km per hour, finds the the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

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your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

(ii) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$? **(17)**

7. (a) Solve $[(x+1)^2 D^2 + (x+1)D - 1]y = ln(x+1)^2 + x - 1$ **(10)**

7. (b) Solve by using the method of variation of parameters.
$dy^2 / dx^2 - 2(dy/dx) = e^x \sin x$. **(10)**

7. (c) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r \cos\left(\theta/\sqrt{2}\right) = a$. **(16)**

7. (d) (i) Find the value of a if the vector $(ax^2y + yz)\,I + (xy^2 - xz^2)\,J + (2xyz - 2x^2y^2)\,K$ has zero divergence. Find the curl of the above vector which has zero divergence.

(ii) For a solenoidal vector F, show that curl curl curl curl $F = \nabla^4 F$ **(14)**

8. (a) By using Laplace transform method
Solve $(D^2 + 2D + 5)\,y = e^{-t} \sin t$, $y(0) = 0$, $y'(0) = 1$ **(15)**

8. (b) A particel slides down the arc of a smooth cycloid hose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half. **(15)**

8. (c) Verify stoke's theorem for $F = (x^2 + y - 4)\,i + 3xyj + (2xz + z^2)k$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 16$ and C is its boundary **(20)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN

AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-II: IAS (M)/25-9-16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 06

Time: Three Hours ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) (i) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \in S_4$. Find the smallest positive integer k such that $\alpha^k = e$.

(ii) In S_6, let $\rho = (123)$ and $\sigma = (456)$. Find a permutation x in S_6 such that $x\rho x^{-1} = \sigma$. **(10)**

1. (b) Is $Z\left[\sqrt{-6}\right] = \left\{a + b\sqrt{-6} \,/\, a, b, \in Z\right\}$ Euclidean domain ? Justify your answer. **(10)**

1. (c) (i) Give an example of an infinite set which is not bounded and having limit points.

(ii) Give an example to show that the intersection of an infinite collection of open sets is not necessarily an open set. **(10)**

1. (d) If $f(z) = u + iv$ is an analytic function of $z = x + iv$. and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z. **(10)**

1. (e) Obtain the dual of the LP problem :

Min. $z = x_1 + x_2 + x_3$. subject to the constraints :

$x_1 - 3x_2 + 4x_3 = 5$, $x_1 - 2x_2 \le 3$, $2x_2 - x_3 \ge 4$; $x_1, x_2 \ge 0$ and x_3 is unrestricted. **(10)**

2. (a) If R is a ring with identity such that $(xy)^2 = x^2y^2$ for all $x, y \in R$, then show that R is commutative. Set an example to show that the above result may be false if R does not have an identity. **(14)**

2. (b) Give an example of a homomorphism $f : R \to R'$ such that 1 is the unity of R, but f(1) is not the unity of R'. **(06)**

2. (c) Let the function f be defined on [0, 1] as follows :

$$f(x) = 2rx \text{ when } \frac{1}{r+1} < x \le \frac{1}{r},\ r = 1, 2, 3, \ldots\ldots$$

Prove that f is R-integrable in [0, 1] and evaluate $\int_0^1 f(x)\,dx$. **(12)**

2. (d) Use the method of contour integration to prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)(x^2 + c^2)^2} = \frac{\pi(b + 2c)}{2bc^3(b + c)^2} \text{ where } b > 0, c > 0. \quad \textbf{(18)}$$

OUR TOPPERS MARKS LIST

- For your final selection, optional subject marks are crucial.
- Choose Optional Subject based on Your Graduation Studies & Score Highest Marks.
- Now Mathematics has become one of the most Cherished Optional Paper among Science Graduates, especially Students with Mathematics background including B.Tech.
- In the new pattern of exam, the average marks of successful candidates in Maths is more than 274 out of 500.
- Mathematics (Opt.) has proven to be the Most Reliable and High Scoring Subject in IAS/IFoS.
- IMS has been successfully providing consistent results since its inception.

Marks are before you and you should analyze yourself!

Kumbhejkar Yogesh Vijay
Rank-8
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	146
General Studies -I (Paper-II)	250	101
General Studies-II (Paper-III)	250	086
General Studies-III (Paper-IV)	250	114
General Studies-IV (Paper-V)	250	100
Optional-I (Mathematics)-(Paper-VI)	126/250	298/500
Optional-II (Mathematics)-(Paper-VII)	172/250	
Written Total	1750	845
Personality Test	275	138
Final Total	2025	983

Ashish Sangwan
Rank-12
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	113
General Studies -I (Paper-II)	250	100
General Studies-II (Paper-III)	250	077
General Studies-III (Paper-IV)	250	098
General Studies-IV (Paper-V)	250	112
Optional-I (Mathematics)-(Paper-VI)	124/250	284/500
Optional-II (Mathematics)-(Paper-VII)	160/250	
Written Total	1750	784
Personality Test	275	195
Final Total	2025	979

Siddhartha Jain
Rank-13
IAS-2015

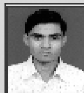
SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	142
General Studies -I (Paper-II)	250	103
General Studies-II (Paper-III)	250	082
General Studies-III (Paper-IV)	250	097
General Studies-IV (Paper-V)	250	099
Optional-I (Mathematics)-(Paper-VI)	114/250	268/500
Optional-II (Mathematics)-(Paper-VII)	154/250	
Written Total	1750	791
Personality Test	275	187
Final Total	2025	978

Pratap Singh
Rank-15
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	133
General Studies -I (Paper-II)	250	107
General Studies-II (Paper-III)	250	082
General Studies-III (Paper-IV)	250	106
General Studies-IV (Paper-V)	250	081
Optional-I (Mathematics)-(Paper-VI)	154/250	283/500
Optional-II (Mathematics)-(Paper-VII)	129/250	
Written Total	1750	792
Personality Test	275	184
Final Total	2025	976

Valluru Kranthi
Rank-65
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	128
General Studies -I (Paper-II)	250	096
General Studies-II (Paper-III)	250	062
General Studies-III (Paper-IV)	250	082
General Studies-IV (Paper-V)	250	086
Optional-I (Mathematics)-(Paper-VI)	154/250	326/500
Optional-II (Mathematics)-(Paper-VII)	172/250	
Written Total	1750	780
Personality Test	275	160
Final Total	2025	940

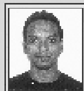


Ketan Bansal
Rank-155
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	114
General Studies -I (Paper-II)	250	089
General Studies-II (Paper-III)	250	078
General Studies-III (Paper-IV)	250	090
General Studies-IV (Paper-V)	250	079
Optional-I (Mathematics)-(Paper-VI)	152/250	308/500
Optional-II (Mathematics)-(Paper-VII)	156/250	
Written Total	1750	758
Personality Test	275	160
Final Total	2025	918

Varun Gunupalli
Rank-183
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	132
General Studies -I (Paper-II)	250	089
General Studies-II (Paper-III)	250	073
General Studies-III (Paper-IV)	250	068
General Studies-IV (Paper-V)	250	091
Optional-I (Mathematics)-(Paper-VI)	133/250	274/500
Optional-II (Mathematics)-(Paper-VII)	141/250	
Written Total	1750	727
Personality Test	275	184
Final Total	2025	911



Padmanabh Baruah
Rank-194
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	120
General Studies -I (Paper-II)	250	084
General Studies-II (Paper-III)	250	065
General Studies-III (Paper-IV)	250	082
General Studies-IV (Paper-V)	250	093
Optional-I (Mathematics)-(Paper-VI)	141/250	284/500
Optional-II (Mathematics)-(Paper-VII)	143/250	
Written Total	1750	728
Personality Test	275	182
Final Total	2025	910

Akhil Goel
Rank-251
IAS-2015

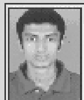
SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	110
General Studies -I (Paper-II)	250	097
General Studies-II (Paper-III)	250	065
General Studies-III (Paper-IV)	250	096
General Studies-IV (Paper-V)	250	087
Optional-I (Mathematics)-(Paper-VI)	142/250	275/500
Optional-II (Mathematics)-(Paper-VII)	133/250	
Written Total	1750	730
Personality Test	275	173
Final Total	2025	903

Parul Kamal
Rank-335
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	113
General Studies -I (Paper-II)	250	095
General Studies-II (Paper-III)	250	069
General Studies-III (Paper-IV)	250	092
General Studies-IV (Paper-V)	250	093
Optional-I (Mathematics)-(Paper-VI)	142/250	282/500
Optional-II (Mathematics)-(Paper-VII)	140/250	
Written Total	1750	744
Personality Test	275	151
Final Total	2025	895

Akshay Godara
Rank-605
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	111
General Studies -I (Paper-II)	250	087
General Studies-II (Paper-III)	250	062
General Studies-III (Paper-IV)	250	087
General Studies-IV (Paper-V)	250	074
Optional-I (Mathematics)-(Paper-VI)	145/250	299/500
Optional-II (Mathematics)-(Paper-VII)	154/250	
Written Total	1750	720
Personality Test	275	154
Final Total	2025	874



Nitish K
Rank-8
IAS-2014

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	142
General Studies -I (Paper-II)	250	073
General Studies-II (Paper-III)	250	076
General Studies-III (Paper-IV)	250	063
General Studies-IV (Paper-V)	250	100
Optional-I (Mathematics)-(Paper-VI)	173/250	346/500
Optional-II (Mathematics)-(Paper-VII)	173/250	
Written Total	1750	800
Personality Test	275	206
Final Total	2025	1006

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For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

3. (a) Let G be the group $\left\{\begin{bmatrix}1 & a\\ 0 & b\end{bmatrix} \middle| \text{where } a, b \in \mathbf{R}, b \neq 0\right\}$ and

$H = \left\{\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} \middle| \text{where } x \in \mathbf{R}\right\}$. Show that H is a subgroup of G. Is H a normal subgroup of G ? Justify your answer. **(15)**

3. (b) (i) Discuss the convergence of the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right).$$

(ii) If $x > 0$, show that $\dfrac{x}{1+x} < \log(1+x) < x$. **(17)**

3. (c) Solve the following linear programming problem by simplex method.
Max. $z = -2x_1 - x_2$, subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$, and $x_1, x_2 \geq 0$. **(18)**

4. (a) (i) Let R denote the ring of all real-valued continuous functions on the closed interval [0, 1]. Is (0) a prime ideal of R ? Justity.
(ii) In **Z**/(8), the ring of integers modulo 8, is the ideal generated by $\overline{2} = 2 + (8)$ a prime ideal ? Is it also maximal ? **(14)**

4. (b) Prove that

$$\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)dx = \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)} \quad \textbf{(12)}$$

4. (c) Find the Taylor's or Laurent's series which represent the function

$$\frac{1}{(1+z^2)(z+2)},$$

(i) when $|z| < 1$
(ii) when $1 < |z| < 2$
(iii) when $|z| > 2$. **(12)**

4. (d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit differentlevels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table :

Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

(12)

SECTION – B

5. (a) Solve $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$. **(10)**

5. (b) Solve the following partial differential equation

$$\left(D^3 - 4D^2D' + 4DD'^2\right) z = 4 \sin (2x + y).$$ **(10)**

5. (c) The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105. **(10)**

5. (d) Show that $\phi = (x - t)(y - t)$ represents the velocity potential of an incompressible two dimensional fluid. Show that the stream lines at time t are the curves.

$(x - t)^2 - (y - t)^2 = \text{constant}$ **(10)**

5. (e) Use Hamilton's equations to find the equation of motion of the simple pendulum. **(10)**

6. (a) Find the general integral of the partial differential equation $(2xy - 1)\, p + (z - 2x^2)\, q = 2(x - yz)$ and also the particular integral which passes through the line $x = 1, y = 0$. **(10)**

6. (b) Reudce $\partial^2 z / \partial x^2 + y^2 \left(\partial^2 z / \partial y^2\right) = y$ to canonical form. **(10)**

6. (c) Form the partial differential equation by eliminating the arbitrary constants a and b from $\log (az - 1) = x + ay + b$. **(10)**

6. (d) The deflection of a vibrating string of length l, is governed by the partial differential equation $y_u = c^2 y_{xx}$. The initial velocity is zero. The initial displacement is given by

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

$$y(x,0)=\begin{cases} x/l, 0<x<l/2 \\ (l-x)/l, l/2<x<l \end{cases} \quad \text{Here } y_u = \partial^2 y/\partial t^2$$

and $y_{xx} = \partial y^2/\partial x^2$.

Find the deflection of the string at any instant of time. **(20)**

7. (a) Solve the following equations by Gauss Seidal method.
$5x_1+x_2+x_3+x_4=4$; $x_1+7x_2+x_3+x_4=12$; $x_1+x_2+6x_3+x_4=-5$;
$x_1+x_2+x_3+4x_4=-6$. **(15)**

7. (b) The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows :

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes. **(10)**

7. (c) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}$ with y(0) = 1 at x = 0.2, 0.4. **(15)**

7. (d) Simplify the following :

(i) $(x+y).x'.y'$ (ii) $x \vee y \wedge y \vee z \wedge y \vee z'$

(iii) $x \vee y \wedge [(x \wedge y') \vee y]'$ **(10)**

8. (a) Determine the motion, of a spherical pendulum, by using Hamilton's equations. **(16)**

8. (b) A uniform straight rod of length 2a is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string of length a to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{(a/g)}$. **(16)**

8. (c) When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal disance from its axis, show that path of each vortex is given by the equation.

$$(r^2 \sin^2\theta - b^2)(r^2-a^2)^2 = 4a^2b^2r^2\sin^2\theta$$

θ being measured from the line through the centre $\perp$ lar to the join of the vortices. **(18)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN

AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

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diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/02/10/16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 07

Time: Three Hours | ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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INSTITUTE FOR IAS/IFOS EXAMINATIONS

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SECTION – A

1. (a) (i) Let W be the vector space of 3 × 3 antisymmetric matrices over K. Show that dim W = 3 by exhibiting a basis of W.

(ii) Find a basis and the dimension of the subspace W of R^4 spanned by $(1, -4, -2, 1), (1, -3, -1, 2)$, and $(3, -8, -2, 7)$ **(10)**

1. (b) For what values of the parameter λ will the following equations for fail to have unique solution

$$3\lambda - y + \lambda z = 1,$$
$$2x + y + z = 2,$$
$$x + 2y - \lambda z = -1 \ ?$$

Will the equations have any solution for these values of λ ? **(10)**

1. (c) Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\left(a^2 \sin^2 x + b^2 \cos^2 x\right)^2}$ **(10)**

1. (d) Let $f(x) = \begin{cases} \dfrac{|x|}{2} + 1 & \text{if} \quad x < 1 \\ \dfrac{x}{2} + 1 & \text{if} \quad 1 \le x < 2 \\ -\dfrac{|x|}{2} + 1 & \text{if} \quad 2 \le x \end{cases}$

What are the points of discontinuity of f, if any ? What are the points where f is not differentiable, if any ? justify yours answers. **(10)**

1. (e) P is a point on the plane $lx + my + nz = p$. A point Q is taken on the line OP such that $OP.OQ = p^2$, prove that the locus of Q is $p(lx + my + nz) = x^2 + y^2 + z^2$. **(10)**

2. (a) Determine whether the following matrices are dependent or independent :

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & -4 \\ 6 & 5 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 8 & -11 \\ 16 & 10 & 9 \end{pmatrix}$$ **(15)**

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For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

2. (b) (i) Show that the volume common to the surface $y^2 + z^2 = 4ax$ and $x^2 + y^2 = 2ax$ is $\frac{2}{3}(3\pi + 8)a^2$.

(ii) If $v = At^{-1/2}e^{-x^2/4a^2t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$ **(20)**

2. (c) A sphere whose centre lies in the positive octant passes through the origin and cuts the planes x = 0, y = 0, z = 0 in circles of radii $a\sqrt{2}, b\sqrt{2}, c\sqrt{2}$ respectively. find the equation of this sphere. **(15)**

3. (a) Find the range, rank, kernel and nullity of the linear transformation $T : R^3 \to R^4$ be defined by
$T(x, y, z) = (x + y + z, x + 2y - 3z, 2x + 3y - 2z, 3x + 4y - z)$. **(14)**

3. (b) (i) Show that if A is a non-singular matrix, then $\det(A^{-1}) = (\det A)^{-1}$.
(ii) If B is non-singular, prove that the matrices A and $B^{-1}AB$ have the same determinant, A and B being both square matrices of order n. **(06)**

3. (c) Let f(x, y) be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Prove that f_{xy} and f_{yx} are not continuous at (0, 0) but $f_{xy}(0, 0) = f_{yx}(0, 0)$. **(14)**

3. (d) Show that the locus of the line of intersection of perpendicular tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ is the cone $a(b + c)x^2 + b(c + a)y^2 + c(a + b)z^2 = 0$. **(16)**

4. (a) Let $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$. Is A diagonalizable ?

If yes find P such that $P^{-1}AP$ is diagonal. **(16)**

4. (b) Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point (3, 1, –1). **(17)**

4. (c) Prove that the tangent planes to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ which are parallel to tangent planes to the cone $\frac{b^2c^2x^2}{c^2-b^2} + \frac{c^2a^2y^2}{c^2-a^2} + \frac{a^2b^2z^2}{a^2+b^2} = 0$ cut the surface in perpendicular generators. **(17)**

SECTION – B

5. (a) Solve $\sqrt{(1+x^2+y^2+x^2y^2)}+xy(dy/dx)=0$ **(10)**

5. (b) Solve $x^3(d^3y/dx^3)+2x^2(d^2y/dx^2)+3x(dy/dx)-3y=x^2+x$. **(10)**

5. (c) Four rods are jointed together to form a parallelogram, the opposite joints are joints by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths. **(10)**

5. (d) If $A=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$ and $B=\sin t\,\mathbf{i}-\cos t\,\mathbf{j}$, find

(i) $\frac{d}{dt}(A\cdot B)$; (ii) $\frac{d}{dt}(A\times B)$; (iii) $\frac{d}{dt}(A\cdot A)$. **(10)**

5. (e) If $F=\cos y\,\mathbf{i}-x\sin y\,\mathbf{j}$, evaluate $\int_C F\cdot dr$ where C is the curve $y=\sqrt{(1-x^2)}$ in the x-y plane from (1, 0) to (0, 1). **(10)**

6. (a) Solve $(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$ **(16)**

6. (b) A uniform beam of length 2a rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and $\beta\,(\alpha>\beta)$, show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by

$$\tan\theta=\frac{1}{2}(\cos\beta-\cot\alpha)$$

and show that the beam is unstable in this position. **(20)**

6. (c) (i) Given that $\rho F=\nabla p$, where ρ, p, F are point functions, prove that $F\bullet \text{curl}\,F=0$.

(ii) Prove that $b\cdot\nabla\left(a\cdot\nabla\frac{1}{r}\right)=\frac{3(a\cdot r)(b\cdot r)}{r^5}-\frac{a\cdot b}{r^3}$ **(14)**

7. (a) Solve $xy_1-y=(x-1)(y_2-x+1)$. **(15)**

7. (b) A particle is moving with central acceleration $\mu(r^5-c^4r)$ being projected from an apse at a distance c with velocity $c^3\sqrt{(2\mu/3)}$, show that its path is the curve $x^4+y^4=c^4$. **(20)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

7. (c) Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\mathbf{F} = 4xz\,\mathbf{i} + xyz^2\,\mathbf{j} + 3z\,\mathbf{k}$. **(15)**

8. (a) Apply the method of variation of parameters to solve $x^2y_2 + 3xy_1 + y = 1/(1-x)^2$. **(12)**

8. (b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3} + 2\sqrt{3}\right]$. where a is the natural length of the string. **(14)**

8. (c) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1, 1, 2)$ has a maximum magnitude 4 in the direction parallel to y-axis. **(07)**

8. (d) Verify Stoke's theorem for the vector $A = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$, where S is the surface of the paaboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. **(17)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN
AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

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diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

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AIR-194 in IAS-2015

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K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 08

Time: Three Hours ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) In S_{10}, let $\beta = (13)(17)(265)(289)$, Find an element in S_{10} that commutes with β but is not a power of β. **(10)**

1. (b) Consider the mapping from $M_2(Z)$ into Z given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a$. Prove or disprove that this is a ring homomorphism. **(10)**

1. (c) A function f is [0, 1] by f(0) = 1,

$$f(x) = (-1)^{n-1} \text{ when } \frac{1}{n+1} < x \le \frac{1}{n} \quad (n = 1, 2, 3, ...).$$

Prove that (i) f is integrable on [0, 1], (ii) $\int_0^1 f = \log(4/e)$. **(10)**

1. (d) Prove that the function $f(z) = u + iv$, where

$$f(z) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} + i\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right) & \text{if } (x, y) \ne (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, but not differentiable at the origin **(10)**

1. (e) Solve graphically the following LPP.

Maximise $Z = 3x_1 + 2x_2$

Subject to $x_1 + 2x_2 \le 6$

$2x_1 + x_2 \le 8$

$-x_1 + x_2 \le 1$

$2x_2 \le 2$

$x_1, x_2 \le 0$ **(10)**

2. (a) (i) $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \mathbf{R}, ad \ne 0 \right\}$. Is H a normal subgroup of GL (2, **R**)?

(ii) Construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field ? Is it an integral domain ? **(18)**

2. (b) (i) Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of non-empty closed intervals such that $I_1 \supset I_2 \supset I_3 \supset ...$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

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For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

(ii) Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of bounded open intervals such that $I_1 \supset I_2 \supset I_3 \supset \ldots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$. **(08)**

2. (c) Show that the series

$$1-\frac{e^{-2x}}{2^2-1}+\frac{e^{-4x}}{4^2-1}-\frac{e^{-6x}}{6^2-1}+\ldots\ldots$$ converges uniformly for all $x \geq 0$. **(08)**

2. (d) Use the method of contour integration to prove that

$$\int_{-\pi}^{\pi}\frac{a\cos\theta}{a+\cos\theta}d\theta = 2\pi a\left\{1-\frac{a}{\sqrt{(a^2-1)}}\right\}, \text{ where } a>1.$$ **(16)**

3. (a) Suppose that a finite group is generated by two elements a and b (that is, every element of the group can be expressed as some product of a's and b's). Given that $a^3 = b^2 = e$ and $ba^2 = ab$, construct the Cayley table for the group. **(06)**

3. (b) Let $H = \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1\end{bmatrix} \middle| a, b, c \in Q\right\}$ under matrix multiplication.

(i) Find Z(H).
(ii) Prove that Z(H) is isomorphic to Q under addition.
(iii) Prove that H/Z(H) is isomorphic to $Q \oplus Q$.
(iv) Are your proofs for parts a and b valid when Q is replaced by **R** ? Are they valid when Q is replaced by Z_p, where p is prime ? **(12)**

3. (c) A function f is defined on (–1, 1) by $f(x) = x^{\alpha}\sin\frac{1}{x^{\beta}}, x \neq 0$
$= 0, x = 0.$

Prove that (i) if $0 < \beta < \alpha - 1$, f' is continuous at 0;

(ii) if $0 < \alpha - 1 \leq \beta$, f' is discontinuous at 0. **(14)**

3. (d) Find the optimal solution of the following transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	

(18)

4. (a) Let F be the field of integers modulo 5. Show that the polynomial x^2+2x+3 is irreducible over F. Use this to construct a field containing 25 elements. **(12)**

4. (b) Prove that the function f defined by

$$f(x)=\sin\frac{1}{x} \qquad \forall x>0$$

is continuous but not uniformly continuous on R^+. **(15)**

4. (c) Evaluate the integral $\int_r \frac{z^2}{(z^2+1)(z-1)^2}dz$, where r is the circle $|z|=2$. **(10)**

4. (d) Show by solving the following LPP by simplex method that the problem has an unbounded solution.

Maximise $Z=107x_1+x_2+2x_3$

subject to $14x_1+x_2-6x_3+3x_4=7$

$16x_1+x_2-6x_3\le 5$

$3x_1-x_2-x_3\le 0$

$x_1,x_2,x_3,x_4\ge 0$ **(13)**

SECTION – B

5. (a) Solve $(y+z)p+(z+x)q=x+y$. **(10)**

5. (b) Solve $(D+D'-1)(D+D'-3)(D+D')z=e^{x+y}\sin(2x+y)$ **(10)**

5. (c) The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
Velocity in km/hr.	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car. **(10)**

5. (d) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved. **(10)**

5. (e) Use Hamilton's equations to find the equations of motion of a projectile in space. **(10)**

6. (a) Form partial differential equation by eliminating arbitrary functions f and g from $z=f(x^2-y)+g(x^2+y)$. **(08)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

6. (b) Find a surface satisfying $r - 2s + t = 6$ and touching the hyperbolic paraboloid $z = xy$ along its section by the plane $y = x$. **(12)**

6. (c) Reduce $y^2(\partial^2 z/\partial y^2) + \partial^2 z/\partial x^2 = 0$ to canonical form **(12)**

6. (d) A tightly stretched elastic string of length l, with fixed end points $x = 0$ and $x = l$ is initially in the position given by $y = C\sin^3(\pi x/l)$, C being constant. It is released from the position of rest. Find the displacement $y(x, t)$. **(18)**

7. (a) Solve the following system of linear equations correct to two decimal places by Gauss-Seidel method.

$10x + 2y + z = 9$
$2x + 20y - 2z = -44$
$-2x + 3y + 10z = 22$ **(15)**

7. (b) Given $\frac{dy}{dx} = 1 + y^2$, where $y = 0$ when $x = 0$, find $y(0.2)$ $y(0.4)$ and $y(0.6)$ **(14)**

7. (c) Convert :
(i) 46655 given to be in the decimal system into one in base 6.
(ii) $(11110.01)_2$ into a number in the decimal system. **(06)**

7. (d) Draw a flow chart for Lagrange's interpolation formula **(15)**

8. (a) Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$. **(16)**

8. (b) If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of the liquid. **(16)**

8. (c) An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $(2/5\mu)^{1/2} c^{5/4}$. **(18)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN
AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

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diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/16/10/16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 09

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SECTION – A

1. **(a)** (i) Show that the diagonal elements of the square of an anti-Hermitian matrix are either zero or negative.
(ii) Prove that the eigen values of a hermitian matrix are always real.
(iii) Use the above result to show that det $(H - 3i\,I)$ can not be zero, if H is a hermitian matrix and I is the unit matrix. **(10)**

1. **(b)** (i) If the vectors (0, 1, a), (1, a, 1), (a, 1, 0) in $\mathbf{R}^3$ (**R**) are linearly dependent, find the value of a.
(ii) Show that the vector (1 + i, 2i), (1, 1 + i) are linearly dependent in $\mathbf{C}^2$ (**C**) and linearly independent in $\mathbf{C}^2$ (**R**). **(10)**

1. **(c)** Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{(2)}} \log\left(\sqrt{2} + 1\right)$ **(10)**

1. **(d)** Evaluate the intrgral $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y}\,dx\,dy$, by changing the order of integration. **(10)**

1. **(e)** Spheres are described to contain the circle z = 0. $x^2 + y^2 = a^2$. Prove that the locus of the extremities of their diameters which are parallel to the x-axis is the rectangular hyperbola $x^2 - z^2 = a^2$, y = 0. **(10)**

2. **(a)** If $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, obtain $\mathbf{A}^2$. Find scalars a and b such that $I + a\mathbf{A} + b\mathbf{A}^2 = \mathbf{O}$, where **I** is the unit matrix and **O** is the null matrix both of order two. **(08)**

2. **(b)** Let T be the linear operator on $\mathbf{R}^3$ which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}.$$ Find the minimal polynomial for T. **(10)**

2. **(c)** (i) Find $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$.
(ii) If $v = At^{-1/2}\ e^{-x^2/4a^2t}$, prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$. **(16)**

2. **(d)** The section of a cone with vertex at P and guiding curve $(x^2/a^2) + (y^2/b^2) = 1$, z = 0 by the plane x = 0 is a rectangular hyperbola. Show that the locus of P is $(x^2/a^2) + \{(y^2 + z^2)/b^2\} = 1$. **(16)**

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

3. (a) (i) Evaluate A^{50} for the matrix $A = \begin{bmatrix} 4/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 \end{bmatrix}$

(ii) Prove that it is impossible to find a matrix P such that

$$P^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}P = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \text{ for any } \alpha, \beta \in \mathbf{R}.$$ **(18)**

3. (b) (i) By using the transformation $x = u(1+v)$, $y = v(1+u)$, prove that

$$\int_0^2\int_0^x \left[(x+y+1)^2 - 4xy\right]^{-1/2} dx\,dy = 2\log 2 - \frac{1}{2}.$$

(ii) Show that the function f defined by setting

$f(x,y) = \dfrac{x^3y}{x^6+y^2}$, when $(x,y) \neq (0,0)$ and $f(0, 0) = 0$, is not continuous a the origin. **(16)**

3. (c) A variable generator meets two generators of the same system through the extremities B and B′ of the minor axis of the principal elliptic section of the hyperboloid in P and P′, prove that $BP \cdot B'P' = b^2 + c^2$. **(16)**

4. (a) Let T be the linear transformation from $\mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

If $\beta = \{(1,0,-1),(1,1,1),(1,0,0)\}$ $\beta' = \{(0,1),(1,0)\}$, be ordered bases of $\mathbf{R}^3$, $\mathbf{R}^2$, respectively, then find the matrix of T relative to β, β'. Also find rank (T) and nullity (T). **(15)**

4. (b) Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values. **(15)**

4. (c) Prove that the lines $\dfrac{x-a+d}{\alpha-\delta} = \dfrac{y-a}{\alpha} = \dfrac{z-a-d}{\alpha+\delta}$ and

$\dfrac{x-b+c}{\beta-\gamma} = \dfrac{y-b}{\beta} = \dfrac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation to the plane in which they lie. **(12)**

4. (d) Show that the plane $8x - 6y - z = 5$ touches the paraboloid $(x^2/2) - (y^2/3) = z$, and find the point of contact. **(08)**

SECTION – B

5. (a) Solve $(x^3 D^3 + 2xD - 2)y = x^2 \ \ell n \ x + 3x.$ **(10)**

5. (b) Find the orthogonal trajectories of cardioids $r = a\,(1 - \cos\theta)$, a being parameter. **(10)**

5. (c) A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight W/2, the weight of the rod being W. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is $W\ell/c$. **(10)**

5. (d) A particle whose mass is m is acted upon by a force $m\mu\left[x + \dfrac{a^4}{x^3}\right]$ towards origin, if it starts from rest at a distance a show that it will arrive at origin in time $\pi/\left(4\sqrt{\mu}\right)$. **(10)**

5. (e) Using Green's theorem, evalualte $\int_C \left(x^2 y\,dx + x^2 dy\right)$ where C is boundary described counter-clock wise of the triangle with vertices (0, 0), (1, 0), (1, 1). **(10)**

6. (a) Solve the equation $xy'' - 2(x+1)y' + (x+2)y = (x-2)e^x$, $(x > 0)$ by changing into normal form. **(16)**

6. (b) Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left\{\frac{3}{\log\left(2+\sqrt{3}\right)} + \frac{4\pi}{3}\right\}$$ **(18)**

6. (c) (i) Show that $r^n \mathbf{r}$ is an irrotational vector for any value of n, but is solenoidal only if $n = -3$ (**r** is position vector of a point).

(ii) Find the value of a, b and c such that

$$F = (3x - 4y + az)\hat{i} + (cx + 5y - 2z)\hat{j} + (x - by + 7z)\hat{k}$$

is irrotational. **(16)**

7. (a) By using Laplace transform method, solve $(D^2 + m^2)\,x = a\cos nt,\ t > 0$ if $x = Dx = 0$ when $t = 0$ **(16)**

7. (b) A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5} - 1$, the equilibrium is stable, whatever be the weight of the particle. **(18)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

7. (c) (i) Find the value of a if $A = a\hat{i} + \hat{j} + \sqrt{5}\hat{k}$ subtends and angle of 60° with $4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}$.

(ii) Find the directional derivative of the scalar function $\phi = 4e^{(2x-y+z)}$ at the point (1, 1, –1) in a direction towards the point. (–3, 5, 6). **(16)**

8. (a) Reduce the equation $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$ to clairaut's form. Hence show that the equation represents a family of conics touching the four sides of a square. **(12)**

8. (b) Solve $\left(2 + 2x^2y^{1/2}\right)y\,dx + \left(x^2y^{1/2} + 2\right)x\,dy = 0$. **(07)**

8. (c) A particle moves with a central acceleration $\mu(r + a^4/r^3)$ being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$. Prove that it describes the curve $r^2\left(2 + \cos\sqrt{3}\theta\right) = 3a^2$. **(15)**

8. (d) Verify Stoke's theorem for the vector field

$A = (3x - 2y)\hat{i} + x^2z\hat{j} + y^2(z+1)\hat{k}$

for a plane rectangular area with vertices at (0, 0), (1, 0) (1, 2), (0, 2) in the $x - y$ plane. **(16)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN
AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-II: IAS (M)/16/10/16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 10

Time: Three Hours | *Maximum Marks: 250*

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) Let H be a subgroup of a group G. If $x^2 \in H$ for all $x \in G$ then prove that H is a normal subgroup of G and G/H is commutative. **(10)**

1. (b) Prove that order of a finite field F is p^n, for some prime p and some positive integer n. **(10)**

1. (c) Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$. **(10)**

1. (d) Locate and name the singularities in the finite z-plane of $\dfrac{ln(z-2)}{(z^2+2z+2)^4}$. **(10)**

1. (e) Does the following L.P.P. has a feasible solution ?

Maximize $Z = x_1 + x_2$,

Subject to $x_1 - x_2 \geq 0, 3x_1 - x_2 \leq -3$

$x_1, x_2 \geq 0$

Show with the help of a graph. **(10)**

2. (a) Find all homomorphisms from $(Z_8, +)$ into $(Z_6, +)$. **(12)**

2. (b) Factorize $x^2 + x + 5$ in F[x], where F is the field of integers mod 11. **(10)**

2. (c) Test for convergence the series

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \ldots. \quad \textbf{(12)}$$

2. (d) Apply calculus of residues to prove that

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right]; a > b > 0, \quad \textbf{(16)}$$

3. (a) Shows that $\mathbf{Z}\left[\sqrt{2}\right] = \left\{m + n\sqrt{2} : m, n \in \mathbf{Z}\right\}$ is a Euclidean domain. **(15)**

3. (b) A function f is defined on [0, 1] by f(0) = 0,

$$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} (n = 0, 1, 2, ...)$$

Prove that (i) f is integrable on [0,1], (ii) $\int_0^1 f = \dfrac{2}{3}$. **(15)**

3. (c) Solve the L.P. Problem

Max. $Z = 6x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

OUR TOPPERS MARKS LIST

- For your final selection, optional subject marks are crucial.
- Choose Optional Subject based on Your Graduation Studies & Score Highest Marks.
- Now Mathematics has become one of the most Cherished Optional Paper among Science Graduates, especially Students with Mathematics background including B.Tech.
- In the new pattern of exam, the average marks of successful candidates in Maths is more than 274 out of 500.
- Mathematics (Opt.) has proven to be the Most Reliable and High Scoring Subject in IAS/IFoS.
- IMS has been successfully providing consistent results since its inception.

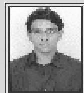
Marks are before you and you should analyze yourself!

Kumbhejkar Yogesh Vijay
Rank-8
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	146
General Studies-I (Paper-II)	250	101
General Studies-II (Paper-III)	250	086
General Studies-III (Paper-IV)	250	114
General Studies-IV (Paper-V)	250	100
Optional-I (Mathematics)-(Paper-VI)	126/250	298/500
Optional-II (Mathematics)-(Paper-VII)	172/250	
Written Total	1750	845
Personality Test	275	138
Final Total	2025	983

Ashish Sangwan
Rank-12
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	113
General Studies-I (Paper-II)	250	100
General Studies-II (Paper-III)	250	077
General Studies-III (Paper-IV)	250	098
General Studies-IV (Paper-V)	250	112
Optional-I (Mathematics)-(Paper-VI)	124/250	284/500
Optional-II (Mathematics)-(Paper-VII)	160/250	
Written Total	1750	784
Personality Test	275	195
Final Total	2025	979



Sidharth Jain
Rank-13
IAS-2015

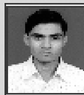
SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	142
General Studies-I (Paper-II)	250	103
General Studies-II (Paper-III)	250	082
General Studies-III (Paper-IV)	250	097
General Studies-IV (Paper-V)	250	099
Optional-I (Mathematics)-(Paper-VI)	114/250	268/500
Optional-II (Mathematics)-(Paper-VII)	154/250	
Written Total	1750	791
Personality Test	275	187
Final Total	2025	978

Pratap Singh
Rank-15
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	133
General Studies-I (Paper-II)	250	107
General Studies-II (Paper-III)	250	082
General Studies-III (Paper-IV)	250	106
General Studies-IV (Paper-V)	250	081
Optional-I (Mathematics)-(Paper-VI)	154/250	283/500
Optional-II (Mathematics)-(Paper-VII)	129/250	
Written Total	1750	792
Personality Test	275	184
Final Total	2025	976

Valluru Kranti
Rank-65
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	128
General Studies-I (Paper-II)	250	096
General Studies-II (Paper-III)	250	062
General Studies-III (Paper-IV)	250	082
General Studies-IV (Paper-V)	250	086
Optional-I (Mathematics)-(Paper-VI)	154/250	326/500
Optional-II (Mathematics)-(Paper-VII)	172/250	
Written Total	1750	780
Personality Test	275	160
Final Total	2025	940



Ketan Bansal
Rank-155
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	114
General Studies-I (Paper-II)	250	089
General Studies-II (Paper-III)	250	078
General Studies-III (Paper-IV)	250	090
General Studies-IV (Paper-V)	250	079
Optional-I (Mathematics)-(Paper-VI)	152/250	308/500
Optional-II (Mathematics)-(Paper-VII)	156/250	
Written Total	1750	758
Personality Test	275	160
Final Total	2025	918

Varun Gunnapalli
Rank-183
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	132
General Studies-I (Paper-II)	250	089
General Studies-II (Paper-III)	250	073
General Studies-III (Paper-IV)	250	068
General Studies-IV (Paper-V)	250	091
Optional-I (Mathematics)-(Paper-VI)	133/250	274/500
Optional-II (Mathematics)-(Paper-VII)	141/250	
Written Total	1750	727
Personality Test	275	184
Final Total	2025	911

Padmanabh Baruah
Rank-194
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	120
General Studies-I (Paper-II)	250	084
General Studies-II (Paper-III)	250	065
General Studies-III (Paper-IV)	250	082
General Studies-IV (Paper-V)	250	093
Optional-I (Mathematics)-(Paper-VI)	141/250	284/500
Optional-II (Mathematics)-(Paper-VII)	143/250	
Written Total	1750	728
Personality Test	275	182
Final Total	2025	910

Akhil Goel
Rank-251
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	110
General Studies-I (Paper-II)	250	097
General Studies-II (Paper-III)	250	065
General Studies-III (Paper-IV)	250	096
General Studies-IV (Paper-V)	250	087
Optional-I (Mathematics)-(Paper-VI)	142/250	275/500
Optional-II (Mathematics)-(Paper-VII)	133/250	
Written Total	1750	730
Personality Test	275	173
Final Total	2025	903

Paul Kumal
Rank-335
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	113
General Studies-I (Paper-II)	250	095
General Studies-II (Paper-III)	250	069
General Studies-III (Paper-IV)	250	092
General Studies-IV (Paper-V)	250	093
Optional-I (Mathematics)-(Paper-VI)	142/250	282/500
Optional-II (Mathematics)-(Paper-VII)	140/250	
Written Total	1750	744
Personality Test	275	151
Final Total	2025	895

Akshay Godara
Rank-605
IAS-2015

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	111
General Studies-I (Paper-II)	250	087
General Studies-II (Paper-III)	250	062
General Studies-III (Paper-IV)	250	087
General Studies-IV (Paper-V)	250	074
Optional-I (Mathematics)-(Paper-VI)	145/250	299/500
Optional-II (Mathematics)-(Paper-VII)	154/250	
Written Total	1750	720
Personality Test	275	154
Final Total	2025	874

Nitish K
Rank-8
IAS-2014

SUBJECT	Max. Marks	Marks Obtained
Essay (Paper-I)	250	142
General Studies-I (Paper-II)	250	073
General Studies-II (Paper-III)	250	076
General Studies-III (Paper-IV)	250	063
General Studies-IV (Paper-V)	250	100
Optional-I (Mathematics)-(Paper-VI)	173/250	346/500
Optional-II (Mathematics)-(Paper-VII)	173/250	
Written Total	1750	800
Personality Test	275	206
Final Total	2025	1006

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Is your answer unique ? If not, give 3 different solutions. **(20)**

4. (a) (i) If $\beta = (1\,2\,3)(1\,4\,5)$, write β^{99} in cycle notation.

(ii) Let $\beta = (1\,3\,5\,7\,9\,8\,6)(2\,4\,10)$ in S_{10}. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$? **(10)**

4. (b) Show that sequence $< f_n >$ where
$f_n(x) = nx(1-x)^n$
does not converge uniformly on [0, 1]. **(15)**

4. (c) The real part of a complex analytic function is $u = x^6 - 3xy^4$. What is its imaginary part ? Express the complex function as a function of $z = x + iy$. **(10)**

4. (d) Solve the assignment problem represented by the following matrix.

	I	II	III	IV	V	VI
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

(15)

SECTION – B

5. (a) Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$. **(10)**

5. (b) Solve $\left(D^2 + DD' - 6D'^2\right)z = x^2 \sin(x + y)$ **(10)**

5. (c) Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four places of decimal. **(10)**

5. (d) Draw a switching circuit that realizes the following switch function. If possible, draw a simpler switching circuit

x	y	z	f(x,y,z)
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

(10)

5. **(e)** Given $u = -\omega y,\ v = \omega x,\ w = 0$; show that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocoty potential does not exist. **(10)**

6. **(a)** Reduce the equation $\partial^2 z/\partial x^2 + 2\left(\partial^2 z/\partial x\partial y\right) + \partial^2 z/\partial y^2 = 0$ to canonical form and hence solve it. **(15)**

6. **(b)** Find a surface passing through the two lines $z = x = 0,\ z - 1 = x - y = 0$ satisfying $r - 4s + 4t = 0$. **(10)**

6. **(c)** Using Newton's forward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (Rs.)	:	Below 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of persons (in thousands)	:	250	120	100	70	50

(13)

6. **(d)** The velocity v os a particle at distance s from a point on its path is given by the table :

s ft	:	0	10	20	30	40	50	60
v ft / sec	:	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft by using Simpson's 1/3 rule. compare the result with Simpson's 3/8 rule. **(12)**

7. **(a)** Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi\left(x + y + z, x^2 + y^2 - z^2\right) = 0$. What is the order of this partial differential equation ? **(07)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

7. (b) A square plate is bounded by the lines $x = 0, y = 0, x = 10$ and $y = 10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 10) = x(10 - x)$ while the other three faces are kept at 0°C. Find the steady state temperature in the plate. **(20)**

7. (c) Solve the following equations by Gauss Seidel Method.
$10x + y + 2z = 13$; $3x + 10y + z = 14$; $2x + 3y + 10z = 15$. **(10)**

7. (d) Draw a flowchart for Regula Falsi Method. **(13)**

8. (a) Write Hamitlton's equations in polar coordinates for a particle of mass m moving in three dimensions in a force field of potential V. **(18)**

8. (b) A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time $\sqrt{\left\{\frac{2M'a}{(M+M')g\sin\alpha}\right\}}$, where a is the length of the plane. **(16)**

8. (c) A source of fluid situated in space of two dimensions is of such strength that $2\pi\rho\mu$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is

$$2\pi\rho\mu^2 a^2 / r(r^2 - a^2),$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure ? **(16)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN

AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/23/10/16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 11

Time: Three Hours ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)
INSTITUTE FOR IAS/IFOS EXAMINATIONS

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Ph: 09999197625, 011-45629987

SECTION – A

1. (a) Let W be the subspace of $\mathbf{R}^5$ spanned by
$u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$,
$u_4 = (1, 4, 5, 1, 8)$, $u_5 = (2, 7, 3, 3, 9)$.
Find a subset of the vectors which form a basis of W. **(10)**

1. (b) (i) If $\mathbf{A}$ is a real skew-symmetric matrix and $\mathbf{A}^2 + I = 0$, then show that $\mathbf{A}$ is orthogonal.
(ii) If $\mathbf{H}$ is a Hermitian matrix, what kind of matrix is e^{iH}? **(10)**

1. (c) Let $f(x, y) = \begin{cases} \dfrac{\sin(x-y)}{|x|+|y|}, & |x|+|y| \neq 0 \\ 0, & (x, y) = (0, 0) \end{cases}$

Is f continuous at the origin ? Justify **(10)**

1. (d) Given $w = (x, y)$ with $x = u + v$, $y = u - v$, show that

$$\frac{\partial^2 w}{\partial u\, \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$ **(10)**

1. (e) Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z + 2 = 0$ in a circle of radius untiy and find the equations of the sphere which has this circle for one of its great circles. **(10)**

2. (a) (i) Let T be a linear operator on C^3 defined by
$T(1, 0, 0) = (1, 0, i)$, $T(0, 1, 0) = (0, 1, 1)$, $T(0, 0, 1) = (i, 1, 0)$.
Is T invertible ? Justify your answer.
(ii) Let $T ; \mathbf{R}^3 \to \mathbf{R}^3$ be a linear operator the matrix A of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Find a basis for the range of T and a bsis for the null space of T. **(18)**

2. (b) A Space probe in the shape of the ellipsoid
$4x^2 + y^2 + 4z^2 = 16$
enters the earth's atmoshere and its surface begins to heat. After one your, the temperature at the point (x, y, z) on the probe's surface is
$T(x, y, z) = 8x^2 + 4yz - 16z + 600$.
Find the hottest point on the probe's surface. **(16)**

2. (c) (i) If the plane $2x - y + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines, find the value of c.

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

(ii) Find the angle between the lines given by $x + y + z = 0$ and

$$\frac{yz}{q-r}+\frac{zx}{r-p}+\frac{xy}{p-q}=0.$$ **(16)**

3. (a) Let T be a linear operator on $\mathbf{R}^3$ which is represented in the standard ordered basis by the matrix

$$A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbf{R}^3$ each vector of which is characteristic vector of T. **(20)**

3. (b) Find the volume in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$. **(15)**

3. (c) Two perpendicular tangent planes to the paraboloid $(x^2/a) + (y^2/b) = 2z$ intersect in a line lying on the plane $x = 0$. Prove that the line touches the parabola $x = 0, y^2 = (a + b)(2z + a)$. **(15)**

4. (a) Let U span $\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$
W = span $\{(1, 2, 2, -2) (2, 3, 2, -3), (1, 3, 4, -3)\}$ be the subspaces of $\mathbf{R}^4$
Find a basis and the dimension of U + W, U, W and $U \cap W$. **(18)**

4. (b) (i) Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(ii) Verify Euler's theorem for $z = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ **(16)**

4. (c) Show that the perpendicular from the origin on hte generator of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ lie on the curve

$$\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(c^2+a^2)^2}{y^2}=\frac{c^2(a^2-b^2)^2}{z^2}$$ **(16)**

SECTION – B

5. (a) Solve $16(x+1)^4 y_4 + 96(x+1)^3 y_3 + 104(x+1)^2 y_2 + 8(x+1) y_1 + y = x^2 + 4x + 3$. **(10)**

5. (b) Find the equation of the system of orthogonal trajectories of the parabolas $r = 2a/(1 + \cos\theta)$, where a is the parameter. **(10)**

5. (c) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable. **(10)**

5. (d) A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. Show that the greatest height attained by the particle is $\frac{1}{2k}\log\frac{g+ku^2}{g}$. **(10)**

5. (e) Apply Greens theorum to evaluate the line integral $\oint_C (4x-2y)dx+(2x-4y)dy$. Where C is the circle $(x-2)^2+(y-2)^2=4$ **(10)**

6. (a) Solve $(x^3+3xy^2)\,dx+(y^3+3x^2y)\,dy=0$. **(08)**

6. (b) Transform the equation
$(2x^2+1)p^2+(x^2+2xy+y^2+2)\,p+2y^2+1=0$.
to Clairaut's form by the substitution $x+y=u$, $xy-1=v$ and interpret it. Find its singular solution also. **(15)**

6. (c) Solve $(1-x^2)y_2+xy_1-y=x\,(1-x^2)^{3/2}$. **(13)**

6. (d) By using Laplace transform method solve
$$\frac{d^2y}{dt^2}+y=t\cos 2t \text{ if } y=0, \frac{dy}{dt}=0 \text{ when } t=0$$ **(14)**

7. (a) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is
$$\mu\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$$
where μ is the cofficient of friction. **(16)**

7. (b) A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$, prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point. Also find the height of the highest point of the parabola subsequently described. **(16)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

7. (c) A prticle moves under a force $m\mu\{3au^4 - 2(a^2 - b^2)u^5\}$, $a > b$ and is projected from an apse at a distance $(a + b)$ with velocity $\sqrt{\mu}/(a+b)$. Show that the equation of its path is $r = a + b\cos\theta$. **(18)**

8. (a) Find the most general differentiable function $f(r)$ so that $f(r)\,\mathbf{r}$ is solenoidal.

(ii) Find $A\times(\nabla\times\mathbf{B})$ and $(\mathbf{A}\times\nabla)\times\mathbf{B}$ at the point $(1, -1, 2)$, if $\mathbf{A} = xz^2\,\mathbf{i} + 2y\,\mathbf{j} - 3xz\,\mathbf{k}$ and $\mathbf{B} = 3xz\,\mathbf{i} + 2yz\,\mathbf{j} - z^2\mathbf{k}$. **(15)**

8. (b) If a is a constant vector, prove that

$$\text{curl}\frac{\mathbf{a}\times\mathbf{r}}{r^3} = -\frac{\mathbf{a}}{r^3} + \frac{3\mathbf{r}}{r^5}(\mathbf{a}\cdot\mathbf{r}). \quad \textbf{(10)}$$

8. (c) If $F = (2y + 3)\,\mathbf{i} + xz\,\mathbf{j} + (yz - x)\,\mathbf{k}$, evaluate $\int_C F\cdot dr$ where C is the path consisting of the straight lines from (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (2, 1, 1). **(10)**

8 (d) Evaluate $\iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k})\cdot\mathbf{n}\,dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane. **(15)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN

AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

TEST SERIES (MAIN)-2016

Test Code: PAPER-II: IAS (M)/30/10/16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 12

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INSTRUCTIONS

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Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) Let a and b belong to a group. $|a| = 12, |b| = 22$, and $\langle a\rangle \cap \langle b\rangle \neq \{e\}$, prove that $a^6 = b^{11}$. **(10)**

1. (b) Show that the set

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a,\ b \text{ are integers} \right\}$$

is a left ideal in the ring M_2 of 2×2 matrices over integers. Further show that S is not a right ideal in M_2. **(10)**

1. (c) Show that the function f defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$$

is integrable on [o, m] m being a positive integer. **(10)**

1. (d) If $f(z) = u + iv$ is an analytic function z, and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$: find f(z) subject to the condition $f\left(\dfrac{\pi}{2}\right) = 0$. **(10)**

1. (e) Find the maximum and minimum value of $z = 5x_1 + 3x_2$, subject to the constraints : $x_1 + x_2 \le 6,\ 2x_1 + 3x_2 \ge 3,\ 0 \le x_1 \ge 3, 0 \le x_2 \ge 3$. **(10)**

2. (a) (i) How many elements of order 5 are there in A_6 ?

(ii) Find group elements α and β in S_5 such that $|\alpha| = 3, |\beta| = 3$, and $|\alpha\beta| = 5$. **(18)**

2. (b) Prove that $\dfrac{x}{1+x} < \log(1+x) < x$ all $x > 0$. Deduce that

$$\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \ldots.. + \frac{1}{2n} < \log 2, \text{ n being a positive integer.}$$

(16)

2. (c) Use a suitable contour integration to show that

$$\int_{-\infty}^{+\infty} \frac{\cos x + x\sin x}{1+x^2}\,dx = \frac{2\pi}{e}.$$ **(16)**

3. (a) (i) If R is a ring with unity 1 and f is a homomorphism of R into an integral domain R' with Ker $f \neq R$, prove that f(1) is the unity of R'

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For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

(ii) Prove that the characteristic of any integral domain is either zero or a prime number. **(20)**

3. (b) Show that the sequence of functions $<f_n>$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on $[0, 1]$. **(15)**

3. (c) Solve the L.P.P. by simplex method

Max. $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$3x_1 + 8x_2 \le 12$

and $x_1, x_2 \ge 0$ **(15)**

4. (a) Show that the centre of a division ring is a field. **(13)**

4. (b) Show that $\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$ exists if and only if $n < m + 1$. **(10)**

4. (c) Find the Taylor's and Laurent's series which represent the function

$$\frac{z^2 - 1}{(z+2)(z+3)}$$

(i) $|z| < 2$ (ii) when $2 < |z| < 3$ (iii) when $|z| > 3$. **(12)**

4. (d) A company has 4 warehouses and 6 stores : the cost of shipping one unit from warehouse i to store j is C_{ij}.

If $C = (c_{ij}) = \begin{pmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{pmatrix}$ and the requirements of the six stores

are 4, 4, 6, 2, 4, 2 and quantities at the warehouse are 5, 6, 2, 9. Find the minimum cost solution. **(15)**

SECTION – B

5. (a) Solve $x(x^2 + 2y^2)\, p - y\,(3x^2 + y^2)\, q = 2z(y^2 - x^2)$. **(10)**

5. (b) Solve the following partial differential equation :

$$\left(D^3 - 4D^2D' + 5DD'^2 - 2D'^3\right) z = e^{y+2x} + (y+x)^{1/2}.$$ **(10)**

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5. (c) The velocity of a particle at distance S from a point on its path is given by the following table:

S (meters)	V (m/sec)
0	47
10	58
20	64
30	65
40	61
50	52
60	38

Estimate the time taken to travel the first 60 meters using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule. **(10)**

5. (d) A majority function is a digital circuit whose output is '1' iff the majority of the inputs are 1. The output is '0' otherwise. Obtain the truth table of a three-input majority function and show that the circuit of a majority function can be obtained with 4 NAND gates. **(10)**

5. (e) Write the Hamiltonian function and equation of motion a compound pendulum. **(10)**

6. (a) Find the integral surface of $x^2p + y^2q + z^2 = 0,\ p = \partial z/\partial x\ , q = \partial z/\partial y$ which passes through the hyperbola $xy = x + y,\ z = 1$. **(10)**

6. (b) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi\left(x^2+y^2+z^2, z^2-2xy\right)=0$. **(07)**

6. (c) Reduce $\partial^2 z/\partial x^2 = (1+y)^2\left(\partial^2 z/\partial y^2\right)$ to canonical from **(15)**

6. (d) A string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\partial y/\partial t\right)_{t=0} = v_0 \sin^3(\pi x/l)$ where $0 < x < \ell$. Find the displacement function. **(18)**

7. (a) The equation $x^2+ax+b=0$ has two real roots α and β show that the iteration method $x_{k+1} = -\dfrac{(ax_k+b)}{x_k}$ is convergent near $x=\alpha$ if $|\alpha|>|\beta|$ and that $x_{k+1}=\dfrac{-b}{x_k+a}$ is convergent near $x=\alpha$ if $|\alpha|<|\beta|$. Show also that iteration method $x_{k+1} = -\dfrac{(x_k{}^2+b)}{a}$ is convergent near $x=\alpha$ if $2|\alpha|<|\alpha+\beta|$ **(12)**

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

7. (b) Compute $y(10)$ using Lagrange's interpolation formula from the following data :

x	3	7	11	17
y	10	15	17	20

(10)

7. (c) Using fourth order classical Runge- Kutta method for the initial value problem. $\frac{du}{dt}=-2tu^2, u(0)=1$ with h=0.2 on the interval [0, 1], calculate u (0.4) correct to six places of decimal. **(14)**

7. (d) Draw a flow chart for Gauss seidel method. **(14)**

8. (a) A particle of mass m moves in a conservative forces field. Find (i) the Lagrangian function and (ii) the equation of motion in cylindrical coordinates (ρ, ϕ, z). **(17)**

8. (b) Show that velocity potential

$$=\frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

gives a possible motion. Determine the form of stream lines and the curves of equal speed. **(17)**

8. (c) Show that the velocity field

$$u(x,y)=\frac{B(x^2-y^2)}{(x^2+y^2)^2},\quad v(x,y)=\frac{2Bxy}{(x^2+y^2)^2}, w=0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with this velocity field. **(16)**

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN

AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

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TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/06/11/16

K. VENKANNA

The person with 16 years of Teaching Experience

MATHEMATICS

FULL LENGTH TEST

Test- 13

Time: Three Hours | ***Maximum Marks: 250***

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

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SECTION – A

1. (a) Let $V=\mathbb{R}^3$ and $\alpha_1=(1,1,2)$, $\alpha_2=(0,1,3)$, $\alpha_3=(2,4,5)$ and $\alpha_4=(-1,0,-1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1,\alpha_2\}$ and $\{\alpha_3,\alpha_4\}$. **(10)**

1. (b) Obtain eigen values and eigen vectors of the differential operator $D:P_2\to P_2$

$D\left(a_0+a_1x+a_2x^2\right)=a_1+2a_2x$, for $a_0, a_1, a_2\in\mathbb{R}$. **(10)**

1. (c) Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r. **(10)**

1. (d) If $V=\log_e \sin\left\{\dfrac{\pi\left(2x^2+y^2+xz\right)^{1/2}}{2\left(x^2+xy+2yz+z^2\right)^{1/3}}\right\}$, find the value of

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} \text{ when } x=0, y=1, z=2.$$ **(10)**

1. (e) A line makes angles $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube; prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$. **(10)**

2. (a) (i) Show that the diagonal entries of a skew symmetric matrix are all zero, but the converse is not true.

(ii) Let $Q'=\{ax^2+bx+c / a\neq 0, a,b,c\in C\}$ **(10)**

2. (b) Show that the three equations

$-2x+y+z=a, x-2y+z=b, x+y-2z=c$

have no solutions unless a + b + c = 0, in which case they have infinitely many solutions. Find these solutions when $a=1, b=1, c=-2$. **(12)**

2. (c) Show that $\displaystyle\iiint\frac{dx\,dy\,dz}{(x+y+z+1)^3}=\frac{1}{2}\left(\log 2-\frac{5}{8}\right)$ throughout the volume bounded by the co-ordinate planes and the plane $x+y+z=1$. **(14)**

2. (d) Show that the equation of the plane containing the line $\dfrac{y}{b}+\dfrac{z}{c}=1$, $x=0$ and parallel to the line $\dfrac{x}{a}-\dfrac{z}{c}=1, y=0$ is $\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}+1=0$ and if 2d is the S.D. show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$. **(14)**

For practising questions which is of utmost importance, I solved all the questions given in the notes (whether solved or unsolved) multiple times in my registers. Besides that, I solved the questions of previous year papers provided by sir, again multiple times. I restricted my preparation upto this point. But if any student faces difficulty in understanding any particular topic or finds notes insufficient for it or wants to practise more, he/she can use any reference book for any particular topic which can easily be found on internet or available in market.

But again a word of caution, try to limit your preparation to the concepts relevant to the syllabus and don't delve into unnecessary theorems or proofs otherwise its a slippery slope to a massive ocean. We tend to skip the proofs of various theorems provided in the syllabus while studying them as they are of not much use. Proofs of theorems are generally not asked in the exams. But still I used to go through each and every proof in a brief manner provided in the notes. The reason being it would give me a better insight of the topic and often helped in me developing solutions of questions.

Test Series:

No optional is complete without writing a test series and it holds true in Maths also. Test Series is as important in your preparation as your notes + books. Firstly, Test Series is the best mode of judging your preparation. You can fairly evaluate your performance with your marks and then focus on the weak topics. Secondly, its a rehearsal of Mains Exam and thus helps you greatly in time management.

Mains exam is nearly a marathon for your hand and thus you get very much trained for facing them. Test Series also provided me another pool of questions to practise. They also helped in developing the ability of answer writing which definitely can't be developed overnight. I attended Test Series of IMS and luckily many questions of Test Series appeared in both IFoS Exam and CSE. I would also request all the candidates to give the test series by coming to classroom if possible and stick to the timelines as it really helps in completion of syllabus.

I hope this writeup clears some of the doubts and gives clarity on maths optional to UPSC IAS aspirants. All the Best

If anyone wants to contact me, please drop me an email - parthjaiswal512@gmail.com. I will be more than happy to help you.

THANK YOU
Parth Jaiswal
AIR-5 in IFoS-2014,
AIR-299 in CSE-2014

3. **(a)** (i) Let A be a 3×3 upper triangular matrix with real entries. If $a_{11}=1, a_{22}=2$ and $a_{33}=3$, determine α,β and γ such that $A^{-1}=\alpha A^2+\beta A+\gamma I$.

(ii) Find a non zero vector common to the space spanned by (1, 2, 3), (3, 2, 1) and the space spanned by (1, 0, 1) and (3, 4, 3). **(16)**

3. **(b)** Find the dimensions of the closed circular can of smallest surface area whose volume is $16\pi\,\text{cm}^3$. **(12)**

3. **(c)** Find $\lim \dfrac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}, (x\to 0)$. **(08)**

3. **(d)** Find the limiting points of coaxial systems defined by the spheres $x^2+y^2+z^2+2x+2y+4z+2=0$ and $x^2+y^2+z^2+x+y+2z+2=0$ **(14)**

4. **(a)** (i) If $P=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$, then find P^{50}.

(ii) Find the dimension of the subspace

$$W=\left\{(x,y,z,w)\in\mathbb{R}^4 \mid x+y+z+w=0, x+y+2z=0, x+3y=0\right\}.$$

(18)

4. **(b)** Examine for continuity at x = a the function f where

$$f(x)=\begin{cases}\dfrac{x^2}{a}-a, & 0<x<a\\ 0\quad, & x=a\\ a-\dfrac{a^3}{x^2}, & a>x\end{cases}$$

Also examine if the function is derivable at a. **(14)**

4. **(c)** Prove that the lines drawn from the origin parallel to the normal $ax^2+by^2+cz^2=1$ at its points of intersection with the plane $\ell x+my+nz=p$ generate the cone $p^2\left(\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}\right)=\left(\dfrac{\ell x}{a}+\dfrac{my}{b}+\dfrac{nz}{c}\right)^2$ **(18)**

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SECTION – B

5. (a) Show that $y_1(x) = x$ and $y_2(x) = |x|$, are linearly independent on the real line, even though the Wronskian cannot be computed. **(10)**

5. (b) Prove that $1/(x+y+1)^4$ is an integrating factor of $(2xy - y^2 - y)dx$ $+(2xy - x^2 - x)\,dy = 0$, and find the solution of this equation. **(10)**

5. (c) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If α, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan\phi = \frac{3}{8} + \tan\theta \cdot$$ **(10)**

5. (d) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where OP = b with velocity v in the direction OP, prove that the time which elapses before it returns to P is

$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$$ **(10)**

5. (e) If $f = \nabla(\vec{a}\cdot\nabla r^{-1})$, show that $div\, f = 0$, and $f = curl\, g$, where $g = -\vec{a}\times\nabla(r^{-1})$. **(10)**

6. (a) Solve $x^3(d^3y/dx^3) + 2x^2(d^2y/dx^2) + 2y = 10(x + 1/x)$. **(10)**

6. (b) Find the solution of the differential equation $y = 2xp - yp^2$ where $p = dy/dx$. Also find the singular solution. **(10)**

6. (c) A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table, show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere. **(15)**

6. (d) (i) In what direction from the point (1, 3, 2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

(ii) Find the most general differentiable function f(r) so that f(r) r is solenoidal. **(15)**

7. (a) Solve $y_2 - 2y_1 + y = xe^x \log x$, $x > 0$ by the method of variation of parameters. **(15)**

7. (b) A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to

Luckily I was able to clear both the examinations in my first attempt. I secured AIR-5 in IFoS-2014 and AIR-299 in CSE-2014. My optional subject was Mathematics. In case of Forest Service Examination, candidate is required to choose 2 Optionals, thus my second optional was Forestry with Mathematics as my first optional. I secured 250/400 (125+125) marks in IFoS Exam and 300/500 (147+153) marks in CSE in Maths. Thus I would give much credit for my success to my correct choice of optional as well as performance in it. I am writing this to share my experience with Maths as an optional subject and would feel happy if I am able to clear some of the doubts as well as apprehensions regarding it which many UPSC aspirants possess.

WHY I CHOSE MATHEMATICS?

I chose Mathematics because of my inherent interest in it from childhood. I have performed well in this in my throughout education and thus was confident enough to handle it well. Another reason for choosing it was, I wanted to have my optional from my background and thus Maths proved to be appropriate choice. Having a science background, I found it much easier to study than any other subject, many of which we have to study for GS prep.

I would like to assert few points regarding it very clearly.

This subject is vast in syllabus and takes more time to study than other optionals.

It also requires consistent practise. But the positive part is - If you are thorough with the subject and have practised it well, you can comfortably attempt complete paper with correct answers and thus gives you a great opportunity to score well in your optional (inspite of the scaling often carried out in it) pushing you above the list.

In this way, this optional gives a bit of security as well as certainty which again comes at a price i.e great amount of hard work. Also IFoS Exam prescribes certain optionals only and Mathematics is one of them. Not all optionals are available for this exam.

So again it gives you the flexibility of giving IFoS Exam.

FROM WHERE TO STUDY?

I attended classroom coaching of IMS, Rajinder Nagar. I restricted my preparation to the handouts provided by Venkanna Sir. Because of the voluminous syllabus, it is necessary to gauge the point where you have to stop. I found that the notes quite comprehensive and provided me a holistic coverage of the syllabus in a highly structured manner. I believe that those notes are sufficient from the theory point of view.

your strong and weak areas and thus helps in adapting preparation to score maximum marks.

I have done self study from various sources. I will share the sources soon.

SOME QUICK TIPS

- Make it a habit to do maths study first thing in the morning as your mind would be most active and fresh at that time.
- While answering, don't write small calculations. Do calculations in the rough area and just write main steps in the answer. This requires a lot of practice.
- It's important to maintain a book of important formulas and theorems.
- Attempt compulsory questions 1 and 5 in about 70 minutes in the beginning.
- In optional questions. If you know all the questions, attempt the tougher questions to get more marks.

Ashish Sangwan

AIR-12 in CSE/IAS-2015

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PARTH JAISWAL

Classroom Student

AIR-5 in IFoS-2014 Examination

AIR-299 in IAS-2014 Examination

MY BACKGROUND

Hello,My name is Parth Jaiswal. I come from Jaipur, Rajasthan. I completed my graduation in Computer Science discipline from IIT Delhi in 2013. Soon afterwards I started preparing for Civil services and Indian Forest Service, aiming for the attempt of year 2014.

the highest point. Show that the reaction between the particle and the wire is zero after a time

$\sqrt{(a/g)}.\log\left(\sqrt{5}+\sqrt{6}\right)$. **(18)**

7. (c) Verify Stoke's theorem for $\text{Å} = (y-z+2)i + (yz+4)j - xzk$, where S is the furface of the cube x = 0, y = 0, z = 0, x = 2, 2, z = 2 above the xy plane. **(17)**

8. (a) By using Laplace transform method solve the differential equation

$\left(D^2 - D - 2\right)y = 20\sin 2t$, subject to initial conditions $y = -1, Dy = 2$ when $t = 0$. **(15)**

8. (b) A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is

$\frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha\cos\theta)\right]$ **(15)**

8. (c) Verify Green's theorem in the plane for $\oint_C \left(2x - y^3\right)dx - xydy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ **(12)**

8. (d) Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field F is given by
$F = (2x - y + 2z)\,\mathbf{i} + (\mathbf{x} + \mathbf{y} - \mathbf{z})\,\mathbf{j} + (\mathbf{3x} - \mathbf{2y} - \mathbf{5z})\,\mathbf{k}$. **(08)**

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PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

KUMBHEJKAR YOGESH VIJAY

Classroom Student

AIR-8 in IAS-2015

AIR-13 in IFoS-2014 Examination &
AIR-143 in IAS-2014 Examination

MY BACKGROUND

I am Yogesh Kumbhejkar. I am an Electrical Engineer from IIT Bombay. I secured AIR 13 in Indian Forest Service Exam (IFoS) 2014 with Mathematics & Physics as the optional subjects. For Civil Service Exam (CSE) also, my optional is Mathematics. In IFoS exam, I scored 231/400 (118 + 113) in maths. In 2013 CSE Mains, my maths score was 250/500 (109 + 141). Hence mathematics has helped me in clearing mains in both CSE and IFoS. I was not selected in the final list of CSE 2013. In my second CSE attempt also I appeared for mains in 2014 with Maths as the optional subject. Now i am awaiting the Mains result. This article is a humble attmept to share my experience of maths optional preparation for CSE/IFoS exam. I would be glad if it helps any UPSC aspirant who is undecided about choosing the optional or those who are already preparing with mathematics as their optional.

WHY MATHEMATICS

It is very important for a UPSC aspirant to have genuine interest in mathematics if he/she wants to choose this optional. Maths used to be my favourite subject in school and in IITB also I had pursued additional courses in mathematics out of interest. Since the syllabus is large & requires considerable practice, it is necessary to have a genuine interest. Apart from my inherent inclination, this optional offers certain advantages which made it an obvious choice. In this optional, the marks you get are almost proportional to your efforts. With proper hard work, a candidate can comfortably attempt all the questions in exam and expect to score around 50% marks even after heavy scaling which can offer the necessary edge in this intense competition. Such candidate generally would not find any question surprising in mains. This kind of certainity is not present in humanities optionals.

THE SYLLABUS

The prescribed syllabus for maths is quite large which makes it necessary to

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

ASHISH SANGWAN
AIR-12(IAS-2015)

Hello, My name is Ashish Sangwan. I have done BTech in computer science from IIT Delhi (2003-2007). After that, I did masters in computer science from Georgia Tech, Atlanta, USA. Then, I worked for 4 years as a research engineer in a couple of startups. I started preparing for civil services exam in January 2013.

I was aiming for CSE 2014 but when the notification came out and they removed one optional, I aimed for CSE 2013 with mathematics optional. I secured AIR 607 in CSE 2013. I got 220/500 in this attempt as my mathematics preparation was average due to lack of time. In CSE 2014, I got 240/500 in maths and couldn't get any rank. Again, there were some loopholes in my preparation which I tried to correct in CSE 2015. In CSE 2015, I got 284/500.

WHY I CHOOSE MATHEMATICS?

I choose Mathematics because of two reasons. First, since childhood I have loved maths. Second, I did my BTech and masters in computer science but computer science is not an optional and the closest optional where I could use my knowledge of computer science was maths.

Maths is a great optional and once you have covered syllabus decently, you can expect basic minimum marks of 220 which are not guaranteed in humanities optionals.

However, syllabus is huge and you require about 1000 hours of study in total (daily 6 hours for 6 months) to complete most of the syllabus. This is enough for getting 220 score given current marking trend.

To score more, you have to consistently do practice. In this regard, joining a test series is must. I did not join test series in my first two attempts and thus was not getting great marks. This time, I joined ims test series and was satisfied with the level of mock tests. Along with practice, test series helps in finding out

My coaching in IMS helped me tremendously because till I had started the coaching, I had absolutely no idea about what to study and how to go about the subject. The benefit of the coaching was that the entire syllabus was covered in a concise way. I did not have to go around searching for books or common questions or any other sort of material. Everything was provided in the material from the coaching centre and my duty was to finish the material and revise them again and again. I depended only on the material provided and did not consult any other book. I kept up with the pace of the classes and thus could finish the syllabus chapter by chapter accordingly as Venkanna sir proceeded.

Once the prelims was over, I started the Test Series with IMS. The test series is very crucial when you are revising and doing the final preparation for the mains. This is because Mathematics is about practicing the same thing again and again - this came with the test series. I gave about 16 tests and in all the tests, I revised the entire syllabus repeatedly.

Therefore, when the Mains Examination came, I did not feel much nervousness as I had already sat for similar tests so many times.

Finally, I would like to say that if you are from a background where you had to deal with mathematics in some way or the other, or you were good at this subject in school or college, you should seriously considering choosing this subject as your optional because if you work hard and are regular with Mathematics, it will pay off handsomely. Mathematics, like any other subject for UPSC CSE can be prepared on your own too; but if you are short in time and would like to finish the subject at the earliest, you can consider taking up a coaching class. This is because, as I mentioned earlier, it gives you all the material and guidance at one place and you do not need to run around searching for the correct book.

I hope some of the things that I said would be of help to those who want to take mathematics.

Padmanabh Baruah

AIR-194 in CSE/IAS-2015

stick to limited sources. I relied on notes provided by Venkanna Sir at IMS for covering the syllabus. Since these notes were very comprehensive, I didn't have to spend time scanning reference books for relevant material. Venkanna Sir's classroom coaching helped me in completing the syllabus in a disciplined manner. Initially I would underline important theorems, formulae, results mentioned in the notes. Then i used to compile them in a notebook and this was useful for revision. So eventually i had a notebook with just the crux of the matter. I would advise all candidates with maths optional to prepare such a summary for all topics. Due to large syllabus, there is a natural tendancy to skip a few chapters. But for the sake of compulsory questions, it is necessary to know at least basics of each chapter. The physics related chapters of statics, dynamics, mechanics are generally left untouched while preparing maths optional. Regarding these chapters, my preparation was such that i would be able to solve the compulsory 10 mark questions. They are quite manageable once you know the basic theory and there is no point in unnecessarily losing marks. The real analysis/calculus & modern algebra chapters are time consuming but candidates can't afford to skip them.

PRACTICE

Just knowing theory is not enough. It needs to be accompanied by consistent problem solving practice. It is best to solve questions that have already been asked in mains. If some problem seems very non-intuitive, it would help if the trick to solve such problem is written in your notebook.

TEST SERIES

Test series is very important for this optional. I had joined IMS test series which helped me in identifying my weak areas. In both CSE and IFoS mains, there were many questions similar to those covered in IMS test series. With enough practice, a candidate can achieve the ability to complete the maths paper in 3 hours. It is important to assess your performance after each test. Necessary steps should be taken to rectify common mistakes that you are commiting in the test series. You should be alert not to repeat the same mistakes again & again. As your performance improves with every test, the actual mains paper will seem just like any other test & you will be able to comfortably complete it. Presentation of your answer matters a lot. Your aim should be to make examiner's life as easy as possible so that he/she will award you maximum marks. Only the final answer doesn't matter. Writing proper steps is also imortant to show the logical flow with which you arrived at the solution. Specifically mention whichever theorem or property you are using in a particular step. Wherever possible, draw neat

diagrams with proper labelling. Such small things will collectively fetch you the extra marks that you are expecting from this optional. The habit of writing such detailed answers will not develope overnight and hence you have to consciously work through the test series in this direction.

DURING MAINS

The mains exam schedule does not provide much gap between General Studies & Maths papers. You will generally have 1 day in between. Your notebook containing important formulae & theorems will be very useful at such times. You will be able to go through this summary of each chapter and it will provide much needed confidence before the actual paper. During the main exam, I would advise completing the compulsory questions 1 & 5 first. Then you can choose 3 out of remaining 6 questions. Easier questions like those from topics like linear programming, numerical analysis, linear algebra etc. should be the priority. Even if you don't know the complete answer to any question, write as many steps as you can since partial marks also matter.

Once you finish paper 1, don't start immediately analyzing your performance. Irrespective of whether you are very happy or deeply unsatisfied about paper 1, try to forget about it and stay calm for paper 2.

INTERVIEW

In the interview, you can expect some questions related to mathematics optional. Generally you won't be asked to solve a problem because that ability has been tested in mains. They would like to see whether you have a genuine curiosity regarding mathematics outside what is mentioned in syllabus. In both my UPSC interviews, I was asked about Ramanujan's work. There were questions on Vedic Mathematics, National Mathematics Day, important Indian Mathematical Institutions, Field medalist Manjula Bhargava etc. Hence while preparing for interview, try to be aware about these non-theorotical aspects of maths as well.

I hope above tips provide some clarity regarding maths optional to UPSC aspirants.

ALL THE BEST!

PREPARATION STRATEGY

for IAS/IFoS MATHEMATICS (Optional) by Successful Candidate of

PADMANABH BARUAH

AIR-194 in IAS-2015

I am Padmanabh Baruah. I graduated in Mechanical Engineering from IIT Guwahati in 2013. I started working in an MNC after my graduation and worked there for about 5 months. After that I went home and stayed for 6 months. It was during this period that I started thinking about what career option I should undertake. After much thinking and deliberation, I came to the conclusion that I would try to get into the Indian Civil Services and it was from here that my journey for Civil Services start.

I came to Delhi in July, 2014 to start my preparation. As I had decided on this career option just before a month or two, I had to start everything from scratch as I had never before prepared for this examination. I took admission into a coaching institute for my GS preparation. I had not decided on my optional as yet. I consulted some seniors regarding which optional I should take and a variety of suggestions came up – geography, psychology, political science etc. But I was not very sure about these subjects as I had never studied them till that time. Then I thought to myself that why not Mathematics. I had been very fond of this subject during my schooling and even in college. But the response I got from those who I consulted was not very positive – everyone kept on saying that it is a very difficult optional subject. Some of the disadvantages they mentioned were – there is no common portion with GS, the syllabus is very vast as compared to humanities, there is no proper guidance etc. Despite this, I had a gut feeling that I should take Mathematics because this is what I had been studying from a long time and if I get some good guidance, I would be able to overcome the difficulties.

It was at this stage that I came to know about IMS mostly through the internet. I enquired in IMS and took up a classroom program in September 2014. Today, when I look back, it seems that it was a very good decision that I took at that time. I cleared the Civil Services Examination 2015 in the first attempt with an AIR 194 only because of my decent marks in Mathematics. My score in GS was quite average, it was only Mathematics which gave me a good rank.

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